# (Non-normal) Conditional expectations in von Neumann algebras

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**Abstract:** One of the best known theorem of Masamichi is the Conditional Expectation Theorem proved in [M. Takesaki, Conditional expectations in von Neumann algebras. JFA 1972] about *normal* conditional expectations. We prove the analogue for *non-normal* conditional expectations. Based on a joint work with J. Bannon and A. Marrakchi in CMP 2020.

# Happy 88 (米壽) to Masamichi!

# Conditional Expectation

Let $N \subset M$ be von Neumann algebras.
Perhaps, we do not want to study the inclusion like $\mathbb{C}1 \otimes N \subset M \otimes \mathbb{B}(\ell_2)$.

**Definition (Umegaki 1954 (& Nakamura, Turumaru,...))**

A *conditional expectation* of $M$ onto $N$ is a unital completely positive map $E\colon M \to N$ which satisfies $E(axb) = aE(x)b$ for $x \in M$ and $a, b \in N$.

Tomiyama '59: If $N \subset M$ admits a *normal* c.e., then $\mathrm{Type}(N) \leq \mathrm{Type}(M)$.

Dixmier '53, Umegaki '54: Any $N \subset M$ with a faithful normal tracial state $\tau$ admits a normal c.e. In fact, it is given by $E\colon L^2(M, \tau) \to L^2(N, \tau|_N)$.

⚠ This is no longer true for a general f.n. state. Takesaki's theorem gives an appropriate generalization.

**Theorem (Takesaki 1972)**

$N \subset M$ admits a normal c.e. $\iff {}_N L^2(N)_N \subset {}_N L^2(M)_N$

# Tomita–Takesaki Theory (1967, 1970∼)

$\phi$ a f.n. state (weight) on $N$

$\rightsquigarrow$ $S_\phi\colon x\xi_\phi \mapsto x^*\xi_\phi$ has polar decomposition $\bar{S}_\phi = J_\phi\Delta_\phi$ on $L^2(N,\phi)$

modular conjugation $J_\phi$ satisfies $J_\phi N J_\phi = N'$

$\Delta_\phi$ defines modular automorphism $\sigma_t^\phi(x) = \Delta^{it}x\Delta^{-it}$ on $N$

characterized by KMS condition

$\phi\circ\sigma_t^\phi = \phi$

$\forall x,y\ \exists F\in A(\mathbb{S})\ F(it) = \phi(\sigma_t^\phi(x)y)$ and $F(1+it) = \phi(y\sigma_t^\phi(x))$

Here $A(\mathbb{S})$ analytic functions on $\mathbb{S} = \{\operatorname{Re} z < 1\}$ that are continuous on $\bar{\mathbb{S}}$

This gives the "right action" of $N$ on $L^2(N,\phi)$;

$$\pi_\phi^{\mathrm{op}}\colon N^{\mathrm{op}} \ni x^{\mathrm{op}} \mapsto J_\phi x^* J_\phi \in \mathbb{B}(L^2(N,\phi))$$

which makes $L^2(N,\phi)$ an $N$-$N$ bimodule;

$$x\xi y := \pi_\phi(x)\pi_\phi^{\mathrm{op}}(y^{\mathrm{op}})\xi\ (= xa\sigma_{-i/2}^\phi(y)\xi_\phi \text{ for } \xi = a\xi_\phi).$$

In fact the $N$-$N$ bimodule $L^2(N,\phi)$ is indep. of $\phi$ (Araki, Connes '74).

So, we simply denote it by $L^2(N)$ and call it the *standard form*,

$$\pi_N\colon N\odot N^{\mathrm{op}} \to \mathbb{B}(L^2(N)).$$

## Conditional expectation theorem

If $N \subset M$ admits a normal c.e., then Tomita–Takesaki theories for $(N, \phi)$ and $(M, \phi \circ E)$ are compatible.

**Theorem (Takesaki 1972)**

$N \subset M$ admits a normal c.e. $\iff {}_N L^2(N)_N \subset {}_N L^2(M)_N$

Easy direction ($\Leftarrow$): For the orthogonal projection $e$ onto $L^2(N)$, put
$$E(x) := exe \in \mathbb{B}(L^2(N)_N) = N.$$
Hard direction ($\Rightarrow$): ${}_N L^2(N, \phi)_N \subset {}_N L^2(M, \phi \circ E)_N$. $\square$

What about the non-normal case?
E.g., If $G \curvearrowright M$ and $G$ is amenable, then $\exists$ c.e. of $M$ onto $M^G$.

**Theorem (BMO 2020 based on Pisier 1995 and Haagerup)**

$N \subset M$ admits a c.e. $\iff {}_N L^2(N)_N \preceq {}_N L^2(M)_N$

The proof relies on complex interpolation theory (à la Pisier) and Tomita–Takesaki theory (à la Haagerup).

# Weak containment and (relative) injectivity

A von Neumann algebra $N \subset \mathbb{B}(\ell_2)$ is *injective* if $\exists$ c.e. of $\mathbb{B}(\ell_2)$ onto $N$.
Hakeda–Tomiyama, Sakai '67: $L(\Gamma)$ is injective $\iff$ $\Gamma$ is amenable.
Connes '76, Wassermann '77: $N$ is injective $\iff$ $N$ is semi-discrete.

A von Neumann algebra $N$ is *semi-discrete* if

$$\mathbb{B}(L^2(N) \mathbin{\bar{\otimes}} L^2(N)) \supset N \otimes N^{\mathrm{op}} \xrightarrow{\pi_N} \mathbb{B}(L^2(N))$$

is continuous. In other words, ${}_N L^2(N)_N \preceq {}_N L^2(N) \mathbin{\bar{\otimes}} L^2(N)_N$.

Note: $N^{\mathrm{op}} \ni x^{\mathrm{op}} \leftrightarrow \bar{x}^* \in \overline{N} \subset \mathbb{B}(\overline{\mathcal{H}})$.

${}_N\mathcal{H}_N$ an $N$-$N$ bimodule, $\pi_{\mathcal{H}} \colon N \odot N^{\mathrm{op}} \to \mathbb{B}(\mathcal{H})$, $x\xi y := \pi_{\mathcal{H}}(x \otimes y^{\mathrm{op}})\xi$

$$\begin{aligned} {}_N\mathcal{H}_N \preceq {}_N\mathcal{K}_N \overset{\text{def}}{\Leftrightarrow} & \ \forall F \Subset N\ \forall \xi \in \mathcal{H}\ \forall \varepsilon > 0\ \exists \eta_1, \ldots, \eta_k \in \mathcal{K} \\ & \text{s.t. } \langle x\xi y, \xi\rangle \approx_\varepsilon \textstyle\sum_i \langle x\eta_i y, \eta_i\rangle\ \forall x, y \in F \\ \Leftrightarrow & \ \mathrm{C}^*(\pi_{\mathcal{K}}(N \odot N^{\mathrm{op}})) \to \mathrm{C}^*(\pi_{\mathcal{H}}(N \odot N^{\mathrm{op}})) \text{ continuous} \end{aligned}$$

**Theorem (BMO 2020 based on Pisier 1995 and Haagerup)**

$N \subset M$ admits a c.e. $\iff$ ${}_N L^2(N)_N \preceq {}_N L^2(M)_N$

i.e., relative injectivity is equivalent to relative semi-discreteness

# Corollaries

$$\Lambda \leq \Gamma \text{ co-amenable} \overset{\text{def}}{\Leftrightarrow} \Gamma/\Lambda \text{ admits } \Gamma\text{-invariant mean}$$
$$\Leftrightarrow \quad \ldots$$
$$\Leftrightarrow L\Lambda \leq L\Gamma \text{ co-amenable}$$

$$N \subset M \text{ is co-injective} \overset{\text{def}}{\Leftrightarrow} M' \subset N' \text{ admits a c.e.}$$
$$\Leftrightarrow M \subset \langle M, e_N \rangle \text{ admits a c.e. (provided } \exists e_N)$$
$$\text{co-semi-discrete} \overset{\text{def}}{\Leftrightarrow} {}_M L^2 M_M \preceq {}_M L^2 M \mathbin{\bar{\otimes}_N} L^2 M_M \; (= L^2\langle M, e_N\rangle)$$

**Corollary (Popa 1986, Anantharaman-Delaroche 1995, BMO 2020)**

co-injectivity $\Leftrightarrow$ co-semi-discreteness

We say it *co-amenable*.

**Corollary**

$N \subset M$ co-amenable and $N \subset P \subset M \Rightarrow P \subset M$ co-amenable

⚠ $N \subset P$ may not! (Monod–Popa 2003)

# Operator space theory and the operator Hilbert space

$\mathrm{Row}_k := \mathbb{M}_{1,k}$ the row Hilbert operator space

$$\|\sum_{i=1}^k x_i \otimes r_i\|_{\mathbb{B}(\ell_2)\otimes\mathrm{Row}_k} = \|\begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix}\|_{\mathbb{M}_{1,k}(\mathbb{B}(\ell_2))} = \|\sum_{i=1}^k x_i x_i^*\|^{1/2}$$

$\mathrm{Col}_k := \mathbb{M}_{k,1}$ the column Hilbert operator space

$$\|\sum_{i=1}^k x_i \otimes c_i\|_{\mathbb{B}(\ell_2)\otimes\mathrm{Col}_k} = \quad \cdots \quad = \|\sum_{i=1}^k x_i^* x_i\|^{1/2}$$

Operator space duality (Effros–Ruan & Blecher–Paulsen): $\mathrm{Col}_k = \overline{\mathrm{Row}_k^*}$

$\mathrm{OH}_k$ the operator Hilbert space (Pisier 1993)

$$\|\sum_{i=1}^k x_i \otimes e_i\|_{\mathbb{B}(\ell_2)\otimes\mathrm{OH}_k} = \|\sum_{i=1}^k x_i \otimes \bar{x}_i\|_{\mathbb{B}(\ell_2\otimes\overline{\ell_2})}^{1/2}$$

Unique o.s. such that $\mathrm{OH}_k \cong \ell_2^k$ (isometric) and $\mathrm{OH}_k \cong \overline{\mathrm{OH}_k^*}$ (c.i.)

$\rightsquigarrow$ complex interpolation formula $\mathrm{OH}_k = (\mathrm{Row}_k, \mathrm{Col}_k)_{1/2}$.

For $(x_1, \ldots, x_k) \in N^k$, define $\Phi\colon T \mapsto \sum_{i=1}^k x_i T x_i^*$.

$\|\Phi\|_{\mathbb{B}(N)}^{1/2} = \|(x_1, \ldots, x_k)\|_{N\otimes\mathrm{Row}_k}$ and $\|\Phi\|_{\mathbb{B}(L^1(N))}^{1/2} = \|(x_1, \ldots, x_k)\|_{N\otimes\mathrm{Col}_k}$

**Theorem (Pisier 1995 and Haagerup)**

$$\|\pi_N(\sum x_i \otimes \bar{x}_i)\|_{\mathbb{B}(L^2(N))}^{1/2} = \|\Phi\|_{\mathbb{B}(L^2(N))}^{1/2} = \|(x_1, \ldots, x_k)\|_{1/2}$$

$\because$ Factorization thm for vN algebra valued analytic functions and so on.

# Proof of Theorem

**Theorem (BMO 2020 based on Pisier 1995 and Haagerup)**

$N \subset M$ admits a c.e. $\iff {}_N L^2(N)_N \preceq {}_N L^2(M)_N$

$(\Leftarrow)$: Extend the $*$-hom $C^*(\pi_M(N \odot N^{\mathrm{op}})) \to C^*(\pi_N(N \odot N^{\mathrm{op}}))$ to a u.c.p. map $\Phi\colon C^*(\pi_M(M \odot N^{\mathrm{op}})) \to \mathbb{B}(L^2(N))$ and $E := \Phi|_M$.

$(\Rightarrow)$: Since $N \subset M$ admits a c.e., the corresp. contraction

$$(N \otimes \mathrm{Row}_k, N \otimes \mathrm{Col}_k)_{1/2} \subset (M \otimes \mathrm{Row}_k, M \otimes \mathrm{Col}_k)_{1/2}$$

is isometric, i.e., for any $x_i \in N \subset M$,

$$\|\pi_N(\textstyle\sum x_i \otimes \bar{x}_i)\|_{\mathbb{B}(L^2(N))} = \|\pi_M(\textstyle\sum x_i \otimes \bar{x}_i)\|_{\mathbb{B}(L^2(M))}.$$

By HB, for any unit vector $\xi$ in $L^2(N)_+$, $\exists$ a state $\psi_\xi$ on $\mathbb{B}(L^2(M))$ s.t.

$$\forall x \in N \quad \langle x\xi x^*, \xi\rangle \leq \psi_\xi(\pi_M(x \otimes \bar{x})).$$

They must be equal by maximality of the *self-polar form* (Connes, Woronowicz '74). Moreover, $\langle x\xi y^*, \xi\rangle = \psi_\xi(\pi_M(x \otimes \bar{y}))$ by polarization. This implies ${}_N L^2(N)_N \preceq {}_N L^2(M)_N$. $\square$