Mirković-Vilonen polytopes and quiver construction of crystal basis in type $A$

Yoshihisa Saito  (Tokyo)

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§ Introduction

Mirković-Vilonen (1997~)
- MV cycles : Algebraic cycles in affine Grassmannian

Kamnitzer (2005~)
- Moment map image $\Rightarrow$ MV polytopes : Polytopes in $\mathfrak{h}_\mathbb{R}(\mathbb{R}^\vee)$
- $\mathcal{M}\mathcal{V}$: the set of all MV polytopes has a crystal structure, and $\mathcal{M}\mathcal{V} \cong B(\infty)$.
- Prove the Anderson-Mirković conjecture in type $A$.
  (It tells us the explicit action of $\tilde{f}_i$ on $\mathcal{M}\mathcal{V}$.)

That is, the Kamnitzer’s result tells us a realization of $B(\infty)$ in terms of MV polytopes.

Today:

In type $A$, compare the Kamnitzer’s realization with
- a realization of $B(\infty)$ in terms of Young tableaux,
- a realization of $B(\infty)$ in terms of irreducible Lagrangians

and, as an application, we will give
- a new proof of AM conjecture.
Notations

\( U_q = U_q(\mathfrak{sl}_{n+1}) = \langle e_i, f_i, t_i^{\pm 1} \mid i \in I \rangle \).

\( B(\infty) \) : the crystal basis of \( U_q^- \)

\(* : U_q \to U_q \) : a \( \mathbb{Q}(q) \)-algebra anti-automorphism

\[ e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad t_i^\pm \mapsto t_i^{\mp}. \]

\( \Rightarrow * : B(\infty) \to B(\infty). \)

Set

\[ \varepsilon_i^*(b) := \varepsilon_i(b^*), \quad \varphi_i^*(b) := \varphi_i(b^*), \]

\[ \widetilde{e}_i^* := * \circ \widetilde{e}_i \circ *, \quad \widetilde{f}_i^* := * \circ \widetilde{f}_i \circ *. \]

\( \Rightarrow \) The set \( B(\infty) \) endowed with maps \( \text{wt, } \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^* \)

is a crystal.

( "the *-crystal structure" on \( B(\infty) \))

That is, \( B(\infty) \) has two crystal structures :

\( (B(\infty) ; \text{wt, } e_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i) \),

\( (B(\infty)^* = B(\infty) ; \text{wt, } e_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*). \)
§ Realization I : Young tableaux

\( \lambda \in P_+ \) : dominant integral weight

\( V(\lambda) \) : irreducible \( U_q \)-module with h.w. \( \lambda \)

\( B(\lambda) \) : crystal basis of \( V(\lambda) \)

**Theorem** (Kashiwara-Nakashima).

\[ B(\lambda) \cong SST(\lambda). \]

Here \( SST(\lambda) \) is the set of semistandard Young tableaux of shape \( \lambda \).

- Take \( \lambda \to \infty \) (w.r.t. \( \lambda \geq \mu \iff \lambda - \mu \in Q_+ \))

\[
\begin{array}{c c c c}
B(\lambda) & \cong & SST(\lambda) \\
\downarrow & & \downarrow \\
B(\infty) & \cong & \mathcal{B}
\end{array}
\]

\( \mathcal{B} \) : The set of all \( n(n+1)/2 \) tuples of non-negative integers

\( a = (a_{i,j})_{1 \leq i < j \leq n+1} \)

We regard \( \mathcal{B} \) as “\( SST(\infty) \)” via

\( a_{i,j} = \) “the number of \( \begin{array}{c} j \end{array} \) in the \( i \)-th row of a tableau”

**Remark .**

(1) The explicit crystal structure of \( \mathcal{B} \) can be determined.

(2) Since \( B(\infty) \) has the \(*\)-crystal structure, \( \mathcal{B} \) also has the induced \(*\)-crystal structure.

(We omit to give them.)
§ Realization II : Lagrangian construction

\( (I, H) \) : (double) quiver of type \( A_n \)
\( (I = \{1, \cdots, n\} : \text{set of vertices, } H : \text{set of arrows}) \)
\( \Omega \subset H : \text{an orientation} \ \Rightarrow \ (I, \Omega) : \text{a quiver of type } A_n. \)

\( V = \bigoplus_{i \in I} V_i : \text{finite dimensional } I\text{-graded complex vector space} \)

\[ E_{V,\Omega} := \bigoplus_{\tau \in \Omega} \text{Hom}_C(V_{\text{out}(\tau)}, V_{\text{in}(\tau)}), \]

\[ X_{V,\Omega} := \bigoplus_{\tau \in H} \text{Hom}_C(V_{\text{out}(\tau)}, V_{\text{in}(\tau)}) = E_{V,\Omega} \oplus E_{V,\overline{\Omega}} \]
\[ \cong T^*E_{V,\Omega}. \]

\( G_V := \prod_{i \in I} GL(V_i) \ltimes E_{V,\Omega} \text{ and } X_V. \)

\( \Rightarrow \mu : X_V \to (\text{Lie} G_V)^* \cong \bigoplus_{i \in I} \text{End}(V_i) : \text{moment map} \)

\[ \Lambda_V := \mu^{-1}(0) : \text{a Lagrangian subvariety of } X_V. \]

\( \text{Irr } \Lambda_V : \text{the set of all irreducible components of } \Lambda_V. \)

**Theorem** (Kashiwara-S).

(1) \( \bigsqcup_V \text{Irr } \Lambda_V \cong B(\infty). \)

(2) \( \bigsqcup_V \text{Irr } \Lambda_V \text{ has a } \ast\text{-crystal structure induced from the map} \)
\[ \ast : B \mapsto {}^tB \ (B \in X_V), \]
and \( \bigsqcup_V \text{Irr } \Lambda_V \cong B(\infty) \) as \( \ast\text{-crystals}. \)

**Remark**. Since \( (I, \Omega) \) is of type \( A_n \), for \( \Lambda \in \Lambda_V \), there is a unique \( G_V\)-orbit \( \mathcal{O} \subset E_{V,\Omega} \) such that
\[ \Lambda = \overline{T^*_\mathcal{O}E_{V,\Omega}}. \]
§ Comparison I & II

○ Preliminaries

- There is a natural one to one correspondence

\[ G_V \text{-orbits in } E_{V,\Omega} \cong \text{isomorphism classes of reps. of } (I, \Omega) \]
with dimension vector = dim V.

- There is a one to one correspondence

\[ \Delta_+ \cong \text{isomorphism classes of indecomposable reps. of } (I, \Omega), \]
where $\Delta_+$ is the set of positive roots (Gabriel’s theorem).

For a positive root $\alpha \in \Delta_+$, we denote by $e(\alpha, \Omega)$ the corresponding indecomposable representation of $(I, \Omega)$.

- $\Delta_+ = \left\{ \alpha_{i,j} := \sum_{k=i}^{j-1} \alpha_k \mid 1 \leq i < j \leq n+1 \right\}$. 
Realization I ⇒ Realization II

Consider the following orientation:

\[ \Omega_0 : \quad 1 \quad 2 \quad 3 \quad \ldots \quad n-2 \quad n-1 \quad n \]

For \( a = (a_{i,j})_{1 \leq i < j \leq n+1} \in \mathcal{B} \), set

\[ e(a, \Omega_0) := \bigoplus_{1 \leq i < j \leq n+1} e(\alpha_{i,j}, \Omega_0)^{\oplus a_{i,j}}. \]

We denote by \( \mathcal{O}_a \subset E_{V, \Omega_0} \) the \( G_V \)-orbit through \( e(a, \Omega_0) \) and let

\[ \Lambda_a := T^*_{\mathcal{O}_a} E_{V, \Omega_0}. \]

**Proposition .**

The map \( \mathcal{B} \to \bigsqcup_V \text{Irr } \Lambda_V \) defined by

\[ a \mapsto \Lambda_a \]

is an isomorphism of crystals in usual sense. Moreover this map is compatible with the \( \ast \)-crystal structures.
§ Realization III : MV polytopes

- Definition of MV polytopes

$K \subset [1, n + 1]$ : a Maya diagram of size $n$

$\mathcal{M}_n$ : the set of all Maya diagrams of size $n$

$\mathcal{M}_n^\times := \mathcal{M}_n \setminus \{\phi, [1, n + 1]\}$

$\mathbf{M} = (M_K)_{K \in \mathcal{M}_n^\times}$ : a family of integers indexed by $\mathcal{M}_n^\times$

- $W = \mathfrak{S}_{n+1} \sim \mathcal{M}_n, \mathcal{M}_n^\times$.

We can identify $\mathcal{M}_n^\times$ with $\Gamma_n := \bigsqcup_{w \in W} \bigcup_{i \in I} W \Lambda_i$ via

$[1, i] \leftrightarrow \Lambda_i$.

- For $\mathbf{M} = (M_K)_{K \in \mathcal{M}_n^\times}$, consider a polytope in $\mathfrak{h}_\mathbb{R}$

$P(\mathbf{M}) := \{h \in \mathfrak{h}_\mathbb{R} \mid \langle h, K \rangle \geq M_K \ (\forall K \in \mathcal{M}_n^\times)\}$.

- A polytope $P(\mathbf{M})$ is called a pseudo-Weyl polytope if it satisfies the following condition:

(BZ-1) for every two indices $i \neq j$ in $[1, n + 1]$ and every $K \in \mathcal{M}_n$ with $K \cap \{i, j\} = \phi$,

$M_{Ki} + M_{Kj} \leq M_{Kij} + M_K$.

Here we denote $Kij = K \cup \{i, j\} \ etc.$, and we set $M_\phi = M_{[1,n+1]} = 0$. 
Remark.

$P(M)$ : a pseudo-Weyl polytope

$\Rightarrow$ $P(M)$ is the convex hull of $\mu := (\mu_w)_{w \in W} \subset \mathfrak{h}_R$ (GGMS datum) where

$$\mu_w := \sum_{i=1}^{n} M_{w\Lambda_i} w\alpha_i^\vee \in \mathfrak{h}_R \quad (w \in W).$$

That is, for a pseudo-Weyl polytope,

$$P(M) \iff M = (M_K)_{K \in \mathcal{M}_n^\star}.$$

Definition.

(1) $M$ is called a BZ datum if it satisfies (BZ-1) and (BZ-2) for every three indices $i < j < k$ in $[1, n + 1]$ and every $K \in \mathcal{M}_n$ with $K \cap \{i, j, k\} = \phi$,

$$M_{Kik} + M_{Kj} = \min \{M_{Kij} + M_{Kk}, M_{Kjk} + M_{Ki}\}.$$  

(2) $P(M)$ is called a MV polytope if $M$ is a BZ datum.

That is,

(BZ-1) $\Rightarrow$ $P(M)$ : a pseudo-Weyl polytope

(BZ-1) \& (BZ-2) $\Rightarrow$ $P(M)$ : a MV polytope
Crystal structure on BZ data

A BZ datum $M$ is called a $w_0$-BZ datum if

$$M_{w_0 \Lambda_i} = M_{[n-i+1,n+1]} = 0 \quad \text{for } 1 \leq \forall i \leq n.$$ 

$BZ^{w_0}$: the set of all $w_0$-BZ data

Let us define a crystal structure on $BZ^{w_0}$. For $M \in BZ^{w_0}$,

$$\text{wt}(M) := \sum_{1 \leq i \leq n} M_{[1,i]} \alpha_i,$$

$$\varepsilon_i(M) := - (M_{[1,i]} + M_{[1,i+1] \setminus \{i\}} - M_{[1,i+1]} - M_{[1,i] \setminus \{i\}}),$$

$$\varphi_i(M) := \varepsilon_i(M) + \langle h_i, \text{wt}(M) \rangle.$$ 

If $\varepsilon_i(M) = 0$, we set $\tilde{e}_i M = 0$.

Otherwise, there exists a unique $w_0$-BZ datum $\tilde{e}_i M$ s.t.

(i) $(\tilde{e}_i M)_{[1,i]} = M_{[1,i]} + 1$,
(ii) $(\tilde{e}_i M)_K = M_K$ for all $K \in M_n^\times \setminus M_n^\times(i)$.

Here $M_n^\times(i) = \{ K \in M_n^\times \mid i \in K \text{ and } i+1 \notin K \} \subset M_n^\times$.

There exists a unique a unique $w_0$-BZ datum $\tilde{f}_i M$ s.t.

(iii) $(\tilde{f}_i M)_{[1,i]} = M_{[1,i]} - 1$,
(iv) $(\tilde{f}_i M)_K = M_K$ for all $K \in M_n^\times \setminus M_n^\times(i)$.

Theorem (Kamniter).

$BZ^{w_0}$ is a crystal which is isomorphic to $B(\infty)$.

- This theorem gives us the 3-rd realization of $B(\infty)$ in terms of BZ data (or MV polytopes).
Anderson and Mirković conjectured the explicit form of the action of $\widetilde{f}_i$ on $BZ^{w_0}$ (AM conjecture). This conjecture is proved by Kamnitzer.

**Theorem** (Kamnitzer).

For each $i \in I$, we have

$$ (f_iM)_K = \begin{cases} \min \{M_K, M_{s_iK} + c_i(M)\} & (K \in M_n^\times(i)), \\ M_K & \text{(otherwise)} \end{cases} $$

Here $c_i(M) = M_{[1,i]} - M_{[1,i+1]\setminus\{i\}}$.

In the last of this talk, we will give a sketch of a new proof of this theorem.
§ From Realization I or II to Realization III

1. e-BZ data and \( w_0 \)-BZ data

To make a bridge form the 1-st or 2-nd realization of \( B(\infty) \) to the 3-rd one, we introduce a notion of e-BZ data.

A BZ datum \( M' \) is called a e-BZ datum if
\[
M'_{\Lambda_i} = M'_{[1,i]} = 0 \quad \text{for } 1 \leq \forall i \leq n.
\]

\( BZ^e \): the set of all e-BZ data

- For \( M = (M_K)_{K \in M_n^\times} \in BZ^{w_0} \), set \( M^* = (M^*_K)_{K \in M_n^\times} \) by
\[
M^*_K := M_{K^c}
\]
where \( K^c := [1, n + 1] \setminus K \) is the compliment of \( K \in M_n^\times \).

Then, it is easy to check \( M^* \in BZ^e \) and the map
\[
*: BZ^{w_0} \rightarrow BZ^e
\]
gives a bijection. The inverse is also denoted by *.

- We can define a crystal structure on \( BZ^e \):
\[
\vec{e}_i^* : BZ^e \rightarrow BZ^{w_0} \rightarrow BZ^{w_0} \rightarrow BZ^e,
\]
\[
\vec{f}_i^* : BZ^e \rightarrow BZ^{w_0} \rightarrow BZ^{w_0} \rightarrow BZ^e,
\]

etc.
From I to III

Definition. Let $K = \{k_1 < k_2 < \cdots < k_l\} \in M_n^\times$ be a Maya diagram. For such $K$, we define a $K$-tableau as an upper-triangular matrix $C = (c_{p,q})_{1 \leq p \leq q \leq l}$ with integer entries satisfying

$$c_{p,p} = k_p \quad (1 \leq p \leq l),$$

and the usual monotonicity conditions for semi-standard tableaux:

$$c_{p,q} \leq c_{p,q+1}, \quad c_{p,q} < c_{p+1,q}.$$

Example.

$K = \{1, 3, 4\} \Rightarrow K$-tableaux are:

$$
\begin{pmatrix}
1 & 1 & 1 \\
3 & 3 & \\
4 &
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 1 & 2 \\
3 & 3 & \\
4 &
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 2 & 2 \\
3 & 3 & \\
4 &
\end{pmatrix}.
$$
Recall $\mathbf{a} = (a_{i,j}) \in \mathcal{B}$: “limit” of semistandard Young tableau.

For a given $\mathbf{a} = (a_{i,j}) \in \mathcal{B}$, let $\mathbf{M}(\mathbf{a}) = (M_K(\mathbf{a}))_{K \in \mathcal{M}^*}$ be a collection of integers defined by

$$M_K(\mathbf{a}) := - \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} a_{i,k_j}$$

$$+ \min \left\{ \sum_{1 \leq p < q \leq l} a_{c_p,q,c_p,q+(q-p)} \right\} \quad C = (c_{p,q}) \text{ is a K-tableau.}$$

and denote the map $\mathbf{a} \mapsto \mathbf{M}(\mathbf{a})$ by $\Psi$.

**Proposition** (Bernstein-Fomin-Zelevinsky).
For any $\mathbf{a} \in \mathcal{B}$, $\Psi(\mathbf{a}) = \mathbf{M}(\mathbf{a})$ is an $e$-BZ datum. Moreover $\Psi : \mathcal{B} \to \mathcal{BZ}^e$ is a bijection.

Moreover we can prove

**Proposition**.
The map $\Psi : \mathcal{B} \xrightarrow{\sim} \mathcal{BZ}^e$ is an isomorphism of $*$-crystals.
From II to III

Any Maya diagram $K \in \mathcal{M}_n^\times$ can be written as a disjoint union of intervals

$$K = [s_1 + 1, t_1] \cup [s_2 + 1, t_2] \cup \cdots \cup [s_l + 1, t_l]$$

$$0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_l < t_l \leq n + 1.$$ 

$K_m = [s_m + 1, t_m]$ $(1 \leq m \leq l)$: the $m$-th component of $K$.

$\text{out}(K) := \{t_m | 1 \leq m \leq l\} \cap [1, n]$,

$\text{in}(K) := \{s_m | 1 \leq m \leq l\} \cap [1, n]$.

$\Omega(K)$: the orientation so that

- an element of $\text{out}(K)$ is a source,
- an element if $\text{in}(K)$ is a sink.

Example. Let $n = 17$ and


Then we have

$\text{out}(K) = \{4, 8, 13\}$, $\text{in}(K) = \{2, 6, 9, 15\}$.

In this case, the orientation $\Omega(K)$ is given as follows:

$$\Omega(K) = \begin{array}{cccccccccccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 4 & 6 & 8 & 9 & 13 & 15 & 17
\end{array}$$

Here $\circ$ is a sink and $\bullet$ is a source. That is,

$\text{sink}(\Omega(K)) = \text{in}(K) = \{2, 6, 9, 15\}$,

$\text{source}(\Omega(K)) = \text{out}(K) \cup \{1, 17\} = \{1, 4, 8, 13, 17\}$.
For $B = (B_\tau)_{\tau \in H} \in X_V$, we set

$$M_K(B) := - \dim_c \text{Coker} \left( \bigoplus_{k \in \text{out}(K)} V_k \xrightarrow{\oplus B_\sigma} \bigoplus_{l \in \text{in}(K)} V_l \right),$$

where $\sigma$ is a path in $\Omega(K)$, and for $\Lambda \in \text{Irr} \Lambda_V$, set

$$M_K(\Lambda) := M_K(B) \quad (B \text{ is a generic point of } \Lambda).$$

**Proposition.** The family of integers $\{M_K(\Lambda)\}_{K \in \mathcal{M}_n^\times}$ is an e-BZ datum and the map $\bigsqcup_V \text{Irr } \Lambda_V \to BZ^e$ defined by

$$\Lambda \mapsto \{M_K(\Lambda)\}_{K \in \mathcal{M}_n^\times}$$

is an isomorphism of $\ast$-crystals.

In particular, for $\Lambda = \Lambda_a$ ($a = (a_{i,j}) \in \mathcal{B}$), we have

$$M_K(a) = M_K(\Lambda_a).$$
\section*{Conclusions}

There are three realizations of $B(\infty)$:

\begin{align*}
B &: \text{ "limit" of SST} \\
& \quad \downarrow \quad \downarrow \\
\mathcal{B}_Z^e &: \text{ e-BZ data} & \bigcup_V \text{ Irr } \Lambda_V &: \text{ Irred. Lagrangians.}
\end{align*}

(a) orbits $\leftrightarrow$ conormal bundles

(b) $B \sim \mathcal{B}_Z^e : a \mapsto M(a) = (M_K(a))_{k \in \mathcal{M}_n}$,

\[ M_K(a) = - \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} a_{i,k_j} \]
\[ + \min \left\{ \sum_{1 \leq p < q \leq l} a_{c_p,q,c_{p,q}+(q-p)} \right\} \]

\[ C = (c_{p,q}) \text{ is a } K\text{-tableau.} \]

(c) $\bigcup_V \text{ Irr } \Lambda_V \sim \mathcal{B}_Z^e : \Lambda \mapsto (M_K(\Lambda))_{K \in \mathcal{M}_n^\times}$,

\[ M_K(\Lambda) = - \dim_{\mathbb{C}} \operatorname{Coker} \left( \bigoplus_{k \in \text{out}(K)} V_k \oplus_{\sigma} V_l \right) \]

- $M_K(a) = M_K(\Lambda_a)$

\[ \Rightarrow \] The above is a commutative diagram.
§ Applications

○ A new proof of AM conjecture

The AM conjecture (proved by Kamnizter) can be re-written as follows:

**Corollary** (e-BZ datum version).

Let $M = (M_K) \in BZ^e$. For each $i \in I$, we have

$$
(f_i^* M)_K = \begin{cases} 
\min \{M_K, M_{s_iK} + c^*_i(M)\} & (K \in M^\times_n(i)^*) \\
M'_K & (\text{otherwise}) 
\end{cases}
$$

Here

$$
M^\times_n(i)^* = \{K \in M^\times_n | i \notin K \text{ and } i + 1 \in K\},
$$

$$
c^*_i(M) = M_{[1,i]^e} - M_{([1, i+1] \setminus \{i\})^e} - 1.
$$

• By using a Lagrangian realization of (e-)BZ data, we can easily check that

$$
(f_i^* M)_K = M_K \quad (K \notin M^\times_n(i)^*).
$$

$\Rightarrow$ The remaining problem is:

$$
(f_i^* M)_K = \min \{M_K, M_{s_iK} + c^*_i(M)\} \quad (K \in M^\times_n(i)^*). \quad (#)
$$
Lemma .

(\#) ⇔ For any $K \in \mathcal{M}_n^\times(i)^*$,
$$M_K(\Lambda) = \min \left\{ M_K(\overline{\Lambda}), M_{s_iK}(\overline{\Lambda}) + \langle h_i, \text{wt}(\overline{\Lambda}) \rangle - \varepsilon_i^*(\Lambda) \right\},$$
where $\overline{\Lambda} = \overline{e}^* \max \Lambda$.

Proposition .
The formula (\#\#) holds for any $K \in \mathcal{M}_n^\times(i)^*$.

Key properties

- $f_i^* \Lambda = \overline{\Lambda}$.
- By the definition, $i$ is a source in $\Omega(s_iK)$.
  $$\Rightarrow M_{s_iK}(\overline{\Lambda}) = M_{s_iK}(\Lambda).$$
- Let $\pi : V \rightarrow \overline{V}$ be a surjective linear map, and $\psi : N \rightarrow \overline{V}$ a linear map. Consider a generic map $\varphi : N \rightarrow V$ such that $\psi = \pi \circ \varphi$.
  $$\begin{array}{c}
  N \\
  \varphi \downarrow
  \psi \\
  V \rightarrow \overline{V}
  \end{array}$$
  $$\Rightarrow \dim \ker \varphi = \max \{ \dim \ker \psi - (\dim V - \dim \overline{V}), 0 \}.$$
§ Future problems

- \( A_{n-1}^{(1)} \) case

- Realization I and II : known

- Realization III :
  There is no corresponding affine Grassmannian.
  \( \Rightarrow \) There is no MV cycle.

But, there exists an affine analogue of BZ datum.
  (Naito-Sagaki’s unpublished result : \( A_\infty \)-case \( \rightarrow n \)-reduction)

In affine case,

- I \( \leftrightarrow \) III : OK.

- II \( \leftrightarrow \) III : not yet (partially done).

- Beck-Nakajima’s affine PBW basis ?
  (In \( A_n \)-case, Realization I is closely related to the theory of PBW basis.)

- Other (finite or affine) types?