Spectral Properties of Wigner Matrices

Benjamin Schlein, University of Bonn

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1. Introduction to Random Matrix Theory

Random matrices are $N \times N$ matrices, whose entries are random variables with a given probability law.

Goal of Random Matrix Theory: establish statistical properties of eigenvalues and eigenvectors of random matrices, in the limit $N \to \infty$.

This is typically a **challenging** task because relation between matrix entries and eigenvalues and eigenvectors is complicated.

We will focus here on **hermitian** and **real symmetric** ensembles. Eigenvalues will always be real.

Gaussian Unitary Ensemble: consists of $N \times N$ hermitian matrices H, with probability density

$$\mathrm{d}P(H) = \text{const} \cdot e^{-\frac{N}{2}\operatorname{Tr} H^2}\,\mathrm{d}H$$

with

$$\mathrm{d}H = \prod_{i<j}^{N} \mathrm{dRe}\ h_{ij}\,\mathrm{dIm}\ h_{ij} \prod_{k=1}^{N} \mathrm{d}h_{kk}$$

Independence: writing $\operatorname{Tr} H^2 = \sum_{i,j} |h_{ij}|^2$, we find

$$dP(H) \sim \prod_{i<j} e^{-N|h_{ij}|^2} \mathrm{dRe} h_{ij}\,\mathrm{dIm} h_{ij} \prod_{j} e^{-\frac{N}{2}h_{jj}^2} \mathrm{d}h_{jj}$$

$\Rightarrow$ Entries are independent Gaussian variables.

Unitary invariance: if H is a GUE matrix and U is unitary and fixed, then UHU^* is also a GUE matrix.

Joint eigenvalue density: explicitly given by:

$$p_N(\lambda_1, \dots, \lambda_N) = \text{const} \cdot \prod_{i<j}^{N} (\lambda_i - \lambda_j)^2 \, e^{-\frac{N}{2} \sum_{j=1}^{N} \lambda_j^2} \, .$$

Correlation functions: we are interested in

$$p_N^{(k)}(\lambda_1, \dots, \lambda_k) = \int \mathrm{d}\lambda_{k+1} \dots \mathrm{d}\lambda_N \, p_N \, (\lambda_1, \dots, \lambda_N)$$

Orthogonal polynomial: $\{\psi_n\}_{n \in \mathbb{N}}$ Hermite functions. Then

$$p_N(\lambda_1, \dots, \lambda_N) = C_N \det \left(\psi_{i-1}(\sqrt{N} \lambda_j) \right)^2_{1 \le i,j \le N} \qquad \text{and}$$

$$p_N^{(k)}(\lambda_1, \dots, \lambda_k) = \frac{(N-k)! N^k}{N!} \det \left(\frac{K^{(N)}(\sqrt{N}\lambda_i, \sqrt{N}\lambda_j)}{\sqrt{N}} \right)_{1 \le i,j \le k}$$

with $K^{(N)}(x,y) = \sum_{k=0}^{N-1} \psi_k(x) \psi_k(y) = \dfrac{\psi_N(x)\psi_{N-1}(y) - \psi_N(y)\psi_{N-1}(x)}{(x-y)}$

One-point function $p_N^{(1)}(\lambda)$ is the **density of states** at λ.

As $N \to \infty$, we find

$$p_N^{(1)}(\lambda) = \frac{K^{(N)}(\sqrt{N}\lambda, \sqrt{N}\lambda)}{\sqrt{N}} \longrightarrow \frac{\mathbf{1}(|\lambda| \le 2)}{2\pi} \sqrt{1 - \frac{\lambda^2}{4}} =: \rho_{\mathrm{sc}}(\lambda)$$

Local statistics: for $k \ge 2$, $p_N^{(k)}(\lambda_1, \ldots, \lambda_k)$ describes eigenvalue correlations. Can only have a limit when $\lambda_1, \ldots, \lambda_k$ are in interval of size $\sim 1/N$. In this case, find Wigner-Dyson distribution

$$\frac{1}{\rho_{\mathrm{sc}}^k(E)} p_N^{(k)}\Big(E + \frac{x_1}{N\rho_{\mathrm{sc}}(E)}, .., E + \frac{x_k}{N\rho_{\mathrm{sc}}(E)}\Big) \to \det\left(\frac{\sin(\pi(x_i - x_j))}{\pi(x_i - x_j)}\right)_{i,j \le k}$$

GOE, GSE: similar formulas can be derived for Gaussian ensembles with different symmetries (orthogonal and symplectic ensembles).

Applications:

- Heavy Nuclei: random matrices have been introduced by Wigner to describe excitation spectra of heavy nuclei.

- Anderson Model: in the isolator phase, the eigenvalues of the Anderson Hamiltonian are Poisson distributed. In the metallic phase, the eigenvalues are expected to follow a Wigner-Dyson distribution.

- Quantum Chaos: integrable classical dynamics should lead to Poisson distribution of energy levels. For chaotic classical motion, the energy level are expected to follow GOE statistics.

Universality Conjecture (vague): the (local) statistics of energy levels of chaotic and disordered systems depend on the symmetries but are independent of further details of the system.

Invariant Ensembles: $N \times N$ hermitian matrices H with probability density

$$\mathrm{d}P(H) = \text{const} \cdot e^{-\frac{N}{2}\mathrm{Tr}\, V(H)} \mathrm{d}H, \qquad \text{where } V(\lambda) \geq 0.$$

For $V(\lambda) = \lambda^2$, this is just GUE. Otherwise, ensemble still invariant w.r.t. unitary conjugation, but entries are not independent.

The joint probability density of the N eigenvalues is given by

$$p(\lambda_1, \ldots, \lambda_N) = \text{const} \cdot \prod_{i<j}^{N} (\lambda_i - \lambda_j)^2 \, e^{-\frac{N}{2} \sum_{j=1}^{N} V(\lambda_j)} .$$

Under appropriate conditions on V, universality for invariant ensembles was proven by Pastur-Shcherbina and by Deift et. al.:

$$\frac{1}{\varrho^k(E)} p^{(k)}\left(E + \frac{x_1}{N\varrho(E)}, .., E + \frac{x_k}{N\varrho(E)}\right) \to \det \left(\frac{\sin(\pi(x_i - x_j))}{\pi(x_i - x_j)}\right)_{i,j \leq k}$$

Question: is it possible to establish universality in situations where the joint probability density is not explicitly known?

2. Wigner Matrices and the Local Semicircle Law

Hermitian Wigner Matrices: $N \times N$ matrices $H = (h_{kj})_{1 \le k,j \le N}$ such that $H^* = H$ and

$$h_{kj} = \frac{1}{\sqrt{N}}\big(x_{kj} + i y_{kj}\big) \qquad \text{for all } 1 \le k < j \le N$$

$$h_{kk} = \frac{2}{\sqrt{N}} x_{kk} \qquad \text{for all } 1 \le k \le N$$

where x_{kj}, y_{kj} and x_{kk} ($1 \le k \le N$) are iid with

$$\mathbb{E}\, x_{jk} = 0, \quad \mathbb{E}\, x_{jk}^2 = \frac{1}{2} \quad \text{and } \mathbb{E}\, e^{\alpha\, x_{ij}^2} < \infty \quad \text{for some } \alpha > 0$$

Remark: scaling so that eigenvalues remain bounded as $N \to \infty$.

$$\mathbb{E} \sum_{\alpha=1}^{N} \lambda_\alpha^2 = \mathbb{E}\,\mathrm{Tr}\, H^2 = \mathbb{E} \sum_{j,k=1}^{N} |h_{jk}|^2 = N^2\, \mathbb{E}\, |h_{jk}|^2$$

$$\Rightarrow \quad \mathbb{E}|h_{jk}|^2 = O(N^{-1})$$

Semicircle Law (Wigner, 1955): for any $\delta > 0$,

$$\lim_{\eta \to 0} \lim_{N \to \infty} \mathbb{P}\left(\left| \frac{\mathcal{N}[E - \frac{\eta}{2}; E + \frac{\eta}{2}]}{N\eta} - \rho_{\text{sc}}(E) \right| \geq \delta \right) = 0$$

where

$$\mathcal{N}[I] = \text{number of eigenvalues in interval } I$$

$$\rho_{\text{sc}}(E) = \frac{1}{2\pi}\sqrt{1 - \frac{E^2}{4}}.$$

Remark 1: semicircle independent of distribution of entries.

Remark 2: Wigner result concerns DOS on macroscopic scales, in intervals containing order N eigenvalues.

Question: What about density of states on smaller scales?

Theorem [Erdős-S.-Yau, 2008]: Fix $|E| < 2$. Then, for any $\delta > 0$,

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathbb{P}\left(\left|\frac{\mathcal{N}\left[E - \frac{K}{2N}; E + \frac{K}{2N}\right]}{K} - \rho_{\text{sc}}(E)\right| \geq \delta\right) = 0$$

Semicircle law holds up to **microscopic** scales.

Intermediate scales: if $\eta(N) \to 0$ such that $N\eta(N) \to \infty$, we have

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\frac{\mathcal{N}\left[E - \frac{\eta(N)}{2}; E + \frac{\eta(N)}{2}\right]}{N\eta(N)} - \rho_{\text{sc}}(E)\right| \geq \delta\right) = 0$$

Previous results by Khorunzhy, Bai-Miao-Tsay, and Guionnet-Zeitouni (up to scales $\eta(N) \simeq N^{-1/2}$).

Main ingredients of proof: upper bound on density and fixed point equation for Stieltjes transform.

Upper bound: states that

$$\mathbb{P}\left(\frac{\mathcal{N}\left[E-\frac{\eta}{2},E+\frac{\eta}{2}\right]}{N\eta}\geq K\right)\lesssim e^{-c\sqrt{KN\eta}}$$

if $\eta=\eta(N)\geq 1/N$.

To show the upper bound we observe that

$$\mathcal{N}[E-\eta/2,E+\eta/2]=\sum_\alpha \mathbf{1}(|\mu_\alpha-E|\leq\eta)$$
$$\lesssim\sum_\alpha\frac{\eta^2}{(\mu_\alpha-E)^2+\eta^2}=\eta\,\mathrm{Im}\;\sum_\alpha\frac{1}{\mu_\alpha-E-i\eta}$$

and hence

$$\rho=\lesssim\frac{1}{N}\mathrm{Im}\;\mathrm{Tr}\,\frac{1}{H-E-i\eta}=\frac{1}{N}\mathrm{Im}\;\sum_{j=1}^N\frac{1}{H-E-i\eta}(j,j)$$

Decomposing H as

$$H = \begin{pmatrix} h_{11} & \mathbf{a}^* \\ \mathbf{a} & B \end{pmatrix}$$

we find (Feshbach map)

$$\frac{1}{H-z}(1,1) = \frac{1}{h_{11} - z - \mathbf{a} \cdot (B-z)^{-1}\mathbf{a}} = \frac{1}{h_{11} - z - \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{\lambda_\alpha - z}}$$

with

$$\xi_\alpha = N|\mathbf{a} \cdot \mathbf{u}_\alpha|^2 \qquad \Rightarrow \qquad \mathbb{E}\,\xi_\alpha = 1$$

where λ_α and $\mathbf{u}_\alpha$ are eigenvalues and eigenvectors of B.

We conclude that, with high probability,

$$\text{Im}\ \frac{1}{H - E - i\eta}(1,1) \lesssim \frac{1}{\text{Im}\ \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{\lambda_\alpha - E - i\eta}}$$

$$\lesssim \frac{1}{\text{Im}\ \frac{1}{N}\text{Tr}\,\frac{1}{B-E-i\eta}} \lesssim \frac{1}{\rho_{\text{minor}}} \simeq \frac{1}{\rho}$$

Fixed point equation: we consider the Stieltjes transform

$$m_N(z) = \frac{1}{N}\mathrm{Tr}\,\frac{1}{H-z}, \qquad m_{\mathrm{sc}}(z) = \int \mathrm{d}y \frac{\rho_{\mathrm{sc}}(y)}{y-z}$$

Convergence of the density follows if we can prove that

$$m_N(z) \to m_{\mathrm{sc}}(z), \qquad \text{for Im } z = \eta \geq K/N.$$

The Stieltjes transform m_{sc} solves the fixed point equation

$$m_{\mathrm{sc}}(z) + \frac{1}{z + m_{\mathrm{sc}}(z)} = 0$$

It is enough to show that, with high probability,

$$\left| m_N(z) + \frac{1}{z + m_N(z)} \right| \leq \delta$$

To this end, we use again

$$m_N(z) = \frac{1}{N} \sum_j \frac{1}{h_{jj} - z - \frac{1}{N}\sum_\alpha \frac{\xi_\alpha^{(j)}}{\lambda_\alpha^{(j)} - z}}$$

3. Delocalization of Eigenvectors

Let $\mathbf{v} = (v_1, \ldots, v_N)$ be an ℓ_2-normalized vector in $\mathbb{C}^N$. Distinguish two extreme cases:

Complete localization: one large component, for example

$$\mathbf{v} = (1, 0, \ldots, 0) \qquad \Rightarrow \qquad \|\mathbf{v}\|_p = 1, \text{ for all } 2 < p \le \infty$$

Complete delocalization: all components have same size,

$$\mathbf{v} = (N^{-1/2}, \ldots, N^{-1/2}) \quad \Rightarrow \quad \|\mathbf{v}\|_p = N^{-\frac{1}{2}+\frac{1}{p}} \ll 1$$

Theorem [Erdős-S.-Yau, 2008]:
Suppose $\mathbb{E}\, e^{\nu|x_{ij}|} < \infty$ for some $\nu > 0$. Fix $\kappa > 0$, $2 < p \le \infty$. Then

$$\mathbb{P}\Big(\exists \mathbf{v} : H\mathbf{v} = \mu\mathbf{v}, \mu \in [-2+\kappa, 2-\kappa], \|\mathbf{v}\|_2 = 1, \|\mathbf{v}\|_p \ge MN^{-\frac{1}{2}+\frac{1}{p}}\Big) \le Ce^{-c\sqrt{M}}$$

for all M, N large enough.

Idea of proof: we write $\mathbf{v} = (v_1, \mathbf{w})$. Hence $H\mathbf{v} = \mu\mathbf{v}$ implies

$$\begin{pmatrix} h-\mu & \mathbf{a}^* \\ \mathbf{a} & B-\mu \end{pmatrix} \begin{pmatrix} v_1 \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \end{pmatrix} \quad \Rightarrow \quad \mathbf{w} = v_1(\mu - B)^{-1}\mathbf{a}$$

By normalization

$$1 = v_1^2 + \mathbf{w}^2 \quad \Rightarrow \quad |v_1|^2 = \frac{1}{1 + \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{(\mu-\lambda_\alpha)^2}} \qquad (\xi_\alpha = N|\mathbf{a}\cdot\mathbf{u}_\alpha|^2),$$

where λ_α and $\mathbf{u}_\alpha$ are the eigenvalues and the eigenvectors of B.

$$|v_1|^2 \leq \frac{1}{\frac{1}{N\eta^2}\sum_{\alpha:|\lambda_\alpha-\mu|\leq\eta}\xi_\alpha} \lesssim \frac{N\eta^2}{|\{\alpha : |\lambda_\alpha - \mu| \leq \eta\}|}$$

Choosing $\eta = K/N$, for a sufficiently large $K > 0$, we find

$$|v_1|^2 \leq \frac{K^2}{N}\frac{1}{|\{\alpha : |\lambda_\alpha - \mu| \leq K/N\}|} \leq c\,\frac{K}{N}$$

with high probability, because, by the local semicircle law, there must be order K eigenvalues λ_α with $|\lambda_\alpha - \mu| \leq K/N$. $\square$

4. Level Repulsion

Theorem [Erdős-S.-Yau, 2008]: Suppose $\mathbb{E}\, e^{\nu |x_{ij}|} < \infty$ for some $\nu > 0$, fix $|E| < 2$.

Fix $k \geq 1$, and assume that the probability density $h(x) = e^{-g(x)}$ of the matrix entries satisfies the bound

$$\left|\widehat{h}(p)\right| \leq \frac{1}{(1 + Cp^2)^{\sigma/2}}, \qquad \left|\widehat{hg''}(p)\right| \leq \frac{1}{(1 + Cp^2)^{\sigma/2}} \quad \text{for } \sigma \geq 5 + k^2.$$

Then there exists a constant $C_k > 0$ such that

$$\mathbb{P}\left(\mathcal{N}\left[E - \frac{\varepsilon}{2N}; E + \frac{\varepsilon}{2N}\right] \geq k\right) \leq C_k\, \varepsilon^{k^2}$$

for all N large enough, and all $\varepsilon > 0$.

Remark: for GUE, we have

$$p(\mu_1, \ldots, \mu_N) \simeq \prod_{i<j} (\mu_i - \mu_j)^2 \quad \Rightarrow \quad \mathbb{P}(\mathcal{N}_\varepsilon \geq k) \simeq \varepsilon^{k^2}$$

5. Universality of hermitian Wigner Matrices

Universality: local eigenvalue statistics in the limit $N \to \infty$ is expected to depend only on symmetry, but to be independent of probability law of matrix entries.

Remark: universality at the edges of the spectrum was established by Soshnikov in 1999 using the moment method. Here I will consider universality in the bulk of the spectrum.

In 2001, Johansson established the validity of bulk universality for ensembles of hermitian Wigner matrices with a Gaussian component (result was later extended by Ben Arous-Péché).

Johansson's approach: consider matrices of the form

$$H = H_0 + t^{\frac{1}{2}} V$$

where V is a GUE-matrix, and H_0 is an arbitrary Wigner matrix.

The matrix H can be obtained by letting every entry of H_0 evolve under a Brownian motion up to time t (more prec. t/N).

The distribution of the eigenvalues of the matrix evolves then according to Dyson's Brownian motion

$$\mathrm{d}\lambda_\alpha = \frac{\mathrm{d}B_\alpha}{\sqrt{N}} + \frac{1}{N} \sum_{\beta \neq \alpha} \frac{1}{\lambda_\alpha - \lambda_\beta} \mathrm{d}t, \qquad 1 \leq \alpha \leq N$$

where $\{B_\alpha : 1 \leq \alpha \leq N\}$ is a collection of independent Brownian motion.

The joint probability distribution of the eigenvalues $\mathbf{x} = (x_1, \ldots, x_N)$ of H is

$$p(\mathbf{x}) = \int \mathrm{d}\mathbf{y}\; q_t(\mathbf{x};\mathbf{y})\; p_0(\mathbf{y})$$

where p_0 is the distribution of the eigenvalues $\mathbf{y} = (y_1, \ldots, y_N)$ of H_0 and

$$q_t(\mathbf{x};\mathbf{y}) = \frac{N^{N/2}}{(2\pi t)^{N/2}} \frac{\Delta_N(\mathbf{x})}{\Delta_N(\mathbf{y})} \;\det\left(e^{-N(x_j-y_k)^2/2t}\right)_{j,k=1}^{N},$$

with the Vandermonde determinant

$$\Delta(\mathbf{x}) = \prod_{i<j}^{N}(x_i - x_j) = \det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ \ldots & \ldots & \ldots & \ldots \\ x_1^N & x_2^N & \ldots & x_N^N \end{pmatrix}$$

This can be proven using the Harish-Chandra/Itzykson-Zuber formula

$$\int_{U(N)} e^{-\frac{N}{2t}\mathrm{Tr}\,(U^* R(\mathbf{x})U - H_0(\mathbf{y}))^2}\mathrm{d}U = \frac{1}{\Delta(\mathbf{x})\Delta(\mathbf{y})}\;\det\left(e^{-\frac{N}{2t}(x_j-y_i)^2}\right)_{1\le i,j\le N}$$

The k-point correlation function of p is therefore given by

$$p^{(k)}(x_1,\ldots,x_k) = \int q_t^{(k)}(x_1,\ldots,x_k;\mathbf{y})\, p_0(\mathbf{y})\mathrm{d}\mathbf{y}$$

where

$$\begin{aligned} q_t^{(k)}(x_1,\ldots,x_k;\mathbf{y}) &= \int q_t(\mathbf{x};\mathbf{y})\,\mathrm{d}x_{k+1}\ldots\mathrm{d}x_N \\ &= \frac{(N-k)!}{N!}\ \det\Big(K_{t,N}(x_i,x_j;\mathbf{y})\Big)_{1\le i,j\le k} \end{aligned}$$

with

$$\begin{aligned} &K_{t,N}(u,v;\mathbf{y}) = \frac{N}{(2\pi i)^2(v-u)t} \\ &\times \int_\gamma \mathrm{d}z \int_\Gamma \mathrm{d}w\,(e^{-N(v-u)(w-r)/t}-1)\prod_{j=1}^{N}\frac{w-y_j}{z-y_j} \\ &\times \frac{1}{w-r}\Big(w-r+z-u-\frac{t}{N}\sum_j \frac{y_j-r}{(w-y_j)(z-y_j)}\Big)e^{N(w^2-2vw-z^2+2uz)/2t} \end{aligned}$$

where γ is the union of two horizontal lines and Γ is a vertical line in the $\mathbb{C}$-plane, and $r\in\mathbb{R}$ is arbitrary.

Convergence of k-point correlation follows from

$$\frac{1}{N\varrho(u)}K_{t,N}\left(u+\frac{x_1}{N\varrho(u)},u+\frac{x_2}{N\varrho(u)};\mathbf{y}\right)\to\frac{\sin\pi(x_2-x_1)}{\pi(x_2-x_1)}\qquad\text{for a.e. }\mathbf{y}$$

To prove convergence of $K_{t,N}$ to sine-kernel Johansson uses

$$\frac{1}{N\varrho(u)}K_{t,N}\Big(u,u+\frac{\tau}{N\varrho};\mathbf{y}\Big)$$
$$=N\int_\gamma\frac{\mathrm{d}z}{2\pi i}\int_\Gamma\frac{\mathrm{d}w}{2\pi i}h_N(w)g_N(z,w)e^{N(f_N(w)-f_N(z))}$$

with

$$f_N(z)=\frac{1}{2t}(z^2-2uz)+\frac{1}{N}\sum_j\log(z-y_j)$$
$$g_N(z,w)=\frac{1}{t(w-r)}[w-r+z-u]-\frac{1}{N(w-r)}\sum_j\frac{y_j-r}{(w-y_j)(z-y_j)}$$
$$h_N(w)=\frac{1}{\tau}\Big(e^{-\tau(w-r)/t\varrho}-1\Big)$$

and performs a detailed asymptotic saddle analysis.

Beyond Johansson: what happens if $t = t(N) \to 0$? Consider

$$t = N^{-1+\varepsilon}$$

Similar integral representation but asymptotic analysis is more delicate and requires microscopic convergence to the semicircle.

Theorem [Erdős-Péché-Ramirez-S.-Yau]: Let $p_N^{(k)}$ be the k-point eigenvalue correlation function for the ensemble $H = H_0 + t^{1/2} V$, where H_0 is an arbitrary Wigner matrix, V is an independent GUE matrix, and $t \geq N^{-1+\varepsilon}$. Then

$$\lim_{N\to\infty} \frac{1}{\rho_{\mathrm{sc}}^k(E)} p_N^{(k)} \left(E + \frac{x_1}{N\rho_{\mathrm{sc}}(E)}, \ldots, E + \frac{x_k}{N\rho_{\mathrm{sc}}(E)} \right) = \det \left(\frac{\sin(\pi(x_i - x_j))}{(\pi(x_i - x_j))} \right)_{i,j=1}^{k}$$

Time reversal to remove Gaussian part: let $h(x)$ be the density of the matrix elements of H_0.

The matrix elements of $H = H_0 + t^{\frac{1}{2}} V$ have density

$$h_t(x) = (e^{tL} h)(x), \qquad \text{with} \qquad L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

Then

$$\int \frac{|h_t(x) - h(x)|^2}{h(x)} dx \leq C t^2$$

Letting $F = h^{\otimes N^2}$ and $F_t = (e^{tL} h)^{\otimes N^2}$ we find

$$\int \frac{|F_t - F|^2}{F} dx_1 \ldots dx_{N^2} \leq C N^2 t^2$$

It is only small for $t \ll N^{-1}$.

Hence $t = N^{-1+\varepsilon}$ is still not enough.

We would like to write

$$h = e^{tL} v_t \qquad \text{with} \qquad v_t = e^{-tL} h$$

But the heat equation cannot be reversed.

$\Rightarrow$ approximate inversion of heat semigroup

Define $v_t = (1 - tL)h$. Then

$$h_t = e^{tL} v_t \simeq h + t^2 L^2 h \qquad \Big(\text{while} \quad e^{tL} h \simeq h + tLh\Big)$$

Therefore

$$\int \frac{|h_t - h|^2}{h} dx \leq C t^4$$

Hence, if $F = h^{\otimes N^2}$ and $F_t = h_t^{\otimes N^2}$, we find

$$\int \frac{|F_t - F|^2}{F} dx_1 \dots dx_{N^2} \leq C N^2 t^4 \ll 1 \qquad \text{for } t = N^{-1+\varepsilon}$$

Theorem [Erdős-Péché-Ramirez-S.-Yau]: Suppose H is a hermitian Wigner matrix, whose entries have law $g = e^{-h}$, for $h \in C^6(\mathbb{R})$. Then,

$$\lim_{N\to\infty} \frac{1}{\rho_{sc}^2(E)} p_N^{(2)} \left(E + \frac{x_1}{N\rho_{sc}(E)}, E + \frac{x_2}{N\rho_{sc}(E)}\right)$$

$$= 1 - \frac{\sin^2(\pi(x_1 - x_2))}{(\pi(x_1 - x_2))^2}$$

The result extends to higher correlation functions, assuming more regularity on h.

Tao-Vu approach: let H and H' be two Wigner matrices whose entries have distribution x, y; assume that typical distance between eigenvalues is order one ($x, y \simeq \sqrt{N}$).

Assume that

$$\mathbb{E}\, x^m = \mathbb{E}\, y^m \qquad \text{for} \qquad 1 \le m \le 4$$

Fix $k \ge 1$ and consider a nice function $G : \mathbb{R}^k \to \mathbb{R}$. Then

$$\left|\mathbb{E}\, G(\lambda_{\alpha_1}(H), \dots, \lambda_{\alpha_k}(H)) - \mathbb{E} G(\lambda_{\alpha_1}(H), \dots, \lambda_{\alpha_k}(H))\right| \to 0$$

as $N \to \infty$.

Idea of proof: change one entry at the time.

$H(z) =$ matrix obtained from H replacing (i, j)-entry with z

$F(z) = G(\lambda_\alpha(H(z)))$ (we take $k = 1$)

$$F(x) = F(0) + xF'(0) + \cdots + \frac{x^5}{5!} F^{(v)}(0) + ..$$

$$F(y) = F(0) + yF'(0) + \cdots + \frac{y^5}{5!} F^{(v)}(0) + ..$$

Therefore

$$|\mathbb{E}F(x) - \mathbb{E}F(y)| \leq \mathbb{E}\,|x|^5 F^{(v)}(0)$$

Observe

$$\mathbb{E}|x|^5 \simeq N^{5/2} \qquad \text{but} \qquad F^{(m)}(0) \simeq N^{-m}$$

In fact

$$F'(0) = G'(\lambda_\alpha(H)) \cdot \frac{\partial \lambda_\alpha}{\partial h_{ij}} = G'(\lambda_\alpha(H)) \cdot \mathbf{v}_\alpha(i)\mathbf{v}_\alpha(j) \simeq N^{-1}$$

Hence

$$|\mathbb{E}F(x) - \mathbb{E}F(y)| \leq CN^{-5/2}$$

Repeating this argument N^2 times, we can replace all entries of H; the total error is $O(N^{-1/2})$.

Universality (Tao-Vu): for given H, find Johansson matrix

$$H_t = e^{-t/2} H_0 + (1 - e^{-t})^{1/2} V$$

such that H and H_t have four matching moments.

This is only possible if entries are supported on at least 3 points.

Universality (Erdős-Ramirez-S.-Tao-Vu-Yau): compare H with the evolved matrix

$$H_t = e^{-t/2} H + (1 - e^{-t})^{1/2} V$$

with $t = N^{-1+\delta}$.

Moments do not match, but they are very close.

6. Universality for Non-Hermitian Ensembles

The local relaxation flow: Dyson Brownian Motion describes evolution of eigenvalues. Equilibrium measure is GUE measure

$$\mu(\mathbf{x})\mathrm{d}\mathbf{x} = \frac{e^{-\mathcal{H}(\mathbf{x})}}{Z}\mathrm{d}\mathbf{x}, \qquad \mathcal{H}(\mathbf{x}) = N\left[\sum_{j=1}^{N}\frac{x_j^2}{2} - \frac{2}{N}\sum_{i<j}\log|x_j - x_i|\right]$$

The evolution of an initial probability density function $f\mu$ w.r.t DBM is described by the heat equation

$$\partial_t f_t = L f_t,$$

with the generator

$$L = \sum_{i=1}^{N}\frac{1}{2N}\partial_i^2 + 2\sum_{i=1}^{N}\left(-\frac{1}{4}x_i + \frac{1}{2N}\sum_{j\neq i}\frac{1}{x_i - x_j}\right)\partial_i$$

Relaxation time of Dyson's Brownian motion given by

$$\frac{1}{2N}\nabla^2\mathcal{H} \geq O(1) \quad \Rightarrow \quad \text{relaxation on times } O(1)$$

Idea: introduce new flow with shorter relaxation time. Define

$$\widetilde{\mathcal{H}}(\mathbf{x}) = N\left[\sum_{j=1}^{N}\left(\frac{x_j^2}{2} + \frac{1}{2R^2}(x_j-\gamma_j)^2\right) - \frac{2}{N}\sum_{i<j}\log|x_j - x_i|\right]$$

$$= \mathcal{H}(\mathbf{x}) + \frac{N}{2R^2}\sum_{j=1}^{N}(x_j-\gamma_j)^2$$

where γ_j is position of the j-th eigenvalue w.r.t. semicircle law, and $R = N^{-\varepsilon} \ll 1$.

Introduce new equilibrium measure $\omega(\mathbf{x}) = e^{-\widetilde{H}(\mathbf{x})}/\widetilde{Z}$ and new evolution

$$\partial_t g_t = \widetilde{L} g_t \qquad \text{with} \qquad \widetilde{L} = L - \frac{1}{R^2}\sum_{j=1}^{N}(x_j - \gamma_j)\,.$$

Observe that

$$\frac{\nabla^2\widetilde{\mathcal{H}}(\mathbf{x})}{N} \geq CR^{-2} \geq N^{2\varepsilon} \gg 1 \quad \Rightarrow \quad \text{relaxation on short times}$$

Hence, if $\mathcal{G}_{i,n}(\mathbf{x}) = G\Big(N(x_i - x_{i+1}), \ldots, N(x_{i+n-1} - x_{i+n})\Big)$, we find

$$\Big|\int \frac{1}{N}\sum_{i\in J}\mathcal{G}_{i,n}\mathrm{d}\omega - \int \frac{1}{N}\sum_{i\in J}\mathcal{G}_{i,n}\, g\,\mathrm{d}\omega\Big| \le C_n\Big(\frac{D_\omega(\sqrt{g})R^2}{N}\Big)^{1/2}$$

with the Dirichlet form

$$D_\omega(h) = \frac{1}{N}\sum_{j=1}^{N}\int \left|\partial_{x_j} h\right|^2 \mathrm{d}\omega$$

On other hand, if difference between generators is small, we expect $f_t\mu \simeq \omega = \psi\mu$. In fact, for $t \gg R^2$, we find that

$$D_\omega(\sqrt{f_t/\psi}) \le CN\Lambda \qquad \text{where} \quad \Lambda = \mathbb{E}_t \sum_j |x_j - \gamma_j|^2 .$$

From microscopic semicircle law, we find $\Lambda \le N^{-\varepsilon}$.

This implies universality for ensembles of the form $H_0 + t^{1/2}V$, if $t \geq N^{-\varepsilon}$, for arbitrary symmetry.

Time-reversal argument implies universality for all matrices whose entries have enough regularity.

Combining with the result of Tao-Vu, we find universality for arbitrary ensembles.

Theorem [Erdős-S.-Yau (2009), Erdős-Yau-Yin (2010)]:
Fix $|E_0| < 2$, $k \in \mathbb{N}$, $\delta > 0$. Then

$$\int_{E_0-\delta}^{E_0+\delta} dE \int dx_1, \dots dx_k \, O(x_1, \dots, x_k) \times \Big[p^{(k)}\Big(E + \frac{x_1}{N\varrho(E)}, .., E + \frac{x_k}{N\varrho(E)}\Big) - p^{(k)}_{\text{Gauss}}\Big(E + \frac{x_1}{N\varrho(E)}, .., E + \frac{x_k}{N\varrho(E)}\Big)\Big] \to 0$$

as $N \to \infty$.

7. Averaged density of states on arbitrarily small scales

Density of states (**DOS**) on intervals of size ε/N:

$$\frac{1}{\varepsilon}\mathcal{N}\left[E-\frac{\varepsilon}{2N};E+\frac{\varepsilon}{2N}\right]=\frac{1}{\varepsilon}\sum_{\alpha=1}^{N}\mathbf{1}(N|\lambda_\alpha-E|\leq\varepsilon/2)$$

For $\varepsilon \lesssim 1$, convergence in probability cannot hold.

Averaged DOS:

$$\frac{1}{\varepsilon}\mathbb{E}\mathcal{N}\left[E-\frac{\varepsilon}{2N};E+\frac{\varepsilon}{2N}\right]=\frac{1}{\varepsilon}\int \mathrm{d}x\,\mathbf{1}(|x|\leq\varepsilon/2)\,p_N^{(1)}\left(E+\frac{x}{N}\right)$$

Universality implies that, as $N\to\infty$,

$$\frac{1}{\varepsilon}\mathbb{E}\mathcal{N}\left[E-\frac{\varepsilon}{2N};E+\frac{\varepsilon}{2N}\right]\to\rho_{\mathrm{sc}}(E)$$

for fixed $\varepsilon > 0$.

Question Does averaged DOS converge to semicircle on smaller intervals?

Theorem [Maltsev-S., 2010]: Let h be the prob. density function of the entries of the hermitian Wigner matrix H. Let

$$\int \left[\left| \frac{h'(s)}{h(s)} \right|^2 + \left| \frac{h''(s)}{h(s)} \right|^2 \right] h(s) ds < \infty$$

Then we have, as $N \to \infty$,

$$\frac{1}{\varepsilon} \mathbb{E} \mathcal{N} \left[E - \frac{\varepsilon}{2N}; E + \frac{\varepsilon}{2N} \right] \to \rho_{\mathrm{sc}}(E)$$

uniformly in $\varepsilon > 0$.

In other words,

$$\lim_{N \to \infty} \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \mathcal{N} \left[E - \frac{\varepsilon}{2N}; E + \frac{\varepsilon}{2N} \right] = \rho_{\mathrm{sc}}(E)$$

and

$$\lim_{N \to \infty} \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \mathcal{N} \left[E - \frac{\varepsilon}{2N}; E + \frac{\varepsilon}{2N} \right] = \rho_{\mathrm{sc}}(E)$$

Upper bound on average DOS (Erdős - S. - Yau, 2008):

we use

$$\mathcal{N}\left[E-\frac{\varepsilon}{2N};E+\frac{\varepsilon}{2N}\right]\lesssim\frac{\varepsilon}{N}\,\text{Im Tr}\,\frac{1}{H-E-i\frac{\varepsilon}{N}}$$

and the representation

$$\frac{1}{H-z}(1,1)=\frac{1}{h_{11}-z-\langle\mathbf{a},(B-z)^{-1}\mathbf{a}\rangle}=\frac{1}{h_{11}-z-\frac{1}{N}\sum_\alpha\frac{\xi_\alpha}{\lambda_\alpha-z}}$$

where

$$\xi_\alpha=N|\mathbf{u}_\alpha\cdot\mathbf{a}|^2\quad\Rightarrow\quad\mathbb{E}\,\xi_\alpha=1$$

We conclude that

$$\begin{aligned}\mathbb{E}\mathcal{N}&\left[E-\frac{\varepsilon}{2N};E+\frac{\varepsilon}{2N}\right]\\&\lesssim\varepsilon\,\mathbb{E}\,\frac{1}{\left((h_{11}-E-\sum_\alpha d_\alpha\xi_\alpha)^2+(\frac{\varepsilon}{N}+\sum_\alpha c_\alpha\xi_\alpha)^2\right)^{1/2}}\end{aligned}$$

with

$$c_\alpha=\frac{\varepsilon}{N^2(\lambda_\alpha-E)^2+\varepsilon^2},\qquad d_\alpha=\frac{N(\lambda_\alpha-E)}{N^2(\lambda_\alpha-E)^2+\varepsilon^2}$$

Convergence to semicircle: define the Stieltjes transform

$$m_N(z) = \frac{1}{N}\mathrm{Tr}\,\frac{1}{H-z} = \frac{1}{N}\sum_\alpha \frac{1}{\mu_\alpha - z}$$

The DOS on scales ε/N is related with the imaginary part

$$\mathrm{Im}\; m_N\left(E + i\frac{\varepsilon}{N}\right) = \sum_\alpha \frac{\varepsilon}{N^2(\mu_\alpha - E)^2 + \varepsilon^2}$$

To prove convergence to semicircle, it is enough to show

$$\frac{1}{\pi}\mathbb{E}\,\mathrm{Im}\; m_N\left(E + i\frac{\varepsilon}{N}\right) \to \rho_{\mathrm{sc}}(E)$$

uniformly in $\varepsilon > 0$.

To this end we show the upper bound on the derivative

$$\left|\frac{\mathrm{d}}{\mathrm{d}E}\mathbb{E}\,\mathrm{Im}\; m_N\left(E + i\frac{\varepsilon}{N}\right)\right| \le CN$$

uniformly in $\varepsilon > 0$.

The upper bound on the derivative implies that, for small but fixed $\kappa > 0$,

$$\frac{1}{\pi}\mathbb{E}\,\text{Im}\ m_N\left(E + i\frac{\varepsilon}{N}\right)$$

$$\simeq \frac{N}{\pi\kappa}\int_{E-\frac{\kappa}{2N}}^{E+\frac{\kappa}{2N}} \mathrm{d}E'\,\mathbb{E}\,\text{Im}\ m_N\left(E' + i\frac{\varepsilon}{N}\right)$$

$$= \frac{1}{\pi\kappa}\mathbb{E}\sum_{\alpha}\left[\text{arctg}\left(\frac{N\left(\mu_\alpha - E - \frac{\kappa}{2N}\right)}{\varepsilon}\right) - \text{arctg}\left(\frac{N\left(\mu_\alpha - E + \frac{\kappa}{2N}\right)}{\varepsilon}\right)\right]$$

$$\simeq \frac{1}{\kappa}\mathbb{E}\mathcal{N}\left[E - \frac{\kappa}{2N}; E + \frac{\kappa}{2N}\right]$$

Hence, letting first $N \to \infty$, and then $\kappa \to 0$,

$$\frac{1}{\pi}\mathbb{E}\,\text{Im}\ m_N\left(E + i\frac{\varepsilon}{N}\right) \simeq \mathbb{E}\frac{1}{\kappa}\mathcal{N}\left[E - \frac{\kappa}{2N}; E + \frac{\kappa}{2N}\right] \to \rho_{\text{sc}}(E)\,. \qquad \square$$