Quantum doubles of the Hecke algebra subfactors and orbifolds

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A. Ocneanu has observed that a mysterious orbifold phenomenon occurs at the system of the M_∞-M_∞ bimodules of the asymptotic inclusion, a subfactor analogue of the quantum double, of the Jones subfactor of type A_{2n+1}.

We show that this is a general phenomenon and identify some of his orbifolds with the ones in our sense by working on the Hecke algebra subfactors of type A of Wenzl. That is, we work on their asymptotic inclusions and show that the M_∞-M_∞ bimodules are described by certain orbifolds (with ghosts) for $SU(3)_{3k}$. We actually compute several examples of the dual principal graphs of the asymptotic inclusions.

As a corollary of the identification of Ocneanu's orbifolds with ours, we show that a non-degenerate braiding exists on the even vertices of D_{2n}, $n > 2$.

We start with a finite braided system of bimodules $\mathcal{M} = \{x_i\}_{i\in I}$ in the sense of Ocneanu. An important example of such a system is obtained from the WZW-models $SU(n)_k$ with Ocneanu's surface bimodule construction. Anothter important example of such a system arises from a subfactor $N \subset M$ with finite index and finite depth.

Recall that an element x in the braided system $\mathcal{M}$ is called *degenerate* in the sense of Ocneanu if it satisfies the identity in Figure 1, where the dashed circle denotes the summation over all the labels $x \in \mathcal{M}$ with coefficient $[x]^{1/2}/[\mathcal{M}]$ as in Figure 2. Such a dashed ring is called a *killing ring* in Ocneanu's terminology.

We recall that we can perform a graphical opeartion called a *handle slide* against a killing ring. We give an example of a handle slide in Figures 3, 4. In this situation here, we assume that the link components on the right hand side are killing rings. (We remark that we have to regard a picture of link as that of a *framed* link now.) Note that this handle slide is valid regardless non-degeneracy condition.

We now suppose that the braiding is non-degenerate in the sense that 0 is the only degenerate element. In the tube algebra, we define the *Ocneanu projection* $p_{a,b}$ for $a, b \in \mathcal{M}$ as in Figure 5. The dashed line again denotes the killing ring.

The following theorem is due to Ocneanu.

Theorem 1 *The above element $p_{a,b}$ gives a system of mutually orthogonal minimal central projections in the tube algebra with $\sum_{a,b} p_{a,b} = 1$.*

[*]joint work with D. E. Evans

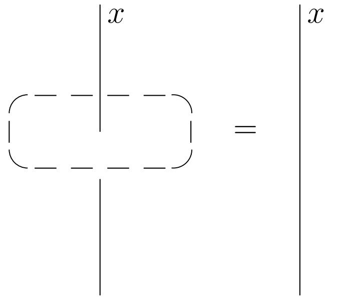

Figure 1. A degenerate element x

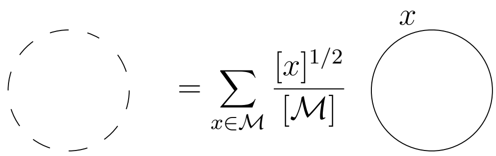

Figure 2. A killing ring

For a subfactor giving a non-degenrate system of M-M bimodules, we get the following proposition.

Proposition 2 *The dual principal graph of the asymptotic inclusion is the fusion graph of the original system, the same as the principal graph.*

We now work on the WZW-model $SU(n)_k$. Let $N \subset M$ be the corresponding Wenzl subfactor. Note that the fusion rule algebra for the WZW-model $SU(n)_k$ has a natural $\mathbf{Z}/n\mathbf{Z}$-grading and that the fusion rule subalgebra given by the grade 0 elements corresponds to the fusion rule algebra of the M-M bimodules arising from this subfactor $N \subset M$. We denote the grading of a primay field a in the model $SU(n)_k$ by $\mathrm{gr}(a)$. Then this system is often degenerate in the above sense. Our next aim is to study the aysmptotic inclusions for these degenerate cases.

We would like to get a full description of the dual principal graph of the asymptotic inclusion in the case of degenerate braiding, and then we have a system of bimodules labeled with pairs of the original bimodules. Then the bimodule $X_{f,f}$ plays a quite subtle role, where f denotes the fixed point of the $\mathbf{Z}/n\mathbf{Z}$-symmetry of the WZW-model $SU(n)_{nk}$. So we first make the following assumption and later prove that this assumption holds in some cases.

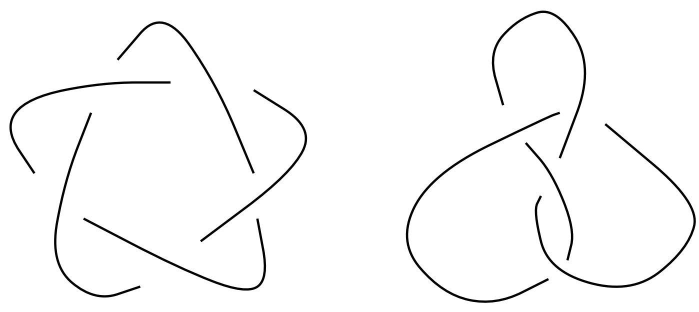

Figure 3. Before a handle slide

Assumption 3 *The M_∞-M_∞ bimodule $X_{f,f}$ decomposes into n irreducible bimodules and each has the same dimension.*

In this assumption, we mean the square root of the Jones index of the corresponding subfactor of a bimodule by the "dimension" of a bimodule. A. Ocneanu has observed this assumption holds for $SU(2)_{2k}$ and we will prove that this also holds for $SU(3)_{3k}$ in a general framework. We conjecture that this assumption holds for any $SU(n)_{nk}$, but combinatorial complexity has prevented us from proving it, so far.

First, we give an answer in the $SU(2)_k$ case.

Theorem 4 *Let $N \subset M$ be the subfactor correspoding to $SU(2)_{2n}$. Then the even vertex $(n,n)_+$ of the dual principal graph of the asymptotic inclusion is connected to the odd vertices $0, 4, \ldots$. The even vertex $(n,n)_-$ of the dual principal graph is connected to the odd vertices $2, 6, \ldots$.*

As a corollary of the above description, we get the following, which was announced by Ocneanu.

Corollary 5 *Let $N \subset M$ be the subfactor correspoding to $SU(2)_k$, that is, the Jones subfactor of type A_{k+1}. Assume $k > 2$. Then the number of irreducible M_∞-M_∞ bimodules arising from the asymptotic inclusion is given as follows.*

$$\begin{array}{ll} \left(\frac{k+1}{2}\right)^2, & \text{if } k \text{ is odd,} \\ \frac{k^2}{4} + \frac{k}{2} + 2, & \text{if } k \text{ is even.} \end{array}$$

Next we work on the asymptotic inclusions of the $SU(3)_{3k}$-subfactors. We have to compute how the central projection $p_{f,f}/3$ decomposes into minimal central projections in the tube algebra Tube $\mathcal{M}$. Counting several kinds of paths on the dual principal graph, we get the following theorem with Unitarity.

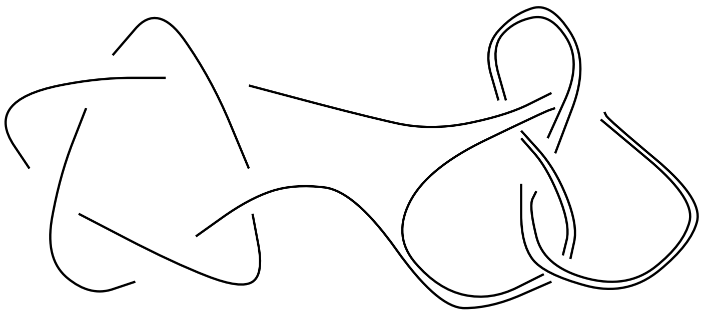

Figure 4. After a handle slide

Theorem 6 *For the subfactor $N \subset M$ arising from the WZW-model $SU(3)_{3k}$, Assumption 0.3 holds.*

This Theorem implies the following by a simple computation. This Corollary is a generalized version of Corollary 0.5.

Corollary 7 *Let $N \subset M$ be the subfactor correspoding to $SU(3)_k$ with $k > 2$. Then the number of the irreducible M_∞-M_∞ bimodules arising from the asymptotic inclusion is given as follows.*

$$\begin{array}{ll} \frac{(k+1)^2(k+2)^2}{36}, & \text{if } k \not\equiv 0 \ mod\ 3, \\ \frac{k^4+6k^3+13k^2+12k+108}{36}, & \text{if } k \equiv 0 \ mod\ 3. \end{array}$$

As examples, we work out the dual principal graphs for small k such as $k = 3, 6$ here.

First, we label the primary fields of $SU(3)_3$ as in Figure 6.

Then the principal graph of the asymptotic inclusion of the subfactor corresponding to $SU(3)_3$ is given as the fusion graph as in the upper half of Figure 7. For the dual principal graph, we know the graph except for the edges connected to the three vertices $(99)_0, (99)_1, (99)_2$. From the Perron–Frobenius property, we can determine these edges as in the bottom half of Figure 7. These edges are marked thick.

Since the subfactor corresponding to $SU(3)_3$ has index 4 and is described as $R \times A_3 \subset R \times A_4$, where A_3 and A_4 are the alternating groups of order 3 and 4 respectively and these groups act freely on the hyperfinite II_1 factor R, the paragroup of the asymptotic inclusion is given by that of the subfactor $R^{A_4 \times A_4} \subset R^{A_4}$, where A_4 is diagonally embedded into $A_4 \times A_4$ and the group A_4 acts freely on R, by Ocneanu's theorem again.

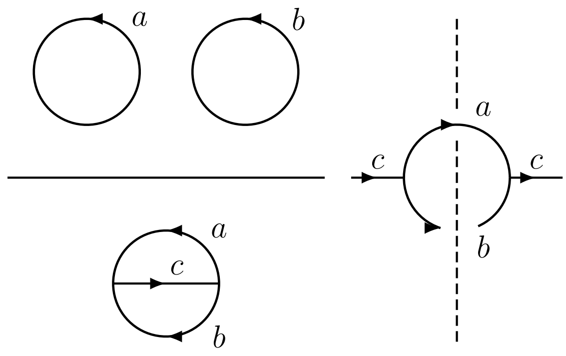

Figure 5. The Ocneanu projection $p_{a,b}$

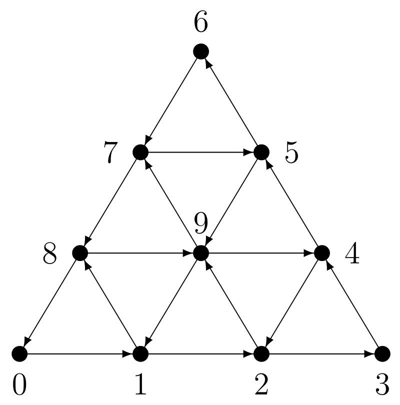

Figure 6. Primary fields for $SU(3)_3$

So the (dual) principal graphs of the asymptotic inclusion can be described with Ocneanu's theorem again. Of course, this method gives the same result as in Figure 7.

The next example is $SU(3)_6$. In this case, the system $\mathcal{M}$ has 10 and thus the principal graph of the asymptotic inclusion has 100 even vertices, and the dual principal graph has 90 even vertices. Since these graphs are too complicated, we draw only the edges concerned with the three even vertices $p^{(0)}_{ff}$, $p^{(1)}_{ff}$, $p^{(2)}_{ff}$. Then the Perron–Frobenius property and counting of paths with unitarity gives the graph as in Figure 8. In this Figure, the symbol (lm) denotes the Young diagram with l boxes in the first row and m boxes in the second row.

In the aobove, we have observed that the even vertices of the dual principal graphs of the asymptotic inclusions are given by merging/splitting of the vertices

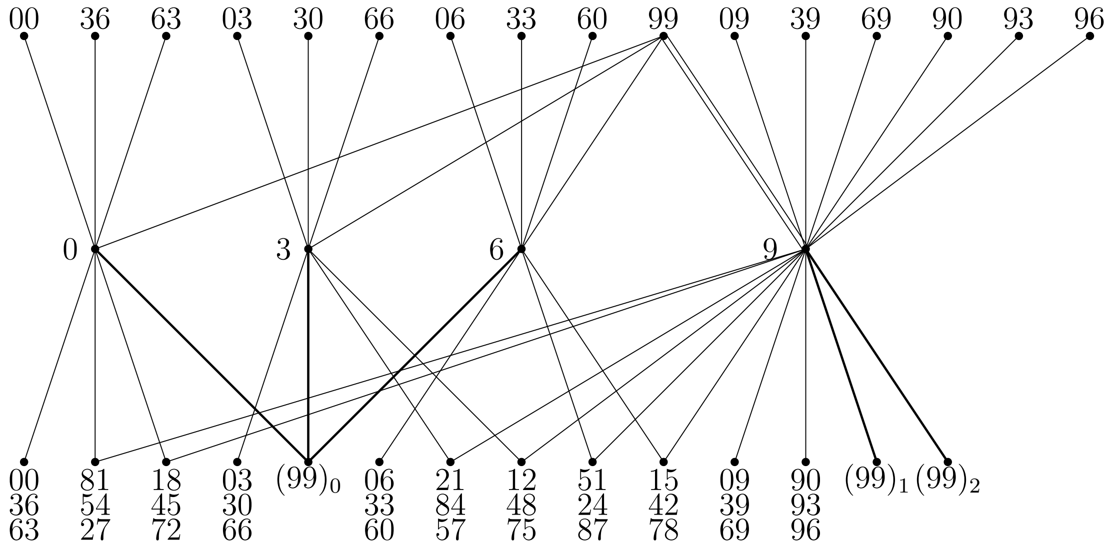

Figure 7. (dual) principal graphs for $SU(3)_3$

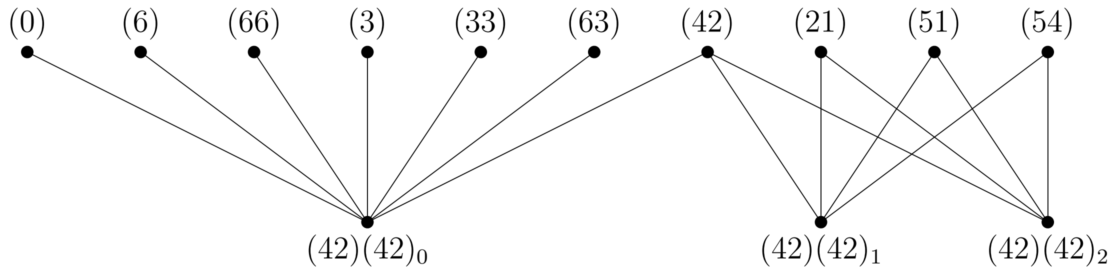

Figure 8. A part of the dual principal graphs for $SU(3)_6$

with symmetries. In the $SU(2)_k$ case, Ocneanu has noticed that this situation is similar to the orbifold construction for subfactors studied by us. However, the dual principal graphs we have studied above *not* orbifold graphs in our sense, because we have merging/splitting of the vertices only for the even vertices. Now we study a relation of this orbifold phenomena of Ocneanu to the oribifold construction in our sense and get the following theorem.

Theorem 8 *Let $N \subset M$ be the Jones subfactor of type A_{4n-3} with $n > 2$. Let $\mathcal{N}_0$ be the subsystem of M_∞-M_∞ bimodules arising from the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ labeled with pairs of even integers as above.*

Let σ be the outer, non-strongly-outer automorphism of order 2 of $N \subset M$. The system $\mathcal{N}_0$ is isomorphic to the system of $(M \otimes M)_{\sigma\otimes\sigma}$-$(M \otimes M)_{\sigma\otimes\sigma}$ bimodules arising from the orbifold subfactor $(N \otimes N)_{\sigma\otimes\sigma} \subset (M \otimes M)_{\sigma\otimes\sigma}$.

As an application, we can prove the following.

Theorem 9 *The system of M-M bimodules arising from a subfactor $N \subset M$ of type D_{2n} has a non-degenerate braiding.*

This gives the following corollary immediately.

Corollary 10 *The dual principal graph of the asymptotic inclusion of the hyperfinite II_1 subfactor $N \subset M$ with principal graph D_{2n} is the fusion graph of the system of the M-M bimodules.*

Remark 11 Ocneanu has constructed a braiding on the even vertices of D_{2n} with an entirely different method. His theory also shows that his braiding and ours must be the same.

Turaev and Wenzl have worked on a similar construction to our orbifold construction in categories of tangles. It seems that their construction, in particular, gives a braiding on the even vertices of D_{2n} and we expect that their braiding is also same as ours, but the actual relation is not clear.

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