On tensor category arising from representation theory of the restricted quantum universal enveloping algebra associated to $\mathfrak{sl}_2$

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§ Introduction

Background

  
  \begin{align*}
  \text{Category of representation of affine Lie algebra } \hat{\mathfrak{g}} & \leftrightarrow \text{representation of } U_q(\mathfrak{g}) \\
  \text{at a root of unity}
  \end{align*}

Main tool: Conformal Field Theory (WZW-model)

- Recently, a “log-version” of the above correspondence is considered.

What is a logarithmic CFT?

- Roughly speaking, a log CFT is a CFT such that “KZ-type equations” have logarithmic singularities.
- But, in mathematical sense, there is no definition. That is, there are only some examples.

As an example of log-CFTs, there is a CFT based on representation of the triplet vertex operator algebra $W(p)$ ($p \in \mathbb{Z}_{\geq 2}$).

Conjecture 1 (Feigin et al.). There is a “log-version” of KL-equivalence. That is, as braided tensor categories,

\begin{align*}
\text{Category of } W(p)\text{-modules} & \leftrightarrow \text{Category of finite dimensional } U_q(\mathfrak{sl}_2)\text{-modules,}
\end{align*}

where $U_q(\mathfrak{sl}_2)$ is the restricted quantum group associated $\mathfrak{sl}_2$ and $q = \exp\left(\frac{\pi \sqrt{-1}}{p}\right)$.

They proved the conjecture for $p = 2$ case.
In 2009, Tsuchiya-Nagatomo proved the following theorem.

**Theorem 2** (Tsuchiya-Nagatomo). As abelian categories, these are equivalent.

- In this talk, we only treat the quantum group side.

**Aim**:
Study tensor structure of $\U_q(\mathfrak{sl}_2)$-$\text{mod}$.

**Main result**:
Indecomposable decomposition of all tensor products of $\U_q(\mathfrak{sl}_2)$-modules is completely determined in explicit formulas.

As a by-product, we show that $\U_q(\mathfrak{sl}_2)$-$\text{mod}$ is not a braided tensor category for $p \geq 3$.

⇒ It needs a rectification for Conjecture 1.

This is a future problem.
§ Preliminaries

Notations

Let $p \geq 2$ be an integer and $q$ be a primitive $2p$-th root of unity. For any integer $n$, we set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Note that $[n] = [p - n]$ for any $n$.

• $\overline{U} = U_q(\mathfrak{sl}_2)$ : The restricted quantum $\mathfrak{sl}_2$

An unital associative $\mathbb{C}$-algebra with generators $E, F, K, K^{-1}$ and relations ;

$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

$K^{2p} = 1, \quad E^p = 0, \quad F^p = 0.$

This is a $2p^3$-dimensional $\mathbb{C}$-algebra and has a Hopf algebra structure defined by

$\Delta: E \mapsto E \otimes K + 1 \otimes E, \quad F \mapsto F \otimes 1 + K^{-1} \otimes F,$

$K \mapsto K \otimes K, \quad K^{-1} \mapsto K^{-1} \otimes K^{-1},$

$\varepsilon: E \mapsto 0, \quad F \mapsto 0, \quad K \mapsto 1, \quad K^{-1} \mapsto 1,$

$S: E \mapsto -EK^{-1}, \quad F \mapsto -KF, \quad K^\pm \mapsto K^{\mp 1}.$

The category $\overline{U}$-$\text{mod}$ of finite-dimensional left $\overline{U}$-modules has a structure of a monoidal category associated with this Hopf algebra structure on $\overline{U}$. 
§ Structure of $\overline{U}$-mod

This is a survey of known results on $\overline{U}$-mod which were proved by Reshetikhin-Turaev, Suter, Xiao, Gunnlaugsdóttir, Feigin-Gainutdinov-Semikhatov-Tipunin, Arike.

Basic algebra

$A$ : an unital associative $\mathbb{C}$-algebra of finite dimension,

$$A = \bigoplus_{i=1}^{n} \mathcal{P}_i^{m_i}$$

: a decomposition of $A$ into indecomposable left ideals where $\mathcal{P}_i \not\cong \mathcal{P}_j$ if $i \neq j$.

For each $i$ take a primitive idempotent $e_i \in A$ such that $A e_i \cong \mathcal{P}_i$, and set $e = \sum_{i=1}^{n} e_i$.

$B_A = e A e$ is called the basic algebra of $A$ which has the following nice properties:

• $B_A$ is Morita-equivalent to $A$.

There is a functor $B_A$-mod $\rightarrow A$-mod defined as

$$\mathcal{Z} \mapsto A e \otimes_{B_A} \mathcal{Z}.$$  

• $B_A$ is described by a quiver with relations.

A $\mathbb{C}$-algebra $B$ is called basic if $B/\text{rad}(B) \cong \mathbb{C}^n$. It is well-known that an basic algebra is described by a quiver with relations and it is easy to see that $B_A$ is basic.

⇒ What is $B_{\overline{U}}$ ?
**Answer:**

The basic algebra $B_{\mathcal{U}}$ of $\mathcal{U}$ is decomposed as a direct product $B_{\mathcal{U}} \cong \prod_{s=0}^{p} B_s$ and one can describe each $B_s$ as follows:

- $B_0 \cong B_p \cong \mathbb{C}$. (1-dimensional algebra)
- For each $s = 1, \ldots, p - 1$, $B_s$ is isomorphic to the 8-dimensional algebra $B$ defined by the following quiver

  ![Quiver Diagram]

  with relations $\tau_i^+ \tau_i^- = 0$ for $i = 1, 2$, and $\tau_1^\pm \tau_2^\mp = \tau_2^\pm \tau_1^\mp$.

**Remark.** To get the basic algebra $B_{\mathcal{U}}$ of $\mathcal{U}$, we need to determine a complete set of mutually orthogonal primitive idempotents of $\mathcal{U}$. The explicit form of it is known, but we omit to give it.

The next problem is:

**What is the structure of $B$-mod?**

In the following, we will give you

- the complete list of indecomposable $B$-modules and
- Auslander-Reiten quiver of $B$-mod.
Classification of indecomposable $B$-modules

We can identify a $B$-module with data

$$\mathcal{Z} = (V^+_Z, V^-_Z; \tau^+_1, Z, \tau^+_2, Z, \tau^-_1, Z, \tau^-_2, Z),$$

where

- $V^\pm_Z$ is a vector space over $\mathbb{C}$ (attached to the vertices $\pm$).
- $\tau^\pm_{i,Z} : V^\pm_Z \to V^\mp_Z$ ($i = 1, 2$) are $\mathbb{C}$-linear maps (attached to the arrows) satisfying $\tau^\pm_{i,Z} \tau^\mp_{i,Z} = 0$, $\tau^\pm_{1,Z} \tau^\mp_{2,Z} = \tau^\pm_{2,Z} \tau^\mp_{1,Z}$.

For positive integers $m, n$ and $i = 1, \ldots, m$, $j = 1, \ldots, n$, we denote the composition of $j$-th projection and $i$-th embedding

$$e_{i,j} : \mathbb{C}^n \to \mathbb{C} \to \mathbb{C}^m.$$

**Proposition 3.** Any indecomposable $B$-module is isomorphic to exactly one of modules in the following list:

- **Simple modules**:
  $$\mathcal{X}^+ = (\mathbb{C}, \{0\}; 0, 0, 0, 0), \quad \mathcal{X}^- = (\{0\}, \mathbb{C}; 0, 0, 0, 0).$$

- **Projective-injective modules**:
  $$\mathcal{P}^+ = (\mathbb{C}^2, \mathbb{C}^2; e_{1,1}, e_{2,1}, e_{2,2}, e_{1,1}) = \begin{array}{c}
\mathbb{C} \\
\mathbb{C}
\end{array} \quad \begin{array}{c} \\
\mathbb{C}
\end{array}, \\
\mathcal{P}^- = (\mathbb{C}^2, \mathbb{C}^2; e_{2,2}, e_{2,1}, e_{1,1}, e_{2,1}) = \begin{array}{c}
\mathbb{C} \\
\mathbb{C}
\end{array} \quad \begin{array}{c} \\
\mathbb{C}
\end{array}.$$
• For each integer \( n \geq 2 \),

\[
\mathcal{M}^+(n) = \left( \mathbb{C}^{n-1}, \mathbb{C}^n; \sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i}, 0, 0 \right)
\]

\[
= \mathbb{C}^{n-1} \xrightarrow{\begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}} \mathbb{C}^n,
\hspace{1cm} (Here \ we \ omit \ 0\text{-arrows}.)
\]

\[
\mathcal{M}^-(n) = \left( \mathbb{C}^n, \mathbb{C}^{n-1}; 0, 0, \sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i} \right),
\]

\[
\mathcal{W}^+(n) = \left( \mathbb{C}^n, \mathbb{C}^{n-1}; \sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i}, 0, 0 \right),
\]

\[
\mathcal{W}^-(n) = \left( \mathbb{C}^{n-1}, \mathbb{C}^n; 0, 0, \sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i} \right).
\]

• For each integer \( n \geq 1 \) and \( \lambda \in \mathbb{P}^1(\mathbb{C}) \),

\[
\mathcal{E}^+(n; \lambda) = \left( \mathbb{C}^n, \mathbb{C}^n; \varphi_1(n; \lambda), \varphi_2(n; \lambda), 0, 0 \right),
\]

\[
\mathcal{E}^-(n; \lambda) = \left( \mathbb{C}^n, \mathbb{C}^n; 0, 0, \varphi_1(n; \lambda), \varphi_2(n; \lambda) \right),
\]

where

\[
\left( \varphi_1(n; \lambda), \varphi_2(n; \lambda) \right) = \begin{cases} 
(\beta \cdot \text{id} + \sum_{i=1}^{n-1} e_{i,i+1}, \text{id}) & (\lambda = [\beta : 1]), \\
(\text{id}, \sum_{i=1}^{n-1} e_{i,i+1}) & (\lambda = [1 : 0]).
\end{cases}
\]

i.e.,

\[
\mathcal{E}^+(n; \lambda) = \begin{cases} 
\mathbb{C}^n \xrightarrow{J(\beta; n)} \mathbb{C}^n & (\lambda = [\beta : 1]), \\
\mathbb{C}^n \xrightarrow{\text{id}} \mathbb{C}^n & (\lambda = [1 : 0]).
\end{cases}
\]

Here \( J(\beta; n) \) is the \((n \times n)\)-Jordan cell with eigenvalue \( \beta \).
Auslander-Reiten quiver of $B$-mod

![Diagram]

Remark. We “divide” the quiver of $B$ into the following two pieces which are isomorphic to the Kronecker quiver:

$$Q^+ := o \xrightarrow{\tau_1^+} o \xleftarrow{\tau_2^+} o \quad \text{“+”} \quad Q^- := o \xleftarrow{\tau_1^-} o \xrightarrow{\tau_2^-} o$$

Consider AR-quivers of $Q^+$ and $Q^-$ (i.e. two copies of AR-quiver of the Kronecker quiver), and “paste” the above two copies.

$\Rightarrow$ AR-quiver of $B$-mod
Structure of $\overline{U}$-mod

Recall a decomposition of the basic algebra $B_\overline{U}$ of $\overline{U}$:

$$B_\overline{U} = \bigoplus_{s=0}^{p} B_s$$

where

$$B_0 \cong B_p \cong \mathbb{C}, \quad B_s \cong B \quad (1 \leq s \leq p - 1).$$

Denote by $\mathcal{C}(s)$ the full subcategory of $\overline{U}$-$\text{mod}$ corresponding to $B_s$-modules (considered as $B_\overline{U}$-modules) for $s = 0, \ldots, p$.

$\Rightarrow$ We have a block decomposition of $\overline{U}$-$\text{mod}$:

$$\overline{U}$-$\text{mod} = \bigoplus_{s=0}^{p} \mathcal{C}(s).$$

• For $s = 1, \ldots, p - 1$, let $\Phi_s$ be the composition of functors

$$\Phi_s : \text{B}-\text{mod} \to B_\overline{U}$-$\text{mod} \to \overline{U}$-$\text{mod}.$$ 

We denote by

$$\mathcal{X}_s^+, \mathcal{X}_{p-s}^-, \mathcal{P}_s^+, \mathcal{P}_{p-s}^-, \mathcal{M}_s^+(n), \mathcal{M}_{p-s}^-(n), \mathcal{W}_s^+(n), \mathcal{W}_{p-s}^-(n),$$

the images of

$$\mathcal{X}^+, \mathcal{X}^-, \mathcal{P}^+, \mathcal{P}^-, \mathcal{M}^+(n), \mathcal{M}^-(n), \mathcal{W}^+(n), \mathcal{W}^-(n),$$

by $\Phi_s$. 
• On the other hand, for \( s = 0 \) or \( p \), let \( \Phi_s \) be the composition of functors

\[
\Phi_s : \mathcal{C}\text{-mod} \to B\bar{U}\text{-mod} \to \bar{U}\text{-mod}.
\]

Let us denote \( \mathcal{X} \cong \mathcal{C} \) the unique simple object in \( \mathcal{C}\text{-mod} \). We denote the corresponding object in \( \mathcal{C}(0) \) and \( \mathcal{C}(p) \) by

\[
\mathcal{X}_p^- := \Phi_0(\mathcal{X}) \in \mathcal{C}(0),
\]
\[
\mathcal{X}_p^+ := \Phi_p(\mathcal{X}) \in \mathcal{C}(p).
\]

We remark that both \( \mathcal{X}_p^- \) and \( \mathcal{X}_p^+ \) are also projective. In that sense, we sometimes denote

\[
\mathcal{P}_p^\pm := \mathcal{X}_p^\pm.
\]
Simple objects in $\mathcal{C}(s)$

The explicit form of $\Phi_s(\mathcal{X}^\pm)$ are given as follows:

$1 \leq s \leq p - 1$

- $\mathcal{X}_s^+ = \Phi_s(\mathcal{X}^+)$ is isomorphic to the $s$-dimensional module defined by basis $\{a_n\}_{n=0,\ldots,s-1}$ and $\mathcal{U}$-action given by

$$K a_n = q^{s-1-2n} a_n,$$

$$E a_n = \begin{cases} [n][s-n]a_{n-1} & (n \neq 0) \\ 0 & (n = 0) \end{cases},$$

$$F a_n = \begin{cases} a_{n+1} & (n \neq s-1) \\ 0 & (n = s-1) \end{cases}.$$

- $\mathcal{X}_{p-s}^- = \Phi_s(\mathcal{X}^-)$ is isomorphic to the $(p-s)$-dimensional module defined by basis $\{a_n\}_{n=0,\ldots,p-s-1}$ and $\mathcal{U}$-action given by

$$K a_n = -q^{p-s-1-2n} a_n,$$

$$E a_n = \begin{cases} -[n][p-s-n]a_{n-1} & (n \neq 0) \\ 0 & (n = 0) \end{cases},$$

$$F a_n = \begin{cases} a_{n+1} & (n \neq p-s-1) \\ 0 & (n = p-s-1) \end{cases}.$$

Remark. Since we consider all finite dimensional $\mathcal{U}$-modules, modules which are not of type $I$ are appeared. For example, $\mathcal{X}_s^+$ is a $\mathcal{U}$-module of type $I$. On the other hand $\mathcal{X}_{p-s}^-$ is not.

$\circ s = 0$ or $p$

$\mathcal{X}_p^+ = \Phi_p(\mathcal{X})$ (resp. $\mathcal{X}_p^- = \Phi_0(\mathcal{X})$) is the $p$-dimensional irreducible module of $\mathcal{U}$ defined as similar way.
Other indecomposable objects in $\mathcal{C}(s)$ ($1 \leq s \leq p - 1$)

- Since $\mathcal{C}(s)$ is equivalent to $B\text{-mod}$ as an abelian category, all information of indecomposable objects in $\mathcal{C}(s)$ can be obtained from one of the corresponding objects in $B\text{-mod}$.

**Example.** In $B\text{-mod}$, the structure of the projective modules $\mathcal{P}^{\pm}$ are given as:

![Diagram of projective modules]

By easy computation, we have

$$\text{Ext}^1_B(\mathcal{X}^\pm, \mathcal{X}^\mp) = \mathbb{C}^2.$$  

We fix basis of $\text{Ext}^1_B(\mathcal{X}^+, \mathcal{X}^-)$ and $\text{Ext}^1_B(\mathcal{X}^-, \mathcal{X}^+)$ by $\{x_1^+, x_2^+\}$ and $\{x_1^-, x_2^-\}$ respectively.

(We omit to give the explicit form of them.)

In the above diagram, we denote $\mathcal{X}_1 \xrightarrow{x} \mathcal{X}_2$ by the extension by $x \in \text{Ext}^1_B(\mathcal{X}_1, \mathcal{X}_2)$.

Applying the functor $\Phi_s$, we have

![Diagram with $\Phi_s$ applied]

As a corollary, we have

$$\dim \mathcal{P}_s^+ = 2p \ (= 2s + 2(p - s)), \quad \dim \mathcal{P}_{p-s}^- = 2p.$$
§ Calculation of tensor products

Main tools
(a) Some (basic) short exact sequences.
   (It is enough to show the existence of them in $B\text{-mod}$.)
(b) Exactness of the functors $- \otimes \mathcal{Z}$ and $\mathcal{Z} \otimes -$.
   ($\vdash \otimes$ in a tensor product over a field $\mathbb{C}$.)
(c) For a projective module $\mathcal{P}$, both $\mathcal{P} \otimes \mathcal{Z}$ and $\mathcal{Z} \otimes \mathcal{P}$ are also projective.
(d) $\mathcal{U}$ is a Frobenius algebra. As a by-product,
   \[ \mathcal{Z} \text{ is projective } \iff \mathcal{Z} \text{ is injective.} \]
(e) Rigidity : For $n \geq 0$,
   \[ \text{Ext}^n_{\mathcal{U}}(\mathcal{Z}_1, \mathcal{Z}_2 \otimes \mathcal{Z}_3) \cong \text{Ext}^n_{\mathcal{U}}(D(\mathcal{Z}_2) \otimes \mathcal{Z}_1, \mathcal{Z}_3), \]
   \[ \text{Ext}^n_{\mathcal{U}}(\mathcal{Z}_1 \otimes \mathcal{Z}_2, \mathcal{Z}_3) \cong \text{Ext}^n_{\mathcal{U}}(\mathcal{Z}_1, \mathcal{Z}_3 \otimes D(\mathcal{Z}_2)). \]
Here $D(\mathcal{Z})$ the standard dual of $\mathcal{Z} \in \mathcal{U}\text{-mod}$. More precisely, define $\mathcal{U}$-module structure on the dual space $D(\mathcal{Z}) := \text{Hom}_\mathbb{C}(\mathcal{Z}, \mathbb{C})$ as:
\[ (a \cdot f)(z) := f(S(a)z) \quad (a \in \mathcal{U}, f \in D(\mathcal{Z}), z \in \mathcal{Z}), \]
where $S$ is the antipode of $\mathcal{U}$.

Remark . It is known that the properties (c), (d) and (e) hold in more general setting. Namely, for any finite dimensional Hopf-algebra $A$ over a field, these properties hold in $A\text{-mod}$.
(Of course, (b) is also satisfied.)
Tensor products of simple modules

The following proposition is proved by Reshetikhin-Turaev.

**Proposition 4 (Reshetikhin-Traev).** For \( s, s' = 1, \ldots, p \),

\[
\mathcal{X}_s^+ \otimes \mathcal{X}_{s'}^+ \cong \begin{cases} 
\bigoplus_{t=|s-s'|+1}^{s+s'-1} \mathcal{X}_t^+ & (s + s' - 1 \leq p), \\
\left( \bigoplus_{t=|s-s'|+1}^{2p-s-s'+1} \mathcal{X}_t^+ \right) \oplus \left( \bigoplus_{t'=2p-s-s'+1}^{p-\delta} \mathcal{P}_{t'}^+ \right) & (s + s' - 1 > p),
\end{cases}
\]

where

\[
\delta = \begin{cases} 
1 & (s + s' - p - 1 \text{ is odd}), \\
0 & (s + s' - p - 1 \text{ is even}).
\end{cases}
\]

- If \( s + s' - 1 \leq p \), the formula is nothing but Clebush-Gordan formula.
- It is easy to see that

\[
\mathcal{X}_s^\pm \otimes \mathcal{X}_1^- \cong \mathcal{X}_1^- \otimes \mathcal{X}_s^\pm \cong \mathcal{X}_s^\mp,
\]

\[
\mathcal{P}_s^\pm \otimes \mathcal{X}_1^- \cong \mathcal{X}_1^- \otimes \mathcal{P}_s^\pm \cong \mathcal{P}_s^\mp.
\]

By the associativity of tensor products, we can calculate other decompositions. For example,

\[
\mathcal{X}_s^- \otimes \mathcal{X}_{s'}^+ \cong (\mathcal{X}_1^- \otimes \mathcal{X}_{s'}^+) \otimes \mathcal{X}_{s'}^+
\]

\[
\cong \mathcal{X}_1^- \otimes (\bigoplus_t \mathcal{X}_t^+) \oplus \bigoplus_t \mathcal{P}_{t'}^+
\]

\[
\cong (\bigoplus_t \mathcal{X}_t^-) \oplus \bigoplus_t \mathcal{P}_{t'}^-.
\]
Main result

**Theorem 5.** Indecomposable decomposition of all tensor products in \(U\)-mod is completely determined in explicit formulas.

Since there are too many indecomposables in \(U\)-mod, we cannot list up all formulas in this talk. In the following, we will give some typical examples.

**Tensor products of \(\mathcal{E}_s^\pm(1; \lambda)\) with simple modules**

By direct calculation, we have the following.

**Proposition 6.** For \(s, s' = 1, \ldots, p - 1, n \geq 1\) and \(\lambda = [\lambda_1 : \lambda_2] \in \mathbb{P}^1(\mathbb{C})\) we have

\[
\mathcal{E}_s^\pm(1; \lambda) \otimes \mathcal{X}_1^- \cong \mathcal{E}_s^\mp(1; -\lambda),
\]

\[
\mathcal{X}_1^- \otimes \mathcal{E}_s^\pm(1; \lambda) \cong \mathcal{E}_s^\mp(1; (-1)^{p-1} \lambda),
\]

\[
\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \cong \mathcal{E}_{s-1}^+(1; \frac{[s]}{s - 1} \lambda) \oplus \mathcal{E}_{s+1}^+(1; \frac{[s]}{s + 1} \lambda),
\]

\[
\mathcal{X}_2^+ \otimes \mathcal{E}_s^+(1; \lambda) \cong \mathcal{E}_{s-1}^+(1; -\frac{[s]}{s - 1} \lambda) \oplus \mathcal{E}_{s+1}^+(1; -\frac{[s]}{s + 1} \lambda).
\]

Here, for \(c \in \mathbb{C}\), we set \(c\lambda = [c\lambda_1 : \lambda_2] \in \mathbb{P}^1(\mathbb{C})\).

**Remark.** This proposition tells us that, in general,

\[
\mathcal{E}_s^\pm(1; \lambda) \otimes \mathcal{X}_1^- \not\cong \mathcal{X}_1^- \otimes \mathcal{E}_s^\pm(1; \lambda),
\]

\[
\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \not\cong \mathcal{X}_2^+ \otimes \mathcal{E}_s^+(1; \lambda).
\]

That is, \(U\)-mod is not a braided tensor category.
Proposition 7. For $s, s' = 1, \ldots, p - 1$ and $\lambda \in \mathbb{P}^1(k)$,

$$\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_{s'}^+ \cong \bigoplus_{t_1 \in I_{s,s'}} \mathcal{E}_{t_1}^+ \left( 1; \frac{s}{t_1} \right) \oplus \bigoplus_{t_2 \in J_{s+s'}} \mathcal{P}_{t_2}^+ \oplus \bigoplus_{t_3 \in J_{p-s+s'}} \mathcal{P}_{t_3}^-.$$ 

Here, $I_{s,s'}, J_{s+s'}, J_{p-s+s'}$ are some sets of integers.

(For $\mathcal{X}_{s'}^+ \otimes \mathcal{E}_s^+(1; \lambda)$, we have a similar formula.)

Proof. There is a (basic) exact sequence in $\mathcal{U}\text{-mod}$:

$$0 \to \mathcal{X}_{p-s} \to \mathcal{E}_s^+(1; \lambda) \to \mathcal{X}_s^+ \to 0.$$

Applying $- \otimes \mathcal{X}_{s'}^+$, we have

$$0 \to \mathcal{X}_{p-s} \otimes \mathcal{X}_{s'}^+ \to \mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_{s'}^+ \to \mathcal{X}_s^+ \otimes \mathcal{X}_{s'}^+ \to 0.$$

By Proposition 4, we have

$$\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_{s'}^+ \cong \bigoplus_{t_1} \mathcal{Z}_{t_1}^+ \oplus \bigoplus_{t_2} \mathcal{P}_{t_2}^+ \oplus \bigoplus_{t_2} \mathcal{P}_{t_3}^-$$

with an exact sequence $0 \to \mathcal{X}_{p-t_1}^- \to \mathcal{Z}_{t_1} \to \mathcal{X}_{t_1}^+ \to 0$ for each $t_1$. We remark that $\mathcal{Z}_{t_1}$ is not projective.

Assume the formula hols for $s'' \leq s' - 1$. Then,

$$(\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_{s'-1}^+) \otimes \mathcal{X}_2^+$$

$$\cong \mathcal{E}_s^+(1; \lambda) \otimes (\mathcal{X}_{s'-1}^+ \otimes \mathcal{X}_2^+)$$

$$\cong \mathcal{E}_s^+(1; \lambda) \otimes (\mathcal{X}_{s'-2}^+ \oplus \mathcal{X}_{s'}^+)$$

$$\cong (\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_{s'-2}^+) \oplus (\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_{s'}^+)$$

tells us that a non-projective indecomposable summand of $\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_{s'}^+$ must be of the form $\mathcal{E}_{t_1}^+ \left( 1; \frac{s}{t_1} \lambda \right)$ with $t = 1, \ldots, p - 1$. Then we have $\mathcal{Z}_{t_1} \cong \mathcal{E}_{t_1}^+ \left( 1; \frac{s}{t_1} \lambda \right)$ since $\mathcal{Z}_{t_1}$ cannot be projective. Thus we have the formula. \qed
Tensor products of $\mathcal{E}_s^\pm(n; \lambda)$ with simple modules

For computing these combination, we need the rigidity.

**Proposition 8.** For $s = 1, \ldots, p - 1$ and $\lambda \in \mathbb{P}^1(k)$,

$$
D(\mathcal{X}_s^\pm) \cong \mathcal{X}_s^\pm, \quad D(\mathcal{E}_s^+(1; \lambda)) \cong \mathcal{E}_{p-s}^-(1; (-1)^s \lambda),
$$

$$
D(\mathcal{E}_s^-(1; \lambda)) \cong \mathcal{E}_{p-s}^+(1; (-1)^{p-s} \lambda).
$$

**Proposition 9.**

$$
D(\mathcal{E}_s^+(n; \lambda)) \cong \mathcal{E}_{p-s}^-(n; (-1)^s \lambda), \quad D(\mathcal{E}_s^-(n; \lambda)) \cong \mathcal{E}_{p-s}^+(n; (-1)^{p-s} \lambda).
$$

**Proof.** Since $\dim \mathcal{E}_s^+(n; \lambda) = pn$ and $D$ preserves direct sum and dimension, $D(\mathcal{E}_s^+(n; \lambda))$ is an indecomposable module of dimension $pn$.

$\Rightarrow$ This is of the form $\mathcal{E}_t^\pm(n; \mu)$ or is projective

(the latter case could occur only if $n \leq 2$).

$$
\text{ext}^1_U(D(\mathcal{E}_s^+(n; \lambda)), \mathcal{X}_s^+) \quad (\text{ext} := \dim \text{Ext})
$$

$$
= \text{ext}^1_U(D(\mathcal{E}_s^+(n; \lambda)) \otimes \mathcal{X}_s^+) = \text{ext}^1_U(\mathcal{X}_1^+, \mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_s^+)
$$

$$
= \text{ext}^1_U(\mathcal{X}_1^+, \mathcal{E}_s^+(n; \lambda) \otimes D(\mathcal{X}_s^+)) = \text{ext}^1_U(\mathcal{X}_1^+ \otimes \mathcal{X}_s^+, \mathcal{E}_s^+(n; \lambda))
$$

$$
= \text{ext}^1_U(\mathcal{X}_s^+, \mathcal{E}_s^+(n; \lambda)) = \text{ext}^1_B(\mathcal{X}^+, \mathcal{E}^+(n; \lambda)) = n.
$$

$\Rightarrow$ $D(\mathcal{E}_s^+(n; \lambda))$ must be of the form $\mathcal{E}_t^\pm(n; \mu)$.

By the similar argument,

$$
\text{ext}^1_U(D(\mathcal{E}_s^+(n; \lambda)), \mathcal{E}_s^+(1; \mu)) = \text{ext}^1_B(\mathcal{E}^-(1; (-1)^s \mu), \mathcal{E}^+(n; \lambda))
$$

$$
= \begin{cases} 
1 & (\mu = -\lambda) \\
0 & (\mu \neq -\lambda)
\end{cases}
$$

$\Rightarrow$ $D(\mathcal{E}_s^+(n; \lambda)) \cong \mathcal{E}_{p-s}^-(n; (-1)^s \lambda)$.

$\square$
Proposition 10.

\[ \mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_s^+ \]

\[ \cong \bigoplus_{t_1 \in I_{s,s'}} \mathcal{E}_{t_1}^+ \left( n; \frac{[s]}{[t_1]} \lambda \right) \oplus \bigoplus_{t_2 \in J_{s+s'}} (\mathcal{P}_{t_2}^+)^n \oplus \bigoplus_{t_3 \in J_{p-s+s'}} (\mathcal{P}_{t_3}^-)^n. \]

(We have a similar formula for \( \mathcal{X}_{s'}^- \otimes \mathcal{E}_s^+(n; \lambda) \).)

**Proof.** The same argument as the case of \( \mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_s^+ \) shows that

\[ \mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_s^+ \cong \bigoplus_{t_1} \mathcal{Z}_{t_1} \oplus \bigoplus_{t_2} (\mathcal{P}_{t_2}^+)^n \oplus \bigoplus_{t_3} (\mathcal{P}_{t_3}^-)^n \]

with an exact sequence \( 0 \to (\mathcal{X}_{p-t_1}^-)^n \to \mathcal{Z}_{t_1} \to (\mathcal{X}_{t_1}^+)^n \to 0 \) for each \( t_1 \). Moreover, by the exact sequence

\[ 0 \to \mathcal{E}_s^+(n-1; \lambda) \to \mathcal{E}_s^+(n; \lambda) \to \mathcal{E}_s^+(1; \lambda) \to 0 \]

and induction on \( n \), we have the following exact sequence

\[ 0 \to \mathcal{E}_{t_1}^+ \left( n-1; \frac{[s]}{[t_1]} \lambda \right) \to \mathcal{Z}_{t_1} \to \mathcal{E}_{t_1}^+ \left( 1; \frac{[s]}{[t_1]} \lambda \right) \to 0. \]

\[ \Rightarrow \mathcal{Z}_{t_1} \in \mathcal{C}(t_1) \text{ and dim}_c \mathcal{Z}_{t_1} = pn. \]

By using the rigidity, we have

\[ \text{ext}^1_{\mathcal{U}}(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_s^+, \mathcal{X}_t^+) = 0, \]

\[ \text{ext}^1_{\mathcal{U}}(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_s^+, \mathcal{X}_{p-t}^-) = n, \]

\[ \text{ext}^1_{\mathcal{U}}(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_s^+, \mathcal{E}_t^+(1; \mu)) = \begin{cases} 1 & (\lambda = \frac{[t]}{[s]} \mu) \\ 0 & (\lambda \neq \frac{[t]}{[s]} \mu) \end{cases}. \]

\[ \Rightarrow \text{By the above properties, } \mathcal{Z}_{t_1} \text{ is uniquely determined. Namely, we have } \mathcal{Z}_{t_1} \cong \mathcal{E}_{t_1}^+ \left( n; \frac{[s]}{[t_1]} \lambda \right). \]
Conclusions

For other combinations, we can compute the explicit formulas by the similar methods.

As a by-product, we have

**Corollary 11.** (1) Let $Z_1$, $Z_2$ be $\mathcal{U}_q(sl_2)$-modules. If $Z_1$ nor $Z_2$ do not have any indecomposable summand of type $\mathcal{E}$, we have $Z_1 \otimes Z_2 \cong Z_2 \otimes Z_1$.

(2) If $p = 2$, for arbitrary $\mathcal{U}_q(sl_2)$-modules $Z_1$, $Z_2$ we have $Z_1 \otimes Z_2 \cong Z_2 \otimes Z_1$.

(3) If $p \geq 3$, there exist $\mathcal{U}_q(sl_2)$-modules $Z_1$, $Z_2$ such that $Z_1 \otimes Z_2 \not\cong Z_2 \otimes Z_1$. In particular, $\mathcal{U}_q(sl_2)$-mod is not a braided tensor category.

**Remark.** These method can be applied only for $sl_2$-case. If $g \neq sl_2$, it is known that $\mathcal{U}_q(g)$-mod has a wild representation type.