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A STABLE DIFFERENCE SCHEME FOR COMPUTING MOTION OF LEVEL SURFACES BY THE MEAN CURVATURE

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ABSTRACT. A difference scheme is introduced for computing the motion of level surfaces moved by the mean curvature. This scheme is proved to be stable in the maximum norm so that the computation can be completed without overflow.

§1. Introduction.

In the research fields of applied sciences like physics, engineering and biology, it is important to track the evolution (motion) of a surface, such as the interface between two kind of materials or two different phases of a certain kind of material. The problem how to track and compute the motion of a surface with a curvature-dependent speed is usually a key point in the studies. Our scheme is based on a level set method. Such a method is developed numerically by Osher and Sethian [OS],[S] and analytically by Chen, Giga and Goto [CGG1] and Evans and Spruck [ES]. See also [CGG2,3]. There are many works now available on this method but we do not try to mention all of them for lack of spaces. If the evolution depends on curvature, the scheme need not be monotone so the convergence to the analytic solution (viscosity solution) is not at all clear (cf. [CL]). In fact as explained later, some of the numerical solutions of the scheme in [OS] may not converge to analytic solutions in uniform topology.

In this paper, we introduce a little bit different scheme reflecting divergence structure and prove its stability in maximum norm for mean curvature flow problems.

We consider the Cauchy problem of the mean curvature flow equation

$$\text{(E)} \qquad u_t = |\nabla u| div \left(\frac{\nabla u}{|\nabla u|} \right), \qquad (t,x) \in Q = (0,\infty) \times \mathbb{R}^N$$

$$\text{(IV)} \qquad u(0,x) = u_0, \qquad x \in \mathbb{R}^N.$$

Let $u = u(t,x)$ be a continuous viscosity solution of (E)-(IV) which takes a negative constant for large $|x|$, which holds true if the initial value u_0 is assumed to have such property. If for each $t \in (0,\infty)$ there is a bounded open set $D(t)$ in $\mathbb{R}^N$ such that $u(t,x) > 0$ for $x \in D(t)$ and $u(t,x) < 0$ for $x \notin \overline{D(t)}$, then the 0-level set $\Gamma_t = \{x; u(t,x) = 0\}$ of $u(t,x)$ determines a closed surface which moves with a speed $V = (n-1)H$ at each point $x \in \Gamma_t$ where $H(t,x)$ is the mean curvature vector at $x \in \Gamma_t$, provided that $\nabla u \neq 0$ on Γ_t. The global existence and uniqueness of the viscosity solution to (E)-(IV) have been proved by Chen, Giga & Goto [CGG1] and Evans & Spruck [ES]. And more important thing is that the level set Γ_t is uniquely determined by its initial data Γ_0 which is independent of the choice of its *defining function* u_0, provided that u_0 is bounded, continuous and $u_0 > 0$ for $x \in D(0), u_0 < 0$ for $x \notin \overline{D(0)}$ and $\Gamma_0 = \{x; u_0(x) = 0\}$. Moreover, in [CGG1] these results are proved for a general geometric evolution.

Here, we discuss the difference methods for computing $u(t,x)$, the viscosity solution of (E)-(IV) To overcome the difficulty of taking 0 value in the denominator which will cause errors in computers and stop the computation, we introduce a parameter $\delta > 0$ and consider the difference approximation of a modified equation

$$(\mathrm{E}_\delta) \qquad u_t = |\nabla u| div \left(\frac{\nabla u}{(|\nabla u|^\sigma + \delta)^{1/\sigma}} \right), \quad (t,x) \in Q = (0,\infty) \times \mathbb{R}^N$$

with the same initial value (IV). Here, $\sigma \geqq 1$ is fixed. It can be shown that the viscosity solution of (E_δ) with (IV) tends to that of (E) when $\delta \to 0$. Thus, it is reasonable to deal with the computation of the solution of (E_δ) as an approximation of the solution of (E) with the same initial value (IV), for a sufficiently small $\delta > 0$ (say, $\delta = 10^{-50}$).

There are several methods now available to compute evolving surfaces moved by mean curvature. Using parametrization of surface a finite element method was studied by Deckelnick and Dziuk[DD]. See also [D]. Although the convergence of the scheme is proved, this scheme does not track the evolution after it develops singularities. A finite element method for level set equations was studied by Walkington [W] based on co-volume method. Another way is to compute Allen-Cahn type reaction diffusion equation or its modification. Such a calculation is done by Nochetto and his collaborators [NPV] and it is very good to track the evolution after it experiences singularities. Another method related to the level set method is introduced by Bence, Merriman and Osher [BMO] where heat equation is used to study the motion by mean curvature. Its convergence is proved by Evans [E] and others.

We thank GARC for giving the opportunity for publication of this short note. The main part of this paper was completed in 1991 but because of personal problems of first two authors the completion of the paper had been delayed.

§2. Difference schemes for the mean curvature flow equation.

Now we introduce our difference scheme for (E_δ), and for simplicity we interpret the scheme here for the two dimensional case $N = 2$. Our difference equation for (E_δ) is given by

$$(1)\qquad \begin{gathered} \frac{u_{jk}^{n+1} - u_{jk}^{n}}{\tau} = g_{jk}^{n} \sum_{i=1}^{2} Di \left(\frac{D_i u_{jk}^{n+\theta}}{((g_{jk}^{n})^{\sigma} + \delta)^{1/\sigma}} \right); \\ j, k = 0, \pm 1, \pm 2, \cdots ;\ n = 0, 1, 2, \cdots ; \\ u_{jk}^{0} = u_0(x_j, y_k), \quad j, k = 0, \pm 1, \pm 2, \cdots . \end{gathered}$$

Here, several notations have been introduced as below. Denoting by x and y the spatial variables in $\mathbb{R}^2$, we use x_j and y_k for the spatial coordinates of the net points.

NOTATION:

- $\tau \equiv \Delta\, t > 0$: increment of the time variable t;
- $t_n = n\tau$: nth time step;
- h_1, h_2 : mesh sizes of x and y directions, respectively;
- $(x_j, y_k) = (jh_1, kh_2)$: net point in $\mathbb{R}^2$, $j, k = 0, \pm 1, \pm 2, \cdots$;
- u_{jk}^{n} : value of the difference solution approximate to $u(t_n, x_j, y_k)$;
- $D_1 u_{jk}^{n} = (u_{j+\frac{1}{2},k}^{n} - u_{j-\frac{1}{2},k}^{n})/h_1, D_2 u_{jk}^{n} = \left(u_{j,k+\frac{1}{2}}^{n} - u_{j,k-\frac{1}{2}}^{n}\right)/h_2$: the approximations to $u_x(t_n, x_j, y_k)$ and $u_y(t_n, x_j, y_k)$ by central difference approach, respectively;
- $g_{jk}^{n} \equiv g(Du_{jk}^{n})$: discretization of $|\nabla u|$ at (t_n, x_j, y_k), which is chosen positive definite for $\{D_i^{\pm} u_{jk}^{n};\ i = 1, \cdots, N\}$, where D^+ and D^- denote the standard forward and backward differences, respectively.

For instance, we may take

$$g_{jk}^{n} = \left(\frac{1}{4} \sum_{i=1}^{N} (|D_i^{+} u_{jk}^{n}| + |D_i^{-} u_{jk}^{n}|)^2 \right)^{\frac{1}{2}}, \quad \text{or}$$

$$g_{jk}^{n} = \max_{1 \leqq i \leqq N} \{ |D_i^{+} u_{jk}^{n}|, |D_i^{-} u_{jk}^{n}| \}.$$

The notation u^θ here denotes $\theta u^{n+1}+(1-\theta)u^n$ for a fixed parameter $\theta \in [0,1]$, and the difference equation (1) is explicit for u^{n+1} if $\theta = 0$, while implicit if $0 < \theta \leqq 1$.

We can prove a sufficient condition for the L^∞ stability of maximum principle type for the difference scheme (1), as the following

Theorem 1. *The difference scheme* (1) *is stable in the sense of* $\|u^n\|_\infty \leqq \|u^0\|_\infty$ *if either* $\theta = 1$ *or* $4\tau(1/h_1^2 + 1/h_2^2) \leqq 1/(1-\theta)$ *when* $0 \leqq \theta < 1$, *where* $\|u^n\|_\infty = \sup_{j,k} |u^n_{jk}|$.

It is sometimes convenient and economic to deal with a surface of rotation in a lower dimension space. Here, to compute the motion of a surface with axisymmetry in $\mathbb{R}^3$, i.e., surfaces of rotation, we rewrite (E) into

$$(\mathrm{E}_r) \qquad u_t = |\tilde{\nabla} u| \widetilde{div}\left(\frac{\tilde{\nabla} u}{|\tilde{\nabla} u|}\right) + \frac{1}{r} u_r, \quad (t,r,z) \in Q = (0,\infty)\times(0,\infty)\times\mathbb{R}$$

where $r = \sqrt{x^2+y^2}$ for $(x,y,z) \in \mathbb{R}^3$, and the differential operators $\tilde{\nabla}$ and $\widetilde{div}$ are those with respect to $(r,z) \in [0,\infty)\times\mathbb{R}$.

For this equation, our difference scheme is constructed as

$$(2) \qquad \begin{aligned} \frac{u^{n+1}_{0k} - u^n_{0k}}{\tau} &= g^n_{0k} \sum_{i=1}^{2} D_i\left(\frac{D_i u^{n+\theta}_{0k}}{[(g^n_{0k})^\sigma + \delta]^{1/\sigma}}\right) + \frac{2(u^{n+\theta}_{1k} - u^{n+\theta}_{0k})}{h_1^2}, \\ & k = 0, \pm 1, \pm 2, \cdots;\ n = 0,1,2,\cdots; \\ \frac{u^{n+1}_{jk} - u^n_{jk}}{\tau} &= g^n_{jk} \sum_{i=1}^{2} D_i\left(\frac{D_i u^{n+\theta}_{jk}}{[(g^n_{jk})^\sigma + \delta]^{1/\sigma}}\right) + \frac{1}{r_j}\cdot\frac{u^{n+\theta}_{j+1,k} - u^{n+\theta}_{jk}}{h_1}, \\ & k = 0, \pm 1, \pm 2, \cdots;\ j = 1,2,\cdots;\ n = 0,1,2,\cdots; \\ u^0_{jk} &= u_0(x_j, y_k), \quad k = 0,\pm 1, \pm 2, \cdots;\ j = 0,1,2,\cdots. \end{aligned}$$

where u^n_{jk} is the approximation of $u(t_n, r_j, z_k), (r_j, z_k) = (jh_1, kh_2)$, and D_1 and D_2 denote the difference operators for ∂_r and ∂_z. Here, the last term of the first equation is used for the approximation of $\frac{u_r}{r}$ on the points of $j = 0$ so that $r_j = 0$. The limit relation $\lim_{r\to 0} \frac{u_r}{r} = u_{rr}$ and the symmetricity $u^n_{-1,k} = u^n_{1k}$ which follows from $u(t,r,z) = u(t,-r,z)$ and thus $u_r(t,0,z) = 0$ are also applied so that

$$\frac{u^{n+\theta}_{1k} - 2u^{n+\theta}_{0k} + u^{n+\theta}_{-1,k}}{h_1^2} = \frac{2(u^{n+\theta}_{1k} - u^{n+\theta}_{0k})}{h_1^2}.$$

On the other hand, the last term of the second equation is derived by the upwind difference approach (with forward difference approximation here) to $\frac{u_r}{r}$ for $j > 0$.

This type of difference approximation is also used in [C] in the discretization of the Laplacian for the axisymmetric solutions.

It both cases, the schemes (1) and (2), the computation is carried out in a restricted domain, say, a rectangular domain. When the initial value function has a negative constant value $-C$ for large $|x|$, so dose the solution $u(t,x)$ with a same constant value. Then we can use this convenient property in our computation to make the work simpler. We have only to compute the value in a (sufficiently large) finite domain, on the boundary of which the solution has a negative constant value. So, we can deal with the boundary condition either as the Dirichlet condition or as the Neumann condition.

The stability condition for the difference scheme (2) is given by

Theorem 2. *The difference equation* (2) *is stable if either*

$$\theta = 1 \quad or \quad 6\frac{\tau}{h_1^2} + 4\frac{\tau}{h_2^2} \leqq \frac{1}{1-\theta} \quad when\ 0 \leqq \theta < 1.$$

It can be seen from Theorems 1 and 2 that the "linearly" full implicit scheme (the case when $\theta = 1$) is absolutely stable, with no restriction to the mesh size and the time increment.

§3. The proof for the stability of the difference scheme.

Here we give the proof of Theorem 2, and Theorem 1 can be proved in the same way.

The difference equations (2) can be rewritten as

$$\frac{u_{0k}^{n+1}-u_{0k}^{n}}{\tau}=g_{0k}^{n}\left\{\frac{2}{h_1^2}\frac{u_{1k}^{n+\theta}-u_{0k}^{n+\theta}}{\left[\left(g_{\frac{1}{2},k}^{n}\right)^{\sigma}+\delta\right]^{1/\sigma}}+\frac{1}{h_2^2}\left(\frac{u_{0,k+1}^{n+\theta}-u_{0k}^{n+\theta}}{\left[\left(g_{0,k+\frac{1}{2}}^{n}\right)^{\sigma}+\delta\right]^{1/\sigma}}-\frac{u_{0k}^{n+\theta}-u_{0,k-1}^{n+\theta}}{\left[\left(g_{0,k-\frac{1}{2}}^{n}\right)^{\sigma}+\delta\right]^{1/\sigma}}\right)\right\}+\frac{2\left(u_{1k}^{n+\theta}-u_{0,k}^{n+\theta}\right)}{h_1^2}$$

$$(n=0,1,2,\cdots;\ k=0,\pm1,\pm2,\cdots),$$

$$\text{(3)}\quad \frac{u_{jk}^{n+1}-u_{jk}^{n}}{\tau}=g_{jk}^{n}\left\{\frac{1}{h_1^2}\left(\frac{u_{j+1,k}^{n+\theta}-u_{jk}^{n+\theta}}{\left[\left(g_{j+\frac{1}{2},k}^{n}\right)^{\sigma}+\delta\right]^{1/\sigma}}-\frac{u_{jk}^{n+\theta}-u_{j-1,k}^{n+\theta}}{\left[\left(g_{j-\frac{1}{2},k}^{n}\right)^{\sigma}+\delta\right]^{1/\sigma}}\right)+\frac{1}{h_2^2}\left(\frac{u_{j,k+1}^{n+\theta}-u_{jk}^{n+\theta}}{\left[\left(g_{j,k+\frac{1}{2}}^{n}\right)^{\sigma}+\delta\right]^{1/\sigma}}-\frac{u_{jk}^{n+\theta}-u_{j,k-1}^{n+\theta}}{\left[\left(g_{j,k-\frac{1}{2}}^{n}\right)^{\sigma}+\delta\right]^{1/\sigma}}\right)\right\}+\frac{1}{r_j}\cdot\frac{u_{j+1,k}^{n+\theta}-u_{jk}^{n+\theta}}{h_1}$$

$$(n=0,1,2,\cdots;\ j=1,2,\cdots;k=0,\pm1,\pm2,\cdots),$$

where the values of g at the fraction points are defined by

$$\text{(4)}\qquad g_{j\pm\frac{1}{2},k}^{n}=\frac{1}{2}(g_{jk}^{n}+g_{j\pm1,k}^{n}),\quad g_{j,k\pm\frac{1}{2}}^{n}=\frac{1}{2}(g_{j,k}^{n}+g_{j,k\pm1}^{n}).$$

By introducing

$$\alpha=\frac{\tau}{h_1^2},\qquad \beta=\frac{\tau}{h_2^2},$$

$$\text{(5)}\qquad a^{\pm}=a_{njk}^{\pm}=\frac{\alpha g_{jk}^{n}}{\left[\left(g_{j\pm\frac{1}{2},k}^{n}\right)^{\sigma}+\delta\right]^{1/\sigma}},$$

$$b^{\pm}=b_{njk}^{\pm}=\frac{\beta g_{jk}^{n}}{[(g_{j,k\pm\frac{1}{2}}^{n})^{\sigma}+\delta]^{1/\sigma}}$$

at each (t_n, r_j, z_k), the equations (3) can be changed to

$$
\begin{aligned}
u_{0k}^{n+1} &= \{1-(1-\theta)(2a^+ + b^+ + b^- + 2_\alpha)\}u_{0k}^n \\
&\quad + (1-\theta)\{(2a^+ + 2\alpha)u_{1k}^n + b^+ u_{0,k+1}^n + b^- u_{0,k-1}^n\} \\
&\quad + \theta\{(2a^+ + 2\alpha)u_{1k}^{n+1} + b^+ u_{0,k+1}^{n+1} + b^- u_{0,k-1}^{n+1} \\
&\qquad - (2a^+ + b^+ + b^- + 2\alpha)u_{0k}^{n+1}\}, \\
&\qquad (n = 0, 1, 2, \cdots;\ k = 0, \pm 1, \pm 2, \cdots), \\
u_{jk}^{n+1} &= \{1-(1-\theta)(a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})\}u_{jk}^n \\
&\quad + (1-\theta)\{(a^+ + \frac{\alpha}{j})u_{j+1,k}^n + a^- u_{j-1,k}^n + b^+ u_{j,k+1}^n + b^- u_{j,k-1}^n\} \\
&\quad + \theta\{(a^+ + \frac{\alpha}{j})u_{j+1,k}^{n+1} + a^- u_{j-1,k}^{n+1} + b^+ u_{j,k+1}^{n+1} + b^- u_{j,k-1}^{n+1} \\
&\qquad - (a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})u_{jk}^{n+1}\}, \\
&\qquad (n = 0, 1, 2, \cdots;\ j = 1, 2, \cdots;\ k = 0, \pm 1, \pm 2, \cdots), \\
u_{jk}^0 &= u_0(r_j, z_k) \quad (j = 0, 1, 2, \cdots; k = 0, \pm 1, \pm 2, \cdots).
\end{aligned} \tag{6}
$$

Noting that $g_{jk}^n \geqq 0$ and $\delta > 0$ we can get the following

Lemma 1.

$$0 \leqq a^\pm = a_{njk}^\pm < 2\alpha, \quad 0 \leqq b^\pm = b_{njk}^\pm < 2\beta.$$

Thus, we can show the following maximum results as

Lemma 2. *If the values of θ, $a^\pm = a_{njk}^\pm$ and $b^\pm = b_{njk}^\pm$ satisfies*

$$1 - (1-\theta)(a^+ + a^- + b^+ + b^- + 2\alpha) \geqq 0, \tag{7}$$

then the difference solution of (2) satisfies

$$
\begin{aligned}
\inf_{j,k} u_{jk}^n &\geqq \inf_{j,k} u_{jk}^0, \\
\max_{j,k} u_{jk}^n &\leqq \max_{j,k} u_{jk}^0, \quad n = 1, 2, \cdots.
\end{aligned}
$$

Proof of Lemma 2. Rewriting the difference equations (6) and noting that $a^+ = a^-$ for $j = 0$, we get

$$
\begin{aligned}
&\{1 + \theta(a^+ + a^- + b^+ + b^- + 2\alpha)\}u_{0k}^{n+1} \\
&\quad = \{1 - (1-\theta)(a^+ + a^- + b^+ + b^- + 2\alpha)\}u_{0k}^{n} \\
&\qquad + (1-\theta)\{(a^+ + a^- + 2\alpha)u_{1k}^{n} + b^+ u_{0,k+1}^{n} + b^- u_{0,k-1}^{n}\} \\
&\qquad + \theta\{(a^+ + a^- + 2\alpha)u_{1k}^{n+1} + b^+ u_{0,k+1}^{n+1} + b^- u_{0,k-1}^{n+1}\}, \\
&\qquad\quad (n = 0,1,2,\cdots; k = 0,\pm1,\pm2,\cdots), \\
&\{1 + \theta(a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})\}u_{jk}^{n+1} \\
&\quad = \{1 - (1-\theta)(a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})\}u_{jk}^{n} \\
&\qquad + (1-\theta)\{(a^+ + \frac{\alpha}{j})u_{j+1,k}^{n} + a^- u_{j-1,k}^{n} + b^+ u_{j,k+1}^{n} + b^- u_{j,k-1}^{n}\} \\
&\qquad + \theta\{(a^+ + \frac{\alpha}{j})u_{j+1,k}^{n+1} + a^- u_{j-1,k}^{n+1} + b^+ u_{j,k+1}^{n+1} + b^- u_{j,k-1}^{n+1}\}, \\
&\qquad\quad (n = 0,1,2,\cdots; j = 1,2,\cdots; k = 0,\pm1,\pm2,\cdots), \\
&u_{jk}^{0} = u_0(r_j, z_k) \quad (j = 0,1,2,\cdots; k = 0,\pm1,\pm2,\cdots).
\end{aligned} \tag{8}
$$

Note that we have done our computation in a certain finite domain and on the boundary of the domain the value of the solution is a constant $-C$. Here, we claim that the maximum and minimum value of the difference solution u_{jk}^n are reached on the boundary, or the initial plane (i.e. $n = 0$). This is shown in the following way. If $j > 0$, since $\alpha/j < 2\alpha$, we have $\{1 - (1-\theta)(a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})\} \geqq 0$ and in a inner (mesh) point of the domain, say (t_n, r_j, z_k), the difference solution satisfies

$$
\begin{aligned}
&\{1+\theta(a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})\}u_{jk}^{n+1} \\
&\quad \leqq \{1 - (1-\theta)(a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})\}\|u^n\| \\
&\qquad + (1-\theta)\{(a^+ + \frac{\alpha}{j})\|u^n\| + a^-\|u^n\| + b^+\|u^n\| + b^-\|u^n\|\} \\
&\qquad + \theta\{(a^+ + \frac{\alpha}{j})\|u^{n+1}\| + a^-\|u^{n+1}\| + b^+\|u^{n+1}\| + b^-\|u^{n+1}\|\}, \\
&\quad = \|u^n\| + \theta(a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})\|u^{n+1}\| \\
&\qquad\qquad (n = 0,1,2,\cdots; j = 1,2,\cdots; k = 0,\pm1,\pm2,\cdots)
\end{aligned}
$$

where $\|\cdot\| = \|\cdot\|_\infty$ here and hereafter. So, we get

$$\begin{aligned}\left\{1+\theta(a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})\right\}\|u^{n+1}\| \\ \leqq \|u^n\| + \theta(a^+ + a^- + b^+ + b^- + \frac{\alpha}{j})\|u^{n+1}\|,\end{aligned}$$

which leads to

$$\|u^{n+1}\| \leqq \|u^n\|.$$

While if $j = 0$, then the estimates become

$$\begin{aligned}\{1+&\theta(2a^+ + b^+ + b^- + 2\alpha)\}u_{0k}^{n+1} \\ &\leqq \{1-(1-\theta)(2a^+ + b^+ + b^- + 2\alpha)\}\|u^n\| \\ &\quad + (1-\theta)\{(2a^+ + 2\alpha)\|u^n\| + b^+\|u^n\| + b^-\|u^n\|\} \\ &\quad + \theta\{(2a^+ + 2\alpha)\|u^{n+1}\| + b^+\|u^{n+1}\| + b^-\|u^{n+1}\|\} \\ &= \|u^n\| + \theta(2a^+ + b^+ + b^- + 2\alpha)\|u^{n+1}\| \\ &\qquad (n = 0, 1, 2, \cdots; k = 0, \pm 1, \pm 2, \cdots),\end{aligned}$$

which also leads to

$$\|u^{n+1}\| \leqq \|u^n\|.$$

Here, if we take max instead of $\|\cdot\|$ in the above argument, then we can obtain

$$\max_{j,k} u_{jk}^n \leqq \max_{j,k} u_{jk}^0.$$

And similarly we can show another inequality

$$\inf_{j,k} u_{jk}^n \geqq \inf_{j,k} u_{jk}^0.$$

Thus we have completed the proof of Lemma 2. □

From Lemma 2, we can obtain our stability condition. Either of the following conditions leads to (7), the condition $\theta = 1$ or

$$a^+ + a^- + b^+ + b^- + 2\alpha \leqq \frac{1}{1-\theta} \quad \text{if } 0 \leqq \theta < 1. \tag{9}$$

While in virtue of Lemma 1, (9) holds true if

$$6\alpha + 4\beta \leqq \frac{1}{1-\theta}$$

is satisfied. This completes the proof for Theorem 2.

To prove Theorem 1, we have only to make a little change in the estimates: simply omit the terms 2α and $\frac{\alpha}{j}$ in the statements.

§4. For the case of generalized mean curvature flow equation.

With the above-mentioned methods, we can construct a stable difference scheme for the so-called generalized mean curvature flow equation

$$(\mathrm{E}') \qquad u_t = |\nabla u| div\left(\frac{\nabla u}{|\nabla u|}\right) + \nu|\nabla u|, \quad (t,x) \in Q = (0,\infty)\times\mathbb{R}^N$$

where ν is a constant (see[CGG1]).

The difference scheme for (E′) is constructed by the following two parts:

(1) the first part of the scheme is constructed as that for (E) in the previous section;

(2) the second part of the scheme is constructed in the way of any kind of stable difference scheme with monotonicity for the Hamilton-Jacob equation $u_t = \nu|\nabla u|$, such as Lax-Friedrichs scheme, Godunov scheme, etc. (see, for example [CL]).

Then, we can show that the obtained difference scheme is stable if the value of $\tau/h_i^2 (i = 1,2)$ are taken sufficiently small ([CGH]).

§5. Some remarks.

1. It is important to note that the stability conditions do not depend on $\delta > 0$ and $\sigma \geqq 1$.
2. If g_{jk}^n is not positive definite for $D^{\pm}u_{jk}^n$, then the difference solution may not converge to the solution of (E_δ), nor to that of (E) when $\delta \to 0$.

Example. It is easy to see that

$$u(t,x) = max\{1 - (2t + |x|^2), -1\}$$

is the unique viscosity solution of the level set equation (E) with initial data

$$u_0(x) = max\{1 - |x|^2, -1\}.$$

Here, if g_{jk}^n in the scheme is replaced by a general central difference approximation, then the value of numerical solution u_{jk}^n at the origin is independent of $n \geqq 0$, because the symmetry forces the central difference at the origin equal to zero.

However, the value of the analytic solution $u(t,x)$ at the origin equals $1-2t$(for $t \leqq \frac{1}{2}$) which is smaller than 1 for $t > 0$. Thus the numerical solution u^n_{jk} does not pointwisely converge to the viscosity solution.

The same remark applies to the scheme in [OS]. Since the central difference approximation is used in [OS], the numerical solution there may not converge to the analytic solution. In that paper, in order to avoid the problem they shifted the grids so that non of the net points agrees with the center of symmetry, for the case when symmetric data are started with.

3. Osher and Sethian discussed some difference constructed in a different way with level surface approach ([OS],[S]). They computed several interesting examples including the torus and dumbbells without discussing the fundamental theory such as stability, etc. In [S], an example of unstable computation of a torus was presented with a quite large Δt but no condition for the stability was given there.
4. In [OS] and [S], the axisymmetric surfaces are computed under the rectilinear coordinates instead of the cylindrical coordinates.

With our stable difference schemes and level surface approach, we investigated motions of several typical surfaces, including the shrink of a torus(surface of a doughnut) and the break of a dumbbell. With this method we can track motions of a surface even after the time when a singularity occurs.

§6. Numerical results.

We present several computation results on the evolution of axisymmetric surfaces here: the "fat torus"and "Hamilton's dumbbell". The computation is carried out by our difference scheme (2) in its explicit form (with $\theta = 0$) while the "safety parameter"δ is taken $\delta = 10^{-50}$ and $\sigma = 2$. We restricted the domain to $(r,z) \in [0,1.5] \times [-1.5,1.5]$ which is discreted into net points with the mesh sizes $h_1 = h_2 = 0.015$ and the time increment $\tau = 0.0000125$. On the axis of symmetry $r = 0$, the boundary condition is treated with the symmetry property of the solution.

1. Fat Torus

As the analytic theory predicts ([SS]), no fattening occurs for generalized solution after it collapses. According to numerical simulation, it becomes convex and shrinks to a point. If the torus is thin, it converges to a "marriage ring"([AI]).

2. Hamilton's Dumbbell

We consider the initial surface of the form

$$\Gamma_0^\lambda = \{(x,y,z); x^2 + y^2 = (1-z^2)(1-\lambda+\lambda z^2)^2\}$$

where $0 \leqq \lambda \leqq 1$ is a parameter. We take the initial data

$$u_0 = (1-z^2)(1-\lambda+\lambda_z^2)^2 - r^2$$

and select the zero level surface to observe. If λ is close to 1, Γ_t^λ pinches into two pieces in a finite time. Then both pieces become convex. Actually, according to Altshuler, Angenent and Giga [AAG], no fattening occurs and Γ_t^λ becomes smooth instantaneously after it pinches. At last each of the pieces shrinks to one point, respectively. See the cases $\lambda = 0.8$ and $\lambda = 0.65$.

From the numerical results we can see that Γ_t^λ becomes convex and then shrinks to a single point for $\lambda = 0.63$.

In between $\lambda = 0.63$ and $\lambda = 0.65$, there may exists a critical value of λ for which the behavior of the shrinking dumbbell is exceptional. The surface does not become convex nor does it pinch. It stays smooth and shrinks to a point. The existence of such a dumbbell is proved analytically in [AAG] by topological argument. See the picture of the case $\lambda = 0.64$, which seems to be the critical value.

Recently, Nochetto et al. [NPV] calculated the evolution of Γ_t^λ by a different method. Their calculation suggests that the critical value be $\lambda = 0.654$, which is very close to ours.

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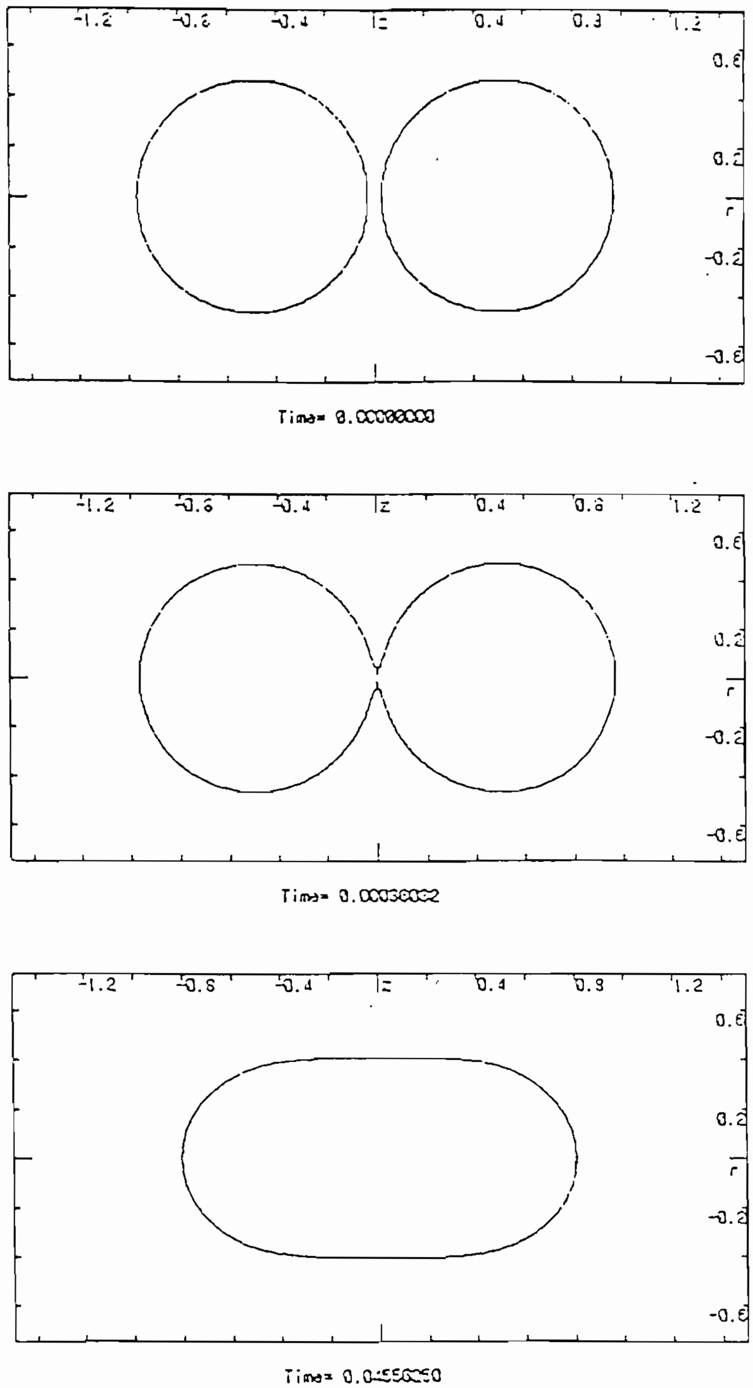

Figure 1. Evolution of a fat torus.

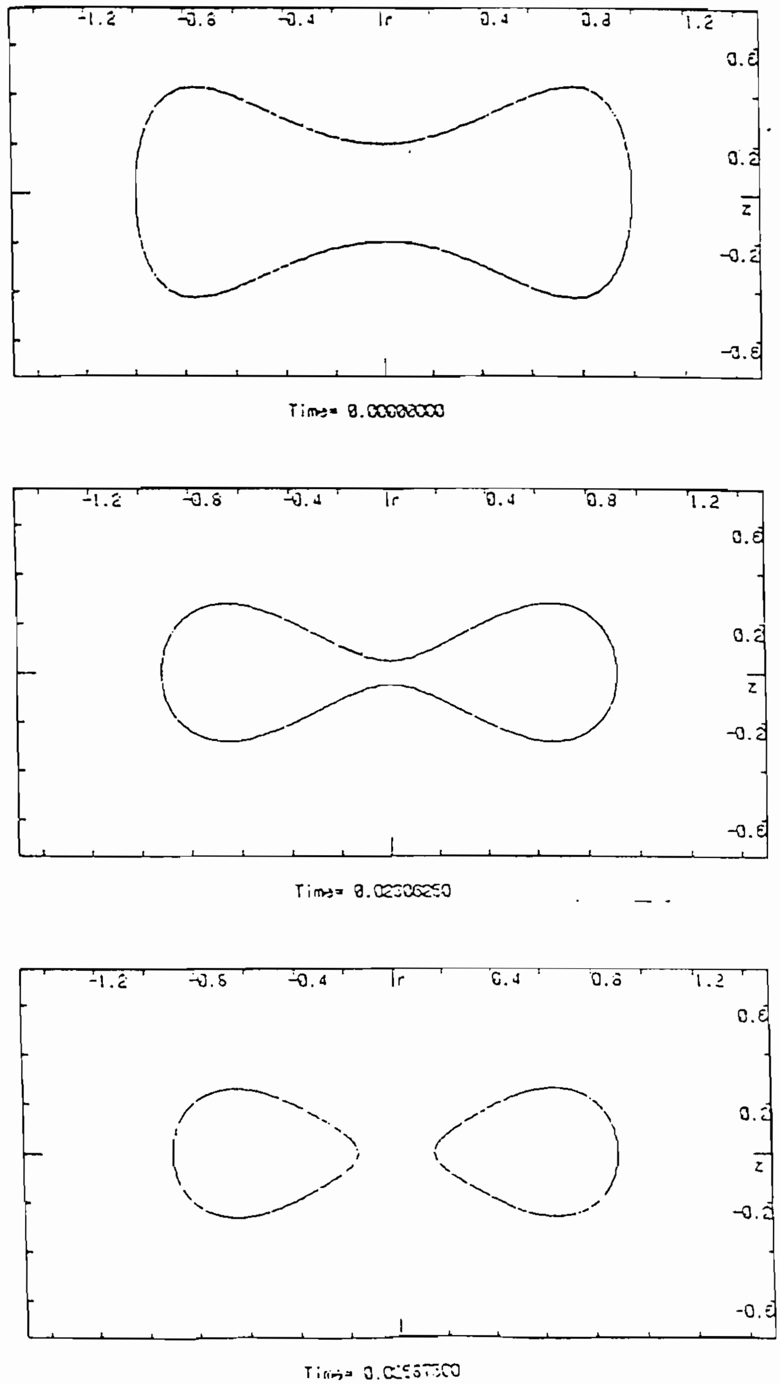

Figure 2. Evolution of Hamilton's dumbbell of the case $\lambda = 0.80$.

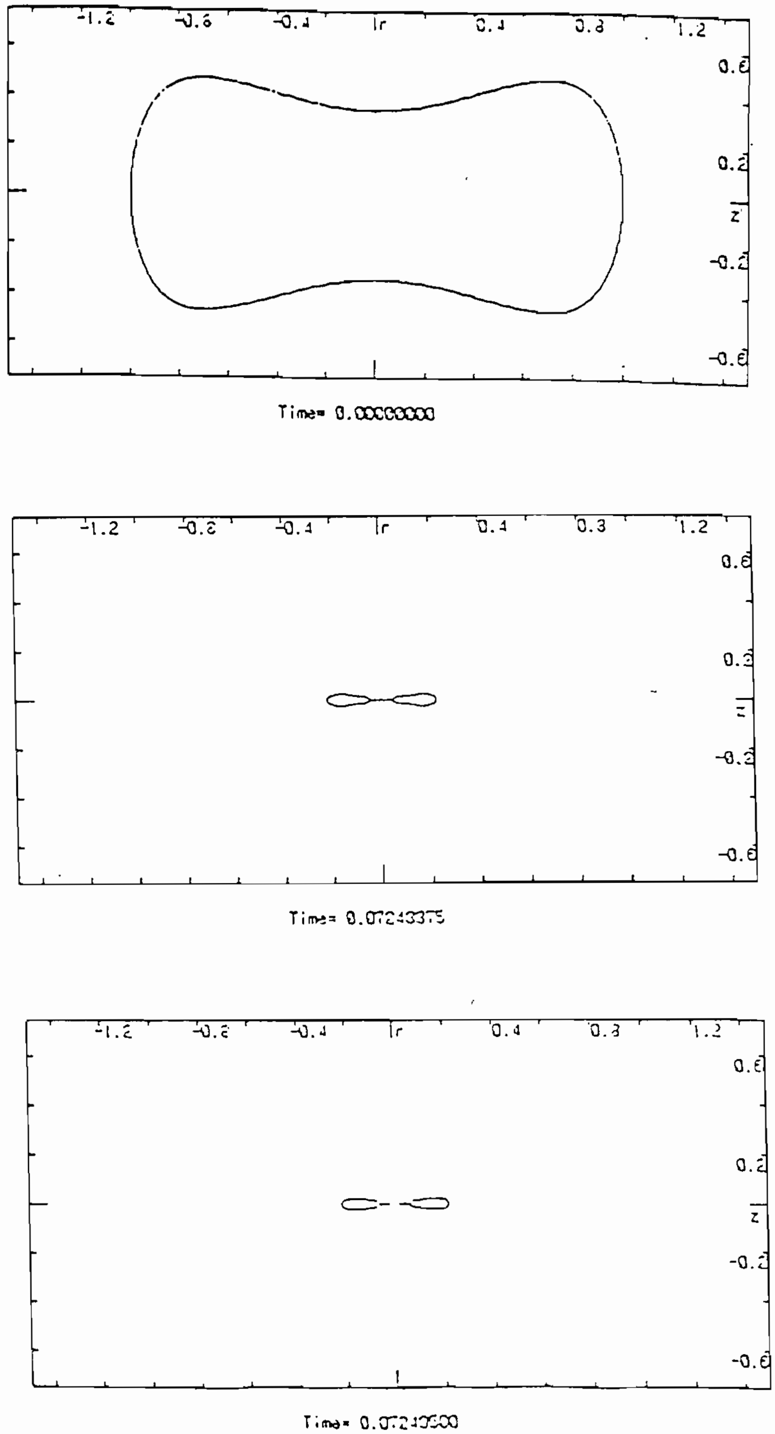

Figure 3. Evolution of Hamilton's dumbbell of the case $\lambda = 0.65$.

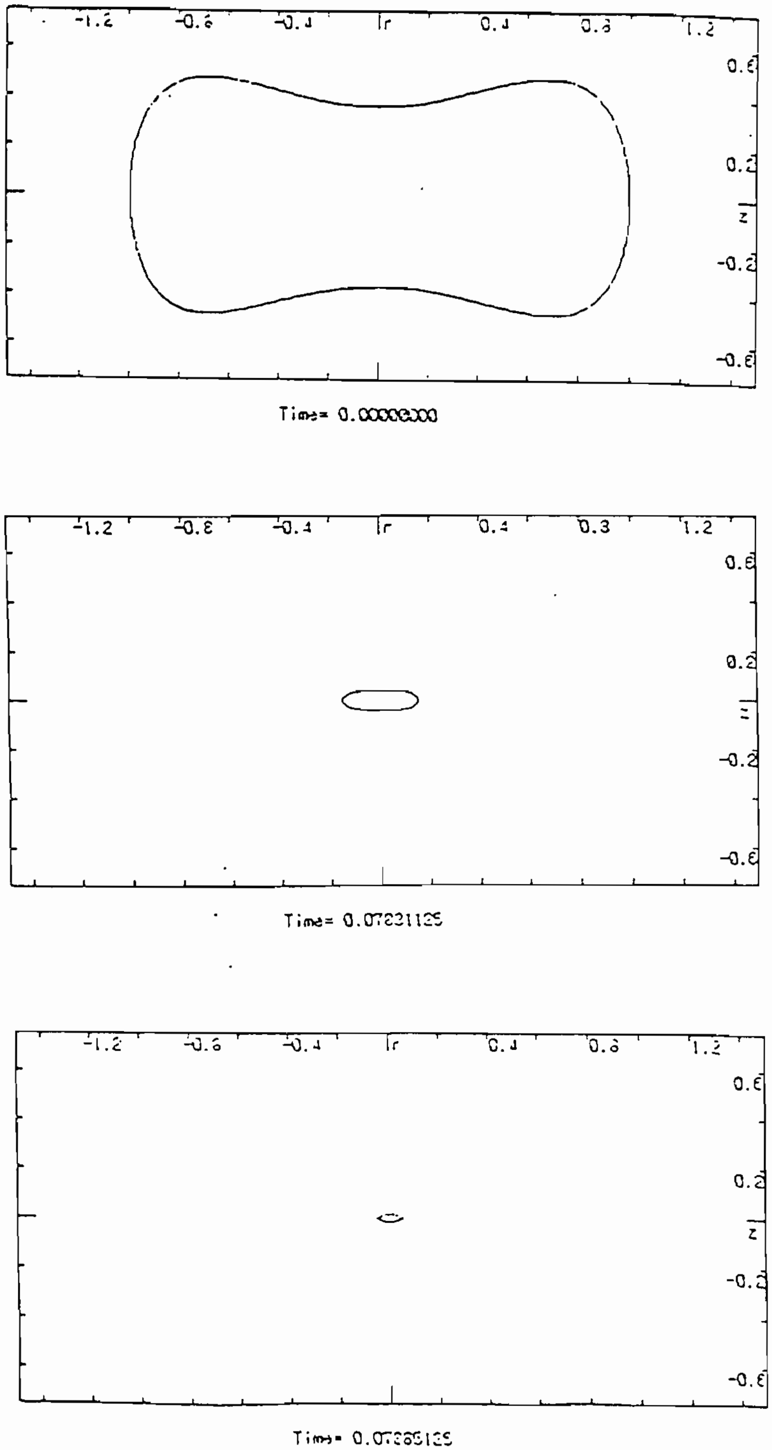

Figure 4. Evolution of Hamilton's dumbbell of the case $\lambda = 0.63$.

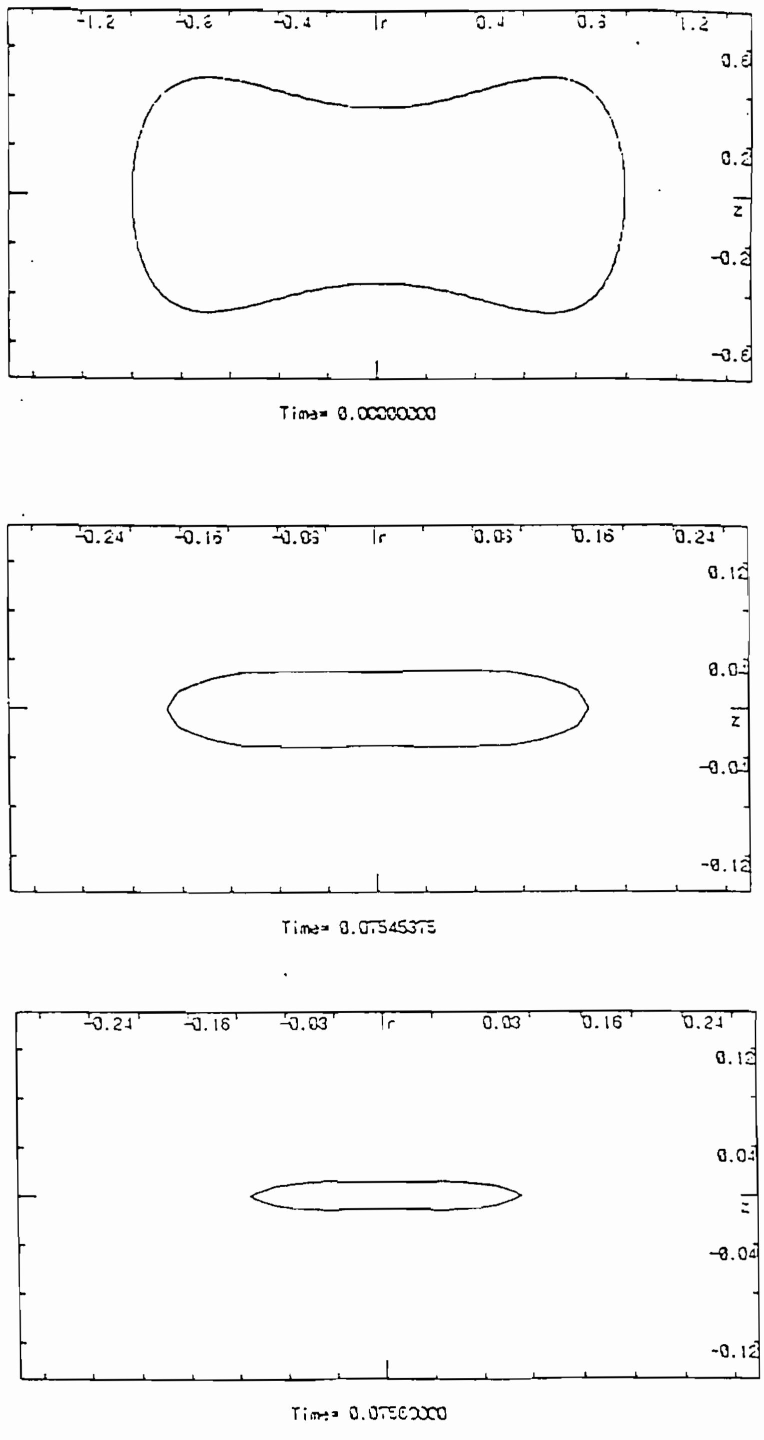

Figure 5. Evolution of Hamilton's dumbbell of the case $\lambda = 0.64$. The scale of the coordinate is changed in the last two pictures to show the details of the shrinking.

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