W^*-rigidity paradigms for embeddings of II_1 factors

Happy 88th birthday Masamichi!

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The isomorphism problem for II_1 factors

- II_1 factors arise from a variety of geometric objects G, such as bipartite graphs, groups (more generally groupoids) and their action on spaces, including actions of groups on the Hilbert space (representations), followed by "geometric operations", like amplifications, tensor products, crossed products, inductive limits.

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• The **many to one** paradigm. E.g.: "All locally finite objects G give rise to *hyperfinite* II_1 factor $R := \overline{\otimes}_n(\mathbb{M}_2(\mathbb{C}), tr)_n$" [MvN43]. More generally: "$L(G) \simeq R$ for all amenable G" [Connes76].

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- The **one to one** paradigm (or **W*-rigidity**). E.g.: "The functor $\Gamma \mapsto L(\Gamma)$ is one to one for $\Gamma \in \mathcal{G} = \{\mathbb{F}_2, \mathbb{F}_2 \times S_\infty\}$" [MvN43].

Both types of results are surprising and quite hard to establish!
Much interest since MvN1943.

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- The hyperfinite II_1 factor R embeds into any other II_1 factor.
- There exist ICC groups Γ, Λ such that $L(\Gamma)$ is not stably isomorphic to $L(\Lambda)$ (one having Gamma the other not), but $L(\Gamma) \hookrightarrow L(\Lambda)$, $L(\Lambda) \hookrightarrow L(\Gamma)$.

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- MvN also make this comment: *The possibility exists that any factor in the case* II_1 *is isomorphic to a sub-ring of any other such factor.*

Some follow up results

• J. Schwartz 1963 + Hakeda-Tomiyama 1967: definition of amenability for tracial vN algebras; hereditarity; $L(\Gamma)$ amenable iff Γ amenable. Thus $L(\Gamma) \hookrightarrow R$ implies Γ amenable, so $L(\mathbb{F}_2) \not\hookrightarrow L(S_\infty) = R$.
• Haagerup 1979: compact approximation property for tracial vN algebras, a hereditary property. Connes-Jones 1984: property (T) for tracial vN algebras; $L(\Gamma)$ has (T) iff Γ has (T); if M has (T), then it cannot have Haagerup unless atomic. Thus $L(SL(3,\mathbb{Z})) \not\hookrightarrow L(\mathbb{F}_2)$.
• Cowling-Haagerup 1988: the invariant $\Lambda_{cb}(M) \in [1, \infty]$; decreasing for embeddings; if $\Gamma_n := Sp(1,n)_{\mathbb{Z}}$, $n \geq 3$, then $L(G_n) \hookrightarrow_s L(G_m)$ iff $n \geq m$.
• Popa 86 separability trick: $\exists$ at most countably many property (T) factors in each virtual isomorhism class
• Ozawa 2002: $\exists$ no maximal separable II_1 factor.
• Ozawa 2003 solidity of $L(\mathbb{F}_2)$: $\exists$ no embeddings of non-amenable, non-prime & prop. Gamma factors into $L(\mathbb{F}_2)$.
• Ozawa-Popa 2007 strong solidity of $L(\mathbb{F}_2)$: $\exists$ no embeddings of non-amenable group measure space factors into $L(\mathbb{F}_2)$.
• W*-rigidity results for embeddings by def./rig. 2001- (many names)

Conspicuous II_1 factor embedding problems

• **Famous embedding conjectures** for specific remarkable II_1 factors:
"If $\Gamma_n := PSL(n, \mathbb{Z})$, then $L(\Gamma_n) \hookrightarrow_s L(\Gamma_m)$ iff $\Rightarrow n \leq m$";
"If $N \hookrightarrow L(\mathbb{F}_2)$ non-amenable, then $N \simeq L(\mathbb{F}_t)$ for some $1 < t \leq \infty$"
"If M non-amenable II_1, then $\exists L(\mathbb{F}_2) \hookrightarrow M$."

• **"Many to one"** type: Produce large families of mutually non-isomorphic (even not virtually isomorphic) II_1 factors that are mutually embeddable.

• **"One-to-one"** type: Produce large families of non-embeddable II_1 factors, e.g., chains, anti-chains. More generally, given a preordered set $(I, \leq)$ with some specific properties (a lattice, a totally ordered set, etc; especially concrete like $\mathbb{R}, \mathbb{Z}$), find $\{G_i\}_{i \in I}$ such that $G_i \mapsto L(G_i)$ satisfies $L(G_i) \hookrightarrow_s L(G_j)$ iff $i \leq j$. Produce complete intervals (...).

"Many to one" paradigm for embeddings

The following result produces a large family of mutually embeddable II_1 factors that are not stably isomorphic ("many to one" type result).

Theorem 1. (P-Vaes 2021)

For each $0 < \alpha \leq 1/2$, let $R_\alpha = R \oplus R$ with trace $\tau_\alpha(x \oplus y)$ $= \alpha\tau_R(x) + (1-\alpha)\tau_R(y)$. Denote $M_\alpha := (R_\alpha, \tau_\alpha)^{\otimes \mathbb{F}_2} \rtimes (\mathbb{F}_2 \times \mathbb{F}_2)$ (left-right Bernoulli action).
Then $\{M_\alpha\}_\alpha$ are stably non-isomorphic (even virtually non-isomorphic), but mutually embeddable.

- Mutual stable embedding of II_1 factors M, N entails an equivalence relation in $\mathbf{II}_1$, which the above shows is much weaker than (stable, resp. virtual) isomorphism.
- If previous conjectures on $L(\mathbb{F}_n)$ hold true, it would show that $L(\mathbb{F}_t)$, $1 < t \leq \infty$, form a single class under the equivalence relation given by mutual stable embedding (and two classes under $\simeq_s$), and that that class is minimal element under preorder $\hookrightarrow_s$ within non-amenable $\mathbf{II}_1$.

W^*-rigidity for embeddings: disjoint families

The next result produces a large **anti-chain** of II_1 factors, i.e., a family of II_1 factors that are mutually non-stably embeddable (they are **disjoint**).

Theorem 2. (P-Vaes 2021)

For each $0 < \alpha \leq 1/2$, let $A_\alpha = \mathbb{C} \oplus \mathbb{C}$ with integral/trace $\tau_\alpha(x \oplus y)$ $= \alpha x + (1-\alpha)y$. Denote $N_\alpha := (A_\alpha, \tau_\alpha)^{\otimes \mathbb{F}_2} \rtimes (\mathbb{F}_2 \times \mathbb{F}_2)$ (left-right Bernoulli action).
Then $\{N_\alpha\}_\alpha$ are disjoint, i.e., if $\alpha \neq \beta$ then $N_\alpha \not\hookrightarrow_s N_\beta$, $N_\beta \not\hookrightarrow_s N_\alpha$.

• Note that the factors in the above family are group-measure space II_1 factors. One would of course like to also have large anti-chains of group factors. Such examples are provided by the order preserving 1-to1 functor from groups to II_1 factors in the next theorem.

W*-rigidity for embeddings: a 1-to-1 functor $\mathcal{G} \to \mathrm{II}_1$

We first construct a 1-to-1 functor $\Gamma \mapsto H_\Gamma$ from the set of countable groups Γ to countable ICC groups, as follows. For each (infinite countable) group Γ, let $\mathbb{F}_{1+|\Gamma|}$ denote the free group with generators $\{a_o\} \cup \{a_g \mid g \in \Gamma\}$. Let $\pi_\Gamma : G_\Gamma \to \mathbb{Z} * \Gamma$ be the group morphism determined by $\pi_\Gamma(a_0) = 1 \in \mathbb{Z}$, $\pi_\Gamma(a_g) = g \in \Gamma$. Let $N_\Gamma \subset G_\Gamma \times G_\Gamma$ be the subgroup of elements (g, g) with $\pi_\Gamma(g) \in \Gamma$. Finally, consider the generalized wreath product group $H_\Gamma := (\mathbb{Z}/2\mathbb{Z})^{((G_\Gamma \times G_\Gamma)/N_\Gamma)} \rtimes (G_\Gamma \times G_\Gamma)$.

The following result shows that the combined functor $\Gamma \mapsto H_\Gamma \mapsto L(H_\Gamma)$, from infinite groups to group II_1 factors, is injective and "order preserving" (w.r.t. the order given by embeddings, for groups resp. II_1 factors).

Theorem 3. (P-Vaes 2021)

If Γ and Λ are arbitrary infinite groups, then $L(H_\Gamma) \simeq_s L(H_\Lambda)$ iff $L(H_\Gamma) \hookrightarrow L(H_\Lambda)$ iff Γ is isomorphic with a subgroup of Λ.
Also, the II_1 factors $L(H_\Gamma)$ and $L(H_\Lambda)$ are virtually isomorphic iff they are stably isomorphic iff they are isomorphic iff $\Gamma \simeq \Lambda$.

W*-rigidity for embeddings: concrete chains

Since the preorder relation $\leq$ on $\mathcal{G}$ (family of all infinite groups) given by embeddings of groups can be very wild, and lots of preordered sets $(I, \leq)$ can be faithfully represented into $(\mathcal{G}, \leq)$, Thm. 3 shows that one can realize many $(I, \leq)$ inside $(\mathbf{II}_1, \hookrightarrow)$ and $(\mathbf{II}_1, \hookrightarrow_s)$. For instance, one gets:

Corollary

There are concrete chains of separable II_1 factors of type $(\mathbb{R}, \leq)$ and $(\omega_1, \leq)$ (the first uncountable ordinal) both in $(\mathbf{II}_1, \hookrightarrow)$ and in $(\mathbf{II}_1, \hookrightarrow_s)$.

W*-rigidity for embeddings: complete intervals

We can actually obtain certain complete intervals inside $(\mathbf{II}_1, \hookrightarrow_s)$ as well, i.e. families of II_1 factors $(M_i)_{i\in I}$ indexed by $(I, \leq)$ such that $M_i \hookrightarrow_s M_j$ iff $i \leq j$ but also so that no other II_1 factor N can sit between M_i, M_j: if $N \in \mathbf{II}_1$ and $M_i \hookrightarrow_s N \hookrightarrow_s M_j$, then $\exists\ i \leq k \leq j$ in I s.t. $N \simeq_s M_k$.

Moreover, one can realize as a complete interval any "discrete" lattice. Recall that a preordered set $(I, \leq)$ is a *lattice* if any pair $a, b \in I$ has an infimum and a supremum. A lattice $(I, \leq)$ is *discrete* if for all $a, b \in I$ the set $\{i \in I \mid a \leq i \leq b\}$ is finite. For instance, $(\mathbb{Z}, \leq)$, or any finite lattice.

Theorem 4. (P-Vaes 2021)

Let $(I, \leq)$ be any countable discrete lattice. There exists a family of separable II_1 factors $(M_i)_{i\in I}$ such that $M_i \hookrightarrow_s M_j$ iff $i \leq j$ iff $M_i \hookrightarrow M_j$ and such that if N is a II_1 factor satisfying $M_i \hookrightarrow_s N \hookrightarrow_s M_j$ (respectively $M_i \hookrightarrow N \hookrightarrow M_j$) then there exists $i \leq k \leq j$ in I such that $N \simeq_s M_k$ (resp. $N \simeq M_k$).