From subfactors to 3-dimensional topological quantum field theories and back

— a detailed account of Ocneanu's theory —

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Abstract

A full proof of Ocneanu's theorem is given that one can produce a rational unitary polyhedral 3-dimensional topological quantum field theory of Turaev-Viro type from a subfactor with finite index and finite depth, and vice versa. The key argument is an equivalence between flatness of a connection in paragroup theory and invariance of a state sum under one of the three local moves of tetrahedra. This was announced by A. Ocneanu and he gave a proof of Frobenius reciprocity and the pentagon relation, which produces a 3-dimensional TQFT via the Turaev-Viro machinery, but he has not published a proof of the converse direction of the equivalence. Details are given here along the lines suggested by him.

1 Introduction

A. Ocneanu claimed that a subfactor with finite index and finite depth produces a rational unitary polyhedral 3-dimensional topological quantum field theory (of Turaev-Viro type), and that from such a theory one can recover such a subfactor. Unfortunately, his announcements [29], [30] lack several details of a proof of this striking announcement. Although he has extensively lectured on his theory, the theory has been inaccessible for general mathematical community. The aim of this paper is to give a fully detailed proof of this claim and help a better understanding of the relation between topological invariants of 3-manifolds, subfactors, rational conformal field theories, and quantum groups.

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The discovery of the celebrated Jones polynomial for links in [20] based on his index theory for subfactors in von Neumann algebra theory [19] was the starting point of an exciting new era in 3-dimensional topology. It has turned out that this aspect of the subfactor theory is deeply related to quantum groups [8], [18] and quantum field theory [42]. Inspired by the Jones polynomial, several authors have introduced invariants of closed 3-manifolds [7], [36], [37], [40], [42]. A. Ocneanu then claimed in [29], [30] that the combinatorial structure appearing in the Turaev-Viro approach to $(2+1)$-dimensional topological quantum field theories is the same as in the paragroup theory, which he introduced for classification of subfactors as a combinatorial characterization of higher relative commutants [26], [28], [21], [22].

Following the pioneering work of Jones [19], the classification of subfactors (of the approximately finite dimensional factor of type II_1) is one of the most important and exciting problems in the theory of operator algebras. A solution to the generating property of subfactors, the most important problem from the analytic viewpoint, was first claimed in [26] in the "finite depth" case, which means finiteness of a fusion algebra arising from the subfactor and is an analogue of rationality of conformal field theory. S. Popa gave a complete proof in this case, and later proved the ultimate result along this line [33], [34], [35].

A. Ocneanu also introduced a new combinatorial structure called "paragroup" in [26]. In a sense, this is a quantization of a finite Galois group. Although [26] lacks details of the paragroup theory, the general theory has been worked out in [21], [22]. Similarity between paragroup theory and theory of exactly solvable models [2], [4] has also been exploited [10], [17], [21], [22]. This paragroup structure is one of the main topics of this paper.

We clarify the relation between axioms of paragroups and axioms of Turaev-Viro type in rational unitary $(2+1)$-dimensional topological quantum field theories based on triangulations of 3-manifolds. In the Turaev-Viro approach [37], we assign a complex number to each labeled tetrahedron appearing in a triangulation of a given 3-manifold. Then we take a weighted product of the values for all the tetrahedra in the triangulation and take a sum of the products over all the possible labelings. To prove topological invariance of this kind of state sums, we need to show that this number does not depend on the triangulation. For this independence, we need tetrahedral symmetry of the assignment of the complex numbers and invariance under the three local moves of tetrahedra as in [37] and [9]. These axioms correspond to the following conditions in paragroup theory and other theories such as solvable lattice models theory or rational conformal field theories. Our aim in this paper is to give details on these correspondences. Basic references on paragroup theory are [26], [28], [21], [22].

(1) Tetrahedral symmetry: We have to prove that we get the same value when we look at a tetrahedron from a different direction. In the bimodule approach of paragroups, this comes from **Frobenius reciprocity** of bimodules. In the string algebra approach of paragroups, this essentially corresponds to the **commuting square** condition in [31], [12]. It can be regarded as a generalization of the **crossing symmetry** in solvable lattice model theory [4] and corresponds to the symmetry condition of the **braiding matrix** in rational conformal field theory [6].

(2) Invariance under the 2nd and 3rd local moves: First, we prove that the state sum is invariant under two of the three local moves. In one of the two moves, two tetrahedra sharing two triangles collapse into two triangles, and in the other, two tetrahedra sharing three triangles collapse into one triangle. Both of these correspond to **unitarity** of a connection in paragroup theory. We have a unitarity condition in all of the following: the bimodule approach to paragroup theory, the string algebra approach to paragroup theory, solvable lattice model theory, and rational conformal field theory.

(3) Invariance under the 1st local move: We also have to prove that the state sum is invariant under the other local move where two tetrahedra sharing one triangle are transformed into three tetrahedra sharing one triangle pairwise. This invariance corresponds to **flatness**, the key notion in paragroup theory. In the case of topological invariants arising from quantum groups, this condition comes from the **pentagon relation**. In solvable lattice models, the flatness condition is closely related to the **Yang-Baxter equation**, and in rational conformal field theory, it comes from the **braiding-fusion relations** as in [6].

With this general machinery, a topological quantum field theory is obtained from each subfactor with finite index and finite depth. See [5], [10], [12], [14], [15], [16], [17], [21], [22], [26], [28], [34], [38], [39], [43] for known examples of such subfactors. See [11] for more relation between subfactors and conformal field theory.

This work began while the first author visited MSRI, Berkeley, and continued while the second author visited University College of Swansea with the financial support from MSRI and the British Council respectively. We wish to thank these institutions for their hospitality, V. F. R. Jones for useful comments on Ocneanu's announcement, T. Kohno for useful comments on topological quantum field theory, R. Nest for useful comments on his work [9], N. Yu. Reshetikhin for his kind explanations on $6j$-symbols and topological invariants, M. Rosso for useful comments on quantum groups and topological invariants, and F. Xu for helpful discussions on rational conformal field theory.

2 From subfactors to generalized $6j$-symbols

Our aim in this section is to produce generalized $6j$-symbols from a given subfactor with finite index and finite depth and show that they satisfy certain axioms. The contents of this section are already in [29], but for the sake of completeness, we list the basic definitions and properties. Proofs for all the statements in this section are found in [29] and obtained by direct computation. (Also see [45] for published proofs.)

We recall basic definitions in the subfactor theory briefly. (See [12] for more details of the basics.) The operator algebras we will use are called II_1 *factors*, which are weakly closed simple $*$-algebras, with a functional called a *trace*, of bounded linear operators on a Hilbert space. We study a *subfactor* $N \subset M$, which is an inclusion of II_1 factors. Jones [19] initiated the study of the index $[M : N]$ which is an analogue of an index of a subgroup in a group. He also introduced a notion *principal graph* of a subfactor as a graph invariant of subfactors. Ocneanu [26] later introduced a notion of *finite depth* as finiteness of the principal graph, which has later turned out to be an analogue of rationality of conformal field theories and q being a root of unity in the quantum group theory.

We fix a subfactor $N \subset M$ of type II_1 with finite index and finite depth. Numerical data will be produced from this subfactor. We do not need hyperfiniteness (approximate finite dimensionality) of the subfactor to get these data, but if the subfactor is hyperfinite, this data, with a certain equivalence relation, gives a complete invariant of the subfactor. (See [21], [22], [26], [28], [33].) We will work on bimodules, that is, Hilbert spaces with left-sided and right-sided actions of von Neumann algebras. Readers are referred to [26], [28], [32], [44], [45] for general theory of bimodules. Also recall that bimodules are analogues in von Neumann algebra theory of representations of Lie groups.

We list the basic properties of bimodules from [29]. (Also see [45].) Let ${}_AX_B$ be an A-B bimodule, where A, B are von Neumann algebras. We define an opposite bimodule $\bar{X} = {}_B\bar{X}_A$, which is the conjugate Hilbert space $\bar{X}$ with the actions $b \cdot \bar{x} \cdot a = \overline{a^* \cdot x \cdot b^*}$, where $a \in A, b \in B, x \in X$. A bimodule ${}_AX_B$ is said to be of *finite type* if $\dim({}_AX) < \infty$, $\dim(X_B) < \infty$. We put $[X] = [{}_AX_B] = \dim({}_AX) \cdot \dim(X_B)$. Then we have the following theorem [29].

Theorem 2.1 *If the bimodules ${}_AX_B$ and ${}_AY_B$ are of finite type, then $Hom({}_AX_B, {}_AY_B)$ is finite dimensional.*

Next, let ${}_AX_B$, ${}_BY_C$, ${}_AZ_C$ be irreducible bimodules of finite type, and define the intertwiner space $\mathcal{H}_{X,Y}^Z = \mathrm{Hom}({}_AX \otimes_B Y_C, {}_AZ_C)$. For the intertwiner $T \in \mathcal{H}_{X,Y}^Z$, we define the right hand side *Frobenius dual* $T^Y \in \mathcal{H}_{Z,\bar{Y}}^X$ by

$$T^Y(z \otimes_B \bar{y}) = (\dim X_B)^{1/2}(\dim Z_C)^{-1/2}\pi_r(y)^*(T^*(z)),$$

where π_r denote the right tensor multiplication. Similarly, we define the left hand side Frobenius dual ${}^XT \in \mathcal{H}_{\bar{X},Z}^Y$ for $T \in \mathcal{H}_{X,Y}^Z$ by

$${}^XT(\bar{x} \otimes_A z) = (\dim_B Y)^{1/2}(\dim_A Z)^{-1/2}\pi_l(x)^*(T^*(z)).$$

Then the following result holds.

Theorem 2.2 *The Frobenius duality map $\cdot^Y : \mathcal{H}_{X,Y}^Z \to \mathcal{H}_{Z,\bar{Y}}^X$ is a conjugate linear isomorphism of left A-modules preserving norms.*

For $T \in \mathcal{H}_{X,Y}^Z$, we define the conjugate intertwiner $\bar{T} \in \mathcal{H}_{\bar{Y},\bar{X}}^{\bar{Z}}$ by $\bar{T}(\bar{y} \otimes_B \bar{x}) = \overline{T(x \otimes_B y)}$. With this, the following theorem also holds.

Theorem 2.3 *For the action of the symmetric group S_3 on the set of von Neumann algebras $\{A, B, C\}$, the Frobenius duality maps gives an action of S_3 by isometric isomorphisms among the following spaces:*

$$\mathcal{H}_{X,Y}^Z, \mathcal{H}_{\bar{Z},X}^{\bar{Y}}, \mathcal{H}_{Y,\bar{Z}}^{\bar{X}}, \mathcal{H}_{\bar{Y},\bar{X}}^{\bar{Z}}, \mathcal{H}_{\bar{X},Z}^{Y}, \mathcal{H}_{Z,\bar{Y}}^{X}.$$

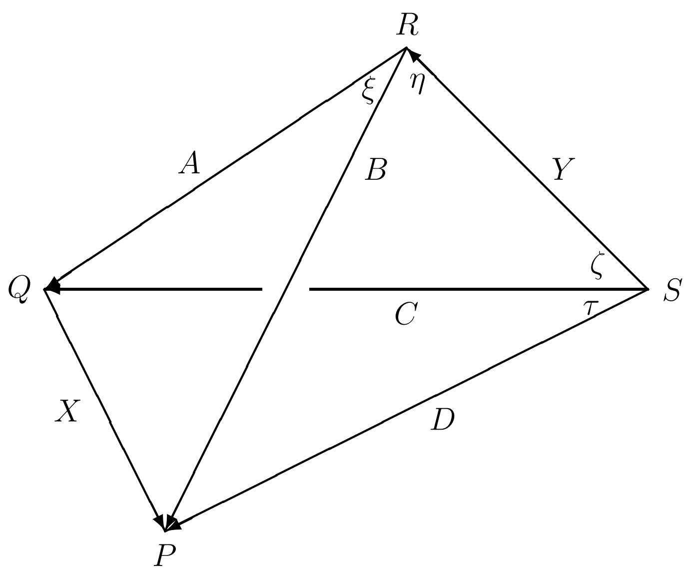
The odd permutations correspond to conjugate linear isomorphisms.

Now we start from a subfactor $N \subset M$. Regard $L^2(M)$ as an N-M-bimodule. For simplicity, we denote this by ${}_N M_M = H$. Similarly, we have ${}_M M_N = \bar{H}$. We take finite tensor powers $\cdots H \otimes \bar{H} \otimes H \otimes \bar{H} \cdots$ and decompose them into irreducible bimodules as in [26, pp. 122–123] or [28, p. 4]. Note that we have four kinds of bimodules; N-N, N-M, M-N, M-M bimodules. The finite depth assumption means that we get only finitely many (equivalence classes of) bimodules in this way. We work on this class of finitely many bimodules (of four kinds).

Among this class, we have two special bimodules. One is ${}_N L^2(N)_N$ and the other is ${}_M L^2(M)_M$. We denote them by ${}_N *_N$ and ${}_M *_M$ respectively. These are identities for tensor product operation in the following sense. We have ${}_A X \otimes_N *_N \cong {}_A X_N$, ${}_N * \otimes_N X_A \cong {}_N X_A$, ${}_A X \otimes_M *_M \cong {}_A X_M$, ${}_M * \otimes_M X_A \cong {}_M X_A$, where $A \in \{M, N\}$.

Take three bimodules X, Y, Z among this class. Put $N_{X,Y}^Z = \dim \mathcal{H}_{X,Y}^Z$ and suppose it is not zero. We choose an orthonormal basis $\{\xi_i\}_i$ for the space $\mathcal{H}_{X,Y}^Z$ consisting of coisometries with mutually orthogonal supports; $\xi_i \xi_j^* = \delta_{ij} 1_Z$, $\sum_i \xi_i^* \xi_i = 1_{X \otimes Y}$. When we refer to "intertwiners" in the remainder of §2 and §3, we always mean intertwiners appearing in these bases. If X or Y is $*$, we choose the intertwiner in a trivial way. For example, the intertwiner from ${}_M X \otimes_N *_N$ to ${}_M X_N$ sends $x \otimes n$ to $x \cdot n$, where $x \in X, n \in N$.

Choose four algebras $P, Q, R, S \in \{M, N\}$, six bimodules ${}_Q A_R$, ${}_P B_R$, ${}_Q C_S$, ${}_P D_S$, ${}_P X_Q$, ${}_R Y_S$, from the above sets of bimodules, and four intertwiners $\xi \in \mathcal{H}_{X,A}^B$, $\eta \in \mathcal{H}_{B,Y}^D$, $\zeta \in \mathcal{H}_{A,Y}^C$, $\tau \in \mathcal{H}_{X,C}^D$. We represent $(P, Q, R, S; A, B, C, D, X, Y; \xi, \eta, \zeta, \tau)$ as an oriented tetrahedron with vertices P, Q, R, S, edges A, B, C, D, X, Y, and faces ξ, η, ζ, τ as in the following figure.



Then the composition $\eta(\xi \otimes 1_Y)(1_X \otimes \zeta)^* \tau^*$ gives an endomorphism of ${}_P D_S$, which is a scalar by irreducibility. The generalized $6j$-*symbol* is defined by

$$W(A, B, C, D, X, Y \mid \xi, \eta, \zeta, \tau) = \eta(\xi \otimes 1_Y)(1_X \otimes \zeta)^* \tau^*,$$

and the normalized $6j$-symbol Z by

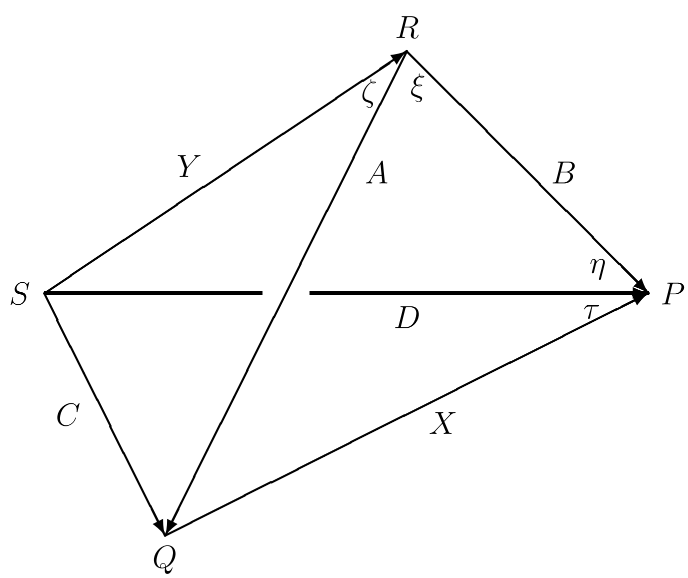
$$Z(A,B,C,D,X,Y \mid \xi,\eta,\zeta,\tau) = [B]^{-1/4}[C]^{-1/4}W(A,B,C,D,X,Y \mid \xi,\eta,\zeta,\tau).$$

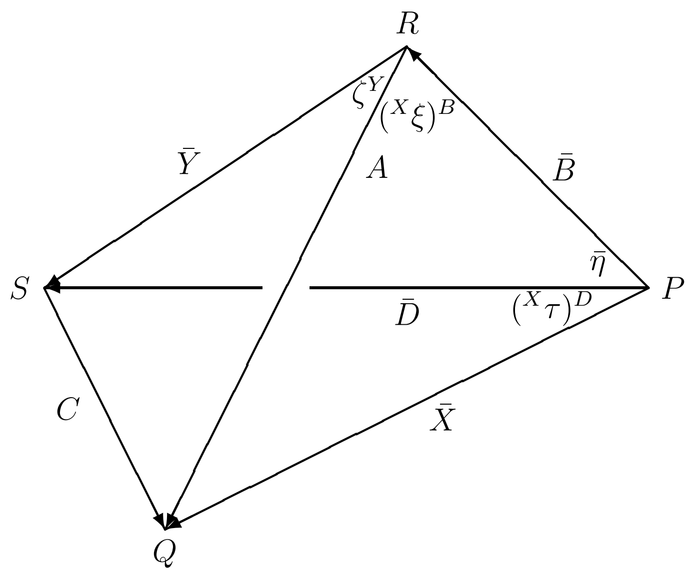
(Ocneanu used slightly different normalizations in [29] and [30]. The above normalization is used here so that our formula is compatible with the original formula of Turaev-Viro [37].)

By direct computations based on Frobenius reciprocity, the following identities are obtained.

$$\begin{aligned}
Z(A,B,C,D,X,Y \mid \xi,\eta,\zeta,\tau) &= Z(\bar{A},\bar{C},\bar{B},\bar{D},\bar{Y},\bar{X}, \mid \bar{\zeta},\bar{\tau},\bar{\xi},\bar{\eta}),\\
Z(\bar{A},X,Y,D,B,C \mid \xi^A,\tau,{}^A\zeta,\eta) &= \overline{Z(A,B,C,D,X,Y \mid \xi,\eta,\zeta,\tau)},\\
Z(B,A,D,C,\bar{X},Y \mid {}^X\xi,\zeta,\eta,{}^X\tau) &= \overline{Z(A,B,C,D,X,Y \mid \xi,\eta,\zeta,\tau)}.
\end{aligned}$$

The symmetric group S_4 may be thought of as acting on the tetrahedron in the above figure. For example, we get the new tetrahedron as in the following figure under one of these permutations. For this new tetrahedron, we reverse some orientation of edges so that the orientations become the same as in the original figure, and we change intertwiners using the Frobenius reciprocity in this procedure as in the next figure.





The above formulae mean that we get the same value of the normalized $6j$-symbol from this new picture. Furthermore, half of S_4 act as orientation reversing symmetries, and we get the complex conjugate as the value of the normalized $6j$-symbols in these cases. Thus the following theorem holds.

Theorem 2.4 (Tetrahedral Symmetry) *The normalized $6j$-symbols Z are invariant under the orientation preserving symmetries of the S_4 action and are changed into their complex conjugate by the orientation reversing symmetries of the S_4 action.*

Because the $6j$-symbols here are defined in terms of intertwiners, ordinary unitarity as in [26] [28] is also valid as follows.

Theorem 2.5 (Unitarity) *The $6j$-symbols satisfy the following unitarity relations.*

$$\begin{aligned}
&\sum_{C,\zeta,\tau} W(A,B,C,D,X,Y \mid \xi,\eta,\zeta,\tau)\overline{W(A,B',C,D,X,Y \mid \xi',\eta',\zeta,\tau)} \\
=\ &\delta_{(B,\xi,\eta),(B',\xi',\eta')}, \\
&\sum_{B,\xi,\eta} W(A,B,C,D,X,Y \mid \xi,\eta,\zeta,\tau)\overline{W(A,B,C',D,X,Y \mid \xi,\eta,\zeta',\tau')} \\
=\ &\delta_{(C,\zeta,\tau),(C',\zeta',\tau')}.
\end{aligned}$$

Another important property is flatness as in [26], [28]. A partition function has the following interpretation in the bimodule approach, and flatness for $6j$-symbols is a generalization of ordinary flatness for connections [26], [28]. (See [21], [22] for the meaning of flatness in the string algebra approach.)

Suppose we have a large diagram as follows.

$$\begin{tikzcd}
 & & \xrightarrow{\cdot\otimes X_1} & & \xrightarrow{\cdot\otimes X_2} & & \xrightarrow{\cdot\otimes X_n} & \\
 & A_1 \arrow[rr, "\xi_n"] \arrow[d, "\tau_m"'] & & A_2 \arrow[rr, "\xi_{n-1}"] & & \cdots \arrow[rr, "\xi_1"] & & B_1 \arrow[d, "\eta_1"] \\
Y_1\otimes\cdot \downarrow & D_m \arrow[d, "\tau_{m-1}"'] & & & & & & B_2 \arrow[d, "\eta_2"] \\
Y_2\otimes\cdot \downarrow & \vdots \arrow[d, "\tau_1"'] & & & & & & \vdots \arrow[d, "\eta_m"] \\
Y_m\otimes\cdot \downarrow & D_1 \arrow[rr, "\zeta_n"'] & & C_n \arrow[rr, "\zeta_{n-1}"'] & & \cdots \arrow[rr, "\zeta_1"'] & & C_1
\end{tikzcd}$$

That is, each edge on the boundary has an assignment of intertwiners, each vertex on the boundary has an assignment of bimodules, and each row and column of the diagram has an assignment of bimodules tensor product from the left and the right respectively. Then the value assigned to this diagram is defined as the sum over all the configurations of the products of all the values of $6j$-symbols in each configuration. This is a direct generalization of partition functions for ordinary connections in [21], [22], [26], [27], [28]. Then by induction, we can regard this value as an inner product of two intertwiners; one is given by the composition of intertwiners at the bottom boundary and those at the left boundary, and the other is given by the composition of those at the right boundary and the top boundary.

Then another important property of the $6j$-symbols is given as follows.

Theorem 2.6 (Flatness) *For the diagram as in the following figure with ${}_N*_N$ or ${}_M*_M$ at all the four corners, we get the value $\delta_{\xi_1,\xi_2}\delta_{\eta_1,\eta_2}$.*

$$\begin{tikzcd}
 & & \cdot\otimes X_1 \rightarrow & \cdot\otimes X_2 \rightarrow & & & \cdot\otimes X_{n+1} \rightarrow & \\
 & * \arrow[r, "\xi_1"] \arrow[d, "\eta_1"'] & A_1 \arrow[r, "\xi_2"] & A_2 & \cdots & A_n \arrow[r, "\xi_{n+1}"] & * \arrow[d, "\eta_1"] \\
Y_1\otimes\cdot \downarrow & B_1 \arrow[d, "\eta_2"'] & & & & & B_1 \arrow[d, "\eta_2"] \\
Y_2\otimes\cdot \downarrow & B_2 & & & & & B_2 \\
 & \vdots & & & & & \vdots \\
 & B_m \arrow[d, "\eta_{m+1}"'] & & & & & B_m \arrow[d, "\eta_{m+1}"] \\
Y_{m+1}\otimes\cdot \downarrow & * \arrow[r, "\xi_1"'] & A_1 \arrow[r, "\xi_2"'] & A_2 & \cdots & A_n \arrow[r, "\xi_{n+1}"'] & *
\end{tikzcd}$$

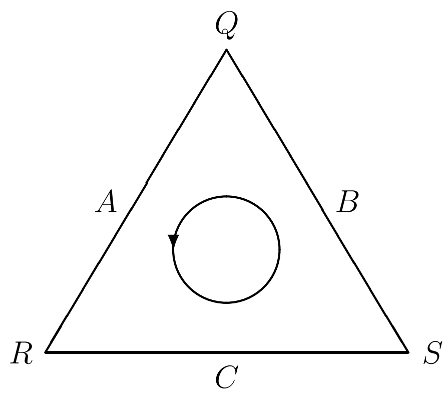
For the proof, it is enough because of unitarity to prove that the value is 1 if $\xi_1 = \xi_2$ and $\eta_1 = \eta_2$. In this case, it is easy to see that the two intertwiners, one from the intertwiners at the bottom and left boundaries and the other from those at the right and top boundaries, are the same. So their inner product has value 1.

3 Definition and topological invariance of the state sum

Using the generalized $6j$-symbols obtained in §2, we construct a topological invariant for oriented compact 3-manifolds without boundary as follows. This construction is similar to those in [37] and [9], but we give the explicit formula including all the normalizing constants, which is missing in [29], [30], because some difference arises from the fact that we have four kinds of bimodules. (We need an orientation of the manifold unlike [37]. See the last part of [9] for a discussion on orientation.) Here we assume that the manifold has no boundary for simplicity, because the case with boundary, which produces $(2+1)$-dimensional topological quantum field theory in the sense of [3], can be handled with a general method based on trivial cobordism as in [37] and [9].

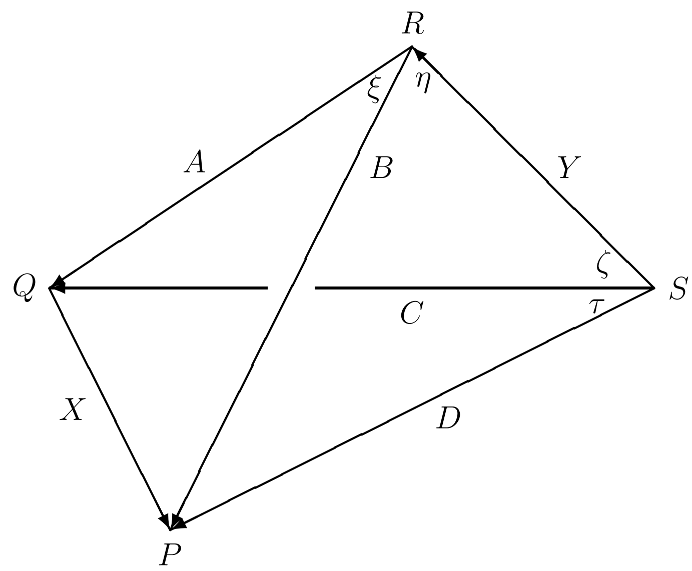
We fix a subfactor $N \subset M$ as in §2 and also fix a compact oriented 3-dimensional manifold P without boundary and triangulation T. We use four kinds of bimodules arising from $N \subset M$ as in §2. (Ocneanu works on a finite system of bimodules in [29], [30] where there are more than two von Neumann algebras in general, but here we work on bimodules arising from a subfactor, because it is notationally simpler on one hand and nothing essential is lost in this limitation on the other.)

We give a label N or M arbitrarily to each vertex of the triangulation T, and denote this labeling by L. Next we define a configuration of bimodules on edges and intertwiners on faces. Choose an oriented triangle with its vertices labeled by algebras $Q, R, S \in \{M, N\}$ as follows.



For the edge connecting the vertices Q and R, we assign a pair of bimodules ${}_R A_Q$ and ${}_Q \bar{A}_R$. For simplicity, we just say this as an assignment of ${}_R A_Q$. Similarly we assign ${}_Q B_S$ and ${}_R C_S$ to the other two edges. This is admissible if and only if $N_{A,B}^C \neq 0$. If this is the case, we assign a triple of intertwiners $\xi : A \otimes B \to C$, $({}^A\xi)^C : B \otimes \bar{C} \to \bar{A}$, ${}^C(\xi^B) : \bar{C} \otimes A \to \bar{B}$. For simplicity again, we just call this assignment of ξ. (This does not depend on the way we project the triangle on the paper by Theorem 2.3.) We choose ξ's so that they make an orthonormal basis, so the number of possible ξ's is $N_{A,B}^C$.

As above, we assign bimodules to edges and intertwiners to faces. For the moment, fix an admissible assignment. We assign a complex number to each labeled tetrahedron appearing in the configuration. Suppose we have a tetrahedron like the following picture and the triangle spanned by edges R, P, S has the counterclockwise orientation.



Then the four faces give the following maps: $\xi : {}_P B \otimes_R Y_S \to {}_P D_S$, $\eta : {}_P X \otimes_Q A_R \to {}_P B_R$, $\zeta : {}_S \bar{Y} \otimes_R \bar{A}_Q \to {}_S \bar{C}_Q$, $\tau : {}_S \bar{C} \otimes_Q \bar{X}_P \to {}_S \bar{D}_P$. Next assign the value

$$Z(A, B, C, D, X, Y \mid \xi, \eta, \bar{\zeta}, \bar{\tau})$$

to this tetrahedron. We assign the value

$$Z(\bar{A}, \bar{C}, \bar{B}, \bar{D}, \bar{Y}, \bar{X} \mid \zeta, \tau, \bar{\xi}, \bar{\eta}),$$

if the tetrahedron has the opposite orientation. By Theorem 2.4, this is the complex conjugate of the above value. Also by Theorem 2.4, this value does not depend on the way we project this tetrahedron, so this assignment is well-defined as a function from labeled tetrahedra to complex numbers.

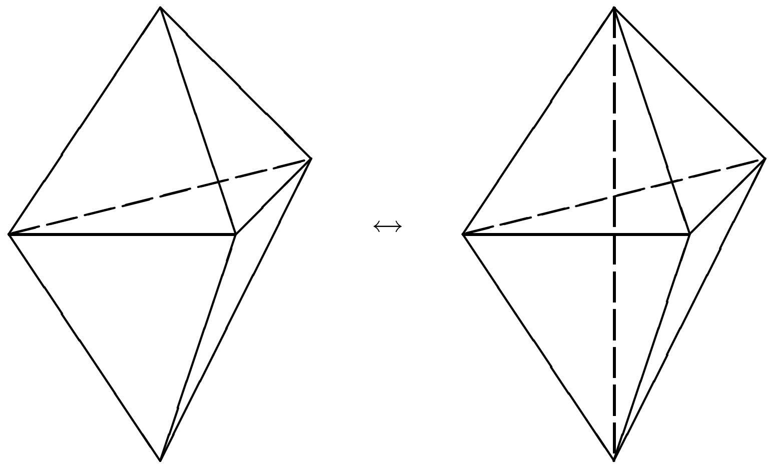
We then define $Z(P, T, L)$ by

$$Z(P, T, L) = \sum_{\text{configurations}} W^{-a} \prod_{\text{bimodules } X} [X]^{1/2} \prod Z(\text{tetrahedron}). \tag{1}$$

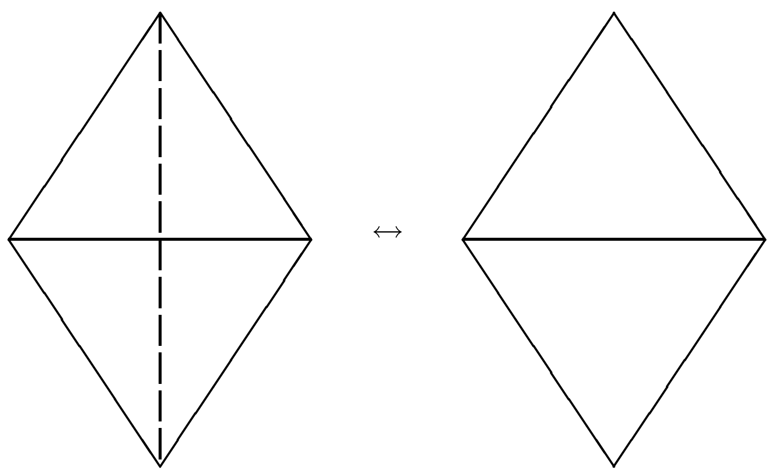
Here the first summation is over all the possible configurations of edges and faces for a given labeling L of vertices, the first product is over all the bimodules appearing on edges, and the second product is over all the tetrahedra in the triangulation T. (For faces, we choose intertwiners from an orthonormal basis as mentioned above.) The numbers W and a are defined by $W = \sum_{{}_N X_N} [X]$ and $a = \#\{\text{vertices appearing in } T\}$. We get the same value for the definition of W, when we use one of the other three kinds of bimodules. Note that our $[X]^{1/2}$ corresponds to w_i^2 in [37] and $1/F_i$ in [9], and W plays the role of w^2 in [37] and F in [9].

The above definition uses a specific triangulation T and a specific labeling L of the vertices. As in [37] and [9], for topological invariance, we have to prove that the above definition does not depend neither on T nor on L. For this purpose, it is sufficient to prove invariance under the following three types of local moves. (For a proof of this sufficiency, see [37] which is based on [1], [25]. Also see [13].) Note that we need to take extra care for independence on L, which was unnecessary in [37], [9].

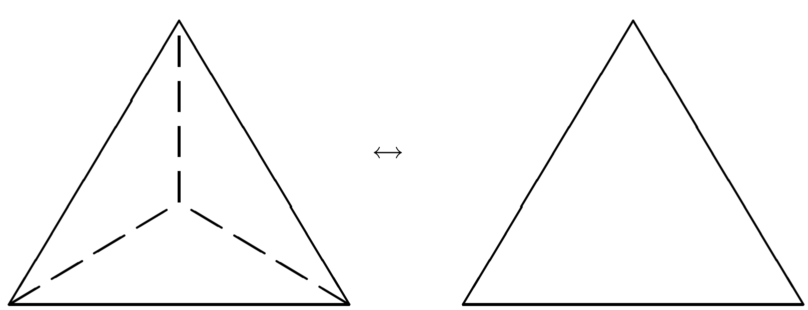
Move I: We have two tetrahedra sharing one common triangle. Then we can split these into three tetrahedra sharing one common triangle pairwise.



Move II: We have two tetrahedra sharing two common triangles. Then these collapse into two triangles.



Move III: We have two tetrahedra sharing three common triangles. Then these collapse into one triangle.



Proofs of invariance under the second and the third moves are essentially the same as those in [37], [9], and follow from unitarity so we just sketch them briefly. The key part of the proof for invariance is that the invariance under the first move follows from flatness and vice versa. We give a detailed account for this equivalence at the end of §3 and §4. This is the most important point of the entire theory and the direction from the invariance to flatness is missing in [29], [30] and different from the arguments in [37], [9].

Proof of the invariance under Move II: This is exactly unitarity. We only have to check the coefficients $[B]^{-1/4}[C]^{-1/4}$ in the definition of Z and the coefficients $[X]^{1/2}$ in that of $Z(P,T,L)$ cancel out. Q.E.D.

Proof of the invariance under Move III: This is again by unitarity. The key observation is that

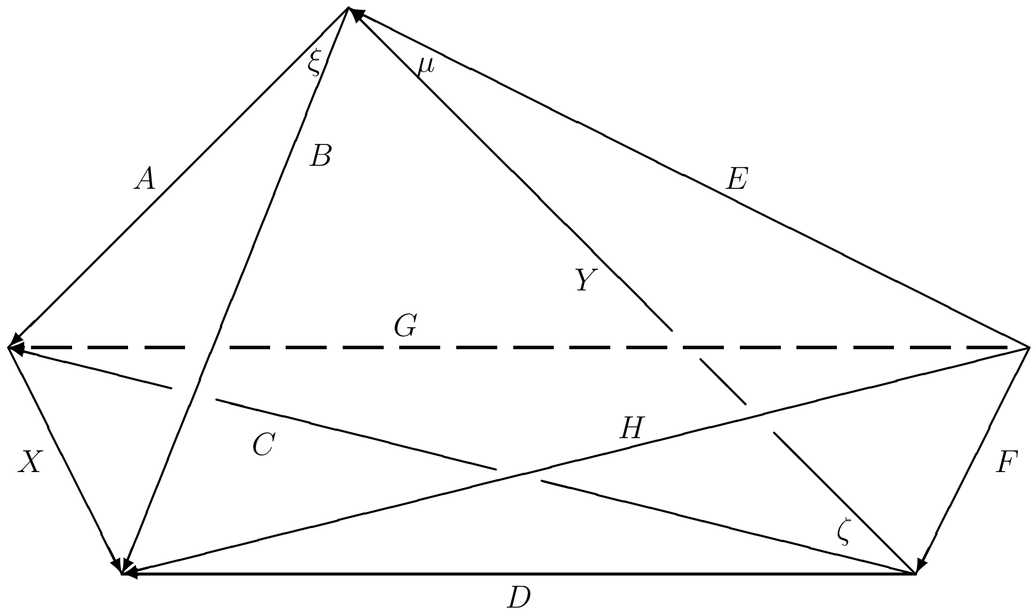
$$W = [{}_Q Z_S]^{-1/2} \sum_{X,Y} N_{X,Y}^Z [{}_Q X_R]^{1/2} [{}_R Y_S]^{1/2}$$

for all ${}_Q Z_S$, where $Q, R, S \in \{M, N\}$. This is proved as in [37, Lemma 1.1.A] and [9, Lemma 4.2]. Q.E.D.

Note that the labeling of the vertex which disappears in this procedure can be either N or M. This fact will be used in the proof of the independence on $Z(P,T,L)$ of L.

Now we come to the most important part of this theory. The invariance under Move I is given by the so-called pentagon relation, and Ocneanu [29] gave a proof of the pentagon relation. We include a proof here for the sake of completeness and for a related argument for the converse direction in the next section.

Proof of the invariance under Move I: We label edges and faces as follows. (We omit labelings for vertices and some faces.)



Then we apply flatness to the following 3×3-diagram to get the value 1.

$$
\begin{tikzcd}
 & {} \arrow[r, "\cdot\otimes Y"] & {} \arrow[r, "\cdot\otimes\bar{F}"] & {} \arrow[r, "\cdot\otimes\bar{E}"] & {} \\
{} \arrow[d, "A\otimes\cdot"'] & * \arrow[r, "1"] \arrow[d, "1"'] & Y \arrow[r, "\mu^F"] & E \arrow[r, "1^E"] & * \arrow[d, "1"] \\
{} \arrow[d, "X\otimes\cdot"'] & A \arrow[d, "\xi"'] & & & A \arrow[d, "\xi"] \\
{} \arrow[d, "\bar{B}\otimes\cdot"'] & B \arrow[d, "{}^{B}1"'] & & & B \arrow[d, "{}^{B}1"] \\
{} & * \arrow[r, "1"'] & Y \arrow[r, "\mu^F"'] & E \arrow[r, "1^E"'] & *
\end{tikzcd}
$$

Now cut the above diagram into two pieces as follows.

$$
\begin{tikzcd}
* \arrow[r] \arrow[d] & Y \arrow[d, "\zeta"] & \\
A \arrow[d] & C \arrow[d] & \\
B \arrow[d] & D \arrow[r] & H \arrow[d] \\
* \arrow[r] & Y \arrow[r] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r] \arrow[d, "\zeta"'] & E \arrow[r] & * \arrow[d] \\
C \arrow[d] & & A \arrow[d] \\
D \arrow[r] & H \arrow[d] & B \arrow[d] \\
& E \arrow[r] & *
\end{tikzcd}
$$

For this cutting, we label the path from $*$ to $Y, C, D, H, E, *$ by ρ and denote the values of the above two diagrams by a_ρ, $\bar{b}_\rho$ respectively. Then flatness gives $\sum_\rho a_\rho \bar{b}_\rho = 1$, unitarity implies $\sum_\rho |a_\rho|^2 = \sum_\rho |b_\rho|^2 = 1$, and so by the Cauchy-Schwarz inequality, we get $a_\rho = b_\rho$.

In the above figure, look at the square at the upper left corner first. The intertwiner from A to C must be ζ to get a non-zero value, and in this case, we get the value 1. The square at the lower left corner has a similar property, and in this case, we get the value $[D]^{1/4}[B]^{-1/4}[Y]^{-1/4}$ because of renormalization. We can compute the values for the upper right and lower right squares for the right diagram in the above figure similarly. By comparing these coefficients and coefficients in the definitions of Z and $Z(P, T, L)$, we can conclude that the equality $a_\rho = b_\rho$ is exactly the invariance under this Move. Note that we still have two squares for a_ρ and three squares for b_ρ. Each of these corresponds to a tetrahedron. Q.E.D.

The above three figures are given in [29], [30].

Finally, we have to prove that $Z(M, T, L)$ does not depend on L. This part is necessary because we also label vertices unlike [37], [9]. This proof is also missing in [29], [30]. A key fact is that one can eliminate a vertex in Move III, and thus the labeling for this vertex can be ignored. It is enough to prove that one can change a labeling of one vertex without changing $Z(M, T, L)$. First we make a triangulation finer if necessary so that we may think that a neighbourhood of the vertex is regarded as an Euclidean ball. Then by repeated use of Move I, we can decrease the number of edges connected the vertex to 3. Then by Move III, we can eliminate the vertex. So we can conclude that the labeling of this vertex does not affect the value $Z(M, T, L)$.

In particular, we can use the labeling assigning N to all the vertices. In this case, we use only N-N bimodules, and then the machinery of [9] produces the same value. Similarly, we can use only M-M bimodules. With this, we can prove that the subfactor

$R \times H \subset R \times G$ gives the same invariant as the same subfactor $R \subset R \times G$, by looking at only M-M bimodules, if H contains no normal subgroup of G. (Here G is a finite group acting on a II_1 factor R outerly and H is a subgroup of G. See [24] for more on this type of subfactors.) V. F. R. Jones informed us that this was also claimed by A. Ocneanu without proof.

Thus the following theorem claimed in [29], [30] has been proved.

Theorem 3.1 *The formula (1) defines a complex-valued topological invariant for oriented closed* 3*-manifolds from a subfactor with finite depth and finite index. If we change the orientation of the manifold or replace the subfactor by an anti-conjugate one, then the number becomes the complex conjugate.*

A. Ocneanu claimed in several talks that the Turaev-Viro invariant [37] is obtained by applying his machinery to the Jones subfactors of type A_n, but here we give two reasons why this is not true. One reason is unitarity. In [37, §8], they work in the case of the Dynkin diagram A_2. As explained there, their two choices of roots of unity produce two different invariants. One of them is not unitary and it does not come from subfactors. Another reason is that we have a $\mathbf{Z}_2$ grading of vertices. As explained above, we can compute the invariant by using only even vertices. In the setting of [37], this corresponds to the case we use only "colors" labeled by integers, while they use all the "colors" labeled by half-integers. This causes a difference of the resulting invariants. (Note that the set of colors labeled by integers is closed under the fusion rule.) This second reason was pointed out to us by T. Kohno.

4 From topological invariants to subfactors

In this section, we prove that if we have a topological invariant of the above type, we can produce (a family of) subfactors with finite index and finite depth. This construction is similar to that from rational conformal field theory in [6], which is based on Witten's lattice gauge theory [42]. But for flatness, the key property of paragroups, we need a different argument, which is missing in [29], [30].

In [9], they construct topological invariants from generalized $6j$-symbols and they show that quantum groups at roots of unity produce examples of their general machinery. The method here produces subfactors from quantum groups via their results, if an additional unitarity condition is satisfied.

First we have to clarify what we mean by the rational unitary polyhedral (2+1)-dimensional topological quantum field theories. The exact definition is as follows.

[1] **Fusion algebra:** We have a *fusion algebra* $\mathcal{A}$ as follows. The associative $\mathbf{C}$-algebra $\mathcal{A}$ is spanned by finitely many X_i's as a $\mathbf{C}$-vector space. Each X_i has a left attribution and a right attribution, and there are two possibilities for attributions, denoted by A and B. By notation ${}_A X_{iB}$ we mean that X_i has the left attribution A and the right attribution B. Each X_j has its conjugate $\bar{X}_j$ among X_i's, and the conjugate operation interchanges right and left attributions. We require $\bar{\bar{X}}_j = X_j$. For X and Y among the X_i's , the product is given by $X \cdot Y = \sum_Z N_{X,Y}^Z Z$, where $N_{X,Y}^Z$ is a non-negative integer and Z is among the X_i's. If the right attribution of X and the left attribution of Y are different,

the product is 0. If Z has non-zero $N_{X,Y}^Z$, its left attribution is the same as that of X and its right attribution is the same as that of Y. We also require

$$N_{X,Y}^{Z} = N_{\bar{Z},X}^{\bar{Y}} = N_{Y,\bar{Z}}^{\bar{X}} = N_{\bar{Y},\bar{X}}^{\bar{Z}} = N_{\bar{X},Z}^{Y} = N_{Z,\bar{Y}}^{X}.$$

We further require that we have two identities ${}_A1_A$ and ${}_B1_B$ with ${}_A\bar{1}_A = {}_A1_A$ and ${}_B\bar{1}_B = {}_B1_B$. The identities satisfy ${}_A1_A \cdot X = X$ if the left attribution of X is A. Similarly, we get $X \cdot {}_A1_A = X$, ${}_B1_B \cdot X = X$, and $X \cdot {}_B1_B = X$, if X has an appropriate attribution in each formula. Each X_i has an assignment $[X]$ of a positive value. The vector $\sum_{{}_A{X_i}_B} [X_i]^{1/2} X_i$ is a simultaneous eigenvector for left multiplications by ${}_A{X_j}_A$. Similar statements hold for all types of attributions.

[2] **Generalized $6j$-symbols:** Choose X, Y, Z from among the X_i's with $N = N_{X,Y}^Z \neq 0$. We have a family of symbols $\xi_j(X,Y,Z)$ with $j = 1, \dots, N$, and in the case of $Y = *$, we simply denote 1_X for $\xi(X,*,X)$, where $* = {}_A*_A$ or $* = {}_B*_B$. For $P,Q,R,S \in \{A,B\}$ and 6 elements ${}_QA_R$, ${}_PB_R$, ${}_QC_S$, ${}_PD_S$, ${}_PX_Q$, ${}_RY_S$ in the fusion algebra and four symbols $\xi = \xi_i(X,A,B)$, $\eta = \xi_j(B,Y,D)$, $\zeta = \xi_k(A,Y,C)$, $\tau = \xi_l(X,C,D)$ for some i,j,k,l, we assign a complex number $W(A,B,C,D,X,Y \mid \xi,\eta,\zeta,\tau)$, which is called a generalized $6j$-symbol. Their normalizations

$$Z(A,B,C,D,X,Y|\xi,\eta,\zeta,\tau) = [B]^{-1/4}[C]^{-1/4} W(A,B,C,D,X,Y|\xi,\eta,\zeta,\tau)$$

satisfy tetrahedral symmetry as in Theorem 2.4. We further assume

$$W(*,A,B,C,A,B \mid 1_A,\xi,1_B,\xi) = 1.$$

[3] **Topological invariance:** The $6j$-symbols Z must satisfy topological invariance if we define the state sum as in (1) for each triangulation. That means that W has to satisfy unitarity and 3×3-flatness as in the proof of invariance under Move I.

The adjective "rational" means the finiteness of X_i's (and N_{ij}^k) in [1], whilst "unitary" means $[X] > 0$ and use of $[X]^{1/4} > 0$. The quantum $6j$-symbol for $U_q(sl_2)$ of Kirillov-Reshetikhin [23] used in [37] is *not always* unitary in this sense. (Our $[X_i]^{1/4}$ corresponds to their w_i, but their w_i's can be non-positive even after some gauge choices.) So their construction is *not always* a rational unitary polyhedral topological quantum field theory in our sense. In [9], their $1/F_i$ corresponds to our $[X_i]^{1/2}$, and they only assume that F_i is real. If these F_i's are chosen to be positive, their data produce a rational unitary polyhedral topological quantum field theory in our sense. This positivity problem is related to a problem of gauge choice in rational conformal field theory.

In many cases as in [37] and [9], we do not have 2-sided labeling of attribution as above. But we can use a trivial modification of our following construction. See Remark 3.2. This will lead us to a self-dual subfactors as in [6].

In [29], [30], Ocneanu handles more general cases where attribution can have more than two possibilities, but we use the above definition, to avoid non-essential complexity.

Suppose we have a topological quantum field theory as above. We first choose an element Y from the fusion algebra. (In subfactor setting, this corresponds to a choice of a minimal projection in a higher relative commutant.) Suppose first that Y has the left attribution A and the right attribution B.

Define the bipartite graphs $\mathcal{G}_1^0$, $\mathcal{G}_2^0$, $\mathcal{H}_1^0$, $\mathcal{H}_2^0$ as follows. Even vertices of $\mathcal{G}_1^0$ and $\mathcal{G}_2^0$ are given by ${}_AX_A$'s in the fusion algebra, odd vertices of $\mathcal{G}_1^0$ and $\mathcal{H}_1^0$ by ${}_AX_B$'s, odd vertices of $\mathcal{G}_2^0$ and $\mathcal{H}_2^0$ by ${}_BX_A$'s, and even vertices of $\mathcal{H}_1^0$ and $\mathcal{H}_2^0$ by ${}_BX_B$'s. The number of edges for $\mathcal{G}_1^0$ between ${}_AX_A$ and ${}_AZ_B$ is given by $N_{X,Y}^Z$. Similarly, we use the numbers $N_{\bar{Y},X}^Z$, $N_{\bar{X},Y}^Z$, $N_{X,Y}^Z$ for $\mathcal{G}_2^0$, $\mathcal{H}_1^0$, $\mathcal{H}_2^0$ respectively. We then look at the connected component containing ${}_A*_A$, and label the resulting four bipartite graphs as $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{H}_1$, $\mathcal{H}_2$. Note that $\mathcal{H}_1$ and $\mathcal{H}_2$ always contain ${}_B*_B$. We then define a connection by

$$\begin{tikzcd} {}_AX_{1A} \arrow[r, "\xi_3"] \arrow[d, "\xi_1"'] & {}_AX_{3B} \arrow[d, "\xi_4"] \\ {}_BX_{2A} \arrow[r, "\xi_2"'] & {}_BX_{4B} \end{tikzcd} = W(X_1, X_2, X_3, X_4, \bar{Y}, Y \mid \xi_1, \xi_2, \xi_3, \xi_4).$$

We also define other types of connections using the crossing symmetry:

$$\begin{tikzcd} X_1 \arrow[r, "\xi_3"] \arrow[d, "\xi_1"'] & X_3 \arrow[d, "\xi_4"] \\ X_2 \arrow[r, "\xi_2"'] & X_4 \end{tikzcd} = \left(\frac{[X_2][X_3]}{[X_1][X_4]}\right)^{1/4} \overline{\begin{tikzcd} X_3 \arrow[r, "\tilde{\xi}_3"] \arrow[d, "\xi_4"'] & X_1 \arrow[d, "\xi_1"] \\ X_4 \arrow[r, "\tilde{\xi}_2"'] & X_2 \end{tikzcd}}$$

$$\begin{tikzcd} X_1 \arrow[r, "\xi_3"] \arrow[d, "\xi_1"'] & X_3 \arrow[d, "\xi_4"] \\ X_2 \arrow[r, "\xi_2"'] & X_4 \end{tikzcd} = \left(\frac{[X_2][X_3]}{[X_1][X_4]}\right)^{1/4} \overline{\begin{tikzcd} X_2 \arrow[r, "\xi_2"] \arrow[d, "\tilde{\xi}_1"'] & X_4 \arrow[d, "\tilde{\xi}_4"] \\ X_1 \arrow[r, "\xi_3"'] & X_3 \end{tikzcd}}$$

In order to prove that this data gives a paragroup, we have to show unitarity and flatness as in [26], [28], [21], [22]. Unitarity for the connection is an immediate consequence of the tetrahedral symmetry and unitarity of the generalized $6j$-symbols. So our aim is to prove flatness for the connection from the special type of flatness for 3×3-diagrams, which is equivalent to invariance under Move I.

This can be done by a graphical method as follows. (In the following diagrams, we omit labeling of edges and vertices for simplicity. Our convention is that parallel edges from $*$ or to $*$ denote the same choices of intertwiners.) First we split the 3×3-diagram in §3 into two pieces as follows.

$$
\begin{tikzcd}
* \arrow[r] \arrow[d] & Y \arrow[r] & E \arrow[d] & E \arrow[r] \arrow[d] & * \arrow[d] \\
A \arrow[d] & & \cdot \arrow[d] & \cdot \arrow[d] & A \arrow[d] \\
B \arrow[d] & & \cdot \arrow[d] & \cdot \arrow[d] & B \arrow[d] \\
* \arrow[r] & Y \arrow[r] & E & E \arrow[r] & *
\end{tikzcd}
$$

As in §3, we label the cutting path by ρ and denote the values of the two diagram by a_ρ and $\bar{b}_\rho$ respectively. By unitarity, we get $\sum_\rho |a_\rho|^2 \leq 1$ and $\sum_\rho |b_\rho|^2 \leq 1$. With these and the formula $\sum_\rho a_\rho \bar{b}_\rho = 1$ expressing 3×3-flatness, we get $a_\rho = b_\rho$ for all ρ. With this formula, we can shrink the size of a large diagram one step at a time. That is, as in the following figure, we can prove that the value for a diagram of $3 \times n$-size is 1.

$$
\begin{tikzcd}
\ast \arrow[r] \arrow[d] & A_1 \arrow[r] & A_2 \arrow[r] & \cdots & \cdots \arrow[r] & A_{n-1} \arrow[r] & \ast \arrow[d] \\
B_1 \arrow[d] & & & & & & B_1 \arrow[d] \\
B_2 \arrow[d] & & & & & & B_2 \arrow[d] \\
\ast \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & \cdots & \cdots \arrow[r] & A_{n-1} \arrow[r] & \ast
\end{tikzcd}
$$

$$
= \begin{tikzcd}
\ast \arrow[r] \arrow[d] & A_1 \arrow[r] & A_2 \arrow[r] & \cdots \arrow[r] & A_{n-2} \arrow[r] & \ast \arrow[d] \\
B_1 \arrow[d] & & & & & B_1 \arrow[d] \\
B_2 \arrow[d] & & & & & B_2 \arrow[d] \\
\ast \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & \cdots \arrow[r] & A_{n-2} \arrow[r] & \ast
\end{tikzcd}
$$

$$= \cdots$$

$$
= \begin{tikzcd}
\ast \arrow[r] \arrow[d] & A_1 \arrow[r] & A_2 \arrow[r] & \ast \arrow[d] \\
B_1 \arrow[d] & & & B_1 \arrow[d] \\
B_2 \arrow[d] & & & B_2 \arrow[d] \\
\ast \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & \ast
\end{tikzcd}
$$

Repeating the same kind of argument vertically, we can conclude that a diagram of any size with $*$ at the four corners and same configurations on parallel edges has a value 1, which is exactly flatness. (We have two kinds of flatness, one for ${}_A{*}_A$ and the other for ${}_B{*}_B$. The both are proved in the same way.)

The case where Y has the attribution ${}_BY_A$ is handled similarly.

If Y has an attribution ${}_AY_A$, then we modify the above construction as follows. For the four corners of the graphs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{H}_1, \mathcal{H}_2$, we now use only ${}_AX_A$'s. Then all the rest are the same as above. The case where Y has the attribution ${}_BY_B$ is handled similarly.

Furthermore, if the graphs $\mathcal{G}_1^0$, $\mathcal{G}_2^0$, $\mathcal{H}_1^0$, $\mathcal{H}_2^0$ are connected for a certain choice of ${}_AY_B$ or ${}_BY_A$, then all the elements of the fusion algebra are identified with bimodules arising from a subfactor. Thus we have proved the following.

Theorem 4.1 *A rational unitary polyhedral topological quantum field theory produces a family of subfactors. If we get connected graphs in the above construction for some choice in the fusion algebra, the topological quantum field theory arises from a subfactor.*

Remark 4.2 Suppose we have no attribution in the fusion algebra as in [9], [37]. Then we choose a Y from the fusion algebra, and go as in the above case where Y has the same right and left attributions. Then the same argument works.

5 Generalized $6j$-symbols and connections for subfactors obtained by basic construction

In this section we prove that the generalized $6j$-symbols correspond to connections obtained from the original subfactor by basic construction of [19] and cutting down by minimal projections in the relative commutants. Of course, this is what we expect. The same kind of argument shows that in the construction of paragroups obtained from rational conformal field theory in [6], their choice of a field corresponds to the choice of a minimal projection in the higher relative commutants. The contents of this section are not in [29], [30].

Suppose we have generalized $6j$-symbols W which come from a subfactor with finite index and finite depth. Because we have a connection from the $6j$-symbols, we may regard the subfactor $N \subset M$ is defined by a double sequence of string algebras $\{A_{kl}\}$ as in [28], [21], [22]. Choose an M-N bimodule X, which corresponds to an odd vertex of the principal graph. Suppose that the vertex has distance n from $*$ and denote a minimal projection in A_{n0} corresponding to this vertex by p. Then we cut A_{nl}'s by p for all l, and look at the inclusions.

$$\begin{array}{ccccccc}
pA_{0,0} & \subset & pA_{0,1} & \subset & pA_{0,2} & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
pA_{n,0}p & \subset & pA_{n,1}p & \subset & pA_{n,2}p & \subset & \cdots
\end{array}$$

It is easy to see that this is obtained by another string algebra construction, which produces a double sequence $\{B_{kl}\}$. First we prove that the connection for B_{kl} is given by $W(\cdot,\cdot,\cdot,\cdot,X,{}_NM_M \mid \cdot,\cdot,\cdot,\cdot)$. By the argument in §4, we know that the following diagram of size $(n+1)\times 2l$ has value 1.

$$
\begin{tikzcd}
* \arrow[r, "\cdot\otimes_N M_M"] \arrow[d, "{}_M M \otimes_N \cdot"'] & A_1 \arrow[r, "\cdot\otimes_M M_N"] & A_2 \arrow[r] & \cdots \arrow[r] & A_{2l-1} \arrow[r] & * \arrow[d] \\
B_1 \arrow[d, "{}_N M \otimes_M \cdot"'] & & & & & B_1 \arrow[d] \\
B_2 \arrow[d] & & & & & B_2 \arrow[d] \\
\vdots \arrow[d] & & & & & \vdots \arrow[d] \\
B_n \arrow[d, "\bar{X}\otimes\cdot"'] & & & & & B_n \arrow[d] \\
* \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & \cdots \arrow[r] & A_{2l-1} \arrow[r] & *
\end{tikzcd}
$$

Then we cut this into two pieces, one is of size $n \times 2l$, and the other is of $1 \times 2l$ and apply the Cauchy-Schwarz inequality again to prove that the two have the same value. This proves that the embedding of $pA_{0,l}$ into $pA_{n,l}p$ based on the original connection is same as the embedding based on the connection arising from $W(\cdot,\cdot,\cdot,\cdot,X,M \mid \cdot,\cdot,\cdot,\cdot)$. We can also prove that the double sequence B_{kl} is flat in the sense that elements in B_{k0} and those in B_{0l} commute.

We can then define a connection using $W(\cdot,\cdot,\cdot,\cdot,X,\bar{X} \mid \cdot,\cdot,\cdot,\cdot)$. With this connection, we can extend the double sequence to the following.

$$
\begin{array}{ccccccccc}
 & & B_{00} & \subset & B_{01} & \subset & B_{02} & \subset \cdots \\
 & & \cap & & \cap & & \cap & \\
B_{1,-1} & \subset & B_{10} & \subset & B_{11} & \subset & B_{12} & \subset \cdots \\
\cap & & \cap & & \cap & & \cap & \\
B_{2,-1} & \subset & B_{20} & \subset & B_{21} & \subset & B_{22} & \subset \cdots \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

That is, we use the connection arising from $W(\cdot,\cdot,\cdot,\cdot,X,\bar{X} \mid \cdot,\cdot,\cdot,\cdot)$ for the embedding

$$
\begin{array}{ccc}
B_{k,-1} & \subset & B_{k,0} \\
\cap & & \cap \\
B_{k+1,-1} & \subset & B_{k+1,0}
\end{array}
$$

Then the following diagram has value 1 by the arguments in §4.

$$
\begin{tikzcd}
* \arrow[r, "\cdot\otimes\bar{X}"] \arrow[d, "X\otimes\cdot"'] & \cdot \arrow[r, "\cdot\otimes M"] & \cdot \arrow[r, "\cdot\otimes M"] & \cdots \arrow[r, "\cdot\otimes M"] & \cdot \arrow[r, "\cdot\otimes X"] & * \arrow[d] \\
\cdot \arrow[d, "\bar{X}\otimes\cdot"'] & & & & & \cdot \arrow[d] \\
\vdots \arrow[d] & & & & & \vdots \arrow[d] \\
* \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdots \arrow[r] & \cdot \arrow[r] & *
\end{tikzcd}
$$

This implies another flatness in the sense that elements in B_{1l} commute with those in $B_{k,-1}$. Thus the higher relative commutants of $B_{0,\infty} \subset B_{1,\infty}$, which is conjugate to $pN \subset pMp$, is obtained by the connection arising from

$$W(\cdot, \cdot, \cdot, \cdot, X, \bar{X} \mid \cdot, \cdot, \cdot, \cdot).$$

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