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Symmetry and quantum chaos in the dissipative SYK model

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さきがけ PRESTO

PRX Quantum (2023) with A.Kulkarni, K.Kawabata, J,Li, S.Ryu (Princeton U)

Plan of the talk

(1)Random matrices and the Altland-Zirnbauer (AZ) class

Level statistics: Symmetry and Universality
10 ensembles: Wigner-Dyson (3) + chiral (3) + BdG (4)

(2)(supersymmetric) Sachdev-Ye-Kitaev (SYK) and AZ

interacting fermionic many-body system.
Nice model to see the AZ & K-theory in contexts of chaos

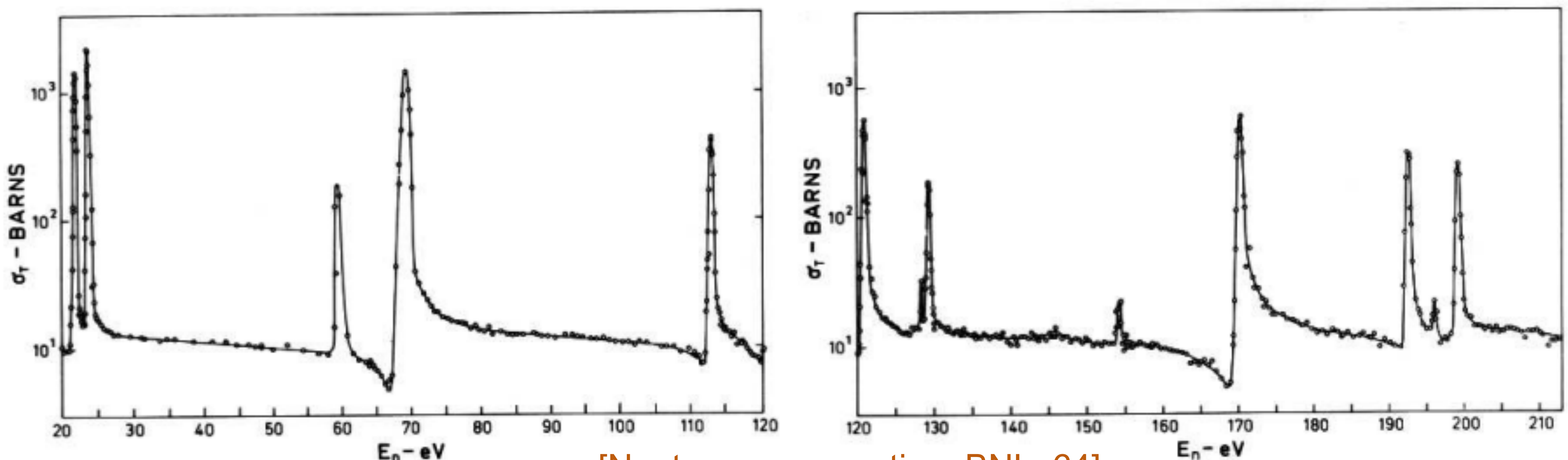
(3)dissipative SYK and symmetry

Fermionic dissipations, classification of fermionic Lindbladians

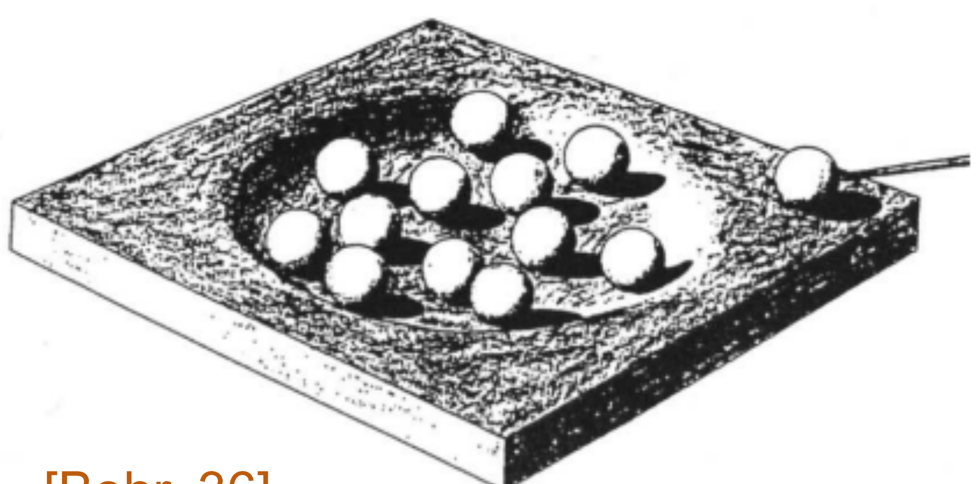
Introduction:Nuclei and complex spectrum

Random matrices are first applied to physics by Wigner.

Motivated by the complicated nuclear spectrum.



[Neutron cross section, BNL, 64]



[Bohr, 36]

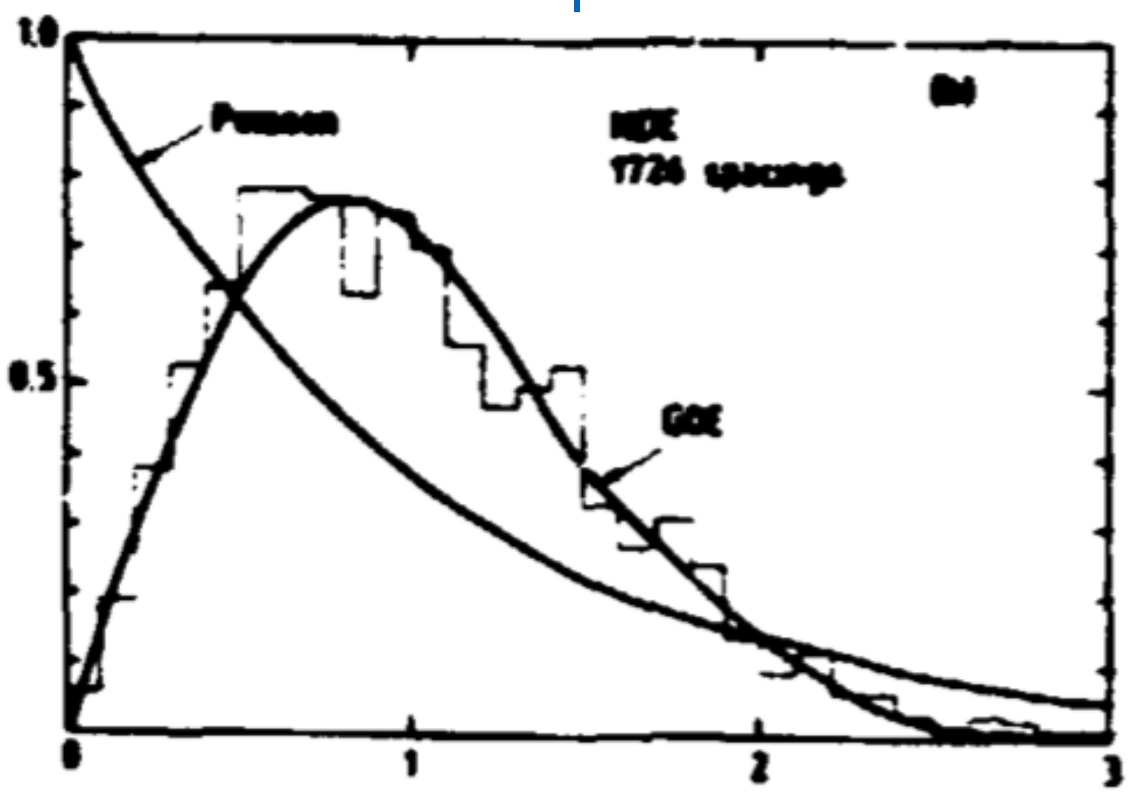
Bohr's wooden toy model.

compounded nuclei are strongly interacting

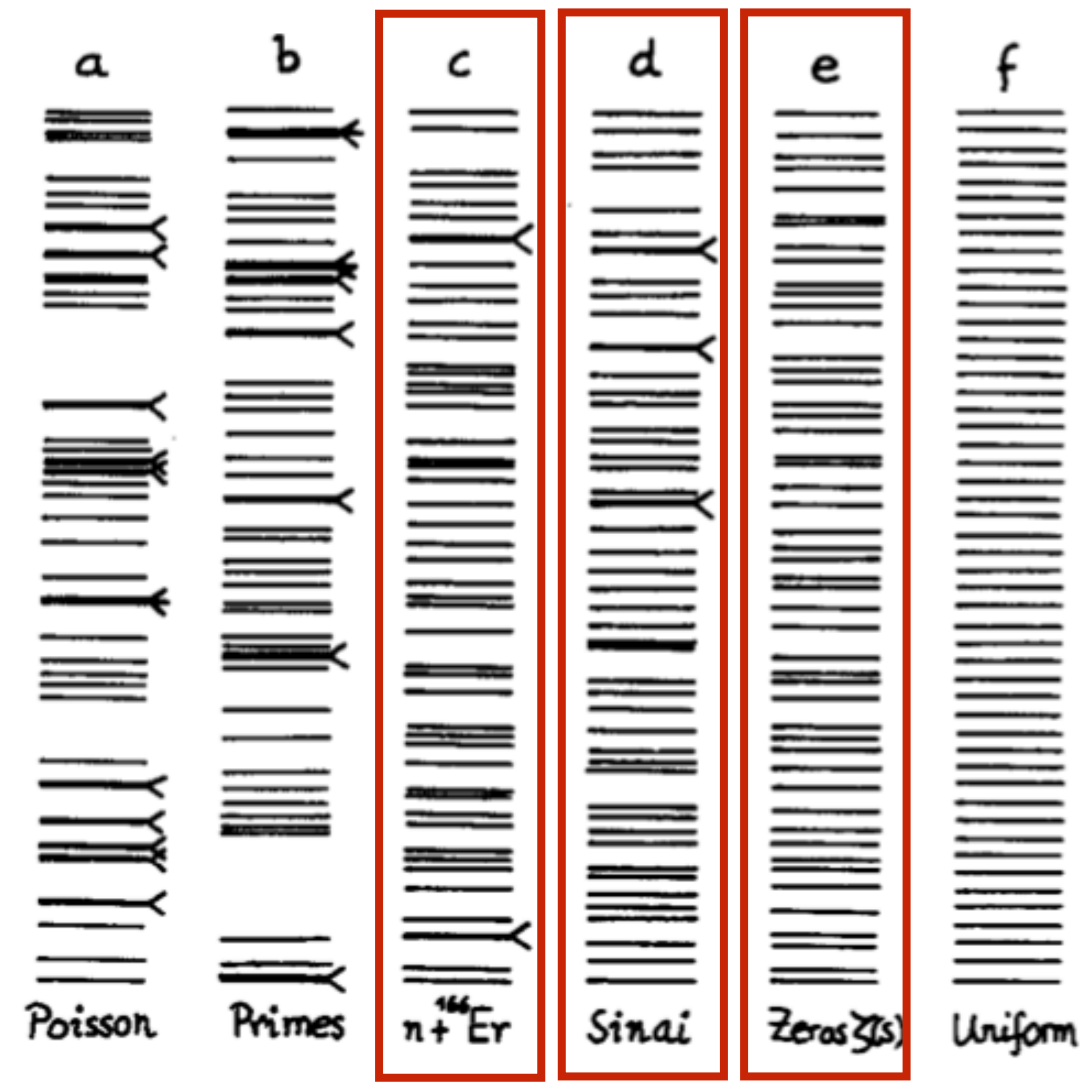
Introduction: Random matrices

The philosophy of a random matrix approach to a spectrum is to focus on the statistical property of the spectrum rather than understanding the specific eigenvalues themselves.

Nuclear spectrum



[Bohigas Haq Pandey, 83]



[Bohigas Giannoni, 84]

Nuclear spectrum, Sinai billiard and zero's of the Riemann zeta have the same statistical property! Very universal.

Ensembles

Wigner-Dyson

Time reversal

Chiral

chiral symmetry (+ Time reversal)

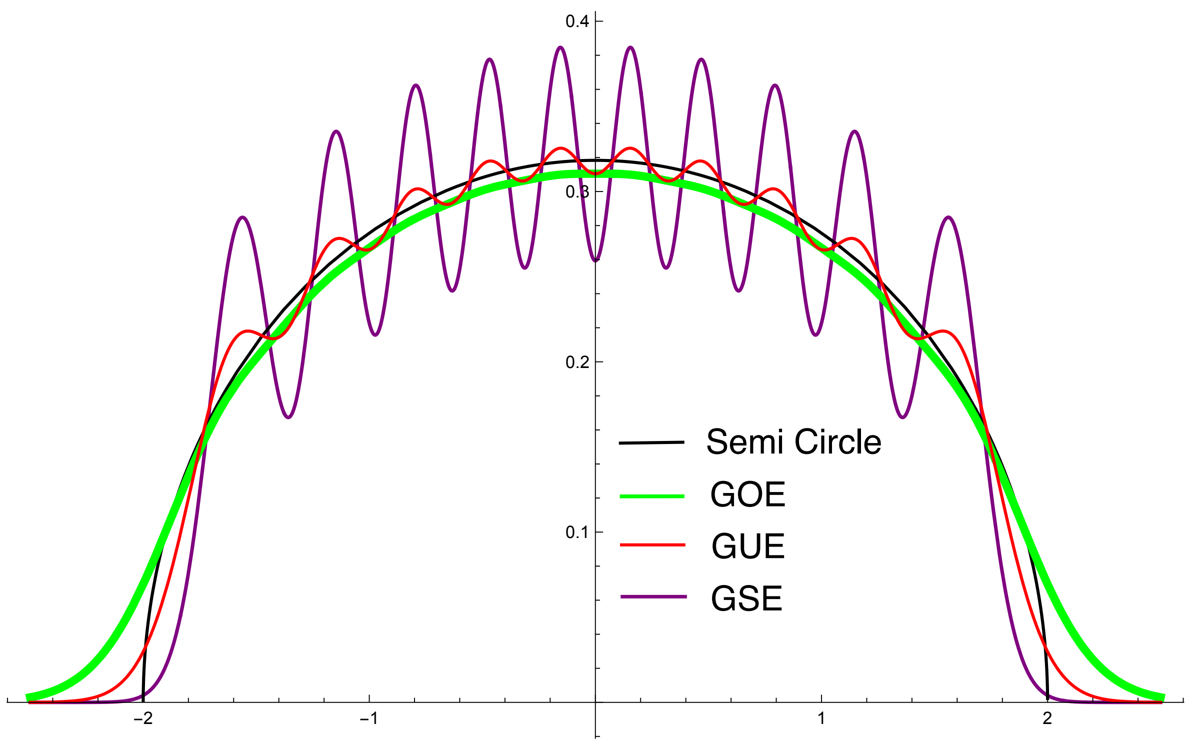
BdG

particle hole symmetry (+ chiral symmetry)

Those complete the 10 Altland-Zirnbauer (AZ) ensembles!

Wigner-Dyson

We consider three ensembles (orthogonal, unitary, symplectic)



10×10 matrices in different ensembles

They approach the famous semi-circle law but also have different "crystal" like structure depending on the ensembles.

To understand that, it is important to know the possible number of off-diagonal components, which is related to time reversal symmetry.

Gaussian unitary ensemble (GUE): 2×2 ex

- Let us consider a generic random 2×2 Hamiltonian

$$H = \begin{pmatrix} c+z & x-iy \\ x+iy & c-z \end{pmatrix} = cI + \boldsymbol{x}\cdot\boldsymbol{\sigma} = cI + r\boldsymbol{n}\cdot\boldsymbol{\sigma}$$

$$\to \lambda = c \pm r$$

$$\boldsymbol{v} = (v^x, v^y, v^z) \qquad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Pauli matrices)

- Then the measure becomes

$$\int \exp(-\frac{1}{2}\mathrm{Tr}H^2)dcdxdydz = \int \exp(-(c^2+x^2+y^2+z^2))dcdxdydz$$

$$= \int \exp(-(r^2+c^2))dcdrr^2 d\Omega_2$$

$$= \# \int drr^2 \exp(-r^2)$$

The probability of having degeneracy is suppressed by r^2

Gaussian orthogonal ensemble (GOE): 2×2 ex

- Let us consider a generic random 2×2 Hamiltonian with reality:

$$H = \begin{pmatrix} c+z & x \\ x & c-z \end{pmatrix} = cI + r(\cos\theta\sigma^x + \sin\theta\sigma^z)$$

$$\to \lambda = c \pm r$$

- Then the measure becomes

  Compared to the unitary ensemble, y is just removed.

$$\int \exp(-\frac{1}{2}\mathrm{Tr}H^2)dcdxdz = \int \exp(-(c^2+x^2+z^2))dcdxdz$$

$$= \int \exp(-(r^2+c^2))rdrd\theta dc$$

$$= \# \int r\exp(-r^2)dr$$

The probability of having degeneracy is suppressed by r

The number of off-diagonal components is reduced.

The probability of having degeneracy is increased, because we only need to tune one parameter to get degeneracy

Gaussian symplectic ensemble (GSE): 2×2 ex

- Let us consider a random 2×2 Hamiltonian but with quaternions:

$$H = \begin{pmatrix} c+v & q \\ \bar{q} & c-v \end{pmatrix} \qquad q = w + \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$$

- Then the measure becomes

$$\int \exp(-\frac{1}{2}\mathrm{Tr}H^2)dcdxdydzdwdv$$

$$= \int \exp(-(c^2+x^2+y^2+z^2+w^2+v^2))dcdxdydzdwdv$$

$$= \int \exp(-(r^2+c^2))r^4 dr d\Omega_4 dc$$

$$= \# \int r^4 \exp(-r^2)dr$$

The probability of having degeneracy is suppressed by r^4

The reason is the same, we have more off-diagonal components.

so far

What we learned is that more off-diagonal components lead to less probability to get degeneracy.

The main message so far is that generic matrices have many *off-diagonal components*

They give a *correlation* among the spectrum.

They are *different* from the random diagonal matrices.

Those correlation patterns can be used to explain the experimental data that looks structureless at first sight.

Time reversal symmetry

The three ensembles (GOE, GUE, GSE) are related to the time reversal symmetry.

The time-reversal symmetry is a symmetry that should reverse the time:

$$\mathcal{T} e^{-itH} \mathcal{T}^{-1} = e^{it(\mathcal{T} H \mathcal{T}^{-1})}$$

Assuming t commutes with the time reversal operator,

the above relation requires

$$\mathcal{T}(-iH)\mathcal{T}^{-1} = i\mathcal{T} H \mathcal{T}^{-1}$$

Therefore the time reversal should *anti-commutes* with the imaginary unit.

These operators are called *anti-linear* operators.

Not a linear operator!

Time reversal and (angular) momentum

The anti-linear nature is compatible with the angular momentum.

Let us consider a quantum particle on a line.

Assuming that the position operator commutes with the time reversal, the momentum anti-commutes with the time reversal:

$$\mathcal{T} p \mathcal{T}^{-1} = \mathcal{T}\Big(-i\frac{d}{dx}\Big)\mathcal{T}^{-1} = i\frac{d}{dx} = -p$$

Going to a particle in a three-dimensional space, the angular momentum the time reversal commutes with the angular momentum,

$$\mathcal{T} \boldsymbol{x} \times \boldsymbol{p} \mathcal{T}^{-1} = \mathcal{T} \boldsymbol{x} \mathcal{T}^{-1} \times \mathcal{T} \boldsymbol{p} \mathcal{T}^{-1} = -\boldsymbol{x} \times \boldsymbol{p}$$

which is again compatible with the time reversal in classical mechanics.

In the position basis, the time reversal is simply the complex conjugate K .

Since $K^2 = 1$, the square of the time reversal is also 1: $\mathcal{T}^2 = 1$

Time reversal and spins

The spin should transform in a same manner with the angular momentum:

$$\mathcal{T} \boldsymbol{s} \mathcal{T}^{-1} = -\boldsymbol{s}$$

Let us consider spin 1/2: $\boldsymbol{s} = \frac{1}{2}\boldsymbol{\sigma}$

We can choose $\mathcal{T} = i\sigma^y K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K$

Now the time reversal satisfies $\mathcal{T}^2 = -1$

This phase -1 is not removed by the redefinition of the phase $\mathcal{T} \to e^{i\theta}\mathcal{T}$

One of consequences is that each state is not invariant under the time-reversal symmetry and always have at least two degenerate state.

$$\mathcal{T}\left|\uparrow\right\rangle = \left|\downarrow\right\rangle, \quad \mathcal{T}\left|\downarrow\right\rangle = -\left|\uparrow\right\rangle$$

(since spin should be flipped under time reversal, up to a sign)

This is the generic property for systems with $\mathcal{T}^2 = -1$
called Kramers' degeneracy.

Time reversal and spins

One example of the time reversal invariant Hamiltonian with spins is

$$H = \boldsymbol{L} \cdot \boldsymbol{s}$$

Hamiltonian is invariant under the time reversal because both the angular momentum $\boldsymbol{L}$ and the spin $\boldsymbol{s}$ change the sign.

For example, when the angular momentum takes the value $J = 1$

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad L_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad L_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{L}$: anti-symmetric

Time reversal and ensemble

Now we see that

No time reversal	$\rightarrow$	GUE
time reversal with $\mathcal{T}^2 = 1$	$\rightarrow$	GOE
time reversal with $\mathcal{T}^2 = -1$	$\rightarrow$	GSE

Gaussian unitary ensemble (GUE)

Without time reversals, there are no constraints on Hamiltonians
The generic Hamiltonian is L×L hermitian matrices

The exponential of the Hamiltonian = evolution operator $\exp(-iHt)$
is an element of the unitary matrices $U(L)$.

In other words, H is an element of the Lie algebra $\mathfrak{u}(L)$

Comparing with the classification of symmetric spaces by Cartan,
These are called *class A*

Gaussian orthogonal ensemble (GOE)

With time reversals $\mathcal{T}^2 = 1$, we can take the basis with $\mathcal{T} = K$

The generic Hamiltonian with the constraint $\mathcal{T} H \mathcal{T}^{-1} = H$

The solution is $H = S$ for a real symmetric matrix S

What is the space of the Hamiltonian ?
Since the generator of $O(L)$ ($\mathfrak{so}(L)$) is the anti-symmetric matrices ,
which is a solution of $\mathcal{T} X \mathcal{T}^{-1} = -X$, it is different from $\mathfrak{so}(L)$
(because of this e^{iX} commutes with the time reversal $\mathcal{T} e^{iX} \mathcal{T}^{-1} = e^{iX}$)

It is rather $"\mathfrak{u}(L) - \mathfrak{so}(L)"$, remembering $H = S + iA$
Then the evolution operator $\exp(-iHt)$ should belong to $U(L)/O(L)$

Comparing with the classification of symmetric spaces by Cartan again,
the space $U(L)/O(L)$ are called *class AI.*

Gaussian symplectic ensemble (GSE)

With time reversals $\mathcal{T}^2 = -1$, we can take the basis with $\mathcal{T} = i\sigma^y K$

(because of the Kramers' degeneracy, the rank always should be even)

$$H = \begin{pmatrix} A+C & B \\ B^\dagger & A-C \end{pmatrix}$$

$$= A \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B_r \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B_i \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + C \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= A \otimes I + B_r \otimes \sigma^x + B_i \otimes \sigma^y + C \otimes \sigma^z$$

Imposing the time reversal $\mathcal{T} H \mathcal{T}^{-1} = H$

$A^* = A$ (real symmetric) $B_1^* = -B_1$ $B_2^* = -B_2$ $C^* = -C$ (real anti-symmetric)

$$H = S_0 \otimes I + iA_x \otimes \sigma^x + iA_y \otimes \sigma^y + iA_z \otimes \sigma^z$$

$$= S_0 \otimes I + i\boldsymbol{A} \cdot \boldsymbol{\sigma}$$

spin-orbit interaction

Gaussian symplectic ensemble (GSE) and quaternion

$$S \otimes I + i\boldsymbol{A}\cdot\boldsymbol{\sigma} = \begin{pmatrix} S + iA_z & A_x - iA_y \\ A_x + iA_y & S - iA_z \end{pmatrix}$$

Another representation

$$I \otimes S_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{A}$$

$$= \begin{pmatrix} S_{11}I & S_{12}I + i\boldsymbol{A}_{12}\cdot\boldsymbol{\sigma} & \cdots & S_{1k}I + i\boldsymbol{A}_{1k}\cdot\boldsymbol{\sigma} \\ S_{12}I - i\boldsymbol{A}_{12}\cdot\boldsymbol{\sigma} & S_{22}I & \cdots & S_{2k}I + i\boldsymbol{A}_{2k}\cdot\boldsymbol{\sigma} \\ \vdots & \vdots & \ddots & \vdots \\ S_{1k}I - i\boldsymbol{A}_{1k}\cdot\boldsymbol{\sigma} & S_{2k}I - i\boldsymbol{A}_{2k}\cdot\boldsymbol{\sigma} & \cdots & S_{kk}I \end{pmatrix}$$

$$= \begin{pmatrix} S_{11} & a_{12} & \cdots & a_{1k} \\ \bar{a}_{12} & S_{22} & \cdots & a_{22} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1k} & \bar{a}_{2k} & \cdots & S_{kk} \end{pmatrix}$$

: quarternion Hermitinan

Gaussian symplectic ensemble (GSE).ctd

Again, the “anti-symmetric” part with $\mathcal{T} X \mathcal{T}^{-1} = -X$ is

$$X = \boldsymbol{S} \cdot \boldsymbol{\sigma} + iA \otimes I$$

(then the exponential commutes with time reversal: $\mathcal{T} \exp(iX) \mathcal{T}^{-1} = \exp(iX)$)

This is a generator $\mathfrak{sp}(k)$ of $Sp(k)$.

(note that for k=1 it reduces to su(2))

Therefore a Hamiltonian in the GSE ensemble is an element of $\mathfrak{u}(2k) - \mathfrak{sp}(k)$

Then the evolution operator takes the value on $U(2k)/Sp(k)$

Comparing with the classification of symmetric spaces by Cartan again, the space $U(2k)/Sp(k)$ are called *class AII.*

Sphere integral and Symmetry

GOE (β=1): $= \int \exp(-(r^2+c^2))rdr\underline{d\theta}dc$

GUE (β=2): $= \int \exp(-(r^2+c^2))dcdrr^2\underline{d\Omega}_2$

GSE (β=4): $= \int \exp(-(r^2+c^2))r^4dr\underline{d\Omega}_4dc$

We get the sphere integral over S^β .

$\boldsymbol{R}^{\beta+1} \sim \boldsymbol{R}_+ \times S^\beta$ determines the spectral statistics .

$$S^1 = O(2)/O(1) \times O(1)$$
$$S^2 = U(2)/U(1) \times U(1)$$
$$S^4 = Sp(2)/Sp(1) \times Sp(1)$$

is a nice representation in our context.

Then we can understand the sphere as a symmetry-breaking pattern that determines the level statistics.

GUE: 2×2

General matrix: $U\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}U^\dagger$ U : 2×2 unitary matrix to diagonalize

In particular, the matrix $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ is not invariant under conjugation by U

but only invariant under diagonal unitary transformation: $U = \begin{pmatrix}e^{i\theta_1} & 0\\ 0 & e^{i\theta_2}\end{pmatrix}$

When the two eigenvalues are degenerate, it is invariant under full U(2):

$$U\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}U^\dagger = \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

We say that the generic matrix configuration breaks the symmetry $U(2)$ to $U(1)\times U(1)$

$U(2)/U(1)\times U(1)$ parametrize the matrix with fixed eigenvalues.

$$U(2)/U(1)\times U(1) \sim SU(2)/U(1) \sim S^2$$

GOE: 2×2

General matrix: $O \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} O^T$ O : 2×2 orthogonal matrix to diagonalize

In particular, the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is not invariant under conjugation by O

but only invariant under diagonal unitary transformation: $O = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$

When the two eigenvalues are degenerate, it is invariant under full O(2):

$$O \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} O^T = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

We say that the generic matrix configuration breaks the symmetry $O(2)$ to $O(1) \times O(1)$

$O(2)/O(1) \times O(1)$ parametrize the matrix with fixed eigenvalues.

$$O(2)/O(1) \times O(1) \sim SO(2)/O(1) \sim S^1$$

GSE: 4×4 (quartanion 2×2)

General matrix: $S\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}S^\dagger$ S: 4×4 symplectic matrix to diagonalize

In particular, the matrix $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ is not invariant under conjugation by S

but only invariant under diagonal unitary transformation: $S=\begin{pmatrix}S_1 & 0\\ 0 & S_2\end{pmatrix}$

$S_i \in Sp(1)$

When the two eigenvalues are degenerate, it is invariant under full Sp(2):

$$S\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}S^\dagger = \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

We say that the generic matrix configuration breaks the symmetry $Sp(2)$ to $Sp(1)\times Sp(1)$

$Sp(2)/Sp(1)\times Sp(1)$ parametrize the matrix with fixed eigenvalues.

$$Sp(2)/Sp(1)\times Sp(1)\sim SO(5)/SO(4)\sim S^4$$

Chiral ensembles

We have focused on the system only with a time-reversal symmetry.
Now we add so-called *chiral symmetry.*

These types of ensembles are interesting, especially from a QCD perspective where the chiral symmetry breaking takes place.

In condensed matter side, sometimes it is called *sublattice symmetry*.

The chiral symmetry $\mathcal{P}$ satisfies

$$\mathcal{P}^2 = 1 \qquad \{H, \mathcal{P}\} = 0$$

Because of the anti-commuting nature, when we have non-zero eigenvalue E_i then there is also a pair $\mathcal{P}\,|E_i\rangle$ with eigenvalue $-E_i$.

Therefore, the origin (E=0) is a special point in those ensembles.

(In other words, the property apart from the origin is determined by the time reversal symmetry)

Symmetry breaking pattern at the origin is an additional data.

Chiral Gaussian unitary ensemble (chGUE)

We take the following form of the chiral symmetry:

$$\mathcal{P} = \begin{pmatrix} I_N & 0 \\ 0 & -I_M \end{pmatrix}$$

A generic Hamiltonian is then

$$\mathcal{H} = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}$$

$$\begin{aligned} A &: N \times N \\ B &: N \times M \\ C &: M \times M \end{aligned}$$

Imposing the chiral symmetry

$$\mathcal{PHP} = \begin{pmatrix} A & -B \\ -B^\dagger & C \end{pmatrix} \quad \rightarrow \quad \boxed{\mathcal{H} = \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix}}$$

class AIII

Therefore a Hamiltonian in the chGUE is an element of

$$\mathfrak{u}(N+M) - (\mathfrak{u}(N) + \mathfrak{u}(M))$$

Then the evolution operator takes the value on $U(N+M)/U(N) \times U(M)$

Comparing with the classification of symmetric spaces by Cartan again, the space $U(N+M)/U(N) \times U(M)$ are called *class AIII.*

Symmetry of chGUE

On the other hand, the symmetry of the ensemble is

$$\mathcal{P}X\mathcal{P} = X \;\rightarrow\; X = \begin{pmatrix} X_N & 0 \\ 0 & X_M \end{pmatrix}$$

$$e^{iX} = \begin{pmatrix} U_N & 0 \\ 0 & V_M \end{pmatrix} \in U(N) \times U(M)$$

$$B \rightarrow U_N B V_M^\dagger$$

Using this symmetry, we can transform the matrix B to

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0\cdots 0 \\ 0 & \lambda_2 & \cdots & 0 & 0\cdots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_N & 0\cdots 0 \end{pmatrix}$$

Eigenvalue of H = singular value of B

chGUE and symmetry enhancement at the origin

Let us consider the simplest non-trivial example of $(N, M) = (1, 1)$

The Hamiltonian is then

$$H = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda e^{i\theta} \\ \lambda e^{-i\theta} & 0 \end{pmatrix} \qquad U = \begin{pmatrix} e^{i\chi_1} & 0 \\ 0 & e^{i\chi_2} \end{pmatrix} \in U(1) \times U(1)$$

$$UHU^\dagger = \begin{pmatrix} 0 & \lambda e^{i(\theta+\chi_1-\chi_2)} \\ \lambda e^{-i(\theta+\chi_1-\chi_2)} & 0 \end{pmatrix}$$

Therefore, the Hamiltonian is invariant under the diagonal subgroup $U(1)$ with $\chi_1 = \chi_2$.

On the other hand, at $\lambda = 0$ the full $U(1) \times U(1)$ symmetry is restored.

Generic configurations break the symmetry $U(1) \times U(1)$ to $U(1)$
Fixed eigenvalue Hamiltonian is parametrized by

$$U(1) \times U(1)/U(1) \sim U(1) \sim S^1$$

On the other hand, the random matrix measure is $dzd\bar{z} = \lambda d\lambda d\theta$

→we get the correct angular part and the power of λ from symmetry!

Chiral Gaussian orthogonal ensemble (chGOE, chGSE)

In exactly the same manner, we get

$$H = \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix}$$

with $B = \begin{cases} \text{N × M real matrix} & \mathcal{T}^2 = 1 \\ \text{N × M quartanion matrix} & \mathcal{T}^2 = -1 \end{cases}$

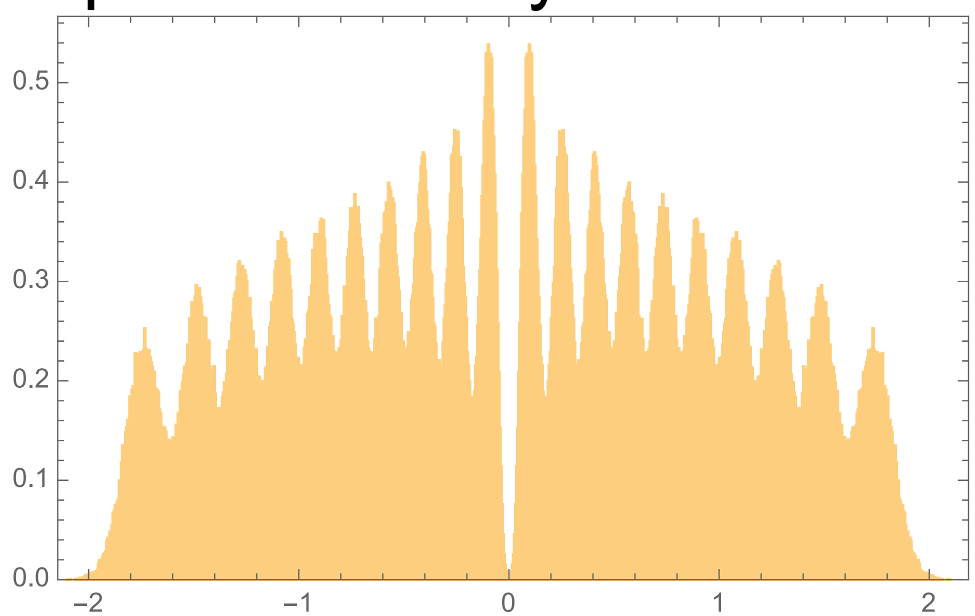
and replacing U by O or Sp.

Comparing with the classification of symmetric spaces by Cartan again,
the space $O(N+M)/O(N) \times O(M)$ is called *class BDI.*
and the space $Sp(N+M)/Sp(N) \times Sp(M)$ is called *class CII.*

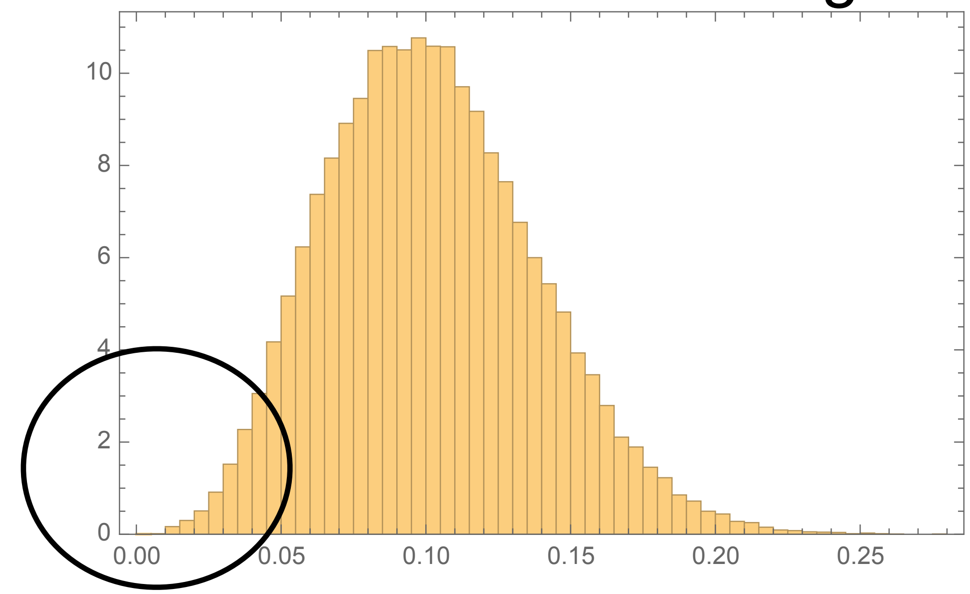
Numerical test of chGSE

L= 40 (k=20) chGSE, 30000 samples

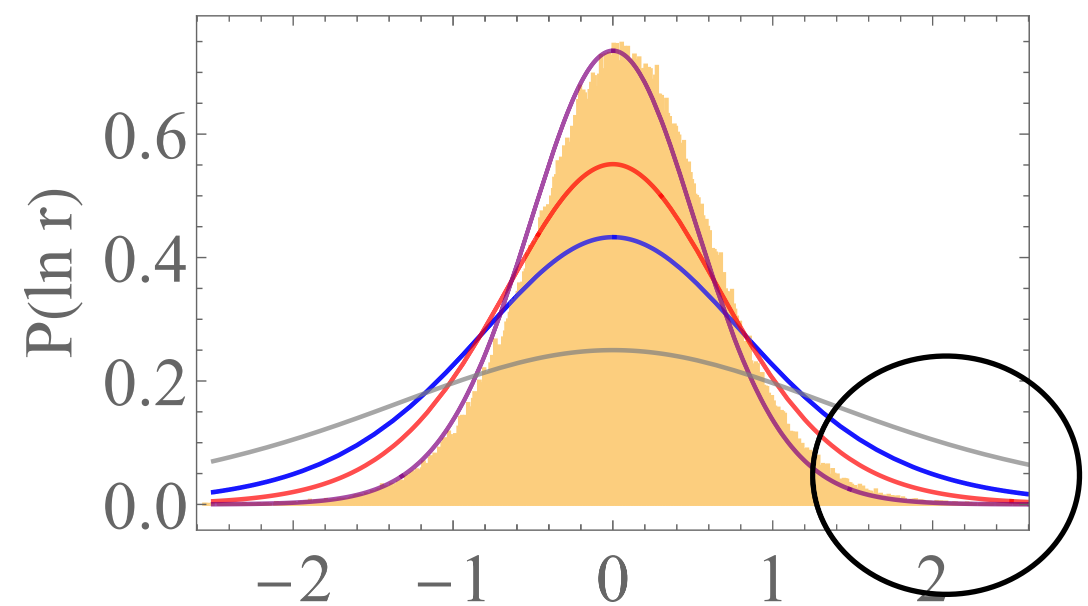
spectral density



Distribution of the first real eigenvalue



Wigner surmise



$$r_n = \frac{E_{n+1} - E_n}{E_{n+2} - E_{n+1}}$$

“adjacent gap ratio”

	Poisson	GOE	GUE	GSE
$p(r)$	$\frac{1}{(1+r)^2}$	$\frac{27}{8}\frac{(r+r^2)}{(1+r+r^2)^{\frac{5}{2}}}$	$\frac{81\sqrt{3}}{4\pi}\frac{(r+r^2)^2}{(1+r+r^2)^4}$	$\frac{729}{4\pi}\frac{(r+r^2)^4}{(1+r+r^2)^7}$
$\langle r\rangle$	$2\log 2 - 1 \approx 0.38629$	$4 - 2\sqrt{3} \approx 0.53590$	$2\frac{\sqrt{3}}{\pi} - \frac{1}{2} \approx 0.60266$	$\frac{32}{15}\frac{\sqrt{3}}{\pi} - \frac{1}{2} \approx 0.67617$

Bogoliubov- de Genne(BdG)

In those ensembles, we have particle-hole symmetry, which anti-commutes with the Hamiltonian.

$$\{H, \mathcal{C}\} = 0$$

These ensembles appear in superconductors.

Because of the anti-commuting nature, when we have non-zero eigenvalue E_i then there is also a pair $\mathcal{C}\,|E_i\rangle$ with eigenvalue $-E_i$.

Therefore, the origin (E=0) is again a special point in those ensembles.
When the state is invariant under particle-hole, they are Majorana zero modes.

Moreover, we can have chiral symmetry.
If those exist, they anti-commute with the particle hole symmetry.

$$\{\mathcal{C}, \mathcal{P}\} = 0$$

Because they do not commute, we could not diagonalize simultaneously them.

When we diagonalize $\mathcal{P}$, the Hamiltonian is $H = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^* \end{pmatrix}$

h: normal state Hamiltonian Δ :order parameter

BdG (C,D)

Now $\mathcal{C}$ is an anti-commuting operator with the Hamiltonian, rather than $\mathcal{T}$

The Hamiltonian then can be thought of as an element of $\mathfrak{so}(L)$ or $\mathfrak{sp}(k)$

(recall that the symmetry e^{iX} that anti-commuters with $\mathcal{T}$ is either in $SO(L)$ or $Sp(k)$)

For $\mathcal{C}^2 = 1$ $\quad H = iA \in \mathfrak{so}(L)$

(class D)

For $\mathcal{C}^2 = -1$ $\quad H = M \in \mathfrak{sp}(k)$

(class C)

In those ensembles, though $\mathcal{C}$ relates two states, $\mathcal{C}$ itself does not constrain the degenerate eigenvalues apart from the origin

Therefore the level statistics in the bulk is that of the class A (GUE).

BdG (C,D) near E=0

We can study the behavior around E=0 using small matrices.

For class D, the smallest matrix is 2×2,

$$H = \begin{pmatrix} 0 & -i\lambda \\ i\lambda & 0 \end{pmatrix}$$

and the symmetry is $O(2)$.

Therefore the symmetry is not broken and there is no level repulsion.

For class C, the smallest matrix is 2×2 and generic matrix is

$$H = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

and the symmetry is $Sp(1) = SU(2)$.

This is essentially the same with the 2×2 unitary matrix and generically the symmetry is broken to $U(1)$

Therefore the fixed eigenvalue Hamiltonians are labelled by

$$SU(2)/U(1) = S^2$$

BdG (DIII)

We have the chiral symmetry $\mathcal{P}$ on top of $\mathcal{C}$ with $\mathcal{C}^2 = 1$

We work in a basis with $\mathcal{C} = 1$ and $\mathcal{P} = \sigma^y$

To impose the chiral symmetry, we expand the matrix in $\mathfrak{so}(2k)$ as

$$H = iA_0 \otimes I + iA_x \otimes \sigma^x + S_y \otimes \sigma^y + iA_z \otimes \sigma^z$$

Then, imposing the chiral symmetry $\{\mathcal{P}, H\} = 0$ we get

$$H = iA_x \otimes \sigma^x + iA_z \otimes \sigma^z = i \begin{pmatrix} A_z & A_x \\ A_x & -A_z \end{pmatrix}$$

On the other hand, the commuting part $iA_0 \otimes I + S_y \otimes \sigma^y$ gives a symmetry

This forms $\mathfrak{u}(k)$. To see clearly this, it is convenient to diagonalize $\mathcal{P} = \sigma^y$

In this basis,

$$iA_0 \otimes I + S_y \otimes \sigma^y \to iA_0 \otimes I + S_y \otimes \sigma^z = \begin{pmatrix} S_y + iA_0 & 0 \\ 0 & -S_y + iA_0 \end{pmatrix}$$

and

$$H \to iA_x \otimes \sigma^y + iA_z \otimes \sigma^x = \begin{pmatrix} 0 & A_x + iA_z \\ A_x - iA_z & 0 \end{pmatrix}$$

BdG (DIII)

Therefore, Hamiltonian is , $\mathfrak{so}(2k) - \mathfrak{u}(k)$ and the symmetry is $U(k)$

The space of the evolution operator is $O(2k)/U(k)$.

Comparing with the classification of symmetric spaces by Cartan again, the space $O(2k)/U(k)$ is called *class DIII.*

$$\begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix} \begin{pmatrix} 0 & A_x + iA_z \\ A_x - iA_z & 0 \end{pmatrix} \begin{pmatrix} U^\dagger & 0 \\ 0 & U^T \end{pmatrix} = \begin{pmatrix} 0 & U(A_x + iA_z)U^T \\ \bar{U}(A_x - iA_z)U^\dagger & 0 \end{pmatrix}$$

Therefore the Hamiltonian transforms as an anti-symmetric tensor of $U(k)$

$\Delta = A_x + iA_z$: order parameter is anti-symmetric

BdG (CI)

We have the chiral symmetry $\mathcal{P}$ on top of $\mathcal{C}$ with $\mathcal{C}^2 = -1$

We work in a basis with $\mathcal{C} = i\sigma^y K$ and $\mathcal{P} = \sigma^y$

To impose the chiral symmetry, we expand the matrix in $\mathfrak{sp}(k)$ as

$$H = iA_0 \otimes I + S_x \otimes \sigma^x + S_y \otimes \sigma^y + S_z \otimes \sigma^z$$

Then, imposing the chiral symmetry $\{\mathcal{P}, H\} = 0$ we get

$$H = S_x \otimes \sigma^x + S_z \otimes \sigma^z = \begin{pmatrix} S_z & S_x \\ S_x & -S_z \end{pmatrix}$$

On the other hand, the commuting part $iA_0 \otimes I + S_y \otimes \sigma^y$ gives a symmetry
This forms $\mathfrak{u}(k)$. To see clearly this, it is convenient to diagonalize $\mathcal{P} = \sigma^y$

In this basis,

$$iA_0 \otimes I + S_y \otimes \sigma^y \to iA_0 \otimes I + S_y \otimes \sigma^z = \begin{pmatrix} S_y + iA_0 & 0 \\ 0 & -S_y + iA_0 \end{pmatrix}$$

and

$$H \to S_x \otimes \sigma^y + S_z \otimes \sigma^x = \begin{pmatrix} 0 & S_x + iS_z \\ S_x - iS_z & 0 \end{pmatrix}$$

BdG (CI)

Therefore, Hamiltonian is , $\mathfrak{sp}(k) - \mathfrak{u}(k)$ and the symmetry is $U(k)$

The space of the evolution operator is $Sp(k)/U(k)$.

Comparing with the classification of symmetric spaces by Cartan again, the space $Sp(k)/U(k)$ is called *class CI.*

$$\begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix} \begin{pmatrix} 0 & S_x + iS_z \\ S_x + iS_z & 0 \end{pmatrix} \begin{pmatrix} U^\dagger & 0 \\ 0 & U^T \end{pmatrix} = \begin{pmatrix} 0 & U(S_x + iS_z)U^T \\ \bar{U}(S_x - iS_z)U^\dagger & 0 \end{pmatrix}$$

Therefore the Hamiltonian transforms as an symmetric tensor of $U(k)$

$\Delta = S_x + iS_z$: order parameter is symmetric

BdG: summary

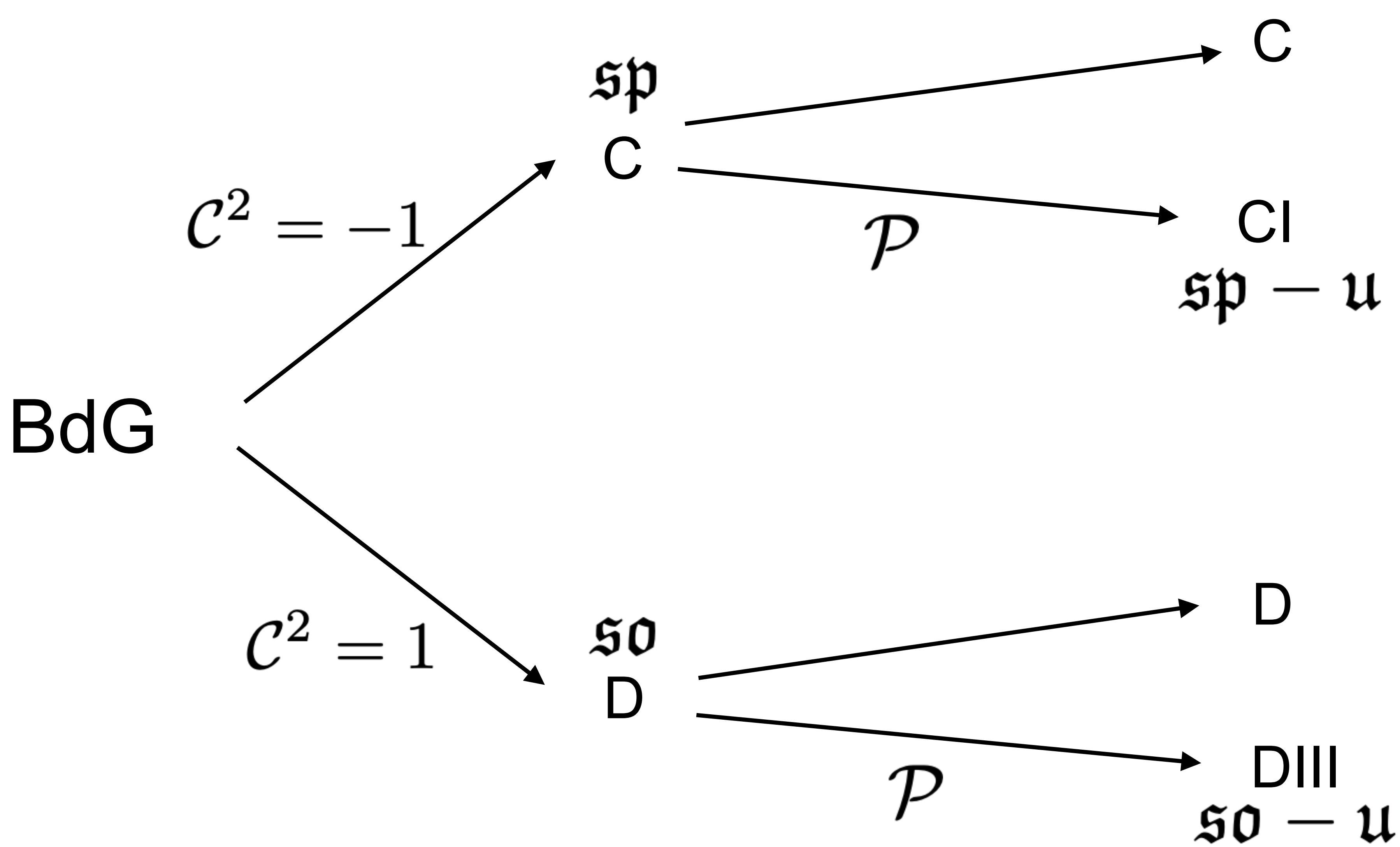
$$H = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^* \end{pmatrix}$$

C: $\Delta^T = \Delta$

D: $\Delta^T = -\Delta$

CI: $\Delta^T = \Delta, \quad h = 0$

DIII: $\Delta^T = -\Delta, \quad h = 0$



BdG (CI) near E=0

We can study the behavior around E=0 using small matrices.

For class CI, the smallest matrix is 2×2 where

$$H = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

and the symmetry is $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in U(1)$

The symmetry is then broken to $O(1)$.

The equal eigenvalue Hamiltonians are parametrized by

$$U(1)/O(1) \sim S^1$$

Note that the eigenvalue is now an order parameter $\lambda = \Delta$.
We are just seeing the symmetry breaking pattern by an order parameter!

BdG (DIII) near E=0

We can study the behavior around E=0 using small matrices.

For class CI, the smallest matrix is 4×4 where

$$H = \begin{pmatrix} 0 & \lambda\sigma^y \\ \lambda\sigma^y & 0 \end{pmatrix}$$

and the symmetry is $\begin{pmatrix} U_2 & 0 \\ 0 & \bar{U}_2 \end{pmatrix} \in U(2)$

The symmetry is then broken by order parameter to $SU(2)$.

The equal eigenvalue Hamiltonians are parametrized by

$$U(2)/SU(2) \sim S^1$$

BdG (CI, DIII) bulk spectral statistics

In those ensembles, we have an anti-unitary symmetry which *commutes* with the Hamiltonian;

$$\mathcal{T} = \mathcal{C}\mathcal{P} \qquad\qquad [\mathcal{T}, H] = 0$$

We also call this anti-uniatry time reversal symmetry.

Since now $\mathcal{T}$ commutes with the Hamiltonian, the bulk level statistics is now changed.

The square of the time reversal is now determined by that of $\mathcal{C}$;

$$\mathcal{T}^2 = (\mathcal{C}\mathcal{P})(\mathcal{C}\mathcal{P}) = -\mathcal{C}^2\mathcal{P}^2 = -\mathcal{C}^2$$

Therefore $\mathcal{T}^2 = -1 \ (\mathcal{C}^2 = 1)$, which means GSE, for DIII

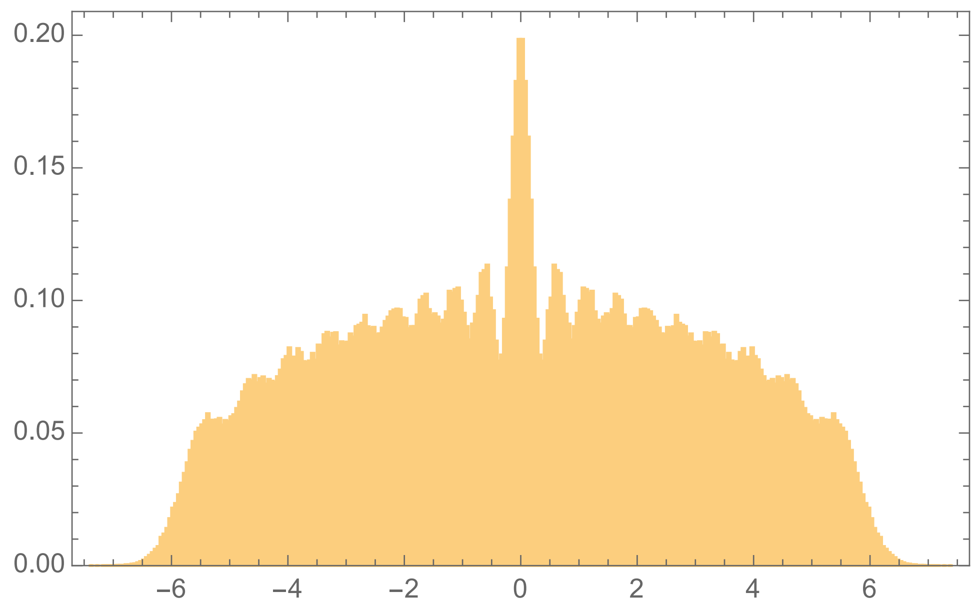
and $\mathcal{T}^2 = 1 \ (\mathcal{C}^2 = -1)$,which gives GOE, for CI.

(The group for the numerator of the coset $G/U(k)$ and O/S of and the bulk statistics are flipped)

Numerical test of D

L= 20 class D, 40000 samples

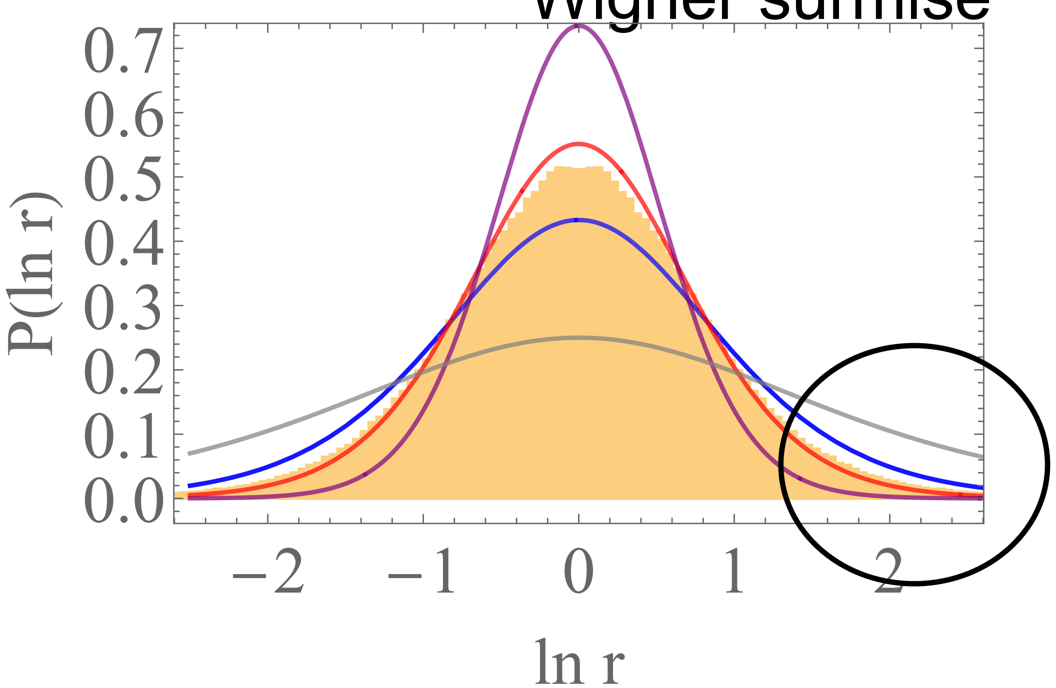
spectral density



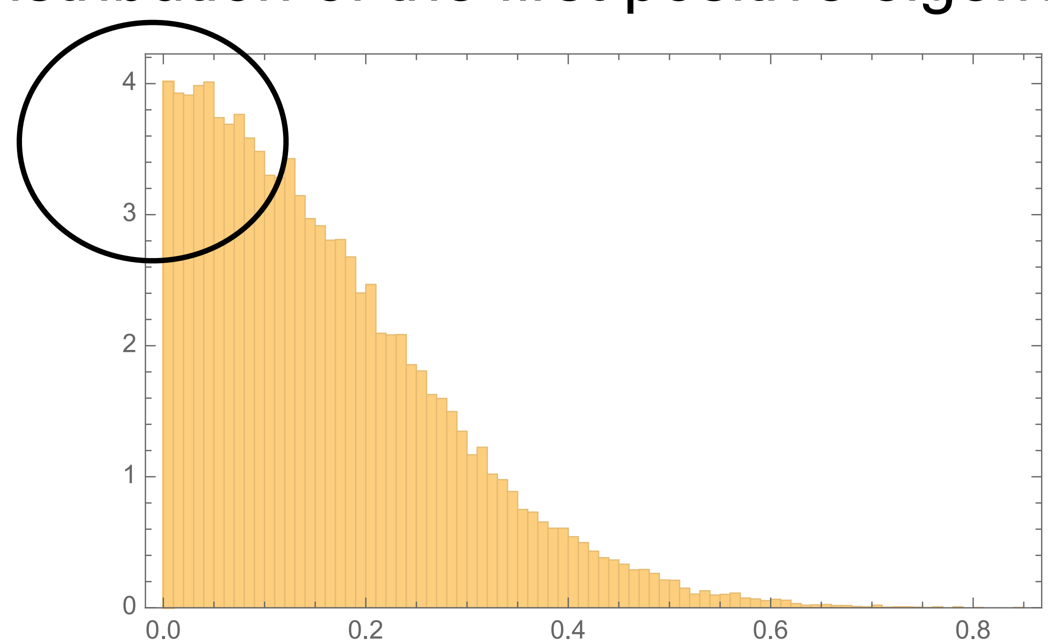
$$r_n = \frac{E_{n+1} - E_n}{E_{n+2} - E_{n+1}}$$

“adjacent gap ratio”

Wigner surmise



Distribution of the first positive eigenvalue



Summary

$$d\lambda_1 \cdots d\lambda_L \prod_{i_1}^{L} |\lambda_i|^{\alpha} \prod_{i<j} (\lambda_i - \lambda_j)^{\beta}$$ **:measure**

Wigner-Dyson

Cartan	$\mathcal{T}^2$	$\mathcal{C}^2$	$[\mathcal{T}(\mathcal{C}), \mathcal{P}]_{\pm}$	sym enhancement $\lambda \to 0$	α	β	symmetry of matrices
A				$1 \to 1$		2	$\mathrm{U}(L)$, adjoint
AI	$+1$			$1 \to 1$		1	$\mathrm{O}(L)$, $\square\square$
AII	-1			$\mathrm{Sp}(1) \to \mathrm{Sp}(1)$		4	$\mathrm{Sp}(L/2)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$

Chiral

Cartan	$\mathcal{T}^2$	$\mathcal{C}^2$	$[\mathcal{T}(\mathcal{C}), \mathcal{P}]_{\pm}$	sym enhancement $\lambda \to 0$	α	β	symmetry of matrices
AIII				$\mathrm{U}(1) \to \mathrm{U}(1)\times\mathrm{U}(1)$	1	2	$\mathrm{U}(L) \times \mathrm{U}(L)$, $(\square,\square)$
BDI	$+1$	$+1$	1	$\mathrm{O}(1) \to \mathrm{O}(1)\times\mathrm{O}(1)$	0	1	$\mathrm{O}(L) \times \mathrm{O}(L)$, $(\square,\square)$
CII	-1	-1	1	$\mathrm{Sp}(1) \to \mathrm{Sp}(1)\times\mathrm{Sp}(1)$	3	4	$\mathrm{Sp}(L/2) \times \mathrm{Sp}(L/2)$, $(\square,\square)$

BdG

Cartan	$\mathcal{T}^2$	$\mathcal{C}^2$	$[\mathcal{T}(\mathcal{C}), \mathcal{P}]_{\pm}$	sym enhancement $\lambda \to 0$	α	β	symmetry of matrices
C		-1		$\mathrm{U}(1) \to \mathrm{Sp}(1)$	2	2	$\mathrm{Sp}(L/2)$, $\square\square$
D		$+1$		$\mathrm{O}(2) \to \mathrm{O}(2)$	0	2	$\mathrm{O}(L)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$
CI	$+1$	-1	-1	$1 \to \mathrm{U}(1)$	1	1	$\mathrm{U}(L)$, $\square\square$
DIII	-1	$+1$	-1	$\mathrm{Sp}(1) \to \mathrm{U}(2)$	1	4	$\mathrm{U}(L)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$

Plan of the talk

(1)Random matrices and the Altland-Zirnbauer (AZ) class

Level statistics: Symmetry and Universality
10 ensembles: Wigner-Dyson (3) + chiral (3) + BdG (4)

(2)(supersymmetric) Sachdev-Ye-Kitaev (SYK) and AZ

interacting fermionic many-body system.
Nice model to see the AZ & K-theory in contexts of chaos

(3)dissipative SYK and symmetry

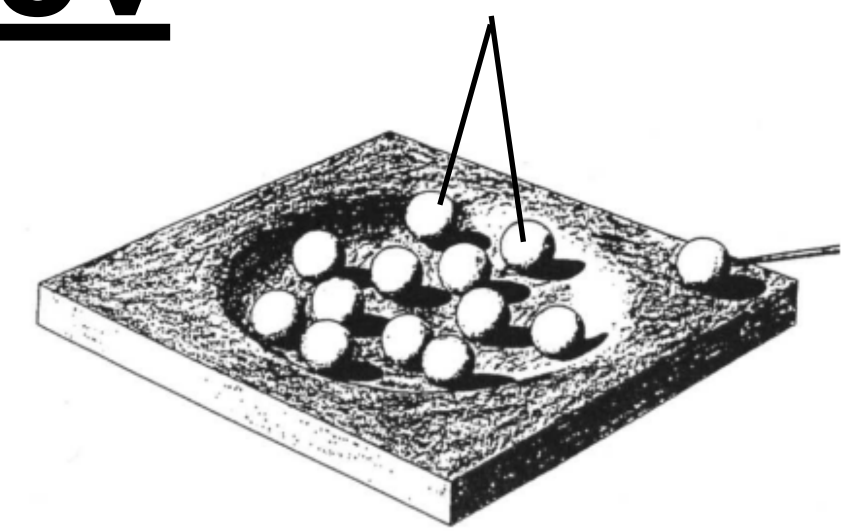
Fermionic dissipations, classification of fermionic Lindbladians

Sachdev-Ye-Kitaev

Original version: complex SYK

$$\sum_{i<j;k<l} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_l$$

[Sachdev-Ye 93]
[Sachdev 15]



$\{c_i^\dagger, c_j\} = \delta_{ij}$ $\{c_i^\dagger, c_j^\dagger\} = \{c_i, c_j\} = 0$:complex fermions

Before Kitaev, and even more before Sachdev and Ye, it was introduced in the context of nuclear physics to study the interaction effect on the level statistics of the nuclear spectrum!

[French-Wong 71]
[Bohigas-Flores 71]

Therefore the SYK is a natural setup to study the level statistics. Kitaev's Majorana version reproduce all the Wigner-Dyson ensemble, and supersymmetric version realizes all the Altland-Zirnbauer ensembles!

Sachdev-Ye-Kitaev

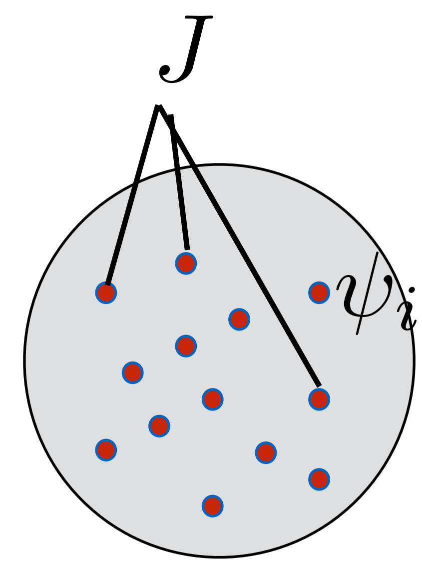
[Sachdev-Ye 93] [Kitaev 15]

N Majorana fermion $\{\psi_i, \psi_j\} = \delta_{ij}$ ($\mathrm{dim}\mathcal{H} = 2^{\frac{N}{2}}$)

Hamiltonian: $H_{SYK} = i^{\frac{q}{2}} \sum_{i_1 < i_2 < \cdots < i_q} J_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$

q : even

with $\langle J_{i_1 i_2 \cdots i_q} \rangle_J = 0$ and $\langle J^2_{i_1 i_2 \cdots i_q} \rangle_J = \frac{\mathcal{J}^2 (q-1)!}{q(2N)^{q-1}}$



・Has the same effective action (Schwarzian) with 2d dilaton (JT) gravity

[Maldacena-Stanford, 16] [Maldacena-Stanford-Yang, 16]

・Very sparse random matrices on many body Hilbert space (Fock space)

$N^q << 2^N$

The density state is different from random matrices, but level statistics agrees.

Jordan-Wigner transformation

Majorana fermions are represented as matrices using the Jordan-Wigner transformation;

even N

$$\psi_{2i-1} = \frac{1}{\sqrt{2}} \underbrace{\sigma^z \otimes \cdots \otimes \sigma^z}_{(i-1)\text{products}} \otimes \overset{i-\text{th}}{\sigma^x} \otimes \underbrace{I \otimes \cdots \otimes I}_{(N/2-i)\text{products}}$$

$$\psi_{2i} = \frac{1}{\sqrt{2}} \sigma^z \otimes \cdots \otimes \sigma^z \otimes \overset{i-\text{th}}{\sigma^y} \otimes I \otimes \cdots \otimes I$$

odd N add $\psi_N = \pm \frac{1}{\sqrt{2}} (2i\psi_1\psi_2)(2i\psi_3\psi_4)\cdots(2i\psi_{N-2}\psi_{N-1})$ to N-1

Then they satisfy $\{\psi_i, \psi_j\} = \delta_{ij}$. Equivalent to *qubits*!

The commutation relation is equivalent to that of the gamma matrices;

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$$

Therefore the mathematics of Majorana fermions are the same with that of the gamma matrices in N dimension and related to spinor representations.

$\chi_i \equiv \sqrt{2}\psi_i$ satisfies completely the same algebra with gamma matrices.

fermion parity symmetry

even N

$(-1)^F = (i\chi_1\chi_2)(i\chi_3\chi_4)\cdots(i\chi_{N-1}\chi_N)$ anti-commutes with all the fermions.

Using the Jordan-Wigner transformation,

$$\chi_{2i-1} = \underbrace{\sigma^z \otimes \cdots \otimes \sigma^z}_{(i-1)\text{products}} \otimes \overset{i-\text{th}}{\sigma^x} \otimes \underbrace{I \otimes \cdots \otimes I}_{(N/2-i)\text{products}}$$

$$\chi_{2i} = \sigma^z \otimes \cdots \otimes \sigma^z \otimes \overset{i-\text{th}}{\sigma^y} \otimes I \otimes \cdots \otimes I$$

$$i\chi_{2i-1}\chi_{2i} = \underbrace{I \otimes \cdots \otimes I}_{(i-1)\text{products}} \otimes \overset{i-\text{th}}{\sigma^z} \otimes \underbrace{I \otimes \cdots \otimes I}_{(N/2-i)\text{products}}$$

spin at site i.

$$(-1)^F = \sigma^z \otimes \cdots \otimes \sigma^z$$

: product of all the spins

odd N

We add $(-1)^F \equiv \chi_N$ to the algebra at N-1.

We do not have fermion parity any more.

Time reversal symmetry : even N

even N

The time-reversal symmetry acts on Majorana fermions as

$$\mathcal{T}\chi_i\mathcal{T}^{-1} = \chi_i$$

In our conventions, the complex conjugate flips the sign of the even fermions:

$$K\chi_{2i-1}K = \chi_{2i-1} \qquad\qquad K\chi_{2i}K = -\chi_{2i}$$

Therefore to ensure the commutativity, we need a modification.

The correct one is

$$\mathcal{T} = \begin{cases} \chi_1\chi_3\chi_5\cdots\chi_{N-1}K & \text{for N = 0 mod 4} \\ \chi_2\chi_4\chi_6\cdots\chi_N K & \text{for N = 2 mod 4} \end{cases}$$

$$= (\text{phase}) \times \sigma^y \otimes \sigma^x \otimes \sigma^y \otimes \cdots K$$

Another anti-unitary operators are given by

$$\mathcal{T}' = \mathcal{T}(-1)^F$$

$$= (\text{phase}) \times \sigma^x \otimes \sigma^y \otimes \sigma^x \otimes \cdots K$$

which anti-commutes with the fermion: $\mathcal{T}'\chi_i\mathcal{T}'^{-1} = -\chi_i$

Time reversal symmetry : even N

property of the time reversal for even N

Explicitly calculating, we obtain

$$\mathcal{T}^2 = \begin{cases} +1 & \text{for N = 0,2 mod 8} \quad (\text{\# of } \sigma^y = \text{even}) \\ -1 & \text{for N = 4,6 mod 8} \quad (\text{\# of } \sigma^y = \text{odd}) \end{cases}$$

and

$$\mathcal{T}(-1)^F = \begin{cases} +(-1)^F\mathcal{T} & \text{for N = 0,4 mod 8} \quad (\text{\# of } \sigma = \text{even}) \\ -(-1)^F\mathcal{T} & \text{for N = 2,6 mod 8} \quad (\text{\# of } \sigma = \text{odd}) \end{cases}$$

From those two, we can deduce

$$\mathcal{T}'^2 = \begin{cases} +1 & \text{for N = 0,6 mod 8} \\ -1 & \text{for N = 2,4 mod 8} \end{cases}$$

N → 8-N mod 8 exchanges $\mathcal{T}$ and $\mathcal{T}'$

Time reversal symmetry : even N

odd N

For even N, we made different anti-unitaries $\mathcal{T}$ and $\mathcal{T}'$ using $(-1)^F$
We cannot do that for odd N since we do have $(-1)^F$

What we have for odd N is

$$\widehat{\mathcal{T}} = \chi_1 \chi_3 \cdots \chi_{N-2} \chi_N K$$

This satisfies

$$\widehat{\mathcal{T}} \chi_i \widehat{\mathcal{T}}^{-1} = \begin{cases} +\chi_i & \text{for N = 1 mod 4} \\ -\chi_i & \text{for N = 3 mod 4} \end{cases} \qquad \begin{matrix} \widehat{\mathcal{T}} = \mathcal{T} \\ \widehat{\mathcal{T}} = \mathcal{T}' \end{matrix}$$

and

$$\widehat{\mathcal{T}}^2 = \begin{cases} +1 & \text{for N = 1,7 mod 8} \quad (\text{\# of } \sigma^y = \text{even }) \\ -1 & \text{for N = 3,5 mod 8} \quad (\text{\# of } \sigma^y = \text{odd }) \end{cases}$$

The property of the time reversal is symmetric under N → 8-N mod 8

Time reversal symmetry :

summary

N mod 8	$\mathcal{T}_+^2$	$\mathcal{T}_-^2$	$[\mathcal{T}, (-1)^F]_\pm$
0	$+1$	$+1$	1
1	$+1$		
2	$+1$	-1	-1
3		-1	
4	-1	-1	1
5	-1		
6	-1	$+1$	-1
7		$+1$	

$$\mathcal{T}_+ \chi_i \mathcal{T}_+^{-1} = \chi_i$$

$$\mathcal{T}_- \chi_i \mathcal{T}_-^{-1} = -\chi_i$$

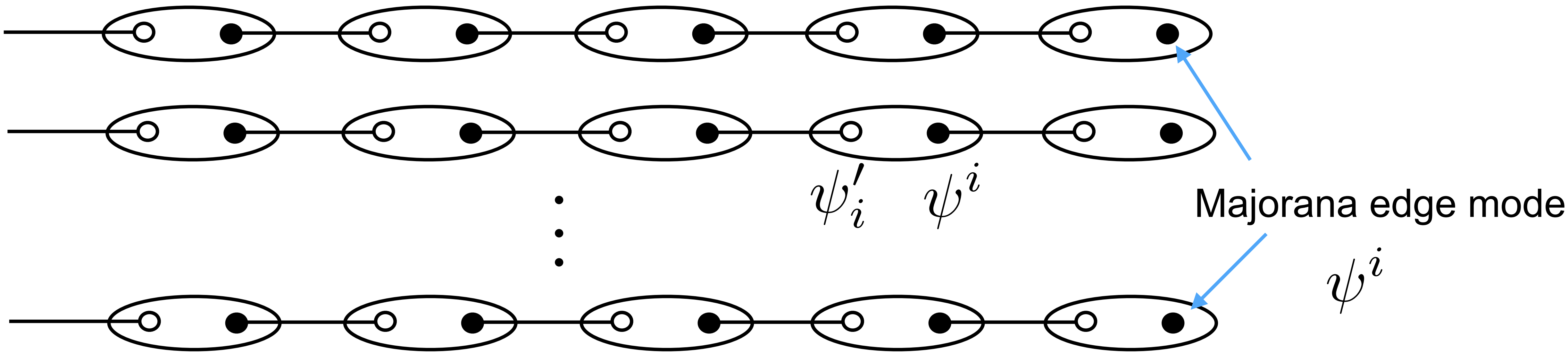
$$(-1)^F \chi_i = -(-1)^F \chi_i$$

SYK Hamiltonian is invariant under both $\mathcal{T}_+$ and $\mathcal{T}_-$

SYK as a boundary of many body MBL-SPT phase:

Kitaev chain [Xu,Ludwig,You 16]

$$\mathcal{T}\psi_i'\mathcal{T}^{-1} = -\psi_i' \quad \mathcal{T}\psi^i\mathcal{T}^{-1} = -\psi^i \quad \mathcal{T}^2 = 1$$



two body int: $i\psi^i\psi^j \rightarrow$ break time reversal

four body int: $\psi^i\psi^j\psi^k\psi^l$ →invariant under time rev [Fidkowski-Kitaev 09]

Consider many body MBL-SPT → SYK is a boundary theory

[Xu,Ludwig,You 16]

A natural generalization of Anderson localization/SPT [Ryu Schnyder Furusaki Ludwig, 09] to interacting systems!

N mod 8 classification of ordinary SYK

[Xu,Ludwig,You 16]

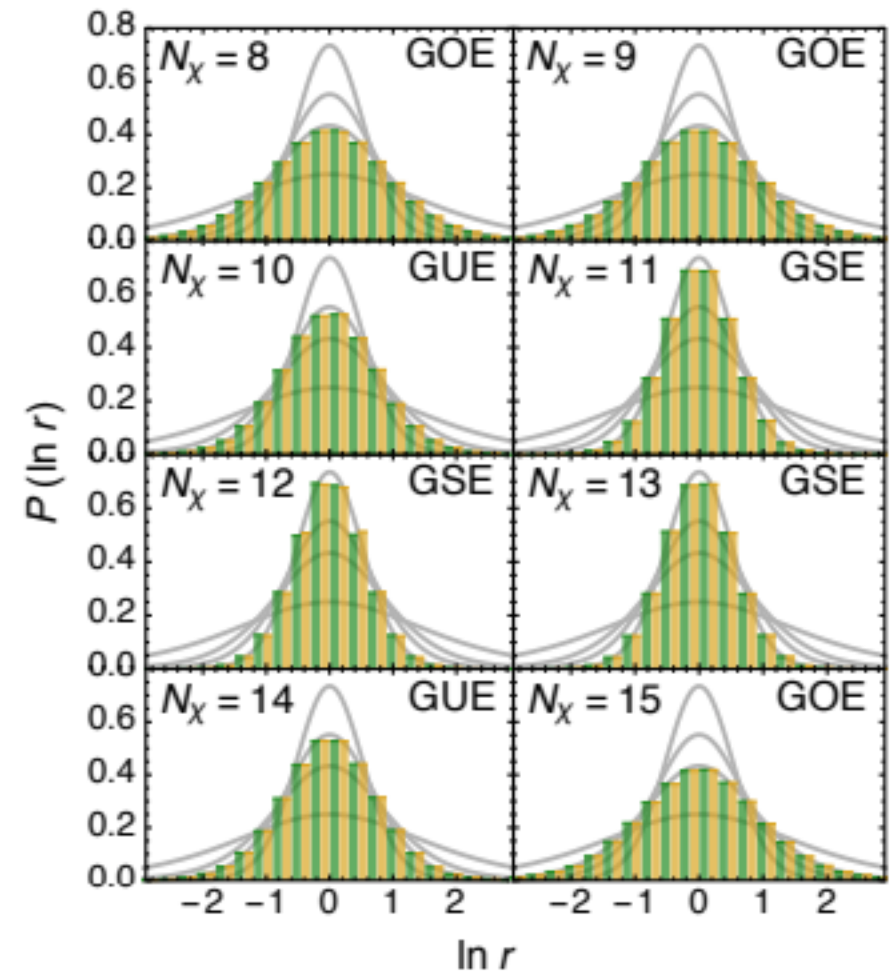
$\mathcal{T}_{\pm}$: time reversal

$(-1)^F$: fermion parity

Gap Ratio $r_n = \dfrac{E_{n+1} - E_n}{E_n - E_{n-1}}$

Table of N mod 8 dependence

N mod 8	$\mathcal{T}_+^2$	$\mathcal{T}_-^2$	$(-1)^F$	$\mathcal{T}(-1)^F = a(-1)^F\mathcal{T}$	Level Stat	qdim
0	$+1$	$+1$	Yes	1	GOE	1
1	$+1$		No		GOE	$\sqrt{2}$
2	$+1$	-1	Yes	-1	GUE	2
3		-1	No		GSE	$2\sqrt{2}$
4	-1	-1	Yes	1	GSE	2
5	-1		No		GSE	$2\sqrt{2}$
6	-1	$+1$	Yes	-1	GUE	2
7		$+1$	No		GOE	$\sqrt{2}$



SYK realize all the Wigner-Dyson statistics!

When $\mathcal{T}$ anti-commutes with $(-1)^F$, $\mathcal{T}$ shuffles two chirality sectors.

To distinguish N and 8-N, we need additional ingredient

Relation to properties of gamma matrices

$$\gamma^{2k-1} = \sigma^z \otimes \cdots \otimes \sigma^z \otimes \overset{k-\text{th}}{\sigma^x} \otimes I \otimes \cdots \otimes I$$

$$\gamma^{2k} = \sigma^z \otimes \cdots \otimes \sigma^z \otimes \overset{k-\text{th}}{\sigma^y} \otimes I \otimes \cdots \otimes I$$

chirality: $\Gamma = (i\gamma^1\gamma^2)(i\gamma^3\gamma^4)\cdots(i\gamma^{N-1}\gamma^N)$ (for even d)

Charge conjugation matrices

$$C_\eta \gamma^\mu C_\eta^{-1} = \eta(\gamma^\mu)^T$$

$$C_\eta^\dagger = C_\eta \quad \eta = \pm 1$$

SYK	**Gamma matrices**
$\mathcal{T}_\eta$	C_η
$\mathcal{T}_\eta^2$	$C_\eta^* C_\eta$
$(-1)^F$	Γ

$\mathcal{T}_\eta \leftrightarrow C_\eta$

$\mathcal{T}_\eta^2 \leftrightarrow C_\eta^* C_\eta$

$(-1)^F \leftrightarrow \Gamma$

Corresponds to Level statistics

d mod 8	$C_+^* C_+$	$C_-^* C_-$	$[C,\Gamma]_\pm$	spinor	reality
0	+1	+1	1	MW	Real
1	+1			M	Real
2	+1	−1	−1	M,W	Complex
3		−1			Pseudo
4	−1	−1	1	W	Pseudo
5	−1				Pseudo
6	−1	+1	−1	M,W	Complex
7		+1		M	Real

M = Majorana
W = Weyl

Supersymmetric SYK

[Gaiotto Fu Maldacena Sachdev, 16]

N Majorana fermion $\{\psi_i, \psi_j\} = \delta_{ij}$ ($\mathrm{dim}\mathcal{H} = 2^{\frac{N}{2}}$)

Supercharge: $Q = i^{\frac{\hat{q}-1}{2}} \sum_{i_1 < \cdots < i_{\hat{q}}} C_{i_1 \cdots i_{\hat{q}}} \psi_{i_1} \cdots \psi_{\hat{q}}$ $\hat{q}$: odd

with $\langle C_{i_1 \cdots i_{\hat{q}}} \rangle_C = 0$ and $\langle C^2_{i_1 \cdots i_{\hat{q}}} \rangle_C = \frac{(\hat{q}-1)!}{N^{\hat{q}-1}} J$

Hamiltonian: $H = Q^2$

- Again solvable at large N though the model is strongly coupled
- Has the similar effective action (super Schwarzian) to 2d JT SUGRA
[Stanford-Witten,17,19]
- Completely reproduce all the AZ classes!
[Kanazawa-Wetting, 17] [Li Liu Xin Zhou, 17] [Sun Ye, 19]

Properties of SUSY SYK

- The Hamiltonian is invariant under $\mathcal{T}_+$, $\mathcal{T}_-$, $(-1)^F$.

  On the other hand, $[\mathcal{T}_+, Q] = 0$, $\{\mathcal{T}_-, Q\} = 0$ $\{(-1)^F, Q\} = 0$

Identifying

susy SYK AZ ensemble

$$\mathcal{T}_+ \to \mathcal{T}$$
$$\mathcal{T}_- \to \mathcal{C}$$
$$(-1)^F \to \mathcal{P}$$
$$Q \to H$$

There is a correspondence between SUSY SYK and AZ class!

Relation to Dirac operators

- Let us consider the $\hat{q} = 1$ (though it corresponds to free fermions)

$$Q = \sum_j C_j \psi^j$$

Rewriting $C_j \to p_\mu$ and $\psi^j \to \gamma^\mu$,supercharge is understood as the Dirac operator in momentum space !

- In SYK side, without changing the symmetry properties we can add

$$\begin{aligned} Q &= Q_1 + Q_5 + Q_9 + \cdots \\ &= \sum_j C_j \psi^j + \sum_{i<j<k<l<m} C_{ijklm} \psi^i \psi^j \psi^k \psi^l \psi^m + \cdots \end{aligned}$$

Including all the terms, we get a generic AZ ensemble matrices.

- We can also consider $\hat{q} = 3$

$$Q = Q_3 + Q_7 + \cdots = i \sum_{i<j<k} C_{ijk} \psi^i \psi^j \psi^k + \cdots$$

$(N, \hat{q})$ and $(8 - N, \hat{q} + 2)$ have the same symmetry property.

Properties of SUSY SYK

[Stanford-Witten, 19]

- The time reversal, fermion parity, and supercharges are summarized as

Cartan	N mod 8	$\mathcal{T}_+^2$	$\mathcal{T}_-^2$	$[\mathcal{T},(-1)^F]_\pm$	α	β	supercharge symmetry
BDI	0	+1	+1	1	0	1	$\mathrm{O}(L)\times\mathrm{O}(L)$, $(\square,\square)$
AI	1	+1				1	$\mathrm{O}(L)$, $\square\square$
CI	2	+1	−1	−1	1	1	$\mathrm{U}(L)$, $\square\square$
C	3		−1		2	2	$\mathrm{Sp}(L/2)$, $\square\square$
CII	4	−1	−1	1	3	4	$\mathrm{Sp}(L/2)\times\mathrm{Sp}(L/2)$, $(\square,\square)$
AII	5	−1				4	$\mathrm{Sp}(L/2)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$
DIII	6	−1	+1	−1	1	4	$\mathrm{U}(L)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$
D	7		+1		0	2	$\mathrm{O}(L)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$

- Of course, the last entry is just writing the tangent of Cartan symmetric space = space of AZ random matrix ensembles, the relation to fermions are unclear.

  The symmetry is understood as the space of rotation matrices in spinor representation and how γ^μ transforms under rotation.

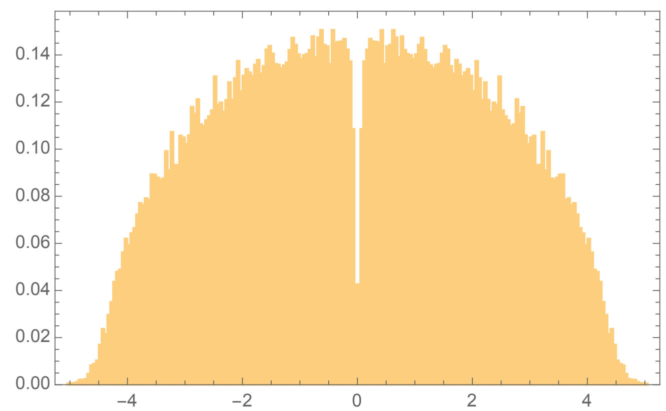
  In lower dimension where spin(d) indeed coincides with other classical groups. (= property of $\mathbb{C}l_{n,0}$)

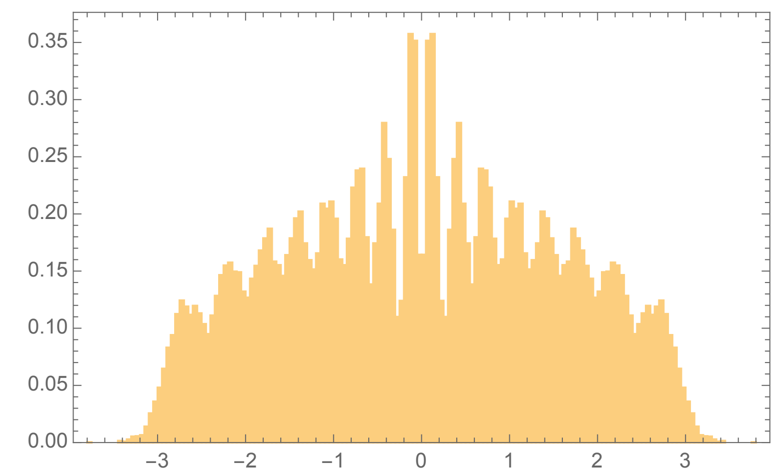
<u>Numerical test of SUSY SYK</u>

$N = 10, \quad \hat{q} = 5$ — DIII

$N = 10, \quad \hat{q} = 3$ — CI

Eigenvalue density of supercharge

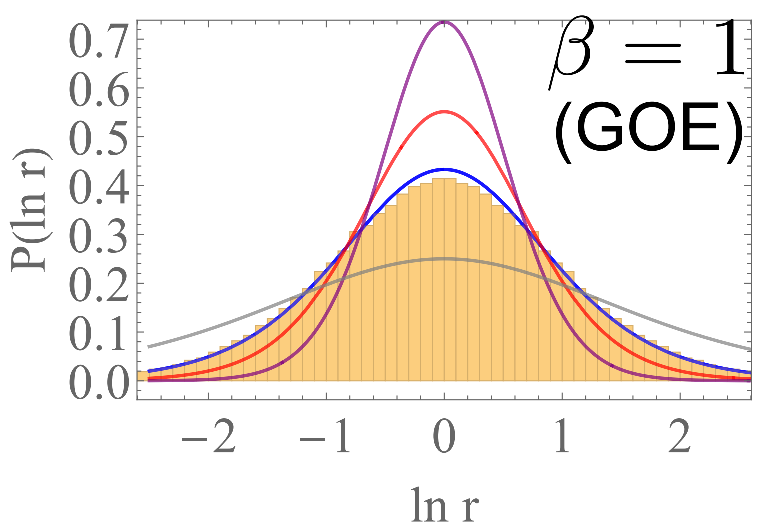


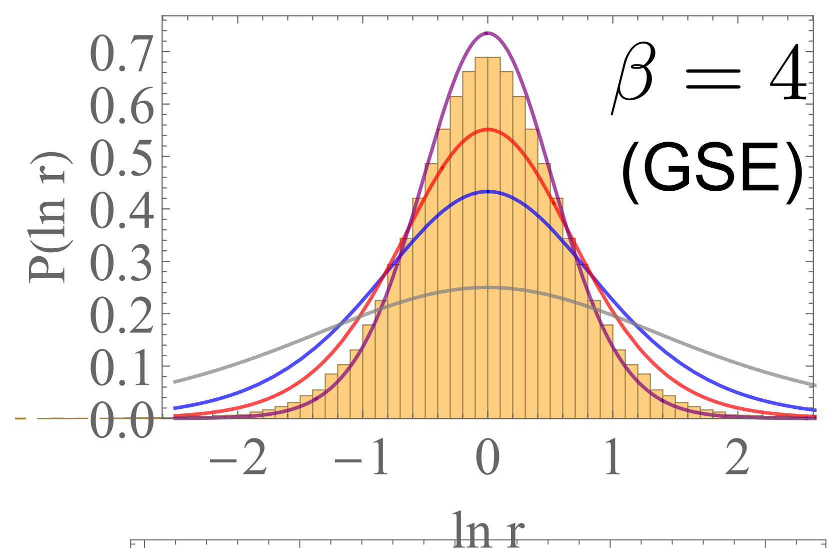


bulk level statistics

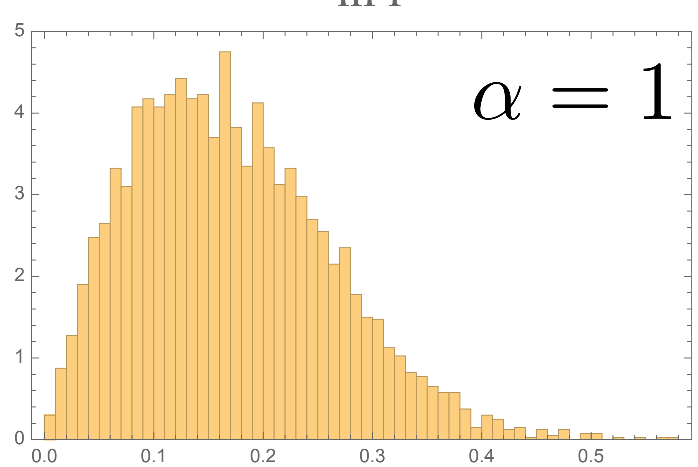
$$r_n = \frac{E_{n+1} - E_n}{E_{n+2} - E_{n+1}}$$

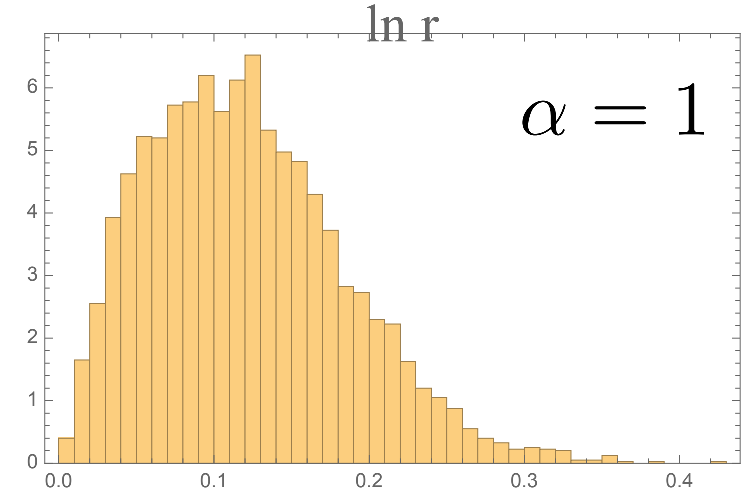
“adjacent gap ratio”





distribution of 1st positive eigenvalue





Summary

SYK

Realize all the Wigner-Dyson ensembles

N mod 8	$\mathcal{T}_+^2$	$\mathcal{T}_-^2$	$(-1)^F$	$\mathcal{T}(-1)^F = a(-1)^F\mathcal{T}$	Level Stat	qdim
0	$+1$	$+1$	Yes	1	GOE	1
1	$+1$		No		GOE	$\sqrt{2}$
2	$+1$	-1	Yes	-1	GUE	2
3		-1	No		GSE	$2\sqrt{2}$
4	-1	-1	Yes	1	GSE	2
5	-1		No		GSE	$2\sqrt{2}$
6	-1	$+1$	Yes	-1	GUE	2
7		$+1$	No		GOE	$\sqrt{2}$

SUSY SYK

Realize all the Altland-Zirnbauer ensembles

Cartan	N mod 8	$\mathcal{T}_+^2$	$\mathcal{T}_-^2$	$[\mathcal{T}, (-1)^F]_\pm$	α	β	supercharge symmetry
BDI	0	$+1$	$+1$	1	0	1	$\mathrm{O}(L) \times \mathrm{O}(L)$, $(\square,\square)$
AI	1	$+1$				1	$\mathrm{O}(L)$, $\square\!\square$
CI	2	$+1$	-1	-1	1	1	$\mathrm{U}(L)$, $\square\!\square$
C	3		-1		2	2	$\mathrm{Sp}(L/2)$, $\square\!\square$
CII	4	-1	-1	1	3	4	$\mathrm{Sp}(L/2) \times \mathrm{Sp}(L/2)$, $(\square,\square)$
AII	5	-1				4	$\mathrm{Sp}(L/2)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$
DIII	6	-1	$+1$	-1	1	4	$\mathrm{U}(L)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$
D	7		$+1$		0	2	$\mathrm{O}(L)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$

Plan of the talk

(1)Random matrices and the Altland-Zirnbauer (AZ) class

Level statistics: Symmetry and Universality
10 ensembles: Wigner-Dyson (3) + chiral (3) + BdG (4)

(2)(supersymmetric) Sachdev-Ye-Kitaev (SYK) and AZ

interacting fermionic many-body system.
Nice model to see the AZ & K-theory in contexts of chaos

(3) dissipative SYK and symmetry

Fermionic dissipations, classification of fermionic Lindbladians

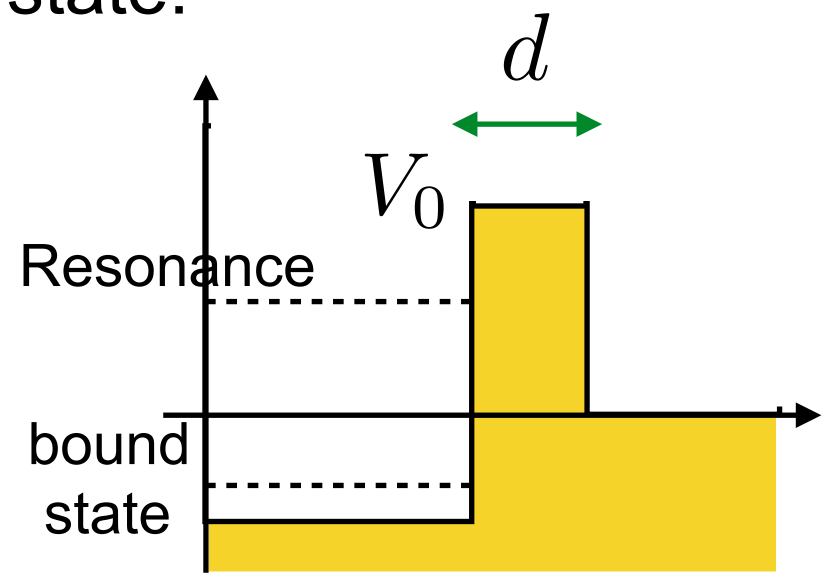
based on PRX Quantum (2023) with A.Kulkarni, K.Kawabata, J,Li, S.Ryu (Princeton U)

Phys.Rev.B 106 (2022) 7 w/ A.Kulkarni and S.Ryu (Princeton U)
Phys.Rev.B 108 (2023) 7 with A.Kulkarni, K.Kawabata, J,Li, S.Ryu (Princeton U)

Complex energy

It is sometimes convenient to complexify energy.
Typically, it is interpreted as a decay rate of the state:

$$\psi(t) \propto e^{-i(E_i - i\gamma_i)t}$$



Consider a particle trapped in a potential, but decay by quantum tunneling.

Gamov' factor $\gamma_i \sim e^{-\frac{2d}{\hbar}\sqrt{2m(V_0 - E_i)}}$

bound state: $e^{-\kappa r} = e^{i(i\kappa)r}$ outgoing wave: pure imaginary momentum

Resonance: $e^{(\gamma+ip)r} = e^{i(p-i\gamma)r}$ outgoing wave: complex momentum

= connection to the environment

The operator acting ***outside of the Hilbert space*** becomes non-Hermitian!

Compatible with the characterization as complex poles in the green functions

$$G(E) = \mathrm{Tr}\frac{1}{E - H}$$

Non-Hermitian matices

General motivation

・Non-Hermitian matrices are more generic than Hermitian matrices!

Ubiquitous.

Technical motivation

・d-dim quantum system = (d+1)-dim classical statistical problems

(evolution operator ↔ transfer matrix)

unitary quantum Hamiltonian → complex probability stat.mech

(eg: finite density QCD, topological terms)

non-Hermitian quantum Hamiltonian ← positive probability stat.mech

(eg: Hatano-Nelson)

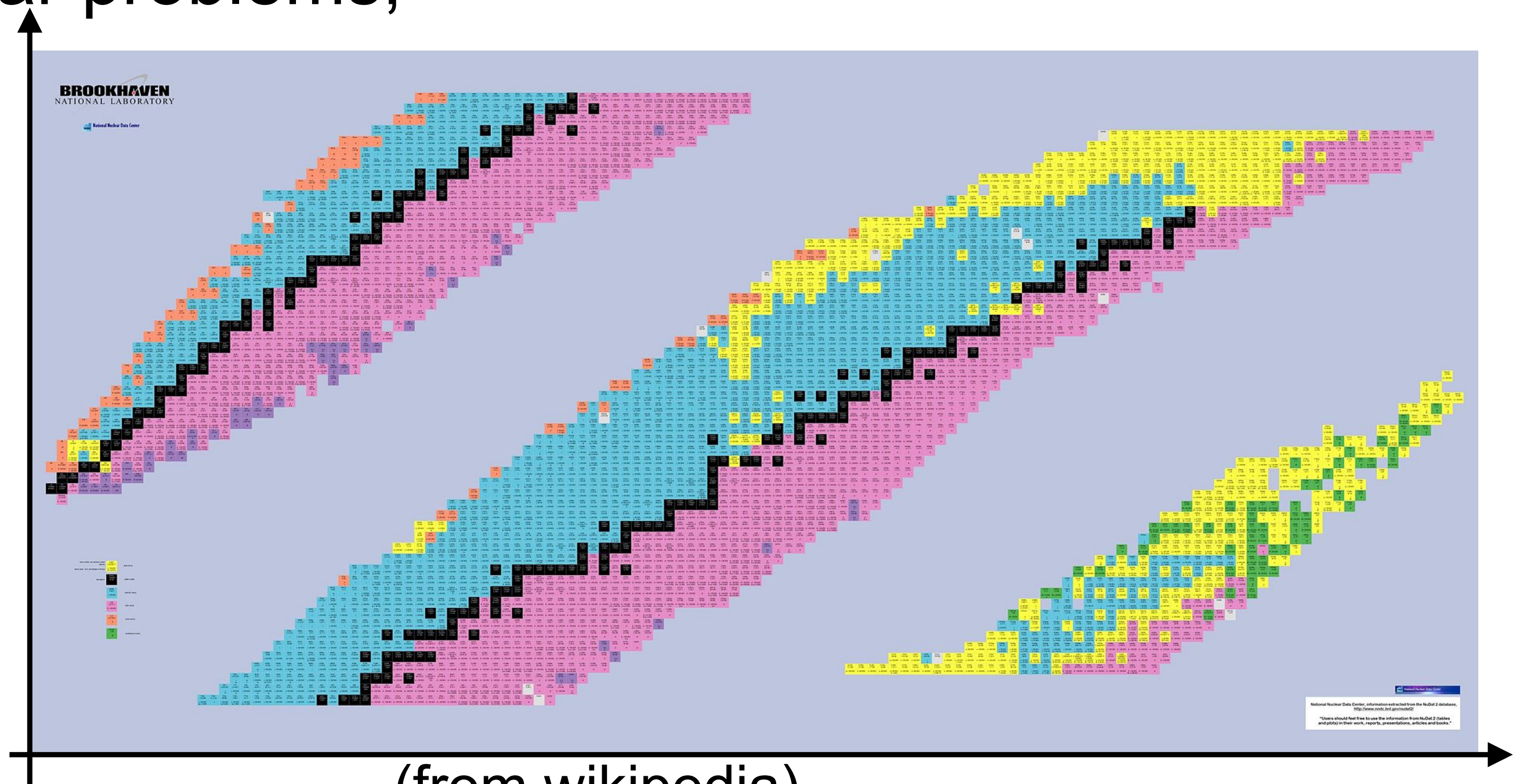
・diffusion eq= imaginary time Schrödinger eq

Motivation from genuinely Quantum systems

・Open quantum systems. The system evolution is not unitary in system-environment setup.

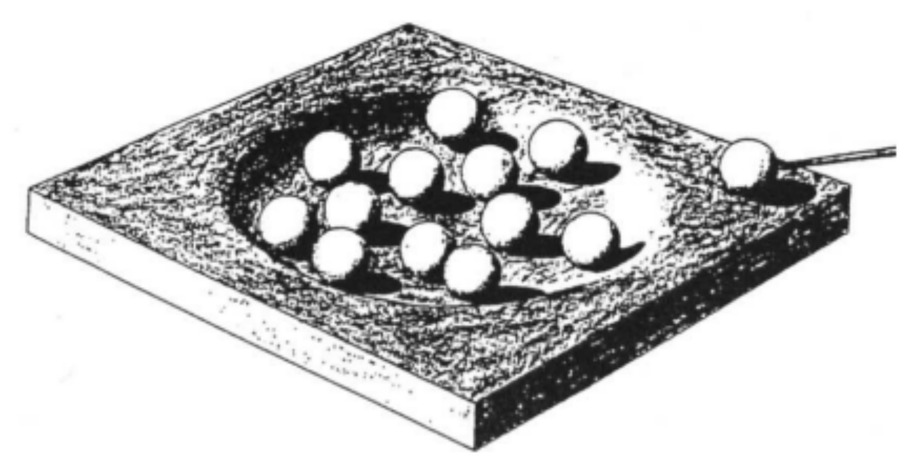
Open Sachdev-Ye-Kitaev

Again going back to the nuclear problems, many nuclei are resonances!



(from wikipedia)

It is natural to think about SYK model coupled to an environment and study the universality of level statistics!



many body: $\sum_{i<j;k<l} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_l$ + dissipation

Another motivation:

a boundary approach to open SPT phases

Non-Hermitian Random matrices

[Kawabata, Shiozaki, Ueda, Sato 19]

38 symmetry classification in non-Hermitian Hamiltonians by symmetry

(10 AZ + (10 - 4) AZ† + (3 + 24 - 5) AZ w/ SLS, we should remove overcounting)

We should be careful about anti-unitaries

Transpose and complex conjugate are different for Non-Hermitian matrices.

Symmetry class		TRS	PHS	TRS†	PHS†	CS
Complex AZ	A	0	0	0	0	0
	AIII	0	0	0	0	1
Real AZ	AI	$+1$	0	0	0	0
	BDI	$+1$	$+1$	0	0	1
	D	0	$+1$	0	0	0
	DIII	-1	$+1$	0	0	1
	AII	-1	0	0	0	0
	CII	-1	-1	0	0	1
	C	0	-1	0	0	0
	CI	$+1$	-1	0	0	1
Real AZ†	AI†	0	0	$+1$	0	0
	BDI†	0	0	$+1$	$+1$	1
	D†	0	0	0	$+1$	0
	DIII†	0	0	-1	$+1$	1
	AII†	0	0	-1	0	0
	CII†	0	0	-1	-1	1
	C†	0	0	0	-1	0
	CI†	0	0	$+1$	-1	1

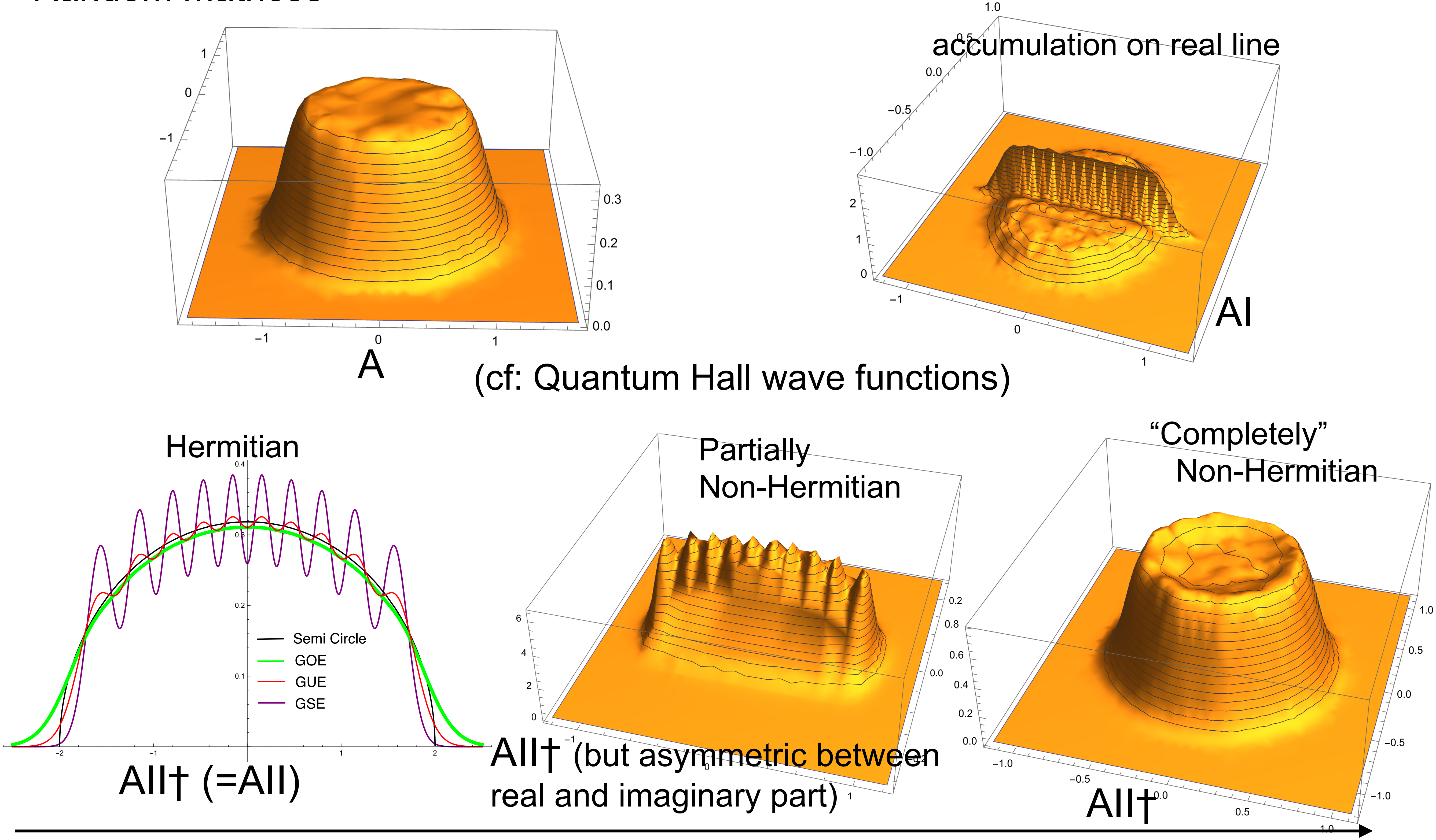
TRS	$\mathcal{T}H\mathcal{T}^{-1}=H$
PHS†	$\mathcal{C}H\mathcal{C}^{-1}=-H$
SLS	$\mathcal{S}H\mathcal{S}^{-1}=-H$
TRS†	$\mathcal{T}H^\dagger\mathcal{T}^{-1}=H$
PHS	$\mathcal{C}H^\dagger\mathcal{C}^{-1}=-H$
CS	$\Gamma H^\dagger\Gamma^{-1}=-H$

s	AZ class	$t=0$	$t=1$
0	A		$\mathcal{S}$
1	AIII	$\mathcal{S}_+$	$\mathcal{S}_-$

s	AZ class	$t=0$	$t=1$	$t=2$	$t=3$
0	AI		$\mathcal{S}_-$		$\mathcal{S}_+$
1	BDI	$\mathcal{S}_{++}$	$\mathcal{S}_{-+}$	$\mathcal{S}_{--}$	$\mathcal{S}_{+-}$
2	D		$\mathcal{S}_+$		$\mathcal{S}_-$
3	DIII	$\mathcal{S}_{--}$	$\mathcal{S}_{-+}$	$\mathcal{S}_{++}$	$\mathcal{S}_{+-}$
4	AII		$\mathcal{S}_-$		$\mathcal{S}_+$
5	CII	$\mathcal{S}_{++}$	$\mathcal{S}_{-+}$	$\mathcal{S}_{--}$	$\mathcal{S}_{+-}$
6	C		$\mathcal{S}_+$		$\mathcal{S}_-$
7	CI	$\mathcal{S}_{--}$	$\mathcal{S}_{-+}$	$\mathcal{S}_{++}$	$\mathcal{S}_{+-}$

Non-Hermitian Random matrices (density of states)

We can find some structures for the distribution/level statistics in non-Hermitian Random matrices

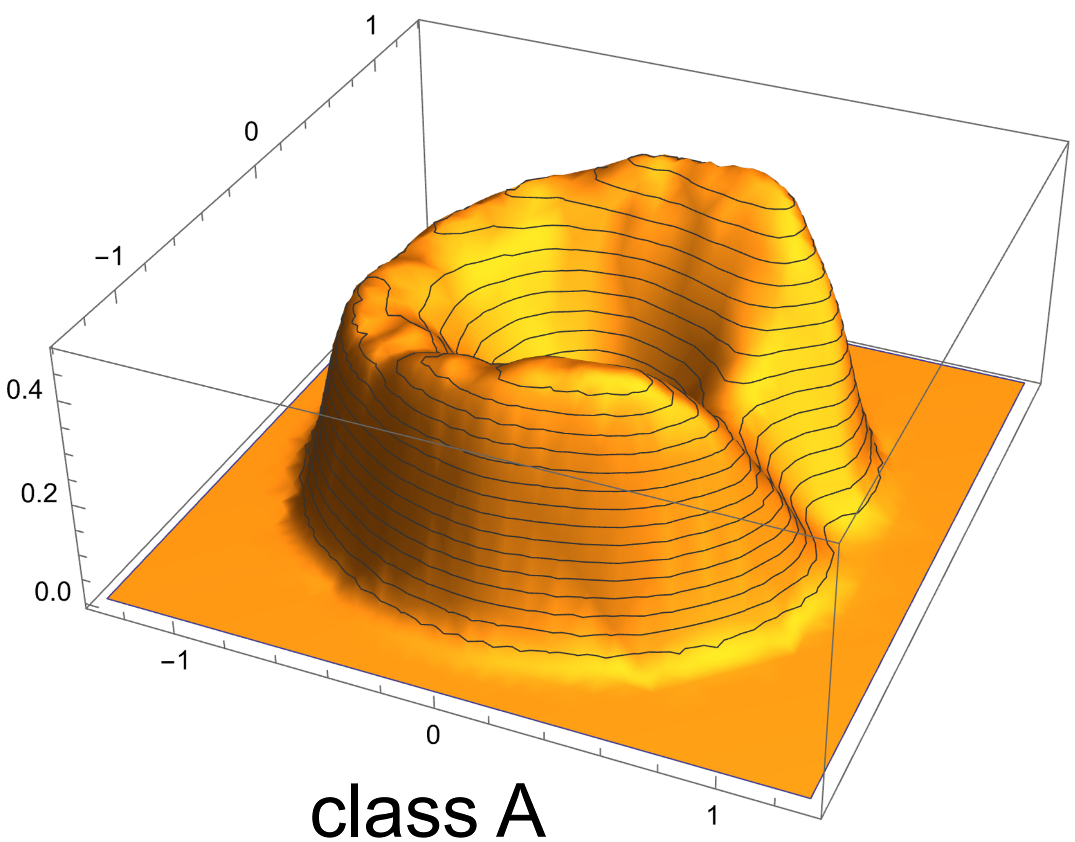


Non-Hermitian Random matrices (level statistics)

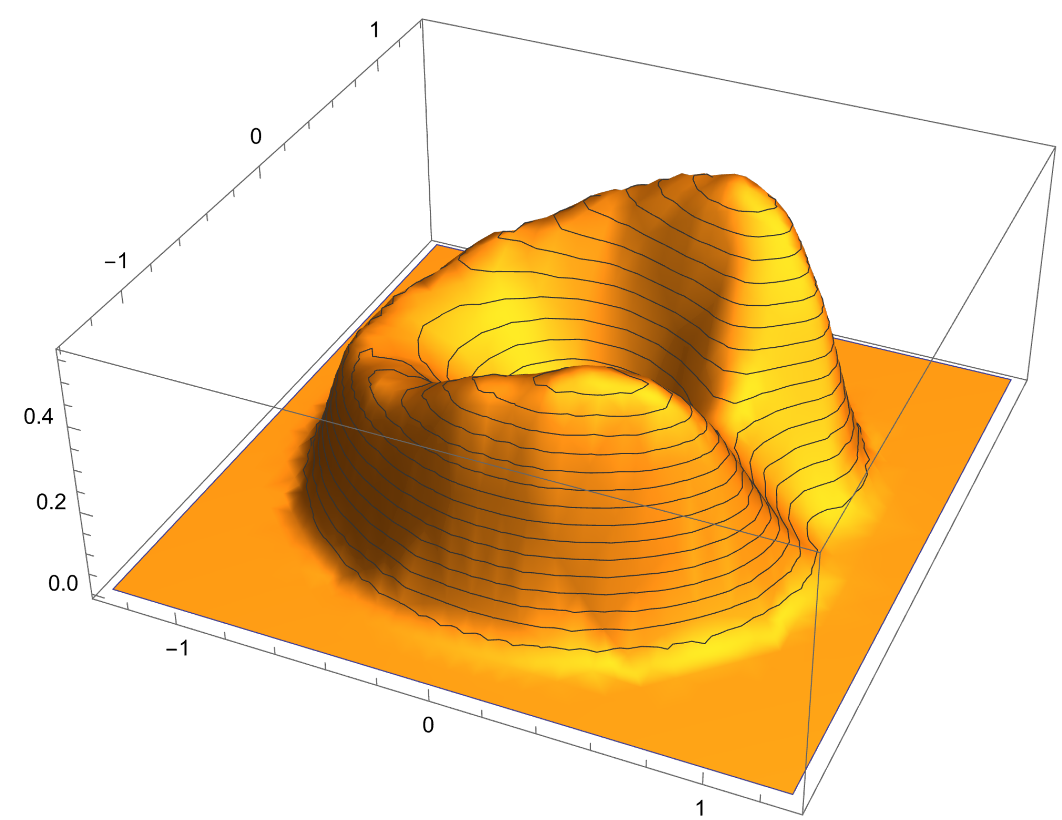
We can find some structures for the distribution/level statistics in non-Hermitian Random matrices

complex spectral gap ratio: $$z := \frac{\lambda - \lambda^{\mathrm{NN}}}{\lambda - \lambda^{\mathrm{NNN}}}$$ [Sa,Ribeiro,Prosen 19]

gives a quantitative measure of level statistics.



class A

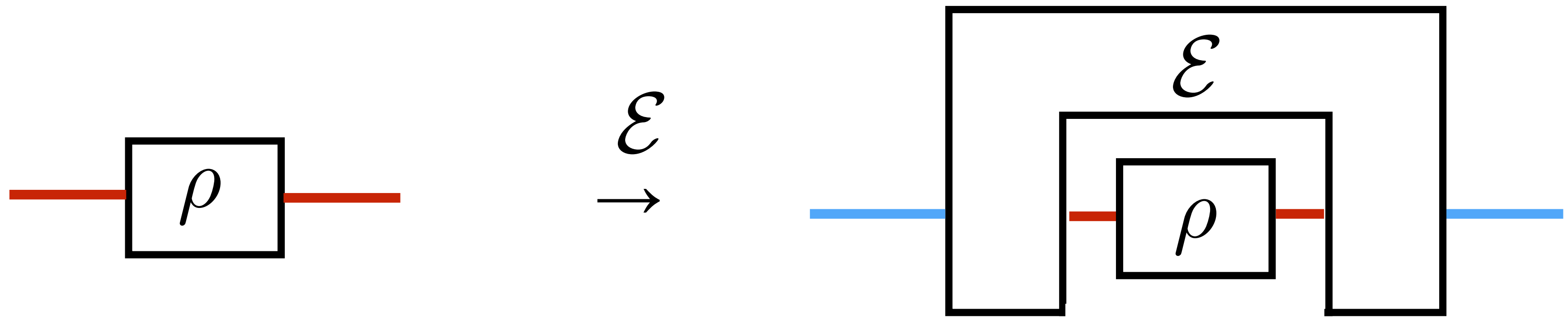


class $\mathrm{AII}^\dagger$

Quantum channels

Completely Positive (CP) and Trace Preserving (TP) (CPTP) map
state ρ in system A (with $\mathcal{H}_A$) to a state in system B (with $\mathcal{H}_B$)

Graphically,

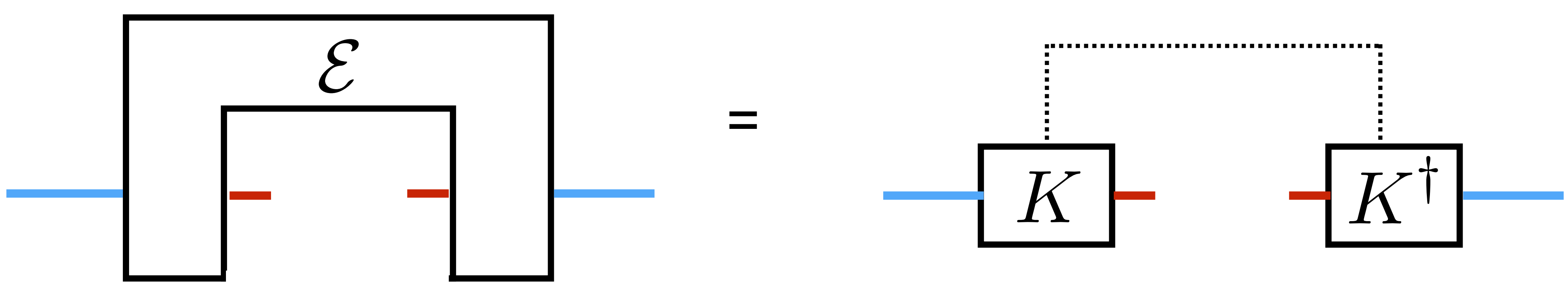


Trace preserving is required since finally we obtain a density matrix.

Complete positivity is needed to guarantee that we get positive operators even when the state is entangled with an environment.

They have so-called Kraus representation:

$$\mathcal{E}(\rho) = \sum_{\mu} K_{\mu} \rho K_{\mu}^{\dagger} \qquad \sum_{\mu} K_{\mu}^{\dagger} K_{\mu} = I$$



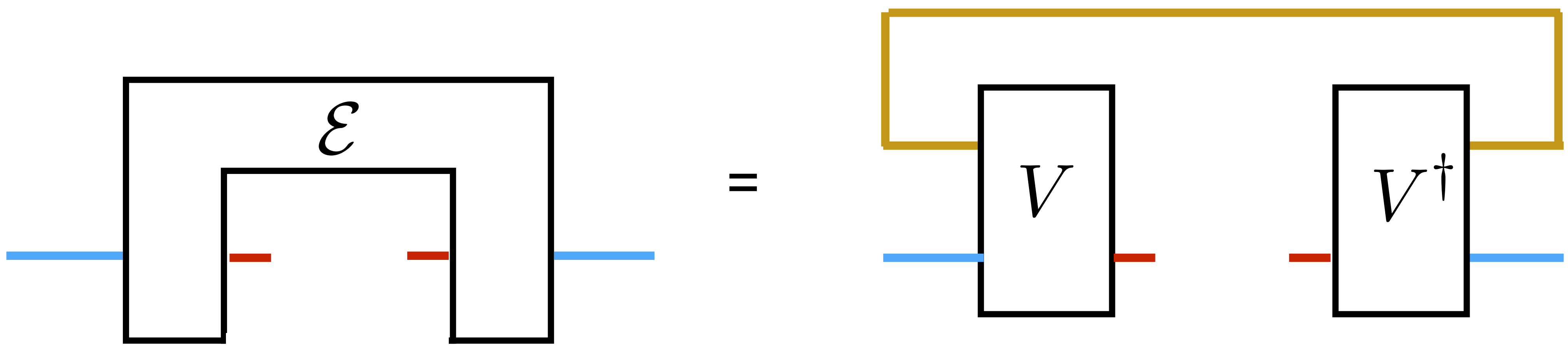
Steinspring representation

We can realize any CPTP map as a unitary conjugation for the system+ environment E

$$\mathcal{E}(\rho) = \mathrm{Tr}_E(V\rho V^\dagger)$$

We can realize all the quantum channels as unitary maps.

Graphically,

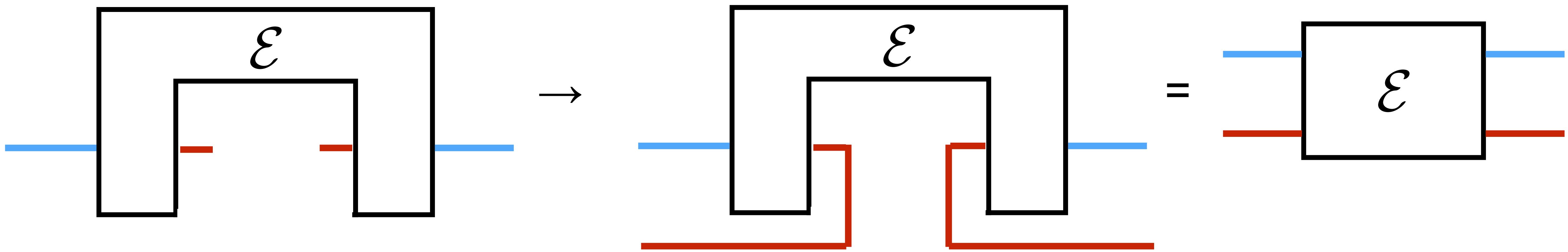


The choice of environments is *not unique*.

Choi isomorphism

Completely positivity is a positivity for entangled states.

Let us consider a maximally entangled $|I\rangle$ state between system & environment and apply a quantum channel: $\mathcal{E}(|I\rangle\langle I|) \ : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$

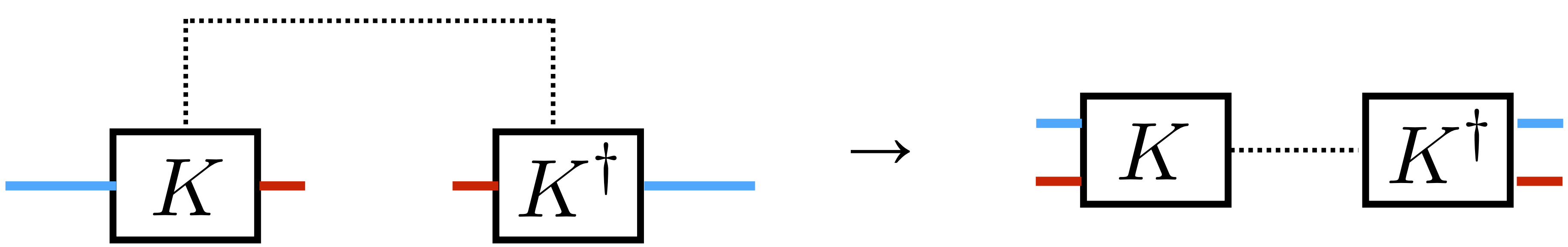


(flips depend on entangled states, but they are only different up to unitaries)

Completely positivity says that $\mathcal{E}(|I\rangle\langle I|)$ is a density matrix.

$$(\mathcal{E}(|I\rangle\langle I|))^\dagger = \mathcal{E}(|I\rangle\langle I|)$$

From the Kraus representation, this is manifest;

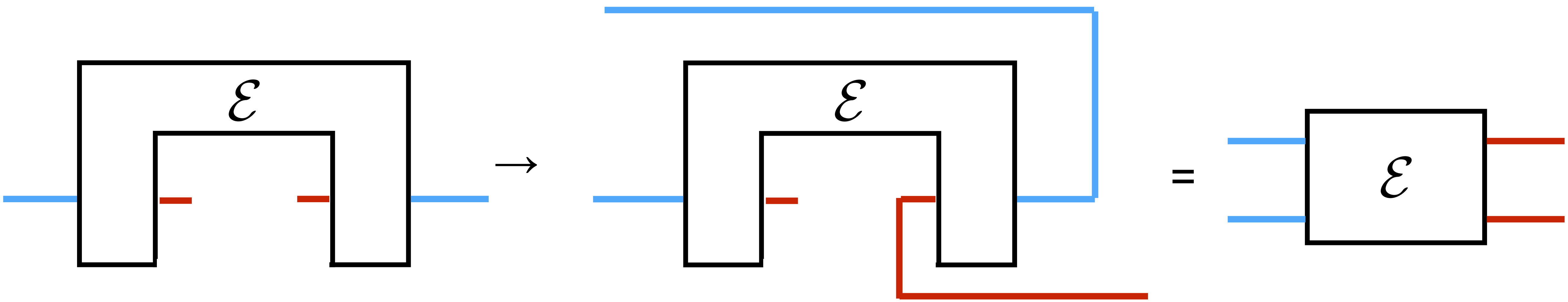


$$= \sum_\mu |K_\mu\rangle\langle K_\mu|$$

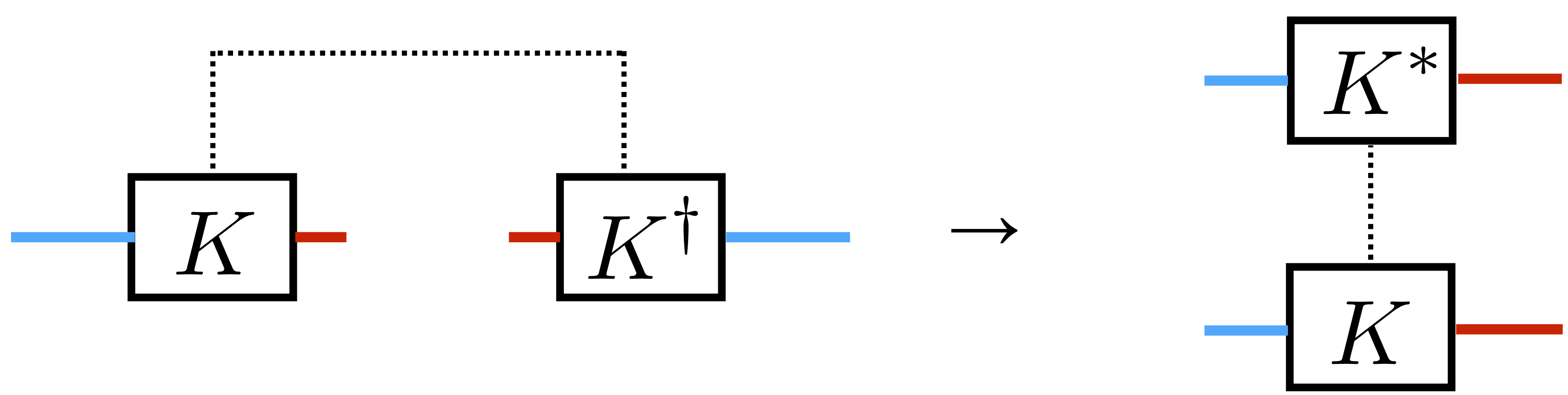
Vectorization

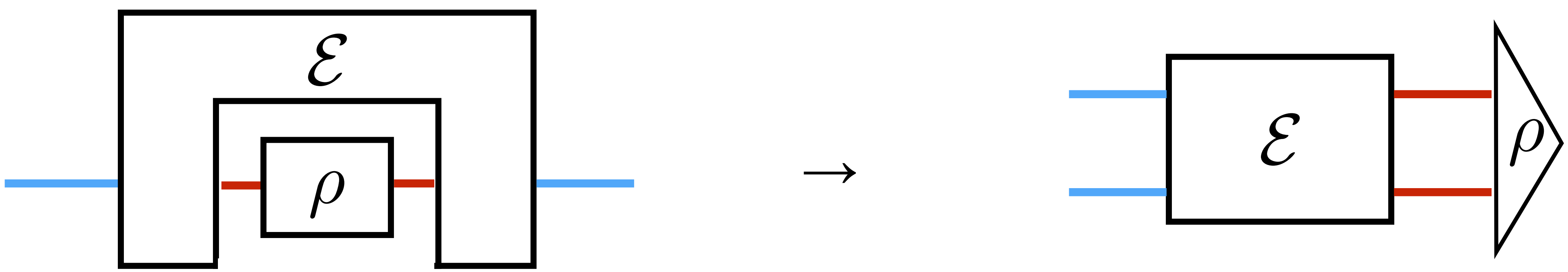
The Choi-isomorphism is not useful when we multiply channels.

Another representation: vectorization



$$\mathcal{E} : \mathcal{H}_A \otimes \mathcal{H}_A^* \to \mathcal{H}_B \otimes \mathcal{H}_B^*$$



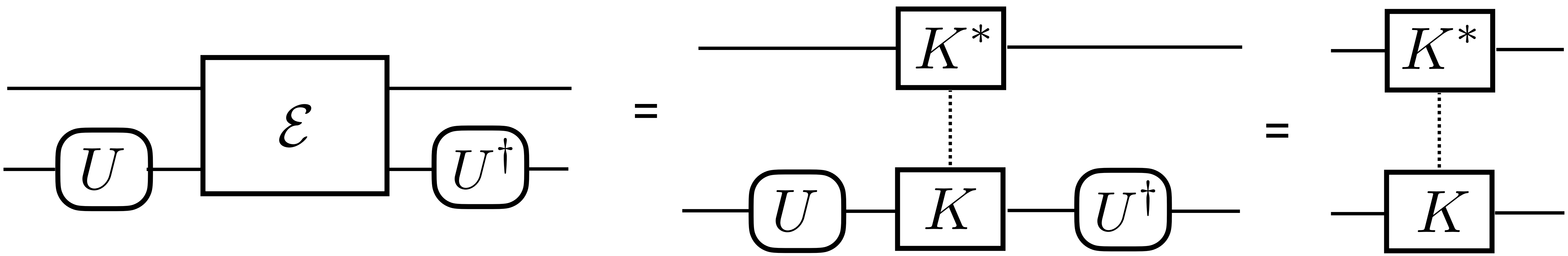


Strong and weak symmetry

assume $\mathcal{H}_A = \mathcal{H}_B$

- strong symmetry

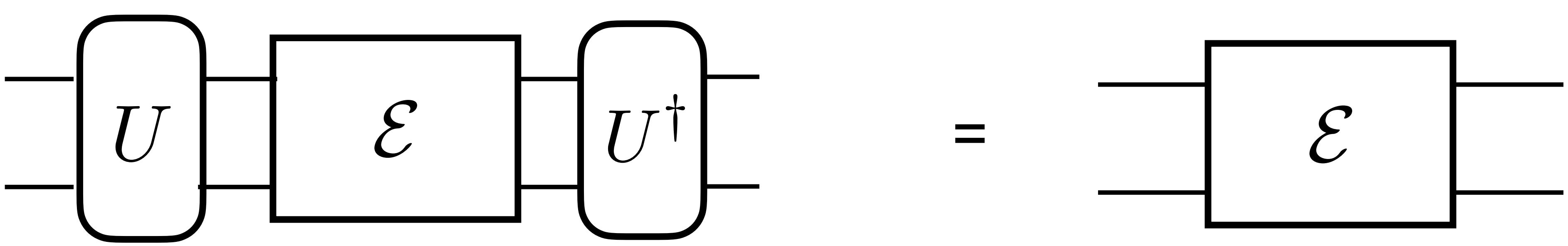
Symmetry of a Hamiltonian is $UHU^\dagger = H$. From this perspective,



is a natural symmetry for quantum channels
This is called strong symmetry.

- weak symmetry

On the other hand, when we think of quantum channels as a single operator



is a symmetry. This is called weak symmetry.

Modular conjugation symmetry

- After vectorization, hermitian conjugate is now represented by the modular conjugation:

$$\mathcal{J}\,|\rho\rangle = |\rho\rangle$$

(modular conjugation for a reference maximally entangled state, not for ρ)

- This is *anti-unitary* operator:

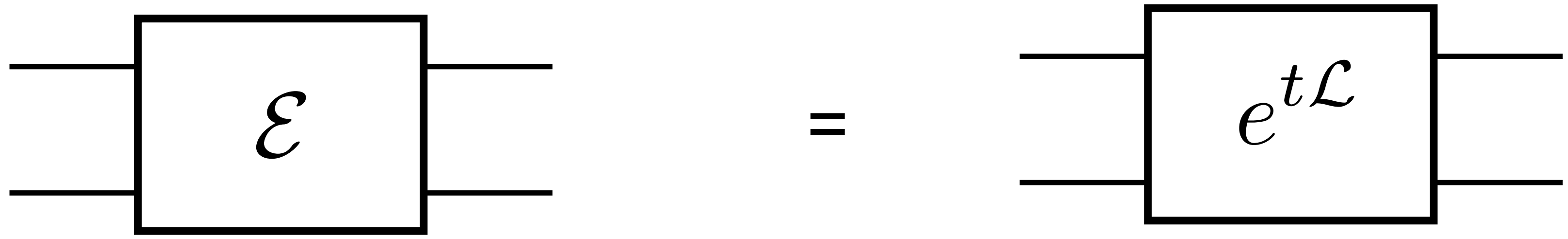
$$\mathcal{J}(A \otimes B)\mathcal{J}^{-1} = B^* \otimes A^*$$

- Any quantum channels are symmetric under the modular conjugation:

$$\mathcal{J}\sum_{\mu}(K_\mu \otimes K_\mu^*)\mathcal{J}^{-1} = \sum_{\mu}(K_\mu \otimes K_\mu^*)$$

Quantum Master equation (Lindblad equation, GKSL):

Let us consider a quantum channel which is described by the exponential of a generator $\mathcal{L}$:



These are called Lindblad equation (GKSL or quantum master equation)

[Lindblad, 76] [Gorini-Kossakowski-Sudarshan, 76]

before vectorization

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \sum_k \left(L_k \rho(t) L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho(t) - \frac{1}{2}\rho(t) L_k^\dagger L_k \right)$$

after vectorization

$$\mathcal{L} = -iH_+ + iH_- + \sum_k L_{k+} \otimes L_{k-}^* - \frac{1}{2}\sum_k L_{k+}^\dagger L_{k+} \mathbb{I}_- - \frac{1}{2}\mathbb{I}_+ \otimes \sum_k L_{k-}^T L_{k-}^*$$

L_k : jump operators

$\mathcal{L}$:Non-Hermitian Hamiltonian on doubled Hilbert space $\mathcal{H}_+ \otimes \mathcal{H}_-$

SYK Lindbladian:

[Kulkarni- **TN** - Ryu, 21]

$$H = H_{SYK}$$

$$L_m = \sum_{i_1 < \cdots < i_p} K_{m,i_1\cdots i_p} \psi_{i_1} \cdots \psi_{i_p} \qquad K_{m,i_1\cdots i_p} \in \mathbb{C}$$

: p-body Jump operators.

We have to double the Hilbert space to vertorize:

ψ_i^+ , ψ_i^- : in total N + N = 2N Majorana fermions.

For example, for $L_i = \sqrt{\mu}\psi^i$

$$\mathcal{L} = -iH^+_{SYK} + i(-1)^{\frac{q}{2}} H^-_{SYK} - i\mu \sum_i \psi^i_+ \psi^i_- - \mu \frac{N}{2} \mathbb{I}_+ \otimes \mathbb{I}_-$$

the reference maximally entangled state (= infinite temp state):

$$\mathcal{L} \left| I \right\rangle = 0 \qquad \psi^i_+ \left| I \right\rangle = -i\psi^i_- \left| I \right\rangle$$

(* we should be careful about tensor product and transpose for fermions or anyons)

[cf: Shiozaki Shapoulian Ryu, 16] [Shapoulian Mong Ryu, 20]
[Yoshida Kudo Katsura Hatsugai, 20]

Symmetry of SYK Lindbladian

Symmetry of the SYK Lindbladian:

[Kawataba,Kulkarni,**TN**, Li, Ryu 22]

always symmetries of Lindbladians

Modular conjugation $\mathcal{J}\psi_i^{\pm}\mathcal{J}^{-1} = \psi_i^{\mp}$ $\quad \mathcal{J}z\mathcal{J}^{-1} = z^*$

weak Fermion Parity $(-1)^{\mathcal{F}}\psi_i^{\pm}(-1)^{\mathcal{F}} = -\psi_i^{\pm}$

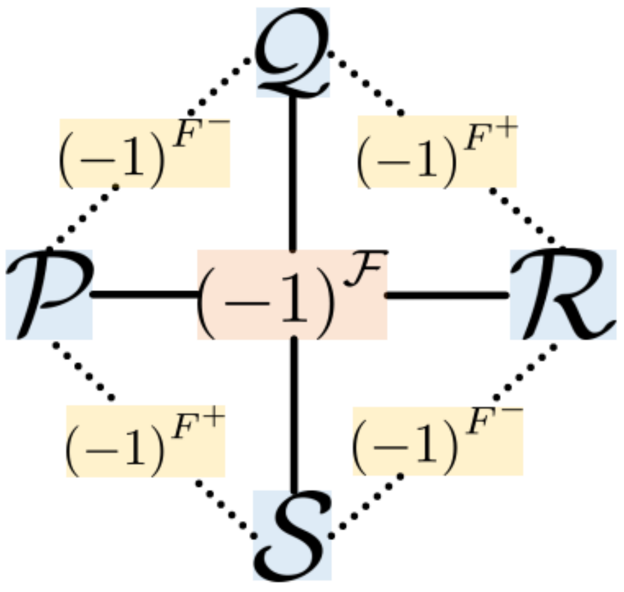
depend on choices of Jump terms/ Hamiltonians

strong Fermion Parity $(-1)^{F_+}\psi_i^{+}(-1)^{F_+} = -\psi_i^{+}$ $\quad (-1)^{F_+}\psi_i^{-}(-1)^{F_+} = \psi_i^{-}$

anti-unitary symmetry $\mathcal{R}\psi_i^{\pm}\mathcal{R}^{-1} = \psi_i^{\pm}$ $\quad \mathcal{R}z\mathcal{R}^{-1} = z^*$

$$\mathcal{R}e^{\mathcal{L}t}\mathcal{R}^{-1} = e^{(\mathcal{R}\mathcal{L}\mathcal{R}^{-1})t}$$: not reverse a time

Similarly to the SYK time reversal, we can combine the fermion parity symmetries to form another anti-unitary symmetry:



We did not find how to define weak anti-unitary symmetry

Algebra of Symmetries

[Kawataba,Kulkarni,**TN,** Li, Ryu 22]

The algebra only depends on $N \mod 4$, in contrast to $N \mod 8$ in SYK

$$\mathcal{J}(-1)^{\mathcal{F}} = a(-1)^{\mathcal{F}}\mathcal{J} \qquad \mathcal{J}\mathcal{R} = b\mathcal{R}\mathcal{J}$$

$N \pmod 4$	0	1	2	3
a	$+1$	-1	$+1$	-1
b	$+1$	$+1$	-1	-1
$\mathcal{R}^2$	$+1$	$+1$	-1	-1

some anti-unitaries are always a symmetry of the SYK Lindbladian

(SYK Hamiltonian is always time reversal invariant or flip the sign)

On the other hand, the strong fermion parity is not always a symmetry

Ex) $L_i = \sqrt{\mu}\psi^i$ model is not invariant under strong Fermion Parity

$L^a = \sum_{i<j} K^a_{ij}\psi^i\psi^j$ model is invariant under strong Fermion Parity

Table of Symmetry classification of Lindbladian (1)

p-body dissipation: $$L_m = \sum_{1 \le i_1 < \cdots < i_p \le N} K_{m;i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p}$$

$p = 1$

$N \pmod 4$	0	1	2	3
fermion parity $(-1)^{\mathcal{F}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
modular conjugation $\mathcal{J}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{P} = \mathcal{R}(-1)^{\mathcal{F}}$	$+1$	0	-1	0
$\mathcal{Q} = \mathcal{R}(-1)^{F^-}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{R}$	$+1$	0	-1	0
$\mathcal{S} = \mathcal{R}(-1)^{F^+}$	$+1$	$+1$	$+1$	$+1$
$q \equiv 0 \pmod 4$ $[K_{m;i}K^*_{m;j} \notin \mathbb{R}]$	$\mathrm{AI} = \mathrm{D}^\dagger$	$\mathrm{AI} = \mathrm{D}^\dagger$	$\mathrm{AI} = \mathrm{D}^\dagger$	$\mathrm{AI} = \mathrm{D}^\dagger$
$q \equiv 0 \pmod 4$ $[K_{m;i}K^*_{m;j} \in \mathbb{R}]$	$\mathrm{BDI}^\dagger + \mathcal{S}_{++}$	$\mathrm{BDI}^\dagger$	$\mathrm{CI}^\dagger + \mathcal{S}_{+-}$	$\mathrm{BDI}^\dagger$
$q \equiv 2 \pmod 4$	BDI	$\mathrm{AI} = \mathrm{D}^\dagger$	CI	$\mathrm{AI} = \mathrm{D}^\dagger$

$p \ge 3$ $\quad \mathcal{Q}$ is a symmetry but $\mathcal{P}$ is not.

this term vanishes. Consequently, we have the following symmetry classification for odd $p \ge 3$: for $q \equiv 0 \pmod 4$ with $K_{m;i}K^*_{m;j} \notin \mathbb{R}$ and $q \equiv 2 \pmod 4$, the symmetry class is class AI (or equivalently class $\mathrm{D}^\dagger$) for arbitrary N; for $q \equiv 0 \pmod 4$ with $K_{m;i}K^*_{m;j} \in \mathbb{R}$, the symmetry class is class $\mathrm{BDI}^\dagger$ for arbitrary N.

Table of Symmetry classification of Lindbladian (2)

p-body dissipation: $$L_m = \sum_{1 \le i_1 < \cdots < i_p \le N} K_{m;i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p}$$

even p

N (mod 4)	0	1	2	3
fermion parity $(-1)^{\mathcal{F}}$, $(-1)^{F^{\pm}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$
modular conjugation $\mathcal{J}$	+1	+1	0	+1
$\mathcal{P}$	+1	0	0	0
$\mathcal{Q}$	+1	+1	0	+1
$\mathcal{R}$	+1	0	0	0
$\mathcal{S}$	+1	+1	0	+1
$q \equiv 0$ (mod 4) $[K_{m;i} K^*_{m;j} \notin \mathbb{R}]$, $q \equiv 2$ (mod 4)	AI=D†	AI=D†	A	AI=D†
$q \equiv 0$ (mod 4) $[K_{m;i} K^*_{m;j} \in \mathbb{R}]$	BDI†	BDI†	A + η = AIII	BDI†

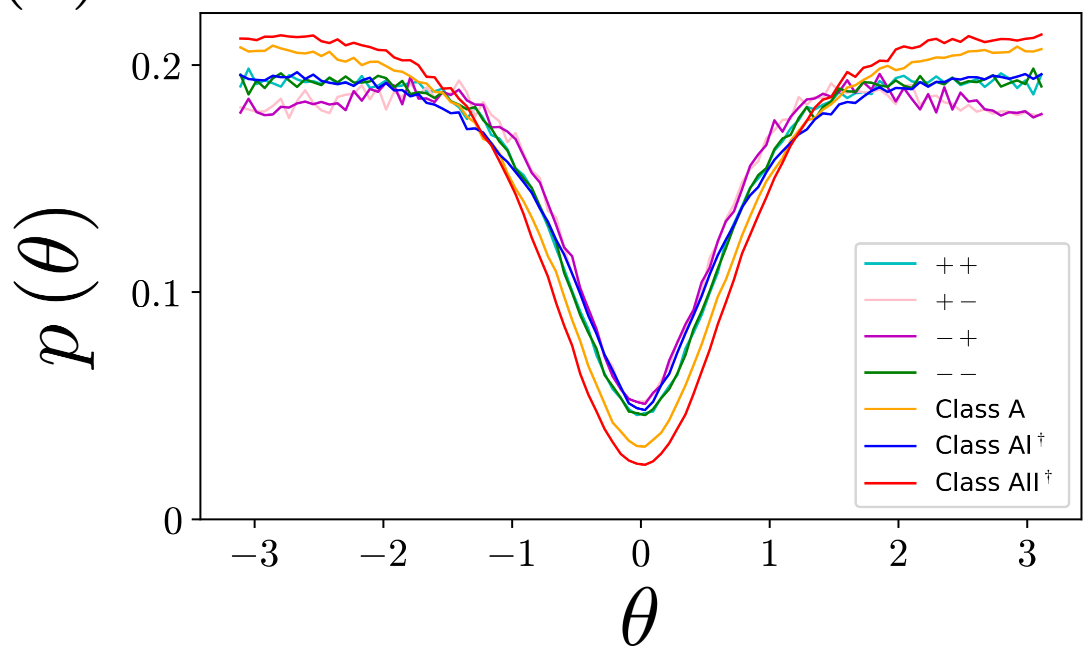
Symmetry and Dissipative many body Chaos: bulk level statistics

complex spectral gap ratio: $z := \dfrac{\lambda - \lambda^{\mathrm{NN}}}{\lambda - \lambda^{\mathrm{NNN}}}$ [Sa,Ribeiro,Prosen 19]

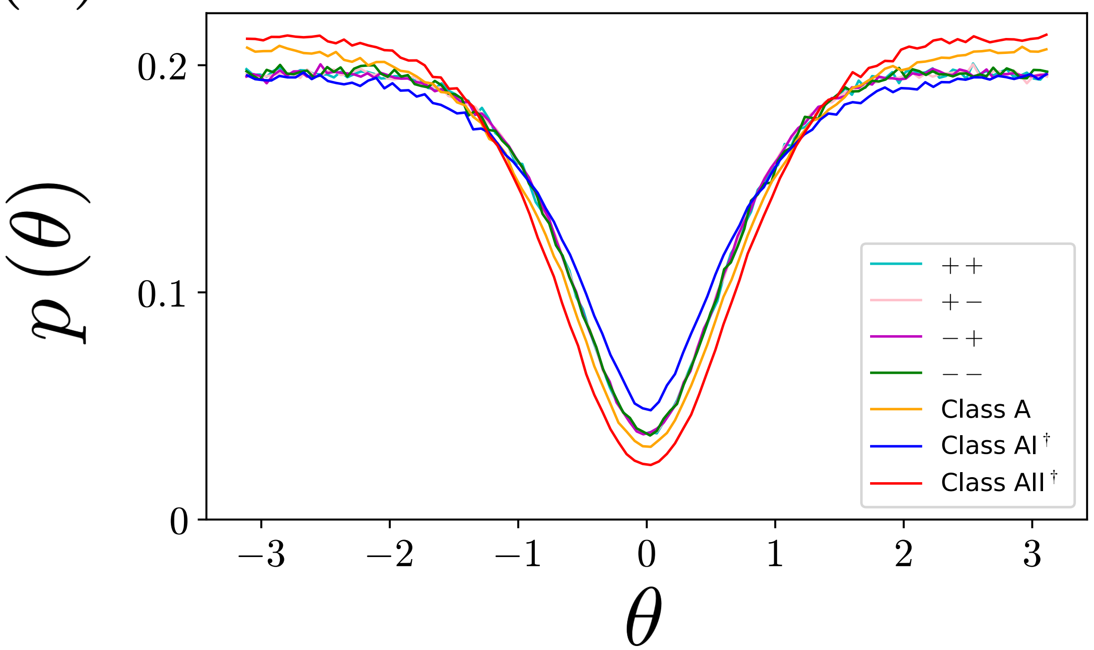
distribution $p(r,\theta)$ of $z = re^{i\theta}$

angle distribution $p(\theta) := \int p(r,\theta)dr$

(a) $p=2 \quad N=10$ (bulk class A) (b) $p=2 \quad N=12$ (bulk class AI)



discrepancy



good agreement

perhaps because Lindbladian is not completely random and see the transition

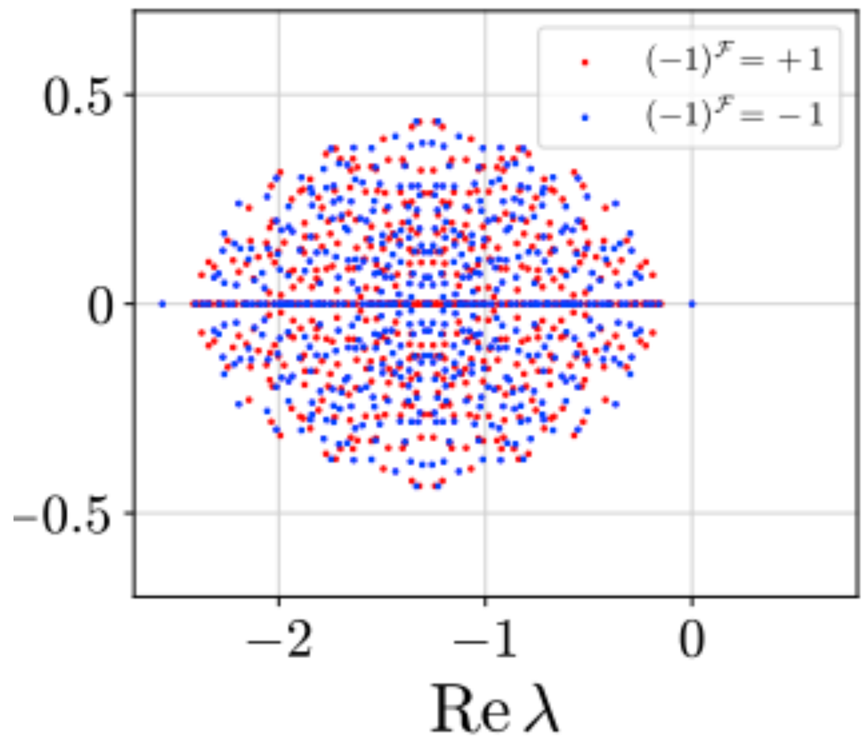
[cf: Garcia-Garcia Loureiro Romero-Bermudez Tezuka, 17]

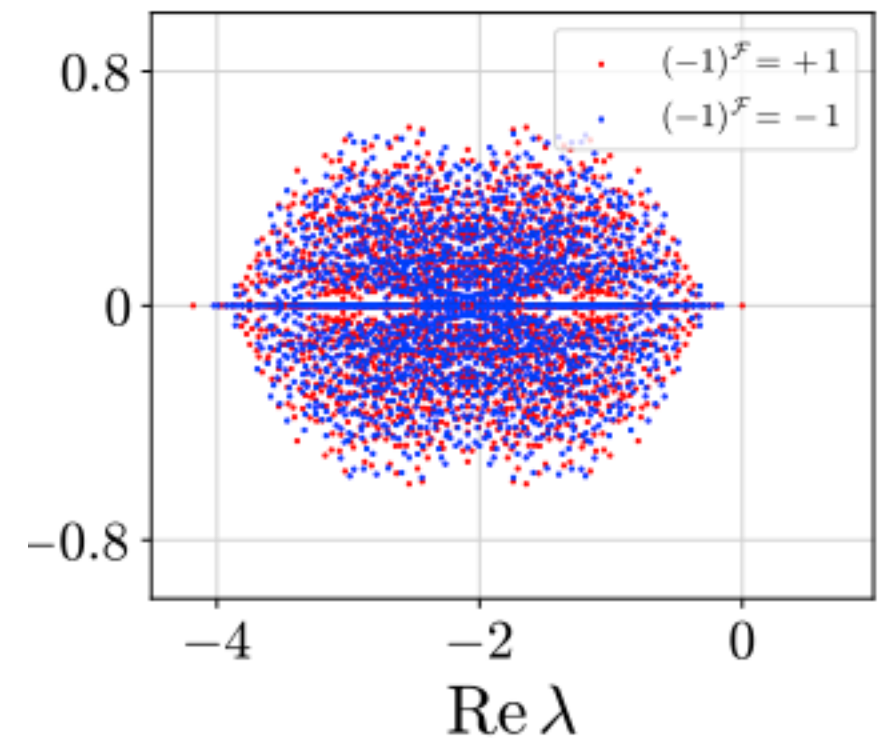
Symmetry and Dissipative many body Chaos: edge level statistics

N=10, p=1 (BDI+ $\mathcal{S}_{-+}$)

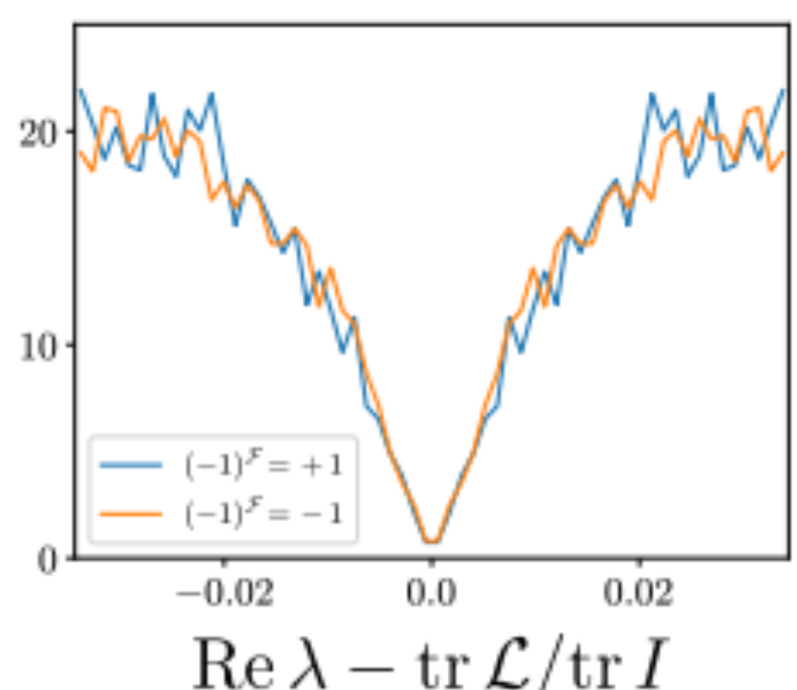
N=12, p=1 (BDI + $\mathcal{S}_{++}$)

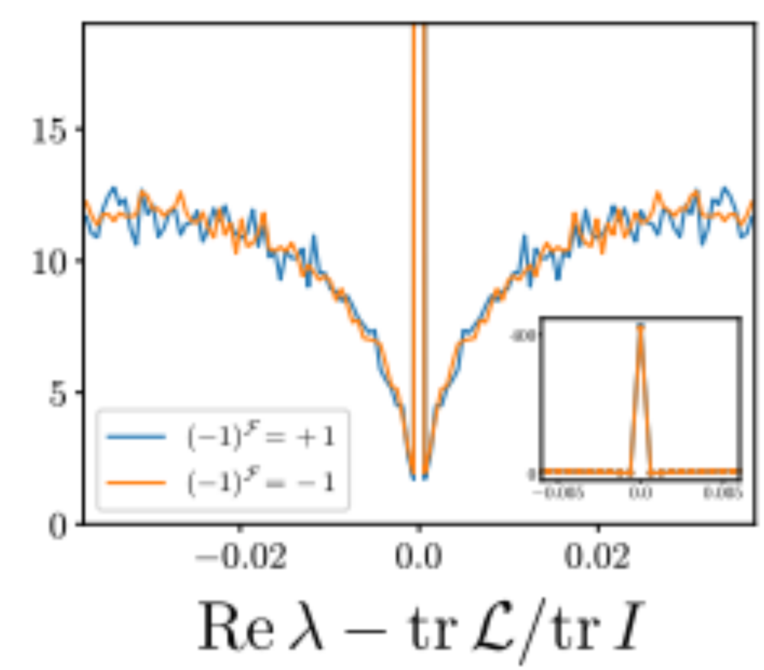
DoS





real axis



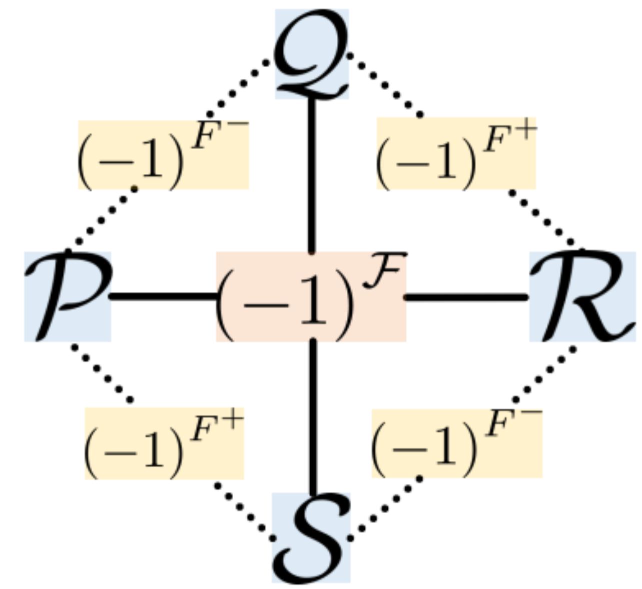


Summary

Symmetry of SYK Lindbladian

$$\mathcal{J}(-1)^{\mathcal{F}} = a(-1)^{\mathcal{F}}\mathcal{J} \qquad \mathcal{J}\mathcal{R} = b\mathcal{R}\mathcal{J}$$

N (mod 4)	0	1	2	3
a	$+1$	-1	$+1$	-1
b	$+1$	$+1$	-1	-1
$\mathcal{R}^2$	$+1$	$+1$	-1	-1



Table

$$p = 1$$

N (mod 4)	0	1	2	3
fermion parity $(-1)^{\mathcal{F}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
modular conjugation $\mathcal{J}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{P}$	$+1$	0	-1	0
$\mathcal{Q}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{R}$	$+1$	0	-1	0
$\mathcal{S}$	$+1$	$+1$	$+1$	$+1$
$q \equiv 0$ (mod 4) $[K_{m;i}K^*_{m;j} \notin \mathbb{R}]$	AI $=$ D†	AI $=$ D†	AI $=$ D†	AI $=$ D†
$q \equiv 0$ (mod 4) $[K_{m;i}K^*_{m;j} \in \mathbb{R}]$	BDI† $+ \mathcal{S}_{++}$	BDI†	CI† $+ \mathcal{S}_{+-}$	BDI†
$q \equiv 2$ (mod 4)	BDI	AI $=$ D†	CI	AI $=$ D†

+p>1

Summary of lectures

・Review of AZ class in random matrix context: $\mathcal{T}$, $\mathcal{P}$, $\mathcal{C}$ determine Cartan class.
symmetry breaking and enhancement fixed point determines universal behaviors.

・Review of the (SUSY) SYK. Realize all the AZ class.
Relation to Majorana fermions and AZ becomes manifest.

・Introduce our SYK Lindbladian model. Natural to study level statistics in the context of nuclear physics and
Anti-unitaries and non-Hermitian AZ class. Reflected to level statistics.

Future problems

・Complete level statistics of non-Hermitian random matrices.
Characterization by symmetry pattern when two eigenvalues collide.

・Understand 38 from Majorana fermions.

・Classification of ("anyonic") quantum channels.

・Maybe related to exceptional groups, coadjoint orbit, etc…

thank you !

Time reversal symmetry

・It is *not* linear, but an *anti-linear* operator.

Dirac's bra-ket notation becomes confusing for non-linear operator.

・A useful braket notation for non-linear operators is

$$A(|\psi\rangle) \equiv |A\psi\rangle$$

An anti-linear operator satisfies

$$A(a\,|\psi\rangle + b\,|\phi\rangle) = a^*\,|A\psi\rangle + b^*\,|A\phi\rangle$$

・The adjoint of an anti-linear operator is defined by

$$\langle\phi|A^\dagger\psi\rangle = \langle\psi|A\phi\rangle$$

(This condition guarantees the anti-unitarity of $A^\dagger$)

・An anti-unitary operator Θ satisfies

$$\langle\Theta\phi|\Theta\psi\rangle = \langle\psi|\phi\rangle$$

In particular, they do not change the norm.

Time reversal symmetry on linear operators

$\mathcal{T} H \mathcal{T}^\dagger$ is a linear operator since

$$\mathcal{T}(H\,|\mathcal{T}^\dagger(a\psi_1 + b\psi_2)\rangle) = a\mathcal{T}(H\,|\mathcal{T}^\dagger\psi_1\rangle) + b\mathcal{T}(H\,|\mathcal{T}^\dagger\psi_2\rangle)$$

A relation between H and $\mathcal{T} H \mathcal{T}^\dagger$ is

$$\mathcal{T} H \mathcal{T}^\dagger = \sum_{m,n} \langle e_m|H|e_n\rangle^* \,|\mathcal{T} e_m\rangle \langle \mathcal{T} e_n|$$

for a basis $|e_n\rangle$.

[see Harlow-TN 23]

Spin(2)

- Isomorphic to U(1)

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \leftrightarrow \quad \begin{aligned} &e^{i\frac{\theta}{2}}(x+iy)e^{i\frac{\theta}{2}} \\ &= e^{i\theta}(x+iy) \end{aligned}$$

SO(2) is represented as a 1×1 d matrix conjugation.

Therefore we say that the vector representation of spin(2) = U(1) appears as the symmetric tensor product of chiral spinor representation.

$$\gamma^1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \Sigma^{xy} = \frac{i}{4}[\gamma^x, \gamma^y] = -\frac{1}{2}\sigma^z$$

2×2 Dirac spinor transforms as $\phi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} \to \exp(i\theta\Sigma^{xy})\phi = \begin{pmatrix} e^{-i\frac{\theta}{2}}\psi \\ e^{i\frac{\theta}{2}}\chi \end{pmatrix}$

$$\phi'^{\dagger}(\gamma^x + i\gamma^y)\phi = \begin{pmatrix} \bar{\psi}' & \bar{\chi}' \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = 2\bar{\psi}'\chi$$

- the space of 2×2 class CI ensemble.

Spin(1)

- Isomorphic to $\mathbb{Z}_2 = O(1)$
- for any real number x, $x = e^{i\theta} x e^{i\theta}$ with $\theta = 0, \pi$
- gamma matrices are 1 dimensional,

$$(\gamma^x)^2 = 2 \qquad \gamma^x = \pm\sqrt{2}$$

(two different representations depending on the sign)

- spinors are real 1 dimensional, $\phi \rightarrow e^{i\theta}\phi$

Symmetric tensor of two spinors are

$$\phi' \gamma^x \phi \rightarrow e^{2i\theta} \phi' \gamma^x \phi = \phi' \gamma^x \phi$$

A trivial example of class AI

Spin(3)

・Isomorphic to SU(2)

$$\boldsymbol{x}\cdot\boldsymbol{\sigma} \quad , \qquad u(\boldsymbol{x}\cdot\boldsymbol{\sigma})u^\dagger \qquad\qquad u \in SU(2)$$

gives a 3d rotation.

Fundamental representation of SU(2) is a spin 1/2 representation of SO(3)
$\boldsymbol{x}\cdot\boldsymbol{\sigma}$ is symmetric matrix.

relation to spinor is

$$\gamma^x = \sigma^x \qquad \gamma^y = \sigma^y \qquad \gamma^z = \sigma^z$$

$$\boldsymbol{x}\cdot\boldsymbol{\sigma} = \sum_i x_i \gamma^i$$

・This is the simplest example of class C

Spin(4)

- Isomorphic to SU(2)×SU(2)

$$h = x_0 + i\boldsymbol{x}\cdot\boldsymbol{\sigma} \quad , \qquad uhv^\dagger \qquad\qquad u, v \in SU(2)$$

gives a 4d rotation.

h is naturally identified with a quaternion.

u,v is then naturally identified with a quaternion with unit norm.

SU(2) = Sp(1) is a more correct interpretation in this context.

The diagonal rotation $u = v$ gives spin(3).
In this sense spin(3) = Sp(1) is better identification.

$$\gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad x_0\gamma^4 + \sum_i x_i\gamma^i = \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}$$

This is the simplest chiral GSE matrix of 4×4.

Spin(5)

- Isomorphic to Sp(2)

$$H = \begin{pmatrix} s & a \\ \bar{a} & -s \end{pmatrix} = \begin{pmatrix} sI & a_4 + i\boldsymbol{a}\cdot\boldsymbol{\sigma} \\ a_4 - i\boldsymbol{a}\cdot\boldsymbol{\sigma} & -sI \end{pmatrix} \qquad U = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \in Sp(2)$$

$UHU^\dagger$ gives a 5d rotation.

$U = \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix}$ gives a 4d rotation before.

$$\gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$H = \sum a_i \gamma^i + a_4\gamma^4 + a_5\gamma^5$$

This is the simplest non trivial GSE matrix of 4×4.

Spin(6)

- Isomorphic to SU(4)

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & \bar{E}_3 & -\bar{E}_2 \\ -E_2 & -\bar{E}_3 & 0 & \bar{E}_1 \\ -E_3 & \bar{E}_2 & -\bar{E}_1 & 0 \end{pmatrix}$$

with the norm $\frac{1}{4}\mathrm{Tr}(FF^\dagger) = |E_1|^2 + |E_2|^2 + |E_3|^2$

$$F \to UFU^T$$

gives a 6d rotation in $(x_1, x_2, x_3, x_4, x_5, x_6) \in \boldsymbol{R}^6$ identified with

$$(E_1, E_2, E_3) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6)$$

First we start with $F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$ and then impose invariance

under the duality and complex conjugate $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{F}^{\rho\sigma}$ ($\boldsymbol{E} = \bar{\boldsymbol{B}}$)

Weyl Spinor is $\mathbf{4}$ of $SU(4)$. F is manifestly anti-symmetric.

Therefore in 6d the vector appears in the anti-symmetric tensor of Weyl spinor.

Spin(8)

- There is a manifestly real representation of the gamma matrices

$$\gamma^1 = \sigma^x \otimes I \otimes I \otimes I \qquad \gamma^5 = \sigma^y \otimes \sigma^x \otimes I \otimes \sigma^y$$
$$\gamma^2 = \sigma^z \otimes I \otimes I \otimes I \qquad \gamma^6 = \sigma^y \otimes \sigma^z \otimes I \otimes \sigma^y$$
$$\gamma^3 = \sigma^y \otimes \sigma^y \otimes \sigma^x \otimes I \qquad \gamma^7 = \sigma^y \otimes I \otimes \sigma^y \otimes \sigma^x$$
$$\gamma^4 = \sigma^y \otimes \sigma^y \otimes \sigma^z \otimes I \qquad \gamma^8 = \sigma^y \otimes I \otimes \sigma^y \otimes \sigma^z$$

Because of this, reality of spin(d+8) agrees with that of spin(d)

(Bott periodicity)

- both Weyl spinors and vectors are 8 dimensional because $2^{\frac{d-1}{2}} = d$.

there is a triality transformation that relates T_{ij} and $\frac{1}{4}[\gamma^i, \gamma^j]$

In that sense, SO(8) spinor coincides with the fundamental rep of itself!

$$T_{12} \to \frac{1}{2}(T_{12} + T_{34} + T_{56} + T_{78})$$ + 27 relations

- This representation is also practically useful to numerical study the SYK for N = 8k.

Spin(7)

- There is an special embedding of spin(7) into spin(8) combining the standard embedding $SO(7) \subset SO(8)$ and the triality

Using this we can find a spin(7) singlet in SO(8) spinor: $\mathbf{8}_s \to \mathbf{1} + \mathbf{7}$

Fidkowski and Kitaev used this fact to construct a gapped pass between 8 layer of SPTs and trivial states [Fidkowski Kitaev, 09].

(spin(7) Casimir is their Hamiltonian)

In our case we can use this map to get SO(7) spinor from SO(8) vectors.

$$\text{SO(8)} \quad \mathbf{8}_v \otimes \mathbf{8}_v \to \mathbf{1} + \mathbf{28} + \mathbf{35}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\text{spin(7)} \quad \mathbf{8} \otimes \mathbf{8} \qquad \mathbf{1} + \boxed{\mathbf{7}} + \mathbf{21} + \cdots$$

Since SO(8) vector is real, spin(7) is also real .

Since $\mathbf{7}$ comes from 8d anti-symmetric matrix $\mathbf{28}$, $(\gamma^\mu C)$ is anti-symmetric.