The Modular Hamiltonian in QFT and Information Theory

Roberto Longo

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Operator Algebras and Applications

On the occasion of the 88th birthday of Masamichi Takesaki

Part I

General introduction

Tomita-Takesaki modular theory

$\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$, $\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$. Embed $\mathcal{M}$ into $\mathcal{H}$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow[\text{isometric}]{X \mapsto X^*} & \mathcal{M} \\ {\scriptstyle X\to X\Omega}\Big\downarrow & & \Big\downarrow{\scriptstyle X\to X\Omega} \\ \mathcal{H} & \xrightarrow[\text{non isometric}]{S_0 : X\Omega \mapsto X^*\Omega} & \mathcal{H} \end{array}$$

$S = \bar{S}_0$, $\Delta = S^*S > 0$ positive selfadjoint

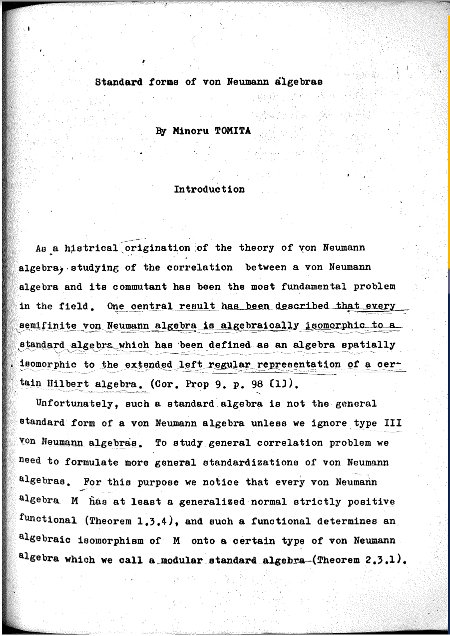
$$t \in \mathbb{R} \mapsto \sigma_t^{\varphi} \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^{\varphi}(X) = \Delta^{it} X \Delta^{-it}$$

intrinsic dynamics associated with φ (*modular automorphisms*).

Moreover, with $S = J\Delta^{1/2}$ the polar decomposition,

$$J\mathcal{M}J = \mathcal{M}'$$



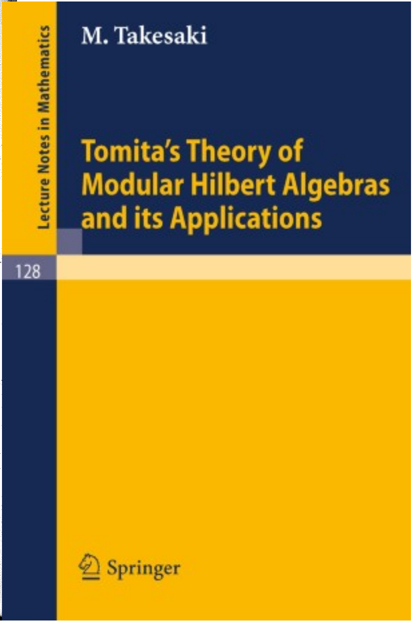
Standard forms of von Neumann algebras

By Minoru TOMITA

Introduction

As a histrical origination of the theory of von Neumann algebra, studying of the correlation between a von Neumann algebra and its commutant has been the most fundamental problem in the field. One central result has been described that every semifinite von Neumann algebra is algebraically isomorphic to a standard algebra which has been defined as an algebra spatially isomorphic to the extended left regular representation of a certain Hilbert algebra. (Cor. Prop 9. p. 98 [1]).

Unfortunately, such a standard algebra is not the general standard form of a von Neumann algebra unless we ignore type III von Neumann algebras. To study general correlation problem we need to formulate more general standardizations of von Neumann algebras. For this purpose we notice that every von Neumann algebra M has at least a generalized normal strictly positive functional (Theorem 1.3.4), and such a functional determines an algebraic isomorphism of M onto a certain type of von Neumann algebra which we call a modular standard algebra (Theorem 2.3.1).

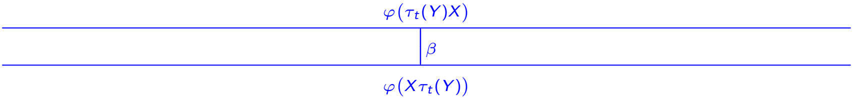


Tomita's preprint by courtesy of Atsushi Inoue

KMS states (Haag-Hugenoltz-Winnink 1967)

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$ $\exists$ function F_{XY} s.t.

(a) $F_{XY}(t) = \varphi\big(X\tau_t(Y)\big)$
(b) $F_{XY}(t + i\beta) = \varphi\big(\tau_t(Y)X\big)$
F_{XY} bounded analytic on $S_\beta = \{0 < \Im z < \beta\}$



KMS states generalise Gibbs states, equilibrium condition for infinite systems

By a remarkable historical accident, Tomita distributed his preprint at the 1967 Baton Rouge conference and, at the same conference, Haag announced the KMS condition.

Masamichi Takesaki participated in Baton Rouge conference. Soon later he completed the modular theory and characterised the modular group by the KMS condition.

• σ^φ is a purely noncommutative object (trivial in the commutative case)

• it is a thermal equilibrium evolution If $\varphi(X) = \mathrm{Tr}(\rho X)$ (type I case) then $\sigma_t^\varphi(X) = \rho^{it} X \rho^{-it}$

• modular time is intrinsic modulo scaling the rescaled group $t \mapsto \sigma^\varphi_{-t/\beta}$ is physical, β^{-1} KMS temperature

• *The Connes Radon-Nikodym cocycle relates the modular groups of different states*

$$u_t = (D\psi : D\varphi)_t \in \mathcal{M} \ , \quad \sigma_t^\psi = u_t \sigma_t^\varphi(\cdot) u_t^*$$

a first step towards Connes' classification of factors.

The modular Hamiltonian

The generator of the modular operator unitary group Δ_φ^{it} is called the modular Hamiltonian $\log \Delta_\varphi$

One may consider the the relative modular operator $\Delta_{\xi,\eta}$, and the more general

modular Hamiltonian $\log \Delta_{\xi,\eta}$

The modular Hamiltonian is the generator of an intrinsic evolution where positivity of the energy is replaced by the KMS condition

Quantum Field Theory

In QFT we have a quantum system with infinitely many degrees of freedom. The system is relativistic and there is particle creation and annihilation.

No mathematically rigorous QFT model with interaction still exists in 3+1 dimensions!

Haag local QFT:

$$O \text{ spacetime regions} \mapsto \text{von Neumann algebras } \mathcal{A}(O)$$

to each region O one associates the observables localised in O.

Local QFT nets

Local net $\mathcal{A}$ on spacetime M: map $O \subset M \mapsto \mathcal{A}(O) \subset B(\mathcal{H})$ s.t.

- *Isotony*, $O_1 \subset O_2 \implies \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- *Locality*, O_1, O_2 spacelike $\implies [\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$
- *Poincaré covariance* (conformal, diffeomorphism) .
- *Positive energy and vacuum vector*.

$O \mapsto \mathcal{A}(O)$: "Noncommutative chart" in QFT

Reeh-Schlieder theorem

The vacuum vector Ω is cyclic and separating for $\mathcal{A}(O)$ if both O and O' have non-empty interior

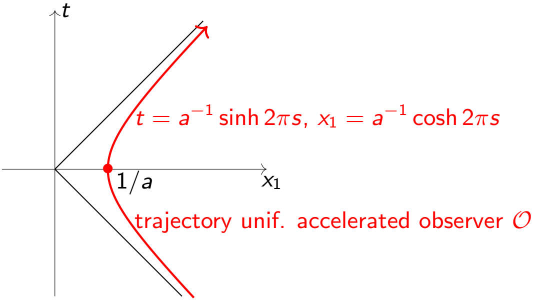
Given such a spacetime region O we may consider the vacuum modular group of O and the

$$\text{the vacuum modular Hamiltonian of } O: \ \log \Delta_O$$

What is the meaning of $\log \Delta_O$?

Bisognano-Wichmann theorem '75, Sewell comment '80

Rindler spacetime (wedge $x_1 > |t|$), vacuum modular group



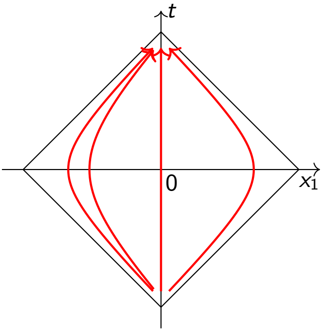
a : uniform acceleration of O
s/a : proper time of O
$\beta = 2\pi/a$: inverse KMS temperature of O

Hawking-Unruh effect!

Time is geodesic, quantum gravitational effect!

Double cone, conformal case

For a bounded region O (double cone, causal envelop of a space ball B), in the conformal case the modular group is given by the geometric transformation (Hislop, L. '81)



local modular trajectories

$$(u, v) \mapsto \big((Z(u,s), Z(v,s)\big)$$

$$Z(z,s) = \frac{(1+z)+e^{-s}(1-z)}{(1+z)-e^{-s}(1-z)}$$

$$u = x_0 + r, \quad v = x_0 - r, \quad r = |\mathbf{x}| \equiv \sqrt{x_1^2 + \cdots + x_d^2}$$

Part II

QFT information and modular theory

Entropy density

Entropy of finite systems

$X = \{x_1, \ldots x_n\}$ a set of events. If x_i occurs with probability p_i, its information is $-\log p_i$

$$\text{Shannon entropy} : S(P) = -\sum p_i \log p_i \ .$$

If $Q = \{q_1, \ldots q_n\}$ other probability distribution (state)

$$\text{Relative entropy} : S(P\|Q) = \sum p_i(\log p_i - \log q_i)$$

mean value in the state P of the difference between the information carried by the state P and the state Q.

Noncommutative entropy: $\varphi = -\mathrm{Tr}(\rho_\varphi\,\cdot)$ state on a matrix algebra

$$\text{von Neumann entropy} : S(\varphi) = \ -\mathrm{Tr}(\rho_\varphi \log \rho_\varphi)$$

Umegaki's relative entropy

$$S(\varphi\|\psi) =: \mathrm{Tr}\big(\rho_\varphi(\log \rho_\varphi - \log \rho_\psi)\big)$$

Araki's relative entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type I so Tr does not exist; however Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is defined in general by

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where ξ, η are cyclic vector representatives of φ, ψ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with ξ, η.

$$S(\varphi|\psi) \geq 0$$

positivity of the relative entropy

Relative entropy is one of the key concepts. We take the view that relative entropy is a primary concept and all entropy notions are derived concepts

In QFT the von Neumann algebra $\mathcal{M} = \mathcal{A}(O)$ is a factor of type III

$\mathcal{M}$ not of type I $\Longrightarrow$ von Neumann entropy $= \infty$

Way out:

• Take $O_1 \subset O_2$ and a type I factor $\mathcal{A}(O_1) \subset \mathcal{F} \subset \mathcal{A}(O_2)$ (split property). Then the von Neumann vacuum entropy on $\mathcal{F}$ is expected to be finite (F. Xu, R.L. for a first model verification)

• Study the relative vacuum entropy $S(\varphi\|vac)$ of any state φ (subtract ultraviolet divergences).

In this talk we deal with the second point

The wave

Figure: Hokusai. *The Great Wave*

The information carried by a classical wave

By a wave (or wave packet), we mean a real solution of the Klein-Gordon equation

$$(\Box + m^2)\Phi = 0 \ ,$$

with compactly supported, smooth Cauchy data $\Phi|_{x^0=0}$, $\Phi'|_{x^0=0}$.

Classical field theory describes Φ by the stress-energy tensor $T_{\mu\nu}$, that provides the energy-momentum density of Φ at any time.

But, how to define the information, or entropy, carried by Φ in a given spacetime region?

We give a classical answer to such a classical question by Operator Algebras and Quantum Field Theory

Standard subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a closed, real linear subspace. *Symplectic complement*:

$$H' = \{\xi \in H : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

H is *cyclic* if $\overline{H + iH} = \mathcal{H}$ and *separating* if $H \cap iH = \{0\}$.

A standard subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic and separating. H is standard iff H' is standard.

H standard subspace $\to$ anti-linear operator $S : D(S) \subset \mathcal{H} \to \mathcal{H}$,

$$S : \xi + i\eta \to \xi - i\eta, \ \xi, \eta \in H$$

$S^2 = 1|_{D(S)}$. S is closed and densely defined, indeed

$$S_H^* = S_{H'}$$

Modular theory beyond von Neumann algebras

Set $S = J\Delta^{1/2}$, polar decomposition of $S = S_H$.

Then J is an anti-unitary involution, $\Delta > 0$ is non-singular and $J\Delta J = \Delta^{-1}$.

$$\Delta^{it}H = H, \quad JH = H'$$

(]one particle Tomita-Takesaki theorem (example: $H = \overline{M_{sa}\xi}$).

cf. Rieffel-van Daele 1977, Leyland-Roberts-Testard (unpublished)

We may assume that H is factorial, namely 1 is not an eigenvalue of Δ,

$$\overline{H + H'} = \mathcal{H}\,, \quad H \cap H' = \{0\}\,,$$

Modular theory for standard subspaces applies to contexts not directly related to von Neumann algebras, e.g. Lie groups, Brunetti, Guido, L.; Neeb, Morinelli, Olafson et al.

Passivity

$\log \Delta_H$ is characterised (up to a proportionality constant) by complete passivity, following Pusz and Woronowicz.

$\mathcal{H}$ a complex Hilbert space, $H \subset \mathcal{H}$ a standard subspace and A a selfadjoint linear operator on $\mathcal{H}$ such that $e^{isA}H = H$, $s \in \mathbb{R}$.

We shall say that A is *passive* with respect to H if

$$-(\xi, A\xi) \geq 0\,, \quad \xi \in D(A) \cap H\,.$$

A is *n-passive* w.r.t. H if the generator of $e^{itA} \otimes e^{itA} \cdots \otimes e^{itA}$ is passive with respect to the n-fold tensor product $H \otimes H \otimes \cdots \otimes H$, A is *completely passive* if A is n-passive for all $n \in \mathbb{N}$.

A is completely passive with respect to H iff $\log \Delta_H = \lambda A$ for some $\lambda \geq 0$.

Entropy of a vector relative to a real linear subspace

Our analysis relies on the concept of *entropy S_k of a vector k in a Hilbert space $\mathcal{H}$ w.r.t. a closed real linear subspace H of $\mathcal{H}$.*

The formula for the entropy S_k is

$$S_k = \Im(k, P_H i \log \Delta_H\, k)$$

Here P_H is the crucial cutting projection $P_H : H + H' \to H$

$$P_H : h + h' \mapsto h$$

In terms of J and Δ,

$$P_H = \Delta^{-1/2}(\Delta^{-1/2} - \Delta^{1/2})^{-1} + J(\Delta^{-1/2} - \Delta^{1/2})^{-1}$$

P_H cuts the Cauchy data, so it is geometric. We thus get a bridge to the geometric interpretation of the modular Hamiltonian

Properties

Some of the main properties of the entropy of a vector are:

- $S_k^H \geq 0$ or $S_k^H = +\infty$ (*positivity*);
- If $K \subset H$, then $S_k^K \leq S_k^H$ (*monotonicity*);
- If $k_n \to k$, then $S_k^H \leq \liminf_n S_{k_n}^H$ (*lower semicontinuity*);
- If $H_n \subset H$ is an increasing sequence with $\overline{\bigcup_n H_n} = H$, then $S_k^{H_n} \to S_k^H$ (*monotone continuity*);
- If $k \in D(\log \Delta_H)$ then $S_k^H < \infty$ (*finiteness on smooth vectors*).

First and second quantisation

First quantisation: map

$$O \subset \mathbb{R}^d \mapsto H(O) \text{ real linear space of } \mathcal{H}$$

local, covariant, etc.

Second quantisation: map

$$O \subset \mathbb{R}^d \mapsto \mathcal{A}(O) \text{ v.N. algebra on } e^{\mathcal{H}}$$

$\mathcal{A}(O) = \mathcal{A}(H(O))$

In our case $H(O)$ is generated by the waves with Cauchy data in B (O double cone with time-zero basis B)

The core of our analysis in this talk will be done in first quantisation, then we apply the second quantisation functor.

Entropy of coherent sectors

H real linear subspace of $\mathcal{H}$ $\rightarrow$ von Neumann algebra on $e^{\mathcal{H}}$

$$\mathcal{A}(H) = \{V(h) : h \in H\}''$$

Given a wave $\Phi \in \mathcal{H}$ consider the automorphism of $\mathcal{A}(H)$

$$\beta_\Phi = \mathrm{Ad}V(\Phi)^*|_{\mathcal{A}(H)} \ .$$

The vacuum relative entropy of Φ on $\mathcal{A}(H)$ is given by the entropy of the vector Φ w.r.t. H. Namely by

$$S(\omega \cdot \beta_\Phi \| \omega) = S_\Phi$$

Araki's relative entropy

Entropy of vector

Entropy of coherent state

Entropy of wave packet

Classical waves and quantum particle waves

The logical dependence in our construction is the following

$$
\begin{tikzcd}
\boxed{\text{Entropy of a vector}} \arrow[r, "\text{one-particle space}", "\textit{classical field theory}"'] \arrow[d, "\text{Fock}"', "\text{space}"] & \boxed{\text{Entropy of a wave}} \arrow[d, "2^{nd}"', "\text{quantisation}"] \\
\boxed{\text{Entropy of a coherent state}} \arrow[r, "\text{local algebras}", "\textit{quantum field theory}"'] & \boxed{\text{Entropy of Klein-Gordon QFT}}
\end{tikzcd}
$$

The entropy of a vector Φ with respect to a real linear subspace $H(O)$ has different interpretations: classically, it measures the information carried by a wave packet in the spacetime region O; from the quantum point of view, it gives the vacuum relative entropy, on the obsevable algebra $\mathcal{A}(O)$ of the coherent state induced by Φ on the Fock space.

Indeed the same mathematical object Φ can describe:

- A classical wave packet
- The wave function of a free quantum particle
- A coherent state in QFT

Entropy of a wave

Waves' time-independent symplectic form

$$\beta(\Phi, \Psi) = \frac{1}{2} \int_{x^0=t} (\Phi'\Psi - \Psi'\Phi) dx \ ,$$

The symplectic form is the imaginary part of complex Hilbert space scalar product (that depends on the mass).

Waves with Cauchy data supported in region O (causal envelop of a space region B) form a real linear subspace $H(O) \equiv H(B)$.

The information carried by the wave Φ in the region O is the entropy S_Φ of the vector Φ w.r.t. $H(O)$

By the Bisognano-Wichmann theorem, we know the modular group of $H(W)$, with W a wedge (B the half-space $x^1 \geq 0$)

Entropy of localised states: $U(1)$-current model

Case of $U(1)$-current: ℓ real function in $S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. We have

$$S(\lambda) = \pi \int_{\lambda}^{+\infty} (x - \lambda)\ell^2(x)\mathrm{d}x \ ,$$

$S(\lambda) =$ entropy of ℓ w.r.t $H(\lambda, \infty)$.

$$S''(\lambda) = \pi \ell^2(\lambda) \geq 0$$

positivity of S''

Quantum Null Energy Condition

The vacuum energy density is $E(\lambda) = \frac{1}{2}\ell^2(\lambda)$ so we have here the QNEC:

$$E(\lambda) = \frac{1}{2\pi} S''(\lambda) \geq 0$$

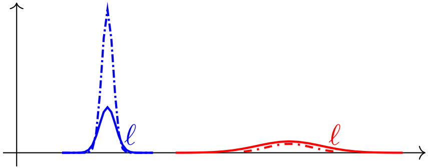


Figure: Two distributions, blue and red, for the same charge $q = \int \ell$. The dashed lines plot the corresponding entropy density rate $S''(t)$: blue high entropy, red low entropy.

Further work in CFT by S. Hollands and L. Panebianco

In QFT, Bousso, Fisher, Liechenauer, and Wall proposed the Quantum Null Energy Condition, QNEC: For null direction deformations

$$\langle T_{uu} \rangle \geq \frac{1}{2\pi} S''_A(\lambda) \ ,$$

here T stress-energy tensor, S_A entanglement entropy relative to the deformed region A, S''_A second derivative of S_A w.r.t. the deformation parameter λ.

Physical arguments give the QNEC from the inequality

$$S''(\lambda) \geq 0$$

with $S(\lambda)$ Araki's relative entropy of every state w.r.t. the vacuum.

QNEC

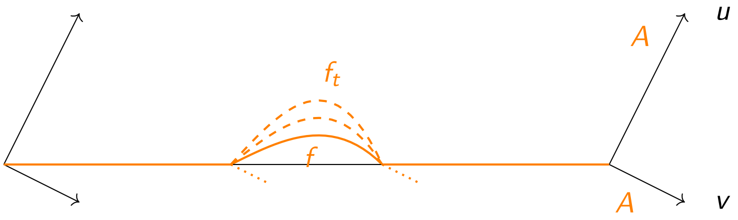


Figure: The function f is the boundary of the deformed region on the null horizon. The entire deformed region is its causal envelop A.

Positivity of the second derivative of the *relative entropy* appears unexpectedly: $S''(\lambda) \geq 0$

The decomposition by Morinelli, Tanimoto and Wegner

In the free field, the modular group pf a deformed wedge is geometric, it acts boostwise on each fiber.

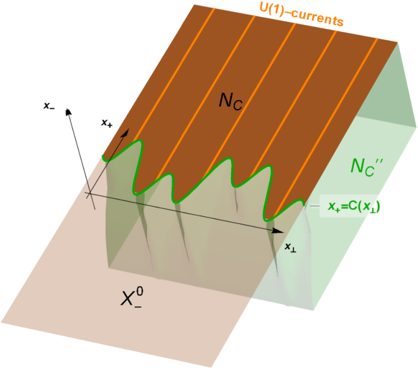


Figure: The fiber decomposion of the modular group

This result complements the Ceyhan and Faulkner result.

Entropy variation

Let Φ be a real Klein-Gordon wave and $H = H(W_\lambda)$. W_λ null translated wedge.

The entropy $S_\Phi(\lambda)$ of Φ w.r.t. the wedge region W_λ is the entropy of the vector Φ w.r.t. the standard subspace $H(W_\lambda)$.

$$S_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1\geq\lambda} (x^1 - \lambda)\, T_{00}(x)\, dx$$

then

$$S''_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1=\lambda} \langle v, Tv\rangle dx \geq 0\ ,$$

where v is the light-like vector $v = (1, 1, 0\ldots, 0)$ *QNEC for coherent states, constant deformations*. (Work with F. Ciolli, G. Ruzzi.)

Here the energy density is

$$\langle T_{00}\rangle_\Phi = \frac{1}{2}\big(\Phi'^2 + |\nabla\Phi|^2 + m^2\Phi^2\big)$$

Entropy and Klein–Gordon field on a globally hyperbolic spacetime

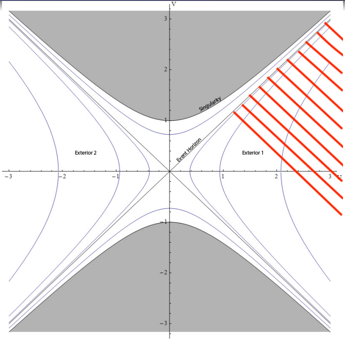


Figure: Schwarzschild-Kruskal spacetime. The red area is a null translated wedge

The convexity of the entropy w.r.t. to the null translation parameter has been obtained very recently for a Klein–Gordon field on a globally hyperbolic spacetime for coherent states (Ciolli, Ranallo, Ruzzi, L.). (Holland, Ishibashi untraslated case)

The abstract result

Let $K \subset H$ be a half-sided modular inclusion of standard subspaces

$$\Delta_H^{-is} K \subset K, \quad s \geq 0,$$

then Δ_H^{is}, Δ_K^{it} generate a positive energy representation of the $ax+b$ group (Borchers, Wiesbrock, Araki, Zsido). Let $H_\lambda = T(\lambda)H$ be the translated subspaces. Then the entropy function

$$\lambda \mapsto S(\lambda) = S_\psi^{H_\lambda} = \Im(\psi, P_{H_\lambda} i \log \Delta_{H_\lambda} \psi) \text{ is convex for all } \psi$$

If $S(\lambda_0) < \infty$ for some $\lambda_0 \in \mathbb{R}$, then

(i) $S(\lambda)$ is finite and C^1 on $[\lambda_0, \infty)$;

(ii) $\frac{d}{d\lambda} S(\lambda)$ is absolutely continuous in $[\lambda_0, \infty)$ with almost everywhere non-negative derivative $\frac{d^2}{d\lambda^2} S(\lambda) \geq 0$.

Entropy density and modular Hamiltonian, massless case

What is the entropy density of a wave? We have to compute the modular Hamiltonian $\log \Delta_B$ of a (unit) space ball B.

In terms of the wave Cauchy data,

$$\log \Delta_B = -2\pi i \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r - D & 0 \end{bmatrix}$$

$D = (d-1)/2$ the scaling dimension, namely

$$\log \Delta_B = -2\pi i \begin{bmatrix} 0 & M \\ L & 0 \end{bmatrix}$$

$$M = \text{ Multiplication operator by } \frac{1}{2}(1-r^2)\,,$$

$$L = \text{ Legendre operator } \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r - D$$

The computation is possible as we know the explicit action of the modular group (Hislop-L.)

In terms of the classical stress-energy tensor T:

$$-(\Phi, \log \Delta_B \Phi) = 2\pi \int_{x_0=0} \frac{1-r^2}{2} \langle T_{00} \rangle_\Phi (x) dx + \pi D \int_{x_0=0} \Phi^2 dx$$

Φ a real wave with smooth, compactly supported Cauchy data.

The right hand side is similar to a formula for the modular Hamiltonian sketched by Casini, Huerta and Myers.

The expression for the massive modular Hamiltonian was a long standing problem! The massless formula suggests to deform (or perturb) the massless set up to describe the massive set up (joint paper with G. Morsella)

One-particle Hilbert space (time zero description)

The symplectic form on β_m is independent of m.

Let $H_m^{\pm}$ the real Hilbert space of $f \in S'(\mathbb{R}^d)$ s.t.

$$||f||_{\pm}^2 = \int_{\mathbb{R}^d} (|\mathbf{p}|^2 + m^2)^{\pm 1/2} |\hat{f}(\mathbf{p})|^2 d\mathbf{p} < +\infty$$

and $\mu_m : H_m^+ \to H_m^-$ the unitary operator

$$\widehat{\mu_m f}(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2}\, \hat{f}(\mathbf{p})\,.$$

Then

$$\imath_m = \begin{bmatrix} 0 & \mu_m^{-1} \\ -\mu_m & 0 \end{bmatrix},$$

is a unitary on $H_m = H_m^+ \oplus H_m^-$ with $\imath_m^2 = -1$,

$\mathcal{H}_m =$ real Hilbert space H_m with complex structure $\imath_m$.

Note that, when $m = 1$, $H_m^{\pm}$ is a usual Sobolev space.

The local massive Hamiltonian $\log \Delta_B$

Add a massive term and an inverse Helmholtz operator term to the massless Hamiltonian

$$m = 0: \quad 2\pi\imath_0 \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r - D & 0 \end{bmatrix}$$

$$m \geq 0: \quad 2\pi\imath_m \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)(\nabla^2 - m^2) - r\partial_r - D - \frac{1}{2}m^2 G_m^B & 0 \end{bmatrix}$$

with

$$G_m^B = E_B \mathrm{H}_m = E_B(-\nabla^2 + m^2)^{-1}.$$

$$\int_B \frac{1-r^2}{2} \langle T_{00}^{(0)}(t,\mathbf{x})\rangle_\Phi d\mathbf{x} \to \int_B \frac{1-r^2}{2} \langle T_{00}^{(m)}(t,\mathbf{x})\rangle_\Phi d\mathbf{x} + \dots$$

The massive modular Hamiltonian is the unique compact perturbation of the massless modular Hamiltonian in a sense to be defined (the Hilbert space is also deformed)

The formula for the local massive Hamiltonian

The modular Hamiltonian $\log \Delta_B$ associated with the unit ball B in the free scalar, mass m QFT is (on Cauchy data)

$$-2\pi A_m = \log \Delta_B \, .$$

$$\log \Delta_B = 2\pi \imath_m \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)(\nabla^2 - m^2) - r\partial_r - D - \frac{1}{2}m^2 G_m^B & 0 \end{bmatrix}$$

with L_m the massive Legendre operator

$$L_m = \frac{1}{2}(1-r^2)(\nabla^2 - m^2) - r\partial_r - D$$

and G_m^B is the Green integral operator with Yukawa potential

$$G_m^B f(\mathbf{x}) = \frac{1}{4\pi} \int_B \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} f(\mathbf{y}) d\mathbf{y} \, ,$$

Yukawa potential

Yukawa predicted in 1934 an exchange particle for the neutron-proton interaction with the potential:

$$G_m = -g^2 \frac{e^{-\alpha m r}}{r}$$

g scaling constant, m mass, r radial distance to the particle, and α constant s.t. $r = 1/\alpha m$ is the approximate range.

This particle was later found, the *pion*, and Yukawa was awarded the Nobel prize.

On the Interaction of Elementary Particles. I.

By Hideki YUKAWA.

(Read Nov. 17, 1934)

§ 1. Introduction

At the present stage of the quantum theory little is known about the nature of interaction of elementary particles. Heisenberg considered the interaction of "Platzwechsel" between the neutron and the proton to be of importance to the nuclear structure.(1)

Recently Fermi treated the problem of β-disintegration on the hypothesis of "neutrino"(2). According to this theory, the neutron and the proton can interact by emitting and absorbing a pair of neutrino and electron. Unfortunately the interaction energy calculated on such assumption is much too small to account for the binding energies of neutrons and protons in the nucleus.(3)

To remove this defect, it seems natural to modify the theory of Heisenberg and Fermi in the following way. The transition of a heavy particle from neutron state to proton state is not always accompanied by the emission of light particles, i. e., a neutrino and an electron, but the energy liberated by the transition is taken up sometimes by another heavy particle, which in turn will be transformed from proton state into neutron state. If the probability of occurrence of the latter process is much larger than that of the former, the interaction between the neutron and the proton will be much larger than in the case of Fermi, whereas the probability of emission of light particles is not affected essentially.

Now such interaction between the elementary particles can be described by means of a field of force, just as the interaction between the charged particles is described by the electromagnetic field. The above considerations show that the interaction of heavy particles with this field is much larger than that of light particles with it.

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(1) W. Heisenberg, Zeit f. Phys. **77**, 1 (1932); **78**, 156 (1932); **80**, 587 (1933). We shall denote the first of them by I.

(2) E. Fermi, ibid. **88**, 161 (1394).

(3) Ig. Tamm, Nature **133**, 981 (1934); D. Iwanenko, ibid. 981 (1934).

Local information in a wave packet

With $S_\Phi(R)$ the entropy of Φ in the radius R ball cantered at $\bar{\mathbf{x}}$, we have

$$
\begin{aligned}
S_\Phi(R) &= \pi \int_{B_R(\bar{\mathbf{x}})} \frac{R^2 - r^2}{R} \langle T^{(m)}_{00}(t,\mathbf{x}) \rangle_\Phi d\mathbf{x} && \text{stress-energy tensor term} \\
&+ \pi \frac{d-1}{2R} \int_{B_R(\bar{\mathbf{x}})} \Phi^2(t,\mathbf{x}) d\mathbf{x} && \text{mass independent normalisation} \\
&+ \pi \frac{m^2}{R} \int_{B_R(\bar{\mathbf{x}})} G_m(\mathbf{x}-\mathbf{y}) \Phi^2(t,\mathbf{x}) d\mathbf{x} d\mathbf{y} && \text{Yukawa potential term}
\end{aligned}
$$

with $r = |\mathbf{x} - \bar{\mathbf{x}}|$

The parabolic distribution

$$S_\Phi(R) = \pi \int_{B_R(\bar{\mathbf{x}})} \underbrace{\frac{R^2 - r^2}{R}}_{parab.\ distr.} \langle T_{00}^{(m)}(t,\mathbf{x})\rangle_\Phi d\mathbf{x} + \cdots$$

The parabolic distribution is higher-dimensional generalization of Wigner semi-circular distribution in three dimensional space (the marginal distribution function of a spherical distribution)

As $R \to 0$

$$S_\Phi(R,\mathbf{x}) = \frac{\pi}{d}\left(\langle T_{00}\rangle_\Phi(t,\mathbf{x}) + D\Phi^2(t,\mathbf{x})\right) A_{d-1}(R) + \ldots$$

Here $A_{d-1}(R) = 2\frac{\pi^{d/2}}{\Gamma(d/2)} R^{d-1}$ is the area of boundary sphere ∂B_R, cf. holographic area theorems, black holes (Bekenstein) and other contexts.

Related work on modular theory and entropy, not treated

Some topics, related to my talk, I could not mention:

• Landauer's bound (relations with Jones' index) R.L.

• Bekenstein's bound, Casini, R.L.

• Entropy distributions of localised states, R.L.

• Relative entropy in CFT, Hollands, Panebianco, R.L.

• Entanglement entropy, Casini, Huerta, Hollands, Sanders, Xu, R.L. (work in progress by Panebianco and Wegener)

• Mutual information for Fermions, Xu, R.L.

• Deformation formulas, Bostelmann, Cadamuro, Del Vecchio, R.L.

Happy birthday Masamichi!

Figure: Souvenir, 2013.