INTRODUCTION TO TOMITA - TAKESAKI THEORY

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§0. Introduction

This is a quick introduction to the theory of Tomita - Takesaki Theory which was completed during the 70's. A von Neumann algebra is dcomposed to the direct sum of those of type I, type II and type III. Before the appearance of T-T theory, a von Neumann algebra was considered as a one which is pathological. In fact, the class of von Neumann algebras is defined by exclusion. Namely, a von Neumann algebra $\mathcal{M}$ is said to be of type III if it does not admit a non-trivial semifinite normal trace. Since the absence of a non-trivial semifinite normal trace was a great handicap before the T-T theory, the class of von Neumann algebras of type III was viewed as pathological objects, just like a measure space where only measurable functions are constant, so that there is no interesting integration theory possible. But in 1963, H. Araki and E.J. Woods showed through their study of free Bose gas that the most of von Neumann algebras appearing in quantum physics are unfortunately of type III. Thus it was no longer possible for an operator algebraist to avoid von Neumann algebras of type III as long as one keeps the option of applying the theory of operator algebras to quantum physics which was one of the main reasons for Murray and von Neumann to develop the theory of von Neumann algebras .

Today, the theory of von Neumann algebras is often viewed as a non-commutative extension of measure theory. It is indeed possible to study the usual measure theory and the theory of integrations as part of operator algebras. In fact, doing so, one can eliminate the pathologies from the measure theory. For example if one starts the measure theory from the theory of σ-algebra of subsets of a given space X, then it is impossible to avoid a measure space $\{X, \Sigma\}$ whose measurable functions are only constant while the space X is far from a singleton set. For example, consider $X = [0, 1]$ and let Σ be the collection of countable or co-countable subsets of X. On this measurable space $\{X, \Sigma\}$ a measurable function has to be constant since X is not countable and does not split into a non-trivial union of a pair of disjoint measurabl subsets. The theory of commutative operator algebras provides a basis for the usual integration theory via the Gelfand and Riesz -

Kakutani theory. However, non-comutative theory is more interesting. Unlike the commutative case, each normal positive linear functional φ on a von Neumann algebra $\mathcal{M}$ gives rise to a time evolution of $\mathcal{M}$, i.e., one parameter group $\{\sigma_t^\varphi : t \in \mathbb{R}\}$, called the modular automorphism group of φ, on the reduced algebra $\mathcal{M}_{s(\varphi)}$, where $s(\varphi)$ is the support of φ, such that

$$x\varphi = \varphi\sigma_{\mathrm{i}}^\varphi(x), \quad x \in \mathcal{M}.$$

Furthermore, its class $\delta_t, t \in \mathbb{R}$, of σ_t^φ in $\mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$ is independent of the choice of φ. In fact if φ and ψ are faithful normal positive linear functionals on $\mathcal{M}$, then there exists a σ-strongly continuous one parameter family of unitaries

$$(\mathrm{D}\varphi : \mathrm{D}\psi)_t \in \mathcal{U}(\mathcal{M}), \quad t \in \mathbb{R},$$

such that

$$\begin{gathered}(\mathrm{D}\varphi : \mathrm{D}\psi)_{s+t} = (\mathrm{D}\varphi : \mathrm{D}\psi)_s \sigma_s^\psi((\mathrm{D}\varphi : \mathrm{D}\psi)_t), \quad s, t \in \mathbb{R}, \\ \sigma_t^\varphi = \mathrm{Ad}((\mathrm{D}\varphi : \mathrm{D}\psi)_t) \circ \sigma_t^\psi.\end{gathered}$$

The family $\{(\mathrm{D}\varphi : \mathrm{D}\psi)_t : t \in \mathbb{R}\}$ is called the Connes cocycle derivative of φ over ψ. These discoveries of the T-T theory provided not only a necessary tool to study a von Neumann algebra $\mathcal{M}$ of type III, but also the reduction of the study of $\mathcal{M}$ to that of a von Neumann algebra $\widetilde{\mathcal{M}}$ of type II equipped with a faithful semi finite normal trace τ and a τ-scaling one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\} \subset \mathrm{Aut}(\widetilde{\mathcal{M}})$ in the sense that $\tau \circ \theta_s = e^{-s}\tau$. More precisely, each von Neumann algebra $\mathcal{M}$ gives rise to a von Neumann algebra $\widetilde{\mathcal{M}}$ equipped with a faithful semi finite normal trace τ and a one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ such that

$$\begin{gathered}\tau \circ \theta_s = e^{-s}\tau, \quad s \in \mathbb{R}, \\ \mathcal{M} = \widetilde{\mathcal{M}}^\theta = \text{The fixed point subalgebra of } \widetilde{\mathcal{M}} \text{ under } \theta.\end{gathered} \tag{1}$$

The system $\left\{\widetilde{\mathcal{M}}, \tau, \mathbb{R}, \theta\right\}$ is called the *core* of $\mathcal{M}$. Furthermore, if a covariant system $\left\{\widetilde{\mathcal{M}}, \tau, \mathbb{R}, \theta\right\}$ satisfies the above two conditions, then the pair $\left\{\widetilde{\mathcal{M}}, \mathcal{M}\right\}$ enjoys the following properties,

$$\begin{gathered}\mathcal{M}' \cap \widetilde{\mathcal{M}} = \mathcal{C} = \text{The Center of } \widetilde{\mathcal{M}}, \\ \mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}_{\tau,\theta}(\widetilde{\mathcal{M}}) = \left\{\alpha \in \mathrm{Aut}(\widetilde{\mathcal{M}}) : \alpha \circ \theta_s = \theta_s \circ \alpha, \tau \circ \alpha = \tau\right\}, \\ \theta(\mathbb{R}) = \mathrm{Aut}(\widetilde{\mathcal{M}}/\mathcal{M}) = \left\{\alpha \in \mathrm{Aut}(\widetilde{\mathcal{M}}) : \alpha|_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}\right\}.\end{gathered} \tag{2}$$

The von Neumann algebra $\mathcal{M}$ is of type III if and only if the flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ does not admit a subsystem equivariant to the translation flow $\{L^\infty(\mathbb{R}), \mathbb{R}, \rho\}$. The von Neumann algebra $\mathcal{M}$ is also isomorphic to the crossed product

$$\mathcal{M} \cong \widetilde{\mathcal{M}} \rtimes_\theta \mathbb{R}.$$

The properties listed in (2) allows us to associate to a each factor $\mathcal{M}$ the following commutativ square of exact sequences called the characteristic square of $\mathcal{M}$:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathcal{U}(\mathcal{C}) & \longrightarrow & \mathrm{B}^1_\theta(\mathbb{R},\mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{U}(\mathcal{M}) & \longrightarrow & \widetilde{\mathcal{U}}(\mathcal{M}) & \longrightarrow & \mathrm{Z}^1_\theta(\mathbb{R},\mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Int}(\mathcal{M}) & \longrightarrow & \mathrm{Cnt}_\mathrm{r}(\mathcal{M}) & \longrightarrow & \mathrm{H}^1_\theta(\mathbb{R},\mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

These facts cleared the mysteries about von Neumann algebras of type III, which allows one to use von Neumann algebras comfortably in quantum physics. Futhermore, the T-T theory is now a vital tool in the recent advances in the conformal quantum field theory of R. Longo and Y. Kawahigashi. This series of lectures will provide a gateway to this noble theory of R. Longo and Y. Kawahigashi.

§1. Preliminaries.

We are going to use freely the notations and the results on the book, [Tk1].

Banach Space Valued Holomorphic Functions: We will treat Banach space valued holomorphic functions frequently. So we summerize the results we are going to use.

THEOREM 1.1. *Let E be a Banach space and G a complex domain. For an E-valued function f on G, the following conditions are equivalent:*

i) *For each $\alpha_0 \in G$ and a sufficiently small $\delta > 0$, there exists a sequence $\{a_n\}$ in E such that*

$$f(\alpha) = \sum_{n=0}^{\infty} (\alpha - \alpha_0)^n a_n, \quad |\alpha - \alpha_0| < \delta \tag{1.1}$$

converges in norm;

ii) *There exists a norm closed subspace F of E^* such that*

$$\|x\| = \sup\{|\langle x, x'\rangle| : x' \in F, \|x'\| \le 1\}, \quad x \in E,$$

and each $x' \in F$ gives rise to a holomorphic function: $\alpha \in G \mapsto \langle f(\alpha), x' \rangle$.

If this is the case, then the sequence $\{a_n\}$ is given by

$$a_n = \frac{1}{n!} f^{(n)}(\alpha_0)\,, \quad n \geq 0\,. \tag{1.2}$$

iii) *The function f is locally bounded in the sense that f is bounded on any compact subset of G and there exists a subspace F, not necessarily norm closed, of E^* such that*

$$\|x\| = \sup\{|\langle x, x' \rangle| : x' \in F, \|x'\| \leq 1\}, \quad x \in E\,,$$

and each $x' \in F$ gives rise to a holomorphic function: $\alpha \in G \mapsto \langle f(\alpha), x' \rangle$.

For the proof, we refer the reader to [Tk2, Appendix A1, pages 463-464]

Densely Defined Closed Operators and Polar Decompostion: Let us remind ourselves the fundamental theory of densely defined closed operators on a Hilbert space, in particular the spectral decompostion theorem and the polar decomposition. Let T be an operator[1] on a Hilbert space $\mathfrak{H}$. Consider the set $\mathfrak{D}^*$ of all those vector $\eta \in \mathfrak{H}$ such that

$$\sup\{|(T\xi \mid \eta)| : \xi \in \mathfrak{D}(T), \|\xi\| \leq 1\} < +\infty.$$

Then $\mathfrak{D}^*$ is a linear subspace. If the domain $\mathfrak{D}(T)$ of T is dense in $\mathfrak{H}$, then each vector $\eta \in \mathfrak{D}^*$ gives rise to another vector η^* such that

$$(T\xi \mid \eta) = (\xi \mid \eta^*), \quad \xi \in \mathfrak{D}(T).$$

The correspondence: $\eta \in \mathfrak{D}^* \mapsto \eta^* \in \mathfrak{H}$ is a linear operator T^* with domain $\mathfrak{D}^*$. The operator T^* is called the adjoint operator of T.

Definition 1.2. An operator T is called *self-adjoint* if

$$T = T^*;$$

normal if

$$T^*T = TT^*;$$

symmetric if

$$\mathfrak{D}(T) \subset \mathfrak{D}(T^*) \quad T^*\xi = T\xi \quad \text{for } \xi \in \mathfrak{D}(T).$$

The graph $\mathfrak{G}(T)$ of an operator T is defined to be the subset of $\mathfrak{H} \oplus \mathfrak{H}$ given by the following

$$\mathfrak{G}(T) = \{\xi \oplus T\xi : \xi \in \mathfrak{D}(T)\} \subset \mathfrak{H} \oplus \mathfrak{H}.$$

[1]We mean by an operator a linear operator on a Hilbert space almost always.

If the closure $\overline{\mathfrak{G}(T)} \subset \mathfrak{H} \oplus \mathfrak{H}$ is the graph of another operator $\bar{T}$, i.e., if

$$\overline{\mathfrak{G}(T)} \cap \Big(\{0\} \oplus \mathfrak{H}\Big) = \{0\},$$

then the operator T is called *preclosed* or *closable*. The operator $\bar{T}$ is naturally called the *closure* of T. For a pair S, T of densely defined operators on $\mathfrak{H}$, we say T is an *extension* of S and write $S \subset T$ if

$$\mathfrak{D}(S) \subset \mathfrak{D}(T) \quad \text{and} \quad T\xi = S\xi \text{ for all } \xi \in \mathfrak{D}(S).$$

Clearly

$$S \subset T \quad \Rightarrow \quad T^* \subset S^*.$$

If $\bar{T}$ is self-adjoint, then it is said to be *essentially self-adjoint* on $\mathfrak{D}(T)$. A self-adjoint operator T is called *positive* if

$$(T\xi \mid \xi) \geq 0, \quad \xi \in \mathfrak{D}(T).$$

THEOREM 1.3. *Let T be a densely defined operator on $\mathfrak{H}$ with domain $\mathfrak{D}(T)$.*

i) *If T is preclosed if and only if its adjoint T^* is densely defined. The closure $\bar{T}$ is given as the second adjoint T^{**} of T.*

ii) *The operator T is normal if and only if there exists a* Proj$(\mathfrak{H})$*-valued Borel measure $E\colon M \in \Sigma \mapsto E(M) \in \mathrm{Proj}(\mathfrak{H})$ on the complex plance $\mathbb{C}$ such that*

$$\mathfrak{D}(T) = \left\{\xi \in \mathfrak{H} : \int_{\mathbb{C}} |\lambda|^2 \mathrm{d}\|E(\lambda)\|^2 < +\infty\right\},$$
$$(T\xi \mid \eta) = \int_{\mathbb{C}} \lambda \mathrm{d}(E(\lambda)\xi \mid \eta), \quad \xi \in \mathfrak{D}(T), \eta \in \mathfrak{H}.$$

The projection valued measure $E(\cdot)$ is uniquely determined by T. Every unitary operator $u \in \mathfrak{U}(\mathfrak{H})$ on $\mathfrak{H}$ commuting with T in the sense that

$$u\mathfrak{D}(T) = \mathfrak{D}(T), \quad Tu\xi = uT\xi, \quad \xi \in \mathfrak{D}(T),$$

commutes with E in the obviouse sense that $uE(M) = E(M)u, M \in \Sigma$. The projection valued measue $E(\cdot)$ is called the spectral measure of T.

iv) *The operator T is self-adjoint if and only if the above spectral measure $E(\cdot)$ is supported by the real line $\mathbb{R}$. Furthermore, the self-adjoint operator T is positive if and only if the spectral measure $E(\cdot)$ is supported by the positive half real line $\mathbb{R}_+ = [0, +\infty)$. In this case, it admits the square root $T^{\frac{1}{2}}$, i.e. the positive self-adjoint operato such that*

$$T = (T^{\frac{1}{2}})^2.$$

v) *If T is a densely defined closed operator on $\mathfrak{H}$, then*

a) *The operator T^*T with domain $\mathfrak{D}(T^*T) = \{\xi \in \mathfrak{D}(T) : T\xi \in \mathfrak{D}(T^*)\}$ is a densely defined self-adjoint positive operator;*

b) *The operator T is the closure of the restriction $T|_{\mathfrak{D}(T^*T)}$ of T on $\mathfrak{D}(T^*T)$;*

c) *There exists uniquely a partial isometry u such that*

$$T = u(T^*T)^{\frac{1}{2}} = (TT^*)^{\frac{1}{2}}u,$$
$$u^*u = \text{The projection to the closure } [T^*\mathfrak{H}] \text{ of } T^*\mathfrak{H},$$
$$uu^* = \text{The projection to the closure } [T\mathfrak{H}] \text{ of } T\mathfrak{H};$$

d) *The domain $\mathfrak{D}(T)$ is precisely $\mathfrak{D}\Big((T^*T)^{\frac{1}{2}}\Big)$.*

The squre root $(T^*T)^{\frac{1}{2}}$ is called the **absolute value** *of T and denoted by $|T|$. The partial isometry u is called the* **phase** *of T.*

Theorem 1.3. *Suppose H is a non-singular positive self-adjoint operator on a Hilbert space $\mathfrak{H}$. A vector $\xi \in \mathfrak{H}$ belongs to the domain $\mathfrak{D}(H^T), T \in \mathbb{R}$, if and only if the $\mathfrak{H}$-valued function $\xi : s \in \mathbb{R} \mapsto \xi(s) = H^{\mathrm{i}s}\xi \in \mathfrak{H}$ admits a continuous bounded extensition to the horizontal stip $\mathbb{D}(T)$ bounded by the real line $\mathbb{R}$ and the parallel line $\mathbb{R}+\mathrm{i}T$ in such a way that the extended function is holomorphic on the interior $\mathbb{D}(T)^\circ$ of the strip $\mathbb{D}(T)$. If this occurs, then we have*

$$H^T\xi = \xi(-\mathrm{i}T).$$

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