Quantum Galois correspondence for subfactors

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Abstract

Ocneanu has obtained a certain type of quantized Galois correspondence for the Jones subfactors of type A_n and his arguments are quite general. By making use of them in a more general context, we define a notion of a subequivalent paragroup and establish a bijective correspondence between generalized intermediate subfactors in the sense of Ocneanu and subequivalent paragroups for a given strongly amenable subfactors of type II_1 in the sense of Popa, by encoding the subequivalence in terms of a commuting square. For this encoding, we generalize Sato's construction of equivalent subfactors of finite depth from a single commuting square, to strongly amenable subfactors.

We also explain a relation between our notion of subequivalent paragroups and sublattices of a Popa system, using open string bimodules.

1 Introduction

Our aim here is to establish the "quantum" version of the Galois correspondence for subfactors using Ocneanu's work in a more general context and encode subequivalence of paragroups in terms of a commuting square.

The Galois correspondence for group actions has been studied in detail in various forms. It gives a bijective correspondence between intermediate algebras of $R^G \subset G$ or $R \subset R \rtimes G$ for a group G acting on a von Neumann algebra R and subgroups of G. (See [12] for one of the most recent forms.) We would like to "quantize" this correspondence for a subfactor $N \subset M$.

Ocneanu's paragroup gives a combinatorial characterization of higher relative commutants for a "good class" of subfactors and a certain quantization of a classical Galois group for a general subfactor of finite index, and this gives a complete invariant for strongly amenable subfactors of type II_1 by Popa's deep classification theorem [24]. So, a natural attempt to quantize the classical Galois correspondence is to establish a bijective correspondence between "subparagroups" of the paragroup for a subfactor and intermediate algebras of $N \subset M$, but it is not clear at all what "subparagroups" mean. We will define an appropriate notion of a subparagroup and call it a *subequivalent paragroup*, since it is based on a notion of equivalence for system of bimodules, due to Ocneanu, and also show that we will also have to "quantize" the notion of intermediate algebras as in [21], in order to get the "Galois correspondence". The essential tools are Ocneanu's several work, particularly [21], and Sato's work on commuting square, but we have to generalize their work first because they have worked on subfactors with finite depth, rather than with strong amenability. This is a natural extension of Ocneanu's work on generalized intermediate subfactors in [21].

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2 Equivalent paragroups

First, we review some basic facts about equivalent paragroups for extremal subfactors $N \subset M$ of type II_1 (with finite index). Here the extremality is in the sense of [23]. Nothing is essentially new here, but our aim here is to give precise definitions, since we do not assume the finite depth condition which is usually assumed in this kind of theory.

Ocneanu has two equivalent axiomatizations of paragroups as in [18], [20]. (See [8, Chapters 10, 12].) For our purpose here, it is more convenient to use the approach in [20] based on fusion rule algebras and quantum $6j$-symbols.

In this setting, a paragroup is a combination of a fusion rule algebra linearly generated by at most countable objects $\{X_i\}$ having two kinds of attributions $\{A, B\}$ on the left and right of each object, a dimension function assigning a positive number $[X_i]^{1/2}$ to each object X_i and giving an algebra homomorphism from the fusion rule algebra to $\mathbf{R}$, quantum $6j$-symbols on this fusion rule algebra, and a generator ${}_AX_B$ which is a finite linear combination of objects $\{X_i\}$ with positive integer coefficients. (See the formulation in [8, Chapter 12], particularly about the axioms of quantum $6j$-symbols. What we really have is an equivalence class of quantum $6j$-symbols, rather than quantum $6j$-symbols themselves.) As in [28], we call such a paragroup an A-B paragroup. Note that it is assumed in [8], [28] that the set $\{X_i\}$ of the objects is finite, but we now allow it to be countable here. It is required that for X and Y among the X_i, the product formula $X \cdot Y = \sum_Z N_{X,Y}^Z Z$ have only finitely many non-zero coefficients $N_{X,Y}^Z$.

We say that an object having a left A-attribution and a right B-attribution is of A-B type. The other types, A-A, B-A, B-B are defined similarly. Since we always use Popa's classification theorem [24], we may and do identify objects with bimodules. For a paragroup π_1, we set the global index $[[\pi_1]]$ of π_1 to be $\sum_{{}_AX_A} [{}_AX_A]$ where the sum is taken over all the objects X of A-A type. If we have infinitely many objects, then the global index is infinity. It was first noted by Ocneanu [20] and is now fairly easy to see that this global index is equal to $\sum_{{}_BY_B} [{}_BY_B]$ where the sum is taken over all the objects Y of B-B type.

If we consider only objects of A-A type [resp. B-B type] for an A-B paragroup, we get a fusion rule algebra with quantum $6j$-symbols. We call such a system the A-A subsystem [resp. B-B subsystem] of the A-B paragroup.

Recall how to construct such a set of data from an extremal subfactor $N \subset M$ of type II_1 with finite index. We first have a bimodule ${}_N L^2(M)_M$. Using the relative tensor product, we get finite tensor powers $\cdots \otimes_N L^2(M) \otimes_M L^2(M) \otimes_N L^2(M) \otimes_M \cdots$ and make irreducible decompositions. We get four kinds of bimodules; N-N, N-M, M-N, M-M. These are the objects we have for the fusion rule algebra. (The *finite depth* assumption means that we get only finitely many equivalence classes in this way.) The additivity and multiplicativity of the dimension function follows from the extremality assumption by [24]. The generator of the fusion rule algebra is ${}_N L^2(M)_M$. We consider this type of paragroups in this paper.

It is easy to see that if we start with an extremal subfactor $N \subset M$ of type II_1 with finite index, we have a paragroup as above. By [25], we know that if we have a paragroup in the above sense, we have an extremal subfactor $N \subset M$ of type II_1 with finite index producing the paragroup as above. We say that a paragroup is finite [resp. strongly amenable] if it arises from a subfactor of finite depth [resp. strong amenability in the sense of Popa [24]]. Popa's major theorem in [24] says that a paragroup is a complete invariant for strongly amenable extremal subfactors of type II_1 with finite index.

We next define equivalent paragroups. If we start with a subfactor $N \subset N \rtimes G = M$ with a finite group G acting on N freely, we get a finite system of N-N bimodules labelled with the group elements in G, and a finite system of M-M bimodules labelled with the elements in the group dual $\hat{G}$. In this example arising from a group crossed product, the systems of N-N bimodules and M-M bimodules are mutually dual in the usual sense in the group theory, and we think that the two systems contain the same amount of information in the sense that we can recover one system from the other. Based on this idea, we make the following definition of equivalence of systems of bimodules.

Definition 2.1 Two fusion rule algebras with dimension functions and quantum $6j$-symbols are said to be *equivalent*, if they appear as an A-A subsystem and a B-B subsystem of a paragroup.

This definition is based on the one of Ocneanu in [20] for the finite depth case. Based on this, we define equivalent paragroups as follows. See Sato's definition of equivalent subfactors with finite depth in [28]. It is not difficult to show that this indeed gives an equivalence relation.

Definition 2.2 An A-B paragroup and a C-D paragroup are said to be *equivalent* if the corresponding A-A subsystem and C-C subsystem are equivalent.

Of course, we could use the B-B subsystem and the D-D subsystem here. We give some basic examples of equivalent paragroups.

Example 2.3 Let $N \subset M$ be a subfactor as above. Let

$$N \subset M \subset M_1 \subset M_2 \subset M_3 \subset \cdots$$

be the corresponding Jones tower. It is then easy to see that the paragroups corresponding to $N \subset M$ and $M_k \subset M_l$ for $k < l$ are equivalent. It is thus trivial to note that the Jones index is not an invariant for this equivalence.

Example 2.4 Let R be a II_1 factor, G a finite group acting on R freely, and H a subgroup of G. Suppose that H does not contain a non-trivial normal subgroup of G. (Such a subgroup is said to be *relatively simple*.) Then the paragroups corresponding to subfactors $R \subset R \rtimes G$ and $R \rtimes H \subset R \rtimes G$ are equivalent.

More generally, the paragroup of a subfactor $N \subset M$ with finite index and finite depth is *often*, but *not always*, equivalent to the paragroup of its intermediate subfactor $P \subset M$ for $P \supset N$.

Example 2.5 Let $N \subset M$ be a subfactor of the hyperfinite II_1 factor with finite index and finite depth. We denote its *opposite* subfactor by $N^{\mathrm{opp}} \subset M^{\mathrm{opp}}$. Then the paragroup of the subfactor $N \otimes N^{\mathrm{opp}} \subset M \otimes M^{\mathrm{opp}}$ is equivalent to the one for Ocneanu's asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$, if the fusion graph of $N \subset M$ is connected. The asymptotic inclusion has been introduced in [18], [19], and it is a subfactor analogue of the quantum double construction of Drinfel′d as noticed by Ocneanu. See [8, Chapters 12, 13].

Ocneanu has generalized in [20] the Turaev-Viro construction of 3-dimensional topological quantum field theory based on triangulation, using quantum $6j$-symbols arising from a subfactor of finite index and finite depth. He has also found in [20] that equivalent paragroups in the above sense produce the same 3-dimensional topological quantum field theory. (See [8, Chapter 12].)

At the end of this section, we mention the work of Sato [26], [27], [28] on a question raised by V. F. R. Jones, since here we will generalize a part of his work. The problem is as follows.

Let

$$\begin{array}{ccc} A_{00} & \subset & A_{01} \\ \cap & & \cap \\ A_{10} & \subset & A_{11} \end{array}$$

be a finite dimensional non-degenerate commuting square with an appropriate trace. Then vertical and horizontal basic constructions are compatible so that we get a double sequence $\{A_{kl}\}_{kl}$ of finite dimensional C^*-algebras. Then using the trace, we get II_1 factors $A_{\infty,l}$ and $A_{k,\infty}$ as the weak closures in appropriate GNS-representations. Jones asked what kind of relations we have for the two "horizontal" and "vertical" subfactors $A_{0,\infty} \subset A_{1,\infty}$ and $A_{\infty,0} \subset A_{\infty,1}$.

Sato's first answer in [26] is that the two subfactors have the same *global indices*, in particular, one is of finite depth if and only if so is the other. The global index of a subfactor $N \subset M$ is a sum of the Jones indices $[{}_N X_N]$ over N-N bimodules X arising from ${}_N M_M$ as above and this is a measure for a size of a paragroup. Note that the global index is finite if and only if the original subfactor is of finite depth. Sato has further proved the following theorem in [27], [28].

Theorem 2.6 *Let $N \subset M$ and $P \subset Q$ be subfactors of the hyperfinite II_1 factor with finite index and finite depth. Then the pair of these two subfactors arise from a single commuting square as above if and only if $N \subset M$ and $P^{\rm opp} \subset Q^{\rm opp}$ have equivalent paragroups.*

3 Subequivalent paragroups and the quantum Galois correspondence

Based on the above notion of equivalence of paragroups, we introduce a notion of subequivalence of paragroups which gives a proper setting for our quantum Galois correspondence.

Definition 3.1 Let π_1, π_2 be an A-B paragroup and a C-D paragroup, respectively. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be the corresponding A-A subsystem and C-C subsystem. If we have an A-C paragroup π_3 such that the A-A subsystem $\tilde{\mathcal{S}}_1$ of π_3 contains a fusion rule subalgebra $\mathcal{S}'_1$ such that the quantum $6j$-symbols restricted on $\mathcal{S}'_1$ is equivalent to those on $\mathcal{S}_1$ and the C-C subsystem of π_3 is isomorphic to $\mathcal{S}_2$ with equivalent quantum $6j$-symbols, then we say that π_1 is *subequivalent* to π_2.

Note that we could use the B-B subsystem or the D-D subsystem in the above definition – we get the same definition. We believe that the above is the right way to define *subparagroups*, but we use the terminology *subequivalence* because it involves equivalence. It may seem that we could or should define a subparagroup by requiring that the A-A subsystem is isomorphic to a subsystem of the C-C subsystem in the above definition, but then using the A-A subsystem and using the B-B subsystem do not give the same definition any more. It may then seem that we could require the A-A subsystem *or* the B-B subsystem is isomorphic to a subsystem of the C-C subsystem *or* the D-D subsystem as a definition of a "subparagroup", but then a "subparagroup" of a "subparagroup" would not be a "subparagroup" of the original paragroup, while we have the following proposition from our definition fairly easily.

Proposition 3.2 *(1) If a paragroup π_1 is subequivalent to a paragroup π_2 and π_2 is subequivalent to a paragroup π_3, then π_1 is subequivalent to π_3.*

(2) If a paragroup π_1 is subequivalent to a paragroup π_2, π_2 is subequivalent to π_1, and π_1 has a finite global index, then π_1 and π_2 are equivalent.

Proof (1) It is easy to show the subequivalence of π_1 to π_3 using appropriate generators of the paragroups.

(2) The finiteness of the global index imply that the global indices of the two paragroups are equal. Then the fusion rule subalgebra in the definition of the subequivalence cannot be proper and we get the conclusion. Q.E.D.

A recent theorem of Ocneanu [22] implies that we have only finitely many isomorphism classes of *irreducible* subequivalent paragroups for a given finite paragroup, since we have an upper bound for the global index. (Here we say that a paragroup is irreducible if the corresponding subfactor has a trivial relative commutant. This condition is equivalent to the requirement that the generator of the paragroup is an irreducible object.) Also see [17] on this finiteness.

It is trivial to have a notion of an intermediate subfactor for a subfactor $N \subset M$, but this notion is not appropriate for our setting. The following notion due to Ocneanu in [21] turns out to be the right "quantization" of the notion of intermediate subfactors.

Definition 3.3 Let $N \subset M$ be a subfactor and

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

the corresponding Jones tower. A subfactor $A \subset B$ is called a *generalized intermediate subfactor* of $N \subset M$ if it is realized as $pN \subset A \subset B \subset pM_kp$ for some non-zero projection $p \in N' \cap M_k$.

Note that the projection p in the above does not have to be a minimal projection. Then we have the following theorem.

Theorem 3.4 *Let $N \subset M$ be a strongly amenable and extremal subfactor of the hyperfinite II_1 factor with finite index and π the corresponding paragroup.*

Then the isomorphism classes of subequivalent paragroups of π are in a bijective correspondence to the isomorphism classes of generalized intermediate subfactors of $N \subset M$. This subequivalence can be "encoded" in a commuting square.

Of course, we rely here on Popa's classification theorem [24]. We give the correspondence in terms of a commuting square in the next section. The correspondence from a generalized intermediate subfactor to a subequivalent paragroup is almost trivial from the definition, so we omit this direction. For the other direction, we use Sato's technique in [28] to construct a certain commuting square and in this way, we can get a combinatorial and concrete description of the subequivalence in terms of commuting squares, rather than just a correspondence in both directions. For this, we need a more subtle estimates of the higher relative commutants than in [28] based on amenability. We give a proof in a more conceptual context in the next section.

Ocneanu has obtained this kind of correspondence for generalized intermediate subfactors of the Jones subfactors of type A_n in [21] by a general argument. We make use of his arguments in a more general context in order to get a specific commuting square for strongly amenable subfactors.

4 Encoding (sub)equivalence in terms of a commuting square

For our proof of Theorem 3.4, we will show that (sub)equivalence of paragroups can be encoded in a single bi-unitary connection, or a commuting square.

We will first need a generalization of Theorem 3.3 in [28] for strongly amenable paragroups. Note that the proof of this theorem in [28] contains a gap, as pointed out by S. Goto, so we need the following type of arguments even in the finite depth case. We start with the following easy lemma.

Lemma 4.1 *Let $N \subset M$ be a strongly amenable and extremal subfactor of type II_1 in the sense of Popa and $N \subset M \subset M_1 \subset M_2 \subset \cdots$ the Jones tower. For any intermediate subfactor $N \subset R \subset M$, the sequence of commuting squares*

$$\begin{array}{ccccc} M'\cap M_k & \subset & R'\cap M_k & \subset & N'\cap M_k \\ \cap & & \cap & & \cap \\ M'\cap M_{k+1} & \subset & R'\cap M_{k+1} & \subset & N'\cap M_{k+1} \\ \cap & & \cap & & \cap \\ \vdots & & \vdots & & \vdots \end{array}$$

recovers $N \subset R \subset M$ uniquely.

Proof Let $\cdots \subset N_2 \subset N_1 \subset N \subset M$ be a tunnel with the generating property in the sense of [24]. Then the inclusion $\bigvee_k (N_k' \cap N) \subset \bigvee_k (N_k' \cap R) \subset \bigvee_k (N_k' \cap M)$ is uniquely determined by the commuting square in the statement of the lemma. Since the smallest and largest algebras here are equal to N and M, respectively, by the generating property, we also have $\bigvee_k (N_k' \cap R) = R$ from the commuting square condition. (See the proof of Theorem 3.3 in [16], for example.) Q.E.D.

We call the above sequence of commuting square the *standard invariant* of $N \subset R \subset M$.

Suppose that we have mutually equivalent strongly amenable paragroups of types A-C and C-D with generators ${}_AX_C$ and ${}_CY_D$, respectively, with system of the C-C bimodules in common. Let ${}_AZ_D = {}_AX \otimes_C Y_D$. We make the following double sequence of finite dimensional commuting squares similarly to the construction in section 3 in [28].

$$\begin{array}{ccccccccccc} & & \mathrm{End}(*) & \subset & \mathrm{End}(\bar{X}) & \subset & \mathrm{End}(\bar{X}X) & \subset & \cdots & \rightarrow & P \\ & & \cap & & \cap & & \cap & & & & \cap \\ \mathrm{End}(*) & \subset & \mathrm{End}(\bar{Y}) & \subset & \mathrm{End}(\bar{Y}\bar{X}) & \subset & \mathrm{End}(\bar{Y}\bar{X}X) & \subset & \cdots & \rightarrow & Q \\ \cap & & \cap & & \cap & & \cap & & & & \cap \\ \mathrm{End}(Y) & \subset & \mathrm{End}(Y\bar{Y}) & \subset & \mathrm{End}(Y\bar{Y}\bar{X}) & \subset & \mathrm{End}(Y\bar{Y}\bar{X}X) & \subset & \cdots & \rightarrow & Q_1 \\ \cap & & \cap & & \cap & & \cap & & & & \cap \\ \mathrm{End}(\bar{Y}Y) & \subset & \mathrm{End}(\bar{Y}Y\bar{Y}) & \subset & \mathrm{End}(\bar{Y}Y\bar{Y}\bar{X}) & \subset & \mathrm{End}(\bar{Y}Y\bar{Y}\bar{X}X) & \subset & \cdots & \rightarrow & Q_2 \\ \cap & & \cap & & \cap & & \cap & & & & \cap \\ \vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \end{array}$$

Here we dropped A, C, D and the symbol $\otimes$ for simplicity. Note that the $*$ in the first line is ${}_C*_C$ and the $*$ in the second line is ${}_D*_D$. For the horizontal direction, we have made relative tensor products with $\bar{Y}, \bar{X}, X, \bar{X}, X, \ldots$, respectively, from the right at each step when we go from the left to the right in the double sequence, and for the vertical direction, we have made relative tensor products with $\bar{Y}, Y, \bar{Y}, Y, \ldots$, respectively, from the left at each step when we go from the top to the bottom in the double sequence. Here the rightmost column has the weak closures in the GNS-representations of the increasing unions with respect to the trace. For saving the space, we now drop "End" and the inclusion symbol, but put attributions in writing, so we mean the above double sequence by the diagram below.

$$
\begin{array}{cccccc}
 & {}_C{*}_C & {}_C\bar{X}_A & {}_C\bar{X}X_C & \cdots \rightarrow & P \\
{}_D{*}_D & {}_D\bar{Y}_C & {}_D\bar{Y}\bar{X}_A & {}_D\bar{Y}\bar{X}X_C & \cdots \rightarrow & Q \\
{}_CY_D & {}_CY\bar{Y}_C & {}_CY\bar{Y}\bar{X}_A & {}_CY\bar{Y}\bar{X}X_C & \cdots \rightarrow & Q_1 \\
\vdots & \vdots & \vdots & \vdots & &
\end{array}
$$

We label this double sequence of finite dimensional algebras as $\{A_{kl}\}_{kl}$ with $k \geq 0, l \geq -1$, $(k,l) \neq (0,-1)$. (The algebra $\mathrm{End}(*)$ in the first line is A_{00} and the one $\mathrm{End}(*)$ in the second line is $A_{1,-1}$.) We then set $B_{kl} = A_{2k,2l}$ for $k, l \geq 0$. Then using $\bar{X}X$, we can extend the double sequence $\{B_{kl}\}_{k,l\geq 0}$ to the one $\{B_{kl}\}_{k+l\geq 0,\, l\geq 0}$. For example, $B_{-1,2} = \mathrm{End}(\bar{X}X)$ and the embedding $B_{-1,2} \subset B_{0,2}$ is given by the left multiplication of $\bar{X}X$. In general the embedding $B_{kl} \subset B_{k,l+1}$ is given by the right multiplication of $\bar{X}X$ and the embedding $B_{kl} \subset B_{k+1,l}$ is given by the left multiplication of $\bar{X}X$ for $k \leq -1$. In the above abbreviated writing, this double sequence is represented as follows.

$$
\begin{array}{ccccc}
 & & & \cdots & \vdots \\
 & & {}_C{*}_C & \cdots \rightarrow & B_{-2,\infty} \\
 & {}_C{*}_C & {}_C\bar{X}X_C & \cdots \rightarrow & B_{-1,\infty} \\
{}_C{*}_C & {}_C\bar{X}X_C & {}_C\bar{X}X\bar{X}X_C & \cdots \rightarrow & B_{0,\infty} \\
{}_CY\bar{Y}_C & {}_CY\bar{Y}\bar{X}X_C & {}_CY\bar{Y}\bar{X}X\bar{X}X_C & \cdots \rightarrow & B_{1,\infty} \\
{}_CY\bar{Y}Y\bar{Y}_C & {}_CY\bar{Y}Y\bar{Y}\bar{X}X_C & {}_CY\bar{Y}Y\bar{Y}\bar{X}X\bar{X}X_C & \cdots \rightarrow & B_{2,\infty} \\
\vdots & \vdots & \vdots & & \vdots
\end{array}
$$

We denote the weak closures in the GNS-representations with respect to the trace by $A_{k,\infty}$, $B_{\infty,l}$, and so on.

Lemma 4.2 *In the above setting, the inclusion $B_{0,\infty} \subset B_{1,\infty}$ is a hyperfinite type II_1 subfactor with finite index and its Jones tower is given by $B_{0,\infty} \subset B_{1,\infty} \subset B_{2,\infty} \subset \cdots$. We also have $B'_{0,\infty} \cap B_{k,\infty} = B_{k,0}$ for $k \geq 0$.*

Proof By the strong amenability of the A-C paragroup, $B_{k,\infty}$ are hyperfinite II_1 factors. Then as usual, we get the first half of the statement.

For the second half, we apply the argument of [19, page 35]. (Also see the proof of Theorem 3.4 in [16], which is quite similar.) The inclusion $B_{k,0} \subset B'_{0,\infty} \cap B_{k,\infty}$ is by the standard flatness argument. (See [8, Sections 10.5, 12.5], for example.) For the converse inclusion, take $x \in B'_{0,\infty} \cap B_{k,\infty}$ and set $x_n = E_{B_{kn}}(x) \in B'_{0n} \cap B_{kn}$ for $n \geq 0$. Then $x_n \in B_{\infty,n}$ is written as a finite sum $\sum_i a_i f_{n-1} b_i$, where $a_i, b_i \in B_{\infty,n-1}$ and f_{n-1} is the horizontal Jones projection, for $n \geq 1$. Then we have

$$\begin{aligned}
\|x - x_n\|_2 &\geq \|E_{B'_{-(n-1),\infty} \cap B_{\infty,\infty}}(x - x_n)\|_2 \\
&= \|x - \sum_i a_i E_{B'_{-(n-1),\infty} \cap B_{\infty,\infty}}(f_{n-1}) b_i\|_2 \\
&= \|x - [X]^{-2} \sum_i a_i b_i\|_2 \\
&= \|x - E_{B_{\infty,n-1}}(x_n)\|_2 \\
&= \|x - x_{n-1}\|_2
\end{aligned}$$

because

$$E_{B'_{-(n-1),\infty} \cap B_{\infty,\infty}}(f_{n-1}) = E_{B'_{-(n-1),\infty} \cap B_{0,\infty}}(f_{n-1}) = [X]^{-2}$$

follows from the strong amenability of the A-C paragroup. Since $\lim_n \|x - x_n\|_2 = 0$, we get $\|x - x_n\|_2 = 0$ for all $n \geq 1$, and in particular $x = x_1 \in B_{k,1}$. Then as in the standard compactness argument of Ocneanu [19] (also see [8, Section 11.4]), we get $x = x_0 \in B_{k,0}$. Q.E.D.

Note that the sequence $\cdots \subset B_{-2,\infty} \subset B_{-1,\infty} \subset B_{0,\infty}$ is a tunnel, but $B_{0,\infty} \subset B_{1,\infty}$ is *not* the basic construction of $B_{-1,\infty} \subset B_{0,\infty}$.

With this lemma, we can prove the following.

Proposition 4.3 *The inclusion $P \subset Q$ constructed as above gives a hyperfinite subfactor of type II_1 and its paragroup is the C-D paragroup we start with.*

Proof Lemma 4.2 shows factoriality of $P, Q_1, Q_3, \ldots$. We also know that $Q' \cap Q \subset P' \cap Q_1 = A_{2,0}$ by Lemma 4.2 and then the center of Q is contained in $A_{1,0}$. It is easy to see that any non-trivial projection in $A_{1,0}$ is not in the center of Q, so we conclude that Q is also a factor. Similarly, we can prove that all Q_k's are factors. Then it is easy to see that $P \subset Q \subset Q_1 \subset Q_2 \subset \cdots$ is the Jones tower of $P \subset Q$. Lemma 4.2 now shows that the higher relative commutants

$$P' \cap P \subset P' \cap Q \subset P' \cap Q_1 \subset P' \cap Q_2 \subset \cdots$$

is given by $\{A_{k,0}\}_{k \geq 0}$.

We next consider $Q' \cap Q_{2k+1}$ whose dimension is equal to that of $P' \cap Q_{2k}$. The standard flatness argument gives $A_{2k+2,-1} \subset Q' \cap Q_{2k+1}$ and we know that these two algebras now have the same dimensions, so we must have the equality $A_{2k+2,-1} = Q' \cap Q_{2k+1}$. This gives the conclusion $A_{k+1,-1} = Q' \cap Q_k$ for all k. Thus the left two columns of the above double sequence gives the higher relative commutants of $P \subset Q$. This proves the lemma. Q.E.D.

For proving Theorem 3.4, we make the following setting. Suppose we have three strongly amenable paragroups of type A-B, A-C, and C-D with generating bimodules ${}_A X_B$, ${}_A Y_C$ and ${}_C Z_D$, respectively. We assume that the A-A systems of the A-B and A-C paragroups are the same and the C-C system of the C-D paragroup is a subsystem of that of the A-C paragroup. In this way, the C-D paragroup is subequivalent to the A-B paragroup. (The equivalence of the A-B paragroup and the A-C paragroup implies that if one of the two is strongly amenable, so is the other by [24]. If these are strongly amenable, we know that the C-D paragroup is amenable by [11], but we do not know whether it is "strongly" amenable or not, so we have assumed in the above that the C-D paragroup is also strongly amenable.) We then construct four hyperfinite II_1 factors $N \subset P \subset Q \subset M$ defined by the following diagram. (The factoriality follows from the argument in the proof of Proposition 4.3.)

$$\begin{array}{ccccccc}
{}_A *_A & {}_A X_B & {}_A X\bar{X}_A & {}_A X\bar{X}X_B & \cdots & \to & N \\
{}_C \bar{Y}_A & {}_C \bar{Y}X_B & {}_C \bar{Y}X\bar{X}_A & {}_C \bar{Y}X\bar{X}X_B & \cdots & \to & P \\
{}_D \bar{Z}\bar{Y}_A & {}_D \bar{Z}\bar{Y}X_B & {}_D \bar{Z}\bar{Y}X\bar{X}_A & {}_D \bar{Z}\bar{Y}X\bar{X}X_B & \cdots & \to & Q \\
{}_B \bar{X}YZ\bar{Z}\bar{Y}_A & {}_B \bar{X}YZ\bar{Z}\bar{Y}X_B & {}_B \bar{X}YZ\bar{Z}\bar{Y}X\bar{X}_A & {}_B \bar{X}YZ\bar{Z}\bar{Y}X\bar{X}X_B & \cdots & \to & M
\end{array}$$

By Lemma 4.2, it is easy to compute the higher relative commutants of $N \subset M$ since the bimodule ${}_B \bar{X}YZ\bar{Z}\bar{Y}_A$ decomposes into a finite sum of B-A bimodules in the A-B paragroup, and then it is also easy to see that we can obtain the subfactor $N \subset M$ as a cut-down of some basic construction of the subfactor corresponding to the A-B paragroup. So in order to complete the proof, all we have to show is that the subfactor $P \subset Q$ has the original C-D paragroup.

We now ignore M in the above construction and compute the "standard invariant" for $N \subset P \subset Q$ considered in Lemma 4.1. For this purpose, we consider the double sequence of commuting squares of finite dimensional algebras as in the following diagram. That is, we add extra algebras to the left and also continue the sequence vertically.

$$\begin{array}{ccccccc}
 & & {}_A *_A & {}_A X_B & \cdots & \to & N \\
 & {}_C *_C & {}_C \bar{Y}_A & {}_C \bar{Y}X_B & \cdots & \to & P \\
{}_D *_D & {}_D \bar{Z}_C & {}_D \bar{Z}\bar{Y}_A & {}_D \bar{Z}\bar{Y}X_B & \cdots & \to & Q \\
{}_A YZ_D & {}_A YZ\bar{Z}_C & {}_A YZ\bar{Z}\bar{Y}_A & {}_A YZ\bar{Z}\bar{Y}X_B & \cdots & \to & Q_1 \\
{}_D \bar{Z}\bar{Y}YZ_D & {}_D \bar{Z}\bar{Y}YZ\bar{Z}_C & {}_D \bar{Z}\bar{Y}YZ\bar{Z}\bar{Y}_A & {}_D \bar{Z}\bar{Y}YZ\bar{Z}\bar{Y}X_B & \cdots & \to & Q_2 \\
\vdots & \vdots & \vdots & \vdots & & & \vdots \\
\downarrow & \downarrow & \downarrow & & & & \downarrow \\
R_0 & R_1 & R_2 & & & & Q_\infty
\end{array}$$

Then as above, we know that $N \subset Q \subset Q_1 \subset Q_2 \subset \cdots$ is the Jones tower of $N \subset Q$. The proof of Propostition 4.3 implies that we have $R_0 = Q' \cap Q_\infty$ and $R_2 = N' \cap Q_\infty$, and thus the subfactor $N \subset Q$ is strongly amenable. The standard flatness argument gives $R_1 \subset P' \cap Q_\infty$. We also know from Lemma 4.1 that the inclusion $N \subset P \subset Q$ is anti-isomorphic to $Q' \cap Q_\infty \subset P' \cap Q_\infty \subset N' \cap Q_\infty$. By comparing the indices of $R_0 \subset R_1 \subset R_2$, which can be computed from the commuting square easily, and those of $Q' \cap Q_\infty \subset P' \cap Q_\infty \subset N' \cap Q_\infty$, we conclude that $R_1 = P' \cap Q_\infty$, that is, the second column from the left in the above diagram gives the relative commutants

$$P' \cap P \subset P' \cap Q \subset P' \cap Q_1 \subset P' \cap Q_2 \subset \cdots.$$

So the three left columns of the above diagram give the "standard invariant" for $N \subset P \subset Q$ considered in Lemma 4.1.

We next construct the following double sequence.

$$\begin{array}{cccccc}
 & & {}_A*_A & {}_AY_C & {}_AY\bar{Y}_A & \cdots \to \tilde{N} \\
 & {}_C*_C & {}_C\bar{Y}_A & {}_C\bar{Y}Y_C & {}_C\bar{Y}Y\bar{Y}_A & \cdots \to \tilde{P} \\
{}_D*_D & {}_D\bar{Z}_C & {}_D\bar{Z}\bar{Y}_A & {}_D\bar{Z}\bar{Y}Y_C & {}_D\bar{Z}\bar{Y}Y\bar{Y}_A & \cdots \to \tilde{Q} \\
{}_AYZ_D & {}_AYZ\bar{Z}_C & {}_AYZ\bar{Z}\bar{Y}_A & {}_AYZ\bar{Z}\bar{Y}Y_C & {}_AYZ\bar{Z}\bar{Y}Y\bar{Y}_A & \cdots \to \tilde{Q}_1 \\
{}_D\bar{Z}\bar{Y}YZ_D & {}_D\bar{Z}\bar{Y}YZ\bar{Z}_C & {}_D\bar{Z}\bar{Y}YZ\bar{Z}\bar{Y}_A & {}_D\bar{Z}\bar{Y}YZ\bar{Z}\bar{Y}Y_C & {}_D\bar{Z}\bar{Y}YZ\bar{Z}\bar{Y}Y\bar{Y}_A & \cdots \to \tilde{Q}_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

It is again easy to see that $\tilde{N} \subset \tilde{Q} \subset \tilde{Q}_1 \subset \tilde{Q}_2 \subset \cdots$ is the Jones tower of $\tilde{N} \subset \tilde{Q}$. Then the proof of Proposition 4.3 shows that three left columns of the above diagram give the standard invariant of $\tilde{N} \subset \tilde{P} \subset \tilde{Q}$ and we see that the standard invariants for $N \subset P \subset Q$ and $\tilde{N} \subset \tilde{P} \subset \tilde{Q}$ are the same. Thus Lemma 4.1 shows that these two inclusions are isomorphic, and in particular, the two subfactors $P \subset Q$ and $\tilde{P} \subset \tilde{Q}$ are isomorphic. Proposition 4.3 then implies that the paragroup of $\tilde{P} \subset \tilde{Q}$ is the original C-D paragroup we have started with. Thus we have completed the proof of Theorem 3.4.

Suppose that the C-C subsystem of the C-D paragroup above now coincides with the C-C subsystem of the A-C paragroup, thus the A-B paragroup and the C-D paragroup are equivalent. Then in the above construction of $P \subset Q$, we can interchange the roles of the two paragroups and get the following corollary, which is a generalization of Theorem 3.3 in [28].

Corollary 4.4 *Suppose the two strongly amenable paragroups are equivalent as above. Then we can construct a double sequence of commuting squares as in [28, Theorem 3.3] so that the "horizontal" and "vertical" subfactors give the C-D paragroup and the opposite of the A-B paragroup, respectively, which we start with.*

In this sense, we may regard the commuting squares used for the construction of $P \subset Q$ "encodes" the (sub)equivalence. Note that if we have only subequivalence, not equivalence, in the above construction, the roles of the A-B paragroup and the C-D paragroup are not symmetric, because in that case the "vertical limits" are not factors in general due to disconnectedness of the Bratteli diagrams.

5 Examples and remarks

The most trivial example is as follows.

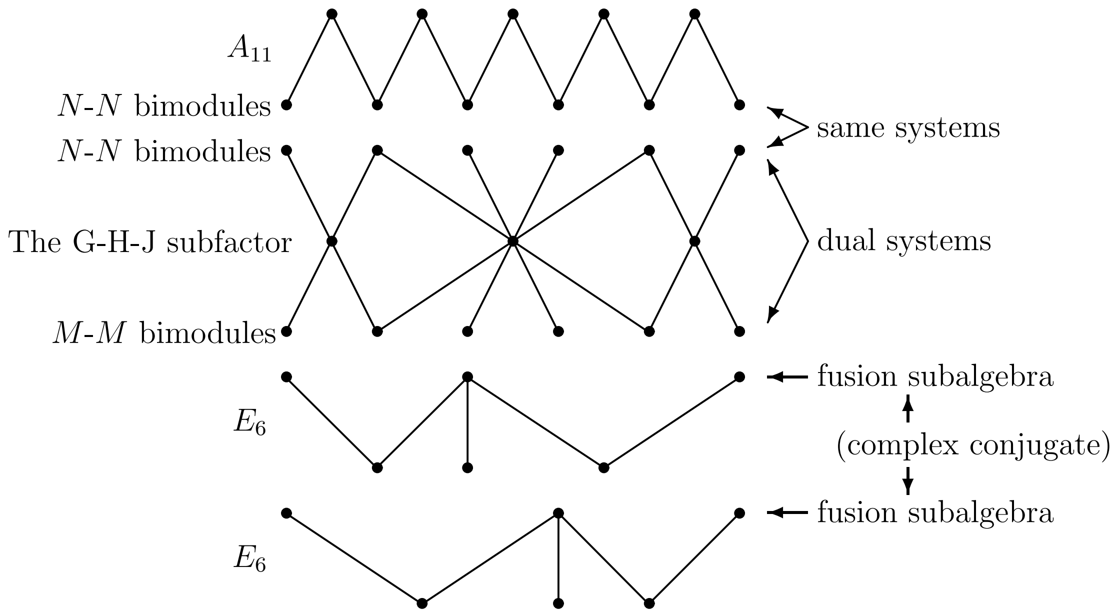

Figure 1: E_6 as a subequivalent paragroup of A_{11}

Example 5.1 Let R be the hyperfinite II_1 factor, G a finite group acting on R freely, and H a subgroup of G. Then subfactors $R \subset R \rtimes H$ and $R \rtimes H \subset R \rtimes G$ are both subequivalent to $R \subset R \rtimes G$.

In this example, the subfactor $R \subset R \rtimes H$ should corresponds to a "subgroup" and $R \rtimes H \subset R \rtimes G$ corresponds to a "group quotient", but in our setting this distinction disappears.

In [6], [15], we have introduced the orbifold construction for subfactors. The easiest case of the orbifold construction is the subfactor of type D_{2n} arising from the one of type A_{4n-3}. In this case, the paragroup of type A_{4n-3} contains a paragroup given by the group A_{4n-3}. The orbifold construction gives D_{2n} as the *quotient* of A_{4n-3} by this $\mathbf{Z}/2\mathbf{Z}$. Both paragroups D_{2n} and $\mathbf{Z}/2\mathbf{Z}$ turn out to be subequivalent paragroups of A_{4n-3} and thus the corresponding subfactors are realized as generalized intermediate subfactors of the subfactor of type A_{4n-3}. Again here, we have no distinction of a subsystem and a quotient.

In [21], Ocneanu has obtained a list of subequivalent systems of bimodules of the system of bimodules arising from the Jones subfactors of type A_n [13]. For example, the paragroups corresponding to the Dynkin diagrams E_6 and E_8 are subequivalent paragroups of those corresponding to E_{11} and A_{29}. The equivalence in these examples are given in terms of the Goodman-de la Harpe-Jones subfactors in [10, Section 4.5]. This example for the case of E_6 and A_{11} was first found in [16] and graphically displayed as in Fig. 1, which appeared in [14] in a slightly different context.

In [7], we have shown that the E_7 commuting squares give a subfactor with the D_{10}. This computation can be interpreted as follows in the above context. The D_{10} fusion rule algebra has a non-trivial symmetry as explained in [5]. This gives a non-trivial equivalence between the two identical systems of the D_{10} paragroup. This equivalence is encoded in the E_7 commuting square.

More examples in connection to conformal inclusions will be discussed in a forthcoming paper [4] based on [2], [3].

6 Sublattices of a standard λ-lattice

Popa [25] has given a complete characterization of double sequences of commuting squares arising as higher relative commutants of extremal subfactors. From the viewpoint of paragroup theory, his axioms give a flat connection on (possibly infinite) graphs. He calls such a double sequence a *standard λ-lattice* for index λ^{-1}. Then we have a natural notion of a sublattice of a standard λ-lattice in the sense that each algebra of the sublattice is a subalgebra of the λ-lattice. Since the flatness condition trivially holds when we pass to a sublattice from a standard λ-lattice, a sublattice of a standard λ-lattice is also a standard λ-lattice in itself. We can naturally define a notion of index for sublattices. Then we have the following.

Theorem 6.1 *Let $N \subset M$ and $P \subset Q$ be subfactors of the hyperfinite II_1 factor with finite index, extremality and strong amenability. Suppose that the standard λ-lattice for $N \subset M$ is a sublattice of that for $P \subset Q$ with finite index. Then the paragroup of the subfactor $P \subset Q$ is subequivalent to that of $N \subset M$.*

Note that the "inclusion" for paragroups is reversed from that for standard λ-lattices. This implies that for a given standard λ-lattice L, we have only finitely many standard λ-lattices *containing* L. This is natural from a viewpoint that *enlarging* a standard λ-lattice is difficult, because the flatness condition gives stronger restrictions.

Proof Suppose that $N \subset M$ and $P \subset Q$ are generated from standard lattices $\{A^0_{00}\}_{kl}$, $\{A^1_{kl}\}_{kl}$, respectively. That is, $\{A^n_{kl}\}_{kl}$ for $n = 0, 1$ are double sequences of commuting squares arising from a flat connection on (possibly infinite) graphs with $A^0_{\infty,0} = N$, $A^0_{\infty,1} = M$, $A^1_{\infty,0} = P$, $A^1_{\infty,1} = Q$, where the meaning of the suffix ∞ is as before. Then by the finiteness assumption of the inclusion of the lattices, we can extend these sequences to a triple sequence $\{A^n_{kl}\}_{nkl}$ of string algebras. Then the flatness of the lattice $\{A^1_{kl}\}_{kl}$ implies that $A^0_{\infty,0}$ and $A^1_{0,\infty}$ commute. Since $A^0_{0,\infty} \subset A^1_{0,\infty} \subset A^2_{0,\infty} \subset \cdots$ is a Jones tower, we conclude that $A^0_{\infty,0}$ and $A^\infty_{0,\infty}$ commute.

We set $\tilde{N} = A^2_{\infty,0}$ and $\tilde{M} = A^2_{\infty,1}$. Then $A^0_{\infty,0} \subset A^2_{\infty,1} \subset A^4_{\infty,2} \subset \cdots$ is a Jones tower, and thus strong amenability of $N \subset M$ implies ${A^0}'_{\infty,0} \cap A^{2l}_{\infty,l} = A^{2l}_{0,l}$, as in the proof of Proposition 4.3. Then we compare two systems of N-N bimodules arising from two subfactors $N \subset M$ and $N \subset \tilde{M}$. From the above computation of the higher relative commutants of $N \subset \tilde{M}$, we have a natural identifications between the sets of even vertices of the principal graphs of $N \subset M$ and $N \subset \tilde{M}$. We can show that this identification indeeds gives an identification of two systems of bimodules as follows.

We represent a system of N-N bimodules as a system of open string bimodules as in [1]. (Open string bimodules were originally introduced in [18] and later generalized in [27]. Here we use a more general form of [1].) In this way, we can realize a system of N-N bimodules as a system of connections on the principal graph of $N \subset M$. (In [1], the finiteness of the graph is used for the compactness argument, but we can now replace it with strong amenability of $N \subset M$ as in the proof of Proposition 4.3, so the possible infiniteness of the graph does not cause a trouble.) Then it is easy to see that the two systems of the connections are the same. Thus we can identify the two systems of N-N bimodules and it shows that the subfactor $N \subset \tilde{M}$ is realized as a generalized intermediate subfactor of $N \subset M$. Since we have an inclusion $N \subset P \subset Q \subset \tilde{M}$ of finite index, we can also conclude that $P \subset Q$ is a generalized intermediate subfactor of $N \subset M$. Q.E.D.

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