Fields Institute Communications
Volume **00**, 0000

# Braiding and extensions of endomorphisms of subfactors

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**Abstract.** We explain methods to extend endomorphisms from a subfactor to a larger factor with (half-)braiding in subfactor theory in connection to conformal field theory and 3-dimensional topological quantum field theory. The most typical examples of such extension are $\alpha$-induction studied by Longo-Rehren, Xu, and Böckenhauer-Evans and Izumi's study of Longo-Rehren construction. As an application, we show that Rehren's new construction of a canonical endomorphism arising from an extension of a system of endomorphisms can be obtained as the dual of the usual Longo-Rehren construction if the extension comes from $\alpha$-induction with non-degenerate braiding.

## 1 Introduction

We discuss methods to extend endomorphisms from a subfactor to a larger factor using some form of braiding and their relations to topological quantum field theory and conformal field theory.

Let us start a classical and elementary example. Let $\alpha$ be an action of a finite group $G$ on a von Neumann algebra $M$ and consider the crossed product $M \times_\alpha G$ with implementing unitaries $u_g$, $g \in G$. Take an automorphism $\pi$ of $M$ and suppose we have the following properties.

$$\begin{aligned} v_g\alpha_g(\pi(x)) &= \pi(\alpha_g(x))v_g, \quad g \in G, x \in M, \\ v_{gh} &= v_g\alpha_g(v_h), \quad g, h \in G. \end{aligned}$$

That is, we assume that $\pi$ and $\alpha_g$ commutes up to a unitary $\alpha$-cocycle $v_g$. Then $\tilde\pi$ defined as follows gives an automorphism of $M \times_\alpha G$ extending $\pi$ on $M$.

$$\begin{aligned} \tilde\pi(x) &= \pi(x), \quad x \in M, \\ \tilde\pi(u_g) &= v_g u_g, \quad g \in G. \end{aligned}$$

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1991 *Mathematics Subject Classification.* Primary 46L37; Secondary 18D10, 46L60, 81T05, 81T40.

We would like to give an analogue of this type of extension for subfactors. (Also note that this type of extension works for continuous group actions. See [13, Section 12] for the case of modular automorphism groups and Connes' Radon-Nikodym cocycles.) We use an analogous method to extend an *endomorphism* from a type III factor $N$ to another factor $M$ containing $N$ as a subfactor of finite index.

For a subfactor $N \subset M$, it is a basic idea in subfactor theory to regard $M$ as a some general form of a "crossed product" of $N$. We then consider endomorphisms of $N$ rather than automorphisms since it is a system of endomorphisms of a factor which produces an interesting algebraic system in connection to topological quantum field theory, conformal field theory and quantum group theory. The cocycle condition is next replaced by a braiding property. These will be explained in detail below.

Such a procedure of extension of endomorphisms from a subfactor $N$ to $M$ is called $\alpha$-induction. This was defined by Longo-Rehren [23] for nets of subfactors based on a suggestion of Roberts [33] and studied in detail by Xu [36, 37]. Xu found several basic properties and very interesting examples arising from conformal inclusions. It was further studied by Böckenhauer-Evans [1, 2, 3], and then later by [5, 6, 7] by identifying it with Ocneanu's graphical construction [28]. We explain here how this method is used for various studies of subfactors.

## 2 Q-systems, systems of endomorphisms and braiding

We first review basic framework for operator algebraic studies of endomorphisms and braiding.

Here we consider a subfactor $N \subset M$ of type III with finite index. In case of subfactor theory of type $\mathrm{II}_1$, we consider the bimodule ${}_N L^2(M)_M$ and irreducible decomposition of its relative tensor powers (with ${}_M L^2(M)_N$). We then study fusion rule algebras, $6j$-symbols, and so on. We refer readers to book [10] for such treatments. Here for subfactors of type III, we use Izumi's framework [14] based on Longo's work [20, 21] to study various "morphisms". (For type III factors $A, B$, we mean by an $A$-$B$ morphism a $*$-homomorphism from $B$ into $A$. We write $\mathrm{Mor}(A, B)$ for the set of $A$-$B$ morphisms.) Basically, this approach based on homomorphisms of type III factors and the one based on bimodules over $\mathrm{II}_1$ factors are equivalent, but here we would like to work on extension of endomorphisms, so we need the type III setting.

Let $\iota : N \to M$ be the embedding map which is regarded as an $M$-$N$ morphism. Then we have its conjugate $\bar\iota : M \to N$ as an $N$-$M$ morphism, which is Longo's canonical endomorphism [20] regarded as a map from $M$ into $N$. By composing them, we have the canonical endomorphism $\gamma = \iota\bar\iota$ as an $M$-$M$ morphism, which is just an endomorphism of $M$. Then we have isometries $v \in \mathrm{Hom}(\mathrm{id}, \gamma)$, $w \in \mathrm{Hom}(\gamma, \gamma^2)$ satisfying

$$v^* w = w^* \gamma(v) \in \mathbf{R}_+, \quad w^* \gamma(w) = w w^*, \quad \gamma(w) w = w^2.$$

Recall here that $A$-$B$ morphisms $\lambda, \rho$, we set

$$\mathrm{Hom}(\lambda, \rho) = \{S \in A \mid S\lambda(x) = \rho(x) S \text{ for all } x \in B\}.$$

Longo's Q-system is a triple $(\gamma, v, w)$ of an endomorphism $\gamma$ of a type III factor $M$ and isometries $v, w$ satisfying the above relations. Longo [22] proved that if we set

$$N = \{x \in M \mid wx = \gamma(x) w, wx^* = \gamma(x^*) w\},$$

for a Q-system, then $N$ is a subfactor of $M$ and $\gamma$ is the canonical endomorphism for $N \subset M$, and this subfactor produces the Q-system we start with. In this sense, specifying a subfactor and specifying a Q-system are equivalent.

We next choose sets of morphisms ${}_N\Delta_N \subset \mathrm{Mor}(N,N)$, ${}_N\Delta_M \subset \mathrm{Mor}(M,N)$, ${}_M\Delta_N \subset \mathrm{Mor}(N,M)$ and ${}_M\Delta_M \subset \mathrm{Mor}(M,M)$ consisting of representative morphisms of irreducible subsectors of sectors of the form $[\bar\iota\iota\cdots\bar\iota\iota]$, $[\bar\iota\iota\cdots\bar\iota]$, $[\iota\cdots\bar\iota\iota]$ and $[\iota\bar\iota\cdots\iota\bar\iota]$ respectively. (We may and do choose $\mathrm{id}_M, \mathrm{id}_N$ in ${}_N\Delta_N, {}_M\Delta_M$ as the endomorphisms representing the trivial sectors.) We also assume that ${}_N\Delta_N$ is finite, that is, the subfactor $N \subset M$ is of finite depth. The sets ${}_N\Delta_N$ and ${}_M\Delta_M$ are systems of endomorphisms of $N$ and $M$, respectively, in the following sense.

**Definition 2.1** Let $M$ be a factor of type III. A set $\Delta \subset \mathrm{End}(M)$ is called a finite system of endomorphisms of $M$ if the following are satisfied.

1. Each $\lambda \in \Delta$ is an irreducible endomorphism of $M$ and the index $[M : \lambda(M)]$ is finite.
2. Endomorphisms in $\Delta$ are mutually inequivalent.
3. The identity of $M$ is in $\Delta$.
4. For any $\lambda \in \Delta$, we have $\mu \in \Delta$ which is a conjugate endomorphism of $\lambda$.
5. For any $\lambda, \mu \in \Delta$, we have $[\lambda][\mu] = \sum_{\nu\in\Delta} N^{\nu}_{\lambda,\mu}[\nu]$ as sectors, where $N^{\nu}_{\lambda\mu}$'s are non-negative integers.

A system of endomorphism naturally gives a fusion rule algebra with composition of endomorphisms as its multiplication, but in general, this multiplication is non-commutative. (When we say a *fusion rule algebra*, the commutativity of the multiplication is not assumed.) Even if the composition of the endomorphisms in the system is commutative (up to inner automorphism of $M$), we may be unable to choose unitary intertwiners in a *compatible* way. We say we have a braiding on a system of endomorphisms if such a compatible choice is possible. The precise definition due to Rehren [30] is as follows.

**Definition 2.2** We say that a system $\Delta$ of endomorphisms of $M$ has a *braiding* if we have unitary operators $\varepsilon(\lambda,\mu) \in \mathrm{Hom}(\lambda\mu, \mu\lambda) \subset M$ for all pairs $\lambda, \mu \in \Delta$ satisfying the following properties.

1. We have $\varepsilon(\mathrm{id},\mu) = \varepsilon(\lambda,\mathrm{id}) = 1$, for any $\lambda, \mu \in \Delta$.
2. Whenever $S \in \mathrm{Hom}(\lambda, \mu\nu)$ we have

$$\begin{aligned} \rho(S)\varepsilon(\lambda,\rho) &= \varepsilon(\mu,\rho)\mu(\varepsilon(\nu,\rho))S, \\ S\varepsilon(\rho,\lambda) &= \mu(\varepsilon_(\rho,\nu))\varepsilon(\rho,\mu)\rho(S), \\ \rho(S)^*\varepsilon(\mu,\rho)\mu(\varepsilon(\nu,\rho)) &= \varepsilon(\lambda,\rho)S^*, \\ S^*\mu(\varepsilon(\rho,\nu))\varepsilon(\rho,\mu) &= \varepsilon(\rho,\lambda)\rho(S)^*, \end{aligned}$$

for any $\lambda, \mu, \nu \in \Delta$.

The unitaries $\varepsilon(\lambda,\mu)$ are called *braiding operators*. We sometimes write $\varepsilon^+$ for $\varepsilon$ and set $\varepsilon^-(\lambda,\mu) = (\varepsilon(\mu,\lambda))^*$. This is called the *opposite* braiding. Such a braiding is not constructed easily, unless we have some extra structure. One such source is theory of quantum groups (see [18], for example). Another is conformal field theory and A. Wassermann [35] gave a braiding structure on a system of endomorphisms corresponding to $SU(n)_k$ using representations of loop groups. We also have Ocneanu's asymptotic inclusions [26, 27] and Longo-Rehren construction

[23, 14] as operator algebraic constructions producing braiding. (These two are essentially the same construction by [24]. Also see [25].)

We also have a notion of non-degeneracy for a braiding. Roughly speaking, this means that $\varepsilon^+$ and $\varepsilon^-$ are "really different" and the precise definition [30] is as follows.

**Definition 2.3** We say that a braiding $\varepsilon$ on a system $\Delta$ of endomorphisms of $M$ is *non-degenerate*, if the equalities $\varepsilon^+(\lambda,\mu)=\varepsilon^-(\lambda,\mu)$ for all morphisms $\mu\in\Delta$ imply $\lambda=\mathrm{id}$.

If we have a braiding on a finite system $\Delta$, we can define $S$- and $T$-matrices whose sizes are the number of endomorphisms in $\Delta$, as in [30]. The above non-degeneracy is equivalent to unitarity of the $S$-matrix as shown in [30], and if it is non-degenerate, the $S$- and $T$-matrices give a unitary representation of $SL(2,\mathbf{Z})$.

## 3 Extension of an endomorphism of a subfactor

We now take a subfactor of $N\subset M$ of type III with finite index. Suppose that the system $\Delta$ of endomorphisms of $N$ arising from the subfactor has a braiding $\varepsilon$. We say the subfactor is braided in such a case. The larger factor $M$ is expressed as $Nv$ with isometry $v$ appearing in the Q-system. Roughly speaking, $M$ is like the crossed product of $N$ by the dual canonical endomorphism $\theta$ and $v$ is like the implementing unitary, though it is not a unitary but an isometry. The braiding property implies that we have $\varepsilon(\theta,\lambda\mu)=\lambda(\varepsilon(\theta,\mu))\varepsilon(\theta,\lambda)$ for $\lambda,\mu\in\Delta$, and this is clearly similar to the cocycle condition mentioned in the introduction. Then for $\lambda\in\Delta$, we can define its extension $\tilde\lambda$ to $M$ by

$$\begin{aligned}\tilde\lambda(x) &= \lambda(x),\quad x\in N,\\ \tilde\lambda(v) &= \varepsilon(\theta,\lambda)v.\end{aligned}$$

This procedure is called $\alpha$-induction. This was defined by [23] for nets of subfactors based on a suggestion of [33] and studied by [36, 37] in a very interesting way for nets of subfactors arising from conformal inclusions. It was further studied by [1, 2, 3] and later by [5, 6, 7] in combination with Ocneanu's graphical method [28]. (Actually, the above extension using $\varepsilon=\varepsilon^+$ is $\alpha^-_\lambda$ in the convention of [1, 2, 3, 5, 6, 7].)

For various conformal inclusions, we consider nets of subfactors $N(I)\subset M(I)$ for intervals $I$ on the circle $S^1$. Then the net structure on $S^1$ naturally produces a braiding on the system of endomorphisms of $N=N(I)$ for a fixed $I$ and then we can apply $\alpha$-induction as above. Then for an endomorphism $\lambda$ of $N$ in the system, we get a subfactor $\alpha_\lambda(M)\subset M$. Xu [36, 37] has produced several interesting examples of subfactors in this way. The subfactors with principal graphs $E_6, E_8$ are among the simplest examples.

Since a braiding $\varepsilon^+$ is always paired with $\varepsilon^-$, we have two ways $\alpha^+_\lambda$ and $\alpha^-_\lambda$ of extending an endomorphism of $N$. These two are indeed different in many cases and by counting the multiplicity of the common irreducible endomorphisms in $\alpha^+_\lambda$ and $\alpha^-_\mu$, we get a matrix $Z=(Z_{\lambda\mu})$ of non-negative integer entries by setting $Z_{\lambda\mu}=\dim\ \mathrm{Hom}(\alpha^+_\lambda,\alpha^-_\mu)$. This is a *modular invariant* in the sense that this matrix commutes with the $S$- and $T$-matrices arising from the braiding. See article [4] in this volume for close relations to theory of modular invariants in conformal field theory.

## 4 Half-braiding and the Longo-Rehren construction

In the above construction of $\alpha_\lambda$, we have assumed that $\lambda$ is in the braided system of endomorphisms, but it is clear that $\lambda$ can be an arbitrary endomorphism of $N$ as long as it commutes with the system of endomorphisms containing the irreducible decomposition of the dual canonical endomorphism up to cocycles. So theory of extending endomorphism can be studies in a much more general setting.

It was Izumi [15] who noticed that study of the Longo-Rehren subfactors [23] can be done as a part of such a general framework. First we make a brief review on the Longo-Rehren construction and related results.

As we have explained above, if we start with an arbitrary subfactor $N \subset M$ with finite index and finite depth, the system of endomorphisms of $N$ (or $N$-$N$ bimodules) is highly unlikely to have a braiding. There is a general machinery to produce a new system with braiding from such a given system and it is analogous to the quantum double construction of Drinfel′d [8] in the sense that this produces a braiding from an arbitrary given system. The first such construction in subfactor theory was Ocneanu's asymptotic inclusion [26, 27] It gives a subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$ from a hyperfinite $\mathrm{II}_1$ subfactor $N \subset M$ with finite index and finite depth. Then the subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$ is of finite index and finite depth and the system of $M_\infty$-$M_\infty$ bimodules arising from this subfactor gives a "quantum double" of the system of $M$-$M$ bimodules (and also that of $N$-$N$ bimodules) of the original subfactor $N \subset M$. (Here $M_\infty$ the GNS-completion of $\bigcup_n M_n$ for the Jones tower with respect to the trace.) Popa [29] has the most general construction of this type called a *symmetric enveloping inclusion*. See [10, Chapter 12] for details about the asymptotic inclusion and its deep relations to topological quantum field theory.

Based on a quite different motivation, Longo-Rehren [23] gave a construction of a subfactor $M \otimes M^{\mathrm{opp}} \subset R$ from a finite system of endomorphism $\{\rho_j\}_j$ of a type III factor $M$. Although this does not look very similar to the construction of the asymptotic inclusion explained above, Masuda [24] has shown that these two are the same construction from a categorical viewpoint. That is, if we start with a finite system of endomorphism $\{\rho_j\}_j$ of a type III factor $M$ and applies the Longo-Rehren construction $M \otimes M^{\mathrm{opp}} \subset R$, we have a natural (finite) system of endomorphisms of $R$. We can always realize the same algebraic structure (that is, the fusion rule algebra and $6j$-symbols) as $\{\rho_j\}_j$ with a hyperfinite $\mathrm{II}_1$ subfactor $N \subset M$ with finite index and finite depth as the system of $M$-$M$ bimodules associated to this subfactor and then we consider the system of $M_\infty$-$M_\infty$ bimodules associated to the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$. Then the system of endomorphisms of $R$ and that of $M_\infty$-$M_\infty$ bimodules are isomorphic (as fusion rule algebras with $6j$-symbols.)

In the Longo-Rehren construction, they show that the endomorphism $\bigoplus \rho_j \otimes \rho_j^{\mathrm{opp}}$ on $M \otimes M^{\mathrm{opp}}$ is a dual canonical endomorphism of some inclusion $M \otimes M^{\mathrm{opp}} \subset R$ by an explicit construction of a $Q$-system. Izumi [15] has shown that an endomorphism of $R$ associated to the inclusion $M \otimes M^{\mathrm{opp}} \subset R$ is realized as an extension of an endomorphism $\rho \otimes id$ on $M \otimes M^{\mathrm{opp}}$ where $\rho$ is a finite direct sum of endomorphisms in the system $\{\rho_j\}_j$ and it has a "half-braiding" [15, Definition 4.2] with respect to the system $\{\rho_j\}_j$ and conversely that such an endomorphism $\rho$ of $M$ with a half-braiding produces an extension of an endomorphism from $M \otimes M^{\mathrm{opp}}$ to $R$ which is associated to the Longo-Rehren subfactor. The definition of the extension [15, Definition 4.4] is very similar to the one of $\alpha$-induction. (Actually, he uses restriction of an endomorphism rather than extension of one, but they are essentially the same. Here we follow the convention in [7, Section 2].) We have the terminology "half"-braiding since the roles of $\rho$ and $\{\rho_j\}_j$ are not symmetric any more. Izumi [15] further showed that such a half-braiding can be studied in terms of Ocneanu's tube algebra [27] and made several explicit computations for very interesting examples in [16]. In [7], we have used a notation $\eta(\sigma, \mathcal{E}_\sigma) \in \mathrm{End}(R)$ for such an extension of $\sigma \in \mathrm{End}(M)$ having a half-braiding $\mathcal{E}_\sigma$ with respect to the system $\{\rho_j\}_j$.

Izumi's work is completely general, but we now consider the case where the system $\{\rho_j\}$ is somehow produced with $\alpha$-induction from a braided subfactor $N \subset M$. That is, we can consider the entire system ${}_M\Delta_M$ of endomorphisms of $M$, the systems ${}_M\Delta_M^\pm$ of those arising from positive/negative inductions $\alpha^\pm$, that of those belonging to the both, and also subsystems of those having "0-grading" (when the original system of endomorphisms of $N$ has a natural grading, e.g. the case of $SU(n)_k$). We have determined in [7] the corresponding system of endomorphisms of $R$ for such cases. (Some of such results have been announced by Ocneanu in his setting of graphical methods [28].) In particular, we have computed the dual principal graph of the asymptotic inclusion of the $E_8$ subfactor and its analogues for $SU(3)_k$ for the first time. Roughly speaking, the main point of the computations in [7] can be summarized as follows. Though the systems produced by $\alpha$-induction do not have a braiding in general (or can be even non-commutative), they still remember the original braiding in the form of a relative braiding [3] between positive and negative subsystems, and this relative braiding gives a half-braiding necessary for Izumi's analysis [15].

## 5 Rehren's new construction

Using the study [7] of Longo-Rehren subfactors arising from $\alpha$-induction, we next study Rehren's new construction in [31]. First we recall Rehren's new construction. Let $\Delta$ be a system of endomorphisms of a type III factor $N$ and consider a subfactor $N \subset M$ with finite index. We call it an $N$-system. An extension of the $N$-system $\Delta$ is a pair $(\iota, \alpha)$ where $\iota$ is the embedding map of $N$ into $M$ and $\alpha$ is a map $\Delta \to \mathrm{End}(M)$, $\lambda \mapsto \alpha_\lambda$ satisfying the following properties.

1. Each $\alpha_\lambda$ has a finite index.
2. We have $\iota\lambda = \alpha_\lambda\iota$ for $\lambda \in \Delta$.
3. We have $\iota(\mathrm{Hom}(\lambda\mu, \nu)) \subset \mathrm{Hom}(\alpha_\lambda\alpha_\mu, \alpha_\nu)$ for $\lambda, \mu, \nu \in \Delta$.

Next let $N_1$, $N_2$ be two subfactors of a type III factor $M$, $(\iota_1, \alpha^1)$ and $(\iota_2, \alpha^2)$ be two extensions of finite systems $\Delta_1, \Delta_2$ of endomorphisms of $N_1, N_2$ to $M$, respectively. For $\lambda \in \Delta_1$ and $\mu \in \Delta_2$, we set $Z_{\lambda,\mu} = \dim\ \mathrm{Hom}(\alpha^1_\lambda, \alpha^2_\mu)$. Then Rehren proved in [31] that we have a subfactor $N_1 \otimes N_2 \subset P$ such that the dual canonical endomorphism on $N_1 \otimes N_2^{\mathrm{opp}}$ has a decomposition $\bigotimes_{\lambda\in\Delta_1,\mu\in\Delta_2} Z_{\lambda,\mu}\lambda \otimes \mu^{\mathrm{opp}}$ by constructing the corresponding Q-system explicitly. This result is quite general in the sense that we do not put any requirement on how $\alpha^1, \alpha^2$ are given and this contains the original Longo-Rehren construction [23] as a special case for $N_1 = N_2 = M$. We, however, do not have many general extensions of systems of endomorphisms and it seems that all known methods are $\alpha$-induction or its variation, except for the original Longo-Rehren case $N_1 = N_2 = M$. In such a case of $\alpha$-induction, we can take $\alpha^1 = \alpha^+$, $\alpha^2 = \alpha^-$ for $\alpha$-induction from $N$ to $M$ based on a braiding $\varepsilon^{\pm}$ on the system of endomorphisms of $N$ and then $(Z_{\lambda,\mu})$ is the modular invariant matrix mentioned above. We study this case of Rehren's new construction with non-degenerate braiding by use of half-braiding explained above.

In such a case, each endomorphism in the entire system ${}_M\Delta_M$ of endomorphisms of $M$ is a direct summand of $\alpha_\lambda^+ \cdot \alpha_\mu^-$ for some $\lambda, \mu \in \Delta$ by [5, Theorem 5.10]. We can verify that each $\alpha_\lambda^{\pm}$ has a half-braiding $\mathcal{E}_\lambda^{\pm}$ with respect to ${}_M\Delta_M$ in such a case. Then Theorem 3.9 in [7] gives that $\{\eta(\alpha_\lambda^+, \mathcal{E}_\lambda^+)\eta^{\mathrm{opp}}(\alpha_\mu^-, \mathcal{E}_\mu^-)\}_{\lambda,\mu\in {}_N\Delta_N}$ gives the entire system of irreducible endomorphisms of $R$ arising from the Longo-Rehren subfactor $M\otimes M^{\mathrm{opp}} \subset R$. (Here $\eta^{\mathrm{opp}}(\alpha_\mu^-, \mathcal{E}_\mu^-)$ is an extension of $\mathrm{id}\otimes(\alpha_\mu^-)^{\mathrm{opp}}$ of $M \otimes M^{\mathrm{opp}}$ to $R$ defined similarly.) This implies that the strict $C^*$-tensor category given by the system of irreducible endomorphisms of $R$ for the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R$ and that given as a direct product of those arising from the systems ${}_N\Delta_N$ and $({}_N\Delta_N)^{\mathrm{opp}}$ are equivalent in the sense of category theory, where $({}_N\Delta_N)^{\mathrm{opp}}$ is naturally interpreted.

This result itself is not new by the following reason. The two systems ${}_N\Delta_N$ and ${}_M\Delta_M$ are equivalent in the sense of Ocneanu, so they produce the same quantum double system by a general theory of Ocneanu. For ${}_N\Delta_N$, the quantum double system is just a direct product of ${}_N\Delta_N$ and $({}_N\Delta_N)^{\mathrm{opp}}$ since we have a non-degenerate braiding. (See [27, 9].) Our methods in [7], however, produces a more detailed structure result even in this relatively easy situation and this gives us a right answer to questions arising from the new construction of Rehren mentioned above.

We now have a concrete description for the endomorphisms of $R$ associated to $M \otimes M^{\mathrm{opp}} \subset R$. Then by methods analogous to [15], we can compute the Q-system of the dual inclusion to this subfactor. Actually, it is very easy to write down the $Q$-system by methods of [15], but then the intertwiners appearing in the description are of rather complicated form. Manipulating these intertwiners in a non-trivial way, we can simplify the expression for the Q-system considerably, and then it turns out that this Q-system is isomorphic to that arising from the new Rehren inclusion $N \otimes N^{\mathrm{opp}} \subset P$ associated with $\alpha$-induction $\alpha^{\pm}$. (Also see "Added in Proof" of [15] about disappearance of a "twist".) Thus we have the following theorem. (Details of the proof will be presented elsewhere.)

**Theorem 5.1** *Let $N \subset M$ be a type III subfactor with finite index and finite depth and suppose that the system ${}_N\Delta_N$ of endomorphisms of $N$ as above has a non-degenerate braiding. Then the Q-system arising from the dual inclusion of the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R$ associated with the system ${}_M\Delta_M$ is isomorphic to that arising from the new Rehren construction $N\otimes N^{\mathrm{opp}} \subset P$ associated with $\alpha$-induction $\alpha^{\pm}$.*

At the end of [31], Rehren asks for Izumi type description of irreducible endomorphisms of $P$ arising from his subfactor $N \otimes N^{\mathrm{opp}} \subset P$ and in particular, he asks whether braiding exists or not on this system of endomorphisms of $P$. The above theorem in particular shows that the system of endomorphisms of $P$ is isomorphic to the direct product system of ${}_M\Delta_M$ and ${}_M\Delta_M^{\mathrm{opp}}$. Thus we solve these problems and the answer to the second question is negative, since this system can be even non-commutative. (Note that [5, Corollary 6.9] gives a criterion for such non-commutativity.)

*Acknowledgment.* We gratefully acknowledge the financial supports of Grant-in-Aid for Scientific Research, Ministry of Education (Japan), Japan Society for the Promotion of Science, the Mitsubishi Foundation and University of Tokyo. A part of this manuscript was written at Mathematical Sciences Research Institute and we thank it for its hospitality.

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