On the Bulk Classification of Non-Hermitian Topological Insulators Modeled by Spectral Operators

Physical Principles to Choose a Physically Meaningful Classification

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2020.12.18@virtualized Yukawa Institute for Theoretical Physics

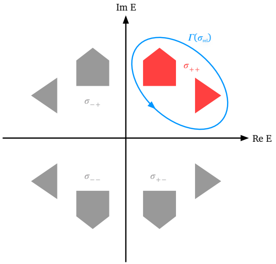
Today's Talk Is Based On

On Choosing a Physically Meaningful Topological Classification for Non-Hermitian Systems and the Issue of Diagonalizability
with Vicente Lenz
Under review, arxiv:2010.09261

Continuity of Spectra of Spectral Operators
with Vicente Lenz
In preparation

Motivated by
Recent works by Kawabata et al. (2019), Zhou & Lee (2019)

- Extends 10-Fold Cartan-Altland-Zirnbauer Classification of **selfadjoint** operators
- Systematic classification of **non-selfadjoint** operators
- 38 symmetry classes + gap-type subclasses



Our Main Results

1. Provides a recipe to classify non-selfadjoint **spectral operators**.

   $\rightsquigarrow$ Inclusion of effects of **disorder**, **perturbations**

2. **Physically meaningful** classification singled out by **identifying relevant states** $\rightsquigarrow$ physical criterion for choice of **line gap**.

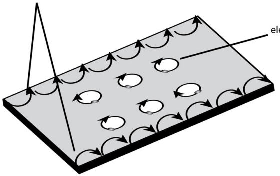
3. Provides **mathematical** recipe for **point gap** classification.

4. **Extends the classification** results by Zhou & Lee, Kawabata et al. from periodic tight-binding to spectral operators.

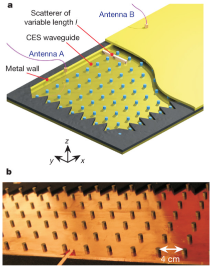
Update since publication of preprint

- Current preprint claims classification result only applies to **diagonalizable** $\subsetneq$ **spectral** operators. $\Longrightarrow$ Range of validity $\uparrow$
- Fixed a mistake: existence of P_{rel} *not* conditional on absence of Jordan blocks.
- Thank you to Vicente Lenz & Masatoshi Sato

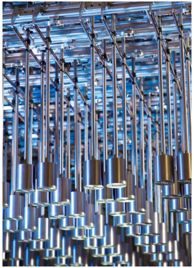
What Are Topological Phenomena?



Quantum Hall Effect



Electromagnetic waves



Coupled Oscillators

What makes a physical effect topological?

Find a *mathematical object* (e. g. projection or vector bundle) *whose topology* manifests itself on the level of physics.

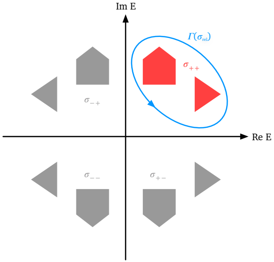
Bulk-Boundary Correspondence

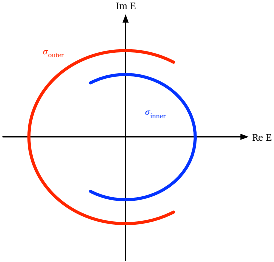
$$O_{\mathrm{bdy}}(t) \approx T_{\mathrm{bdy}} = f(T_{\mathrm{bulk}})$$

Step 1: Bulk Classification

- Classify systems with certain symmetries
- Identify all topological invariants

General Properties of Non-Selfadjoint Operators





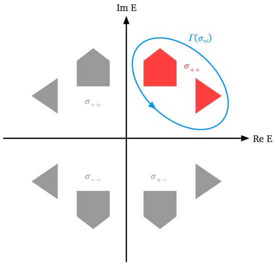
Differences between selfadjoint and non-selfadjoint operators?

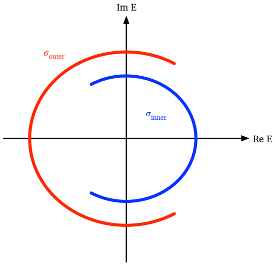
1. Spectrum may be **complex**.
2. They need **not** be **diagonalizable**/have Jordan blocks.*

⇝ Intuition can only be made precise for spectral operators!

*What that means mathematically will be clarified later in the talk.

Point Gap vs. Line Gaps





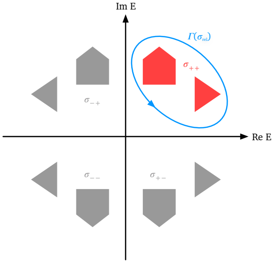
Gap types for selfadjoint ⟶ non-selfadjoint operators

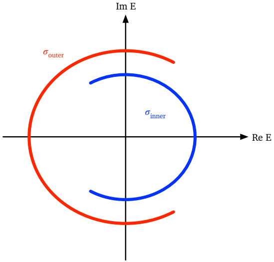
Co-dimension 1 ⟶ { Co-dimension 1 (line gaps), Co-dimension 2 (point gaps) }

Differences between point and line gaps

- Point gaps just enforce bounded invertibility
- Line gaps prevent states from crossing the relevant line.
- Real and imaginary line gaps
  ⇝ imaginary and real axis (order reversed!)
- H has line gap $\Longrightarrow$ H has point gap

Relevant Symmetries of Non-Selfadjoint Operators



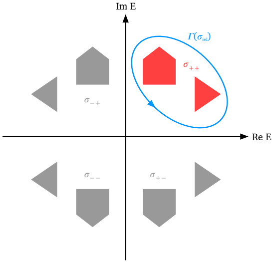


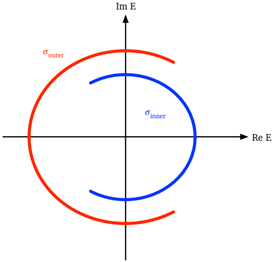
Symmetries for **selfadjoint** $\longrightarrow$ **non-selfadjoint** operators

$$U\,H\,U^{-1} = \pm H \quad \longrightarrow \quad \begin{cases} U\,H\,U^{-1} = \pm H \\ U_*\,H\,U_*^{-1} = \pm H^* \end{cases}$$

where U and U_* are (anti)linear, invertible maps with $U^2 = \pm \mathbb{1}$

Relevant Symmetries of Non-Selfadjoint Operators

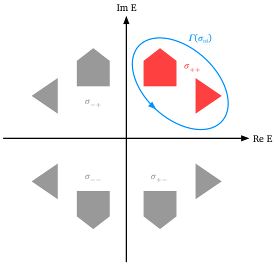


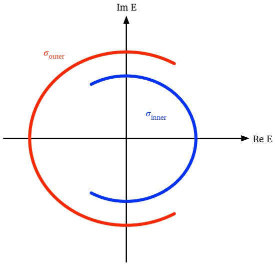


Type	*Condition on* H	$\sigma(H) =$
ordinary	$V\,H\,V^{-1} = +H$	$+\sigma(H)$
chiral	$S\,H\,S^{-1} = -H$	$-\sigma(H)$
$\pm$TR	$T\,H\,T^{-1} = +H$	$+\overline{\sigma(H)}$
$\pm$PH	$C\,H\,C^{-1} = -H$	$-\overline{\sigma(H)}$
pseudo	$V_*\,H\,V_*^{-1} = +H^*$	$+\overline{\sigma(H)}$
chiral*	$S_*\,H\,S_*^{-1} = -H^*$	$-\overline{\sigma(H)}$
$\pm$TR*	$T_*\,H\,T_*^{-1} = +H^*$	$+\sigma(H)$
$\pm$PH*	$C_*\,H\,C_*^{-1} = -H^*$	$-\sigma(H)$

(Compared with Kawabata et al. this uses different nomenclature for the symmetries.)

38-Fold Classification of Non-Selfadjoint Topological Insulators



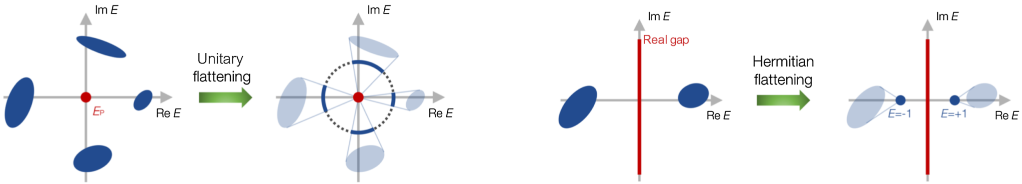


Bare basics

- $\left.\begin{array}{c}\text{Symmetries of } H \\ \text{Gap type}\end{array}\right\} \longleftrightarrow$ Topological class of H
- Topological class $= \bigcup \{$Topological phases$\}$
- Topological phase = Operators connected by symmetry- and gap-preserving continuous deformations
- Homotopy definition of topological phase (usually first-principles starting point)
- Phases labeled by a finite set of topological invariants
- Number and nature of topological invariants depends on topological class

38-Fold Classification of Non-Selfadjoint Topological Insulators

Normalization of Non-Selfadjoint Operatosr



Point gap

- H homotopically deformed to unitary
- *Idea:* polar decomposition $H = V_H \, M$
- Unless H is normal $[V_H, M] \neq 0$

(Real) line gap

- H homotopically deformed to spectrally flattened hamiltonian Q
- $Q = Q^* = \mathbb{1} - 2P \longleftrightarrow P = P^2 = P^*$
- In general P *not* a spectral projection of H

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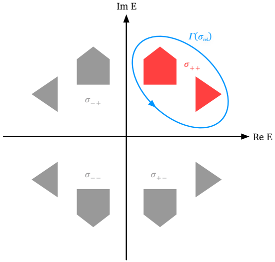
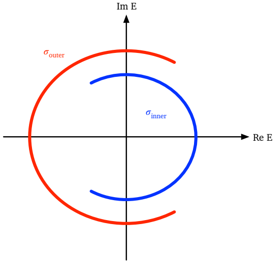




Classification result by Kawabata et al. (2019)

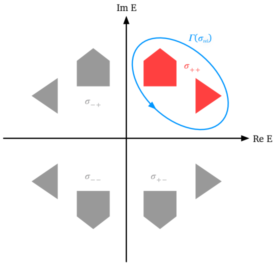
- 38 topological symmetry classes + gap-type subclasses
- Eliminated doubly counted cases
- Determined coarse* topological classification by computing twisted equivariant K-groups $\rightsquigarrow$ classification tables
- Applies to periodic tight-binding operators

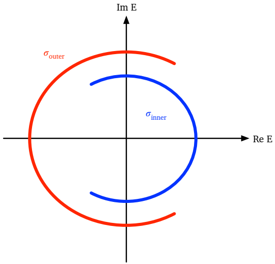
Example

SLS	AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$\mathcal{S}_{++}$	BDI	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
		$\mathrm{L_r}$	$\mathcal{R}_1 \times \mathcal{R}_1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
		$\mathrm{L_i}$	$\mathcal{R}_1 \times \mathcal{R}_1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\mathcal{S}_{--}$	DIII	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
		$\mathrm{L_r}$	$\mathcal{R}_3 \times \mathcal{R}_3$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$
		$\mathrm{L_i}$	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

* Coarse means not all topological invariants are captured by those specific K-groups

38-Fold Classification of Non-Selfadjoint Topological Insulators





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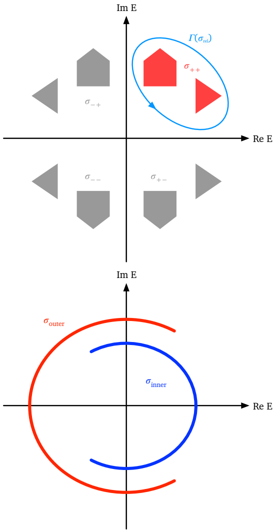
$\mathcal{S}_{--}$	DIII	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
		L_r	$\mathcal{R}_3 \times \mathcal{R}_3$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
		L_i	$\mathcal{C}_1$	0	$\mathbb{Z}$	0

Zoomed in

* Coarse means not all topological invariants are captured by those specific K-groups

38-Fold Classification of Non-Selfadjoint Topological Insulators



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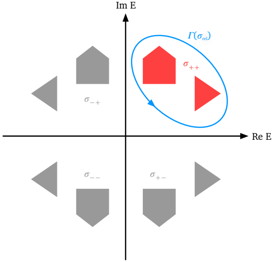
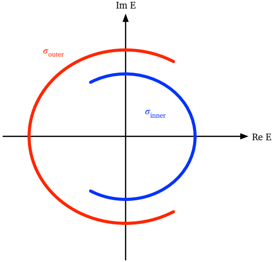
Example

Problem solved!?

* Coarse means not all topological invariants are captured by those specific K-groups

Questions that Motivated Our Work





1. Do **Jordan blocks** matter in the topological classification?

   Answer: For spectral operators, **no**.

   For non-spectral operators, the notion of Jordan block is ill-defined.

2. What **physical data** determine which of the mathematical classifications is relevant for physical phenomena?

   Answer for "line gaps": Physically relevant states

   Answer for "point gaps": I do not know.

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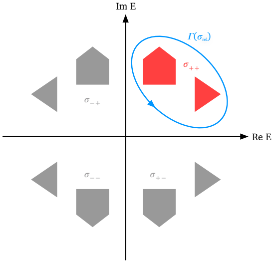

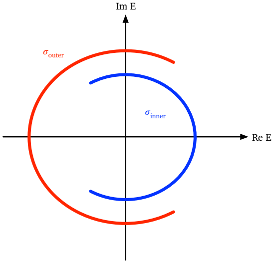

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Definition of Spectral Operators

Definition (Spectral Operator)

A spectral operator $H \in \mathcal{B}(\mathcal{X})$ is a bounded operator on a Banach space $\mathcal{X}$ that possesses a **projection-valued measure** $\{P(\Omega)\}_{\Omega \in \mathfrak{B}(\mathbb{C})}$ on $\mathbb{C}$ with the following properties: for all Borel sets $\Omega \in \mathfrak{B}(\mathbb{C})$ we have

a $[H, P(\Omega)] = 0$ and

b $\sigma\big(H|_{\mathrm{ran}\,P(\Omega)}\big) \subseteq \overline{\Omega}$.

- Definition goes back to Dunford, Schwartz, Bade, Kakutani & Wermer (series of papers in 1954!)
- Theory developed across 700 (!) pages in Part III of Dunford & Schwartz's book $\rightsquigarrow$ Re-discovered by my collaborator Vicente Lenz
- Projection-valued measure takes values in oblique projections

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b. $\sigma\big(H|_{\mathrm{ran}\, P(\Omega)}\big) \subseteq \overline{\Omega}$.

- Periodic tight-bining operators are spectral operators
  $\rightsquigarrow$ Link to works by Kawabata et al.
- Not all operators are spectral!

Decompositions of Spectral Operators

Theorem (Jordan normal form)

*Any spectral operator $H = S + N$ can be **uniquely** decomposed into a **scalar part***

$$S = \int_{\mathbb{C}} E \, \mathrm{d}P(E)$$

*and a **quasi-nilpotent part** (i. e. $\sigma(N) = \{0\}$)*

$$N = H - S$$

*that **commute** $[S, N] = 0$.*

Dunford & Schwartz, *Linear Operators, Part III, Spectral Operators*, Chapter XV, Wiley-Interscience, 1988

Decompositions of Spectral Operators

Theorem (Cartesian decomposition)

Any spectral operator $H = H_{\mathrm{Re}} + \mathrm{i} H_{\mathrm{Im}}$ *can be* ***uniquely*** *decomposed into a real and imaginary part with the following properties:*

a. $[H_{\mathrm{Re}}, H_{\mathrm{Im}}] = 0$
b. $\sigma(H_{\mathrm{Re}}), \sigma(H_{\mathrm{Im}}) \subseteq \mathbb{R}$
c. H_{Re} *is a scalar operator and* H_{Im} *a spectral operator.*
d. *The Boolean algebra of projections generated by the projection-valued measures of* H_{Re} *and* H_{Im} *is bounded.*

Dunford & Schwartz, *Linear Operators, Part III, Spectral Operators*, Chapter XV, Wiley-Interscience, 1988

Decompositions of Spectral Operators

Theorem (Polar decomposition)

Any spectral operator $H = U\,M$ can be decomposed into a phase U and a modulus M with the following properties:

a. $[U, M] = 0$
b. $\sigma(U) \subseteq \mathbb{S}^1$, $\sigma(M) \subseteq [0, \infty)$
c. *U is a scalar operator and M a spectral operator.*
d. *The Boolean algebra of projections generated by the projection-valued measures of U and M is bounded.*

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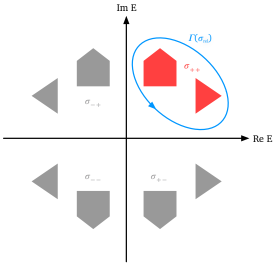
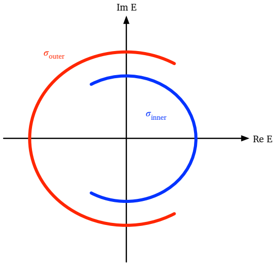
Spectral Operators Admit an Analytic Functional Calculus





For functions f that are analytic on some set $\Omega \supset \sigma(H)$, we set

$$f(H) := \sum_{n=0}^{\infty} \frac{N^n}{n!} \int_{\sigma(H)} f^{(n)}(E) \, \mathrm{d}P(E) = \frac{\mathrm{i}}{2\pi} \int_{\Gamma(\sigma(H))} \mathrm{d}z \, f(z) \, (H - z)^{-1}$$

where N is the quasi-nilpotent part.

Important

When σ_{rel} is an isolated spectral island, then

$$1_{\sigma_{\mathrm{rel}}}(H) = 1_{\sigma_{\mathrm{rel}}}(S) = 1_{\sigma_{\mathrm{rel}}}(S + N')$$

where N' is *any* quasi-nilpotent operator with $[S, N'] = 0$

Relation to Diagonalizable Operators

Definition (Diagonalizable operator)

$H \in \mathcal{B}(\mathcal{H})$: $\exists$ similarity transform $G \in \mathcal{B}(\mathcal{H})^{-1}$ that makes

$$G \, H \, G^{-1} = \int_{\mathbb{C}} E \, \mathrm{d}P(E)$$

normal, where $P(\Omega)$ is the projection-valued measure.

Relation to Diagonalizable Operators

Proposition

$H \in \mathcal{B}(\mathcal{H})$ ***diagonalizable***

$\Longleftrightarrow$

$H = S$ ***scalar*** *operator on a* ***Hilbert space***

$\rightsquigarrow$ Dunford & Schwartz, Theorem XV.6.4

Relation to Diagonalizable Operators

Corollary

Suppose $H \in \mathcal{B}(\mathcal{H})$ is a spectral operator on a Hilbert space.

1. *The **scalar part** S of $H = S + N$ defines a **normal operator** with respect to a suitably chosen scalar product.*
2. *The **real part** H_{Re} of $H = H_{\mathrm{Re}} + \mathrm{i} H_{\mathrm{Im}}$ defines a **selfadjoint operator** with respect to a suitably chosen scalar product.*
3. *If in addition $H = U\, M \in \mathcal{B}(\mathcal{H})^{-1}$, then the **phase** U is a uniquely defined **unitary operator** with respect to a suitably chosen scalar product.*

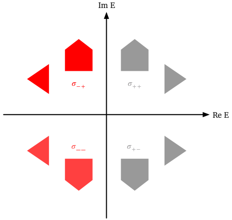
Continuity of Spectra

Conjecture

*Let $\lambda \mapsto H(\lambda)$ be a continuous path in the set of spectral operators. Then the spectrum $\sigma(H(\lambda))$ is **inner and outer** (upper and lower) **continuous** in λ.*

- Outer/upper semicontinuity is for free.
  (Kato, Chapter VI.3.1, Theorem 3.1)
- Proof is work-in-progress.
- Not sure whether the result is contained in Dunford & Schwartz (almost 700 pages, so the answer could be yes).
- **Proceeding under the assumption that this conjecture is true.**

Definition of the Relevant Projection P_{rel}



Our approach

- Pick **physically relevant states** $\rightsquigarrow \sigma_{\mathrm{rel}}$
- Define projection onto relevant states

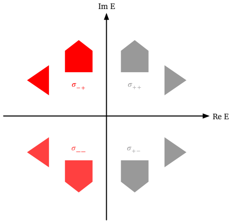
$$P_{\mathrm{rel}} := \frac{\mathrm{i}}{2\pi} \int_{\Gamma(\sigma_{\mathrm{rel}})} \mathrm{d}z \, (H - z)^{-1}$$

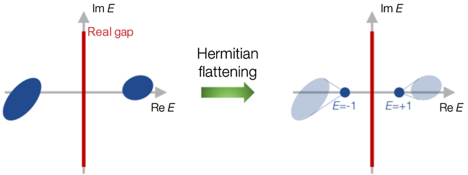
- Symmetries of H and σ_{rel}
  $\Longrightarrow$ symmetries and constraints of P_{rel}

Kawabata et al. (real line gap)

- H homotopically deformed to spectrally flattened hamiltonian Q
- $Q = Q^* = \mathbb{1} - 2P \longleftrightarrow P = P^2 = P^*$
- In general P *not* a spectral projection of H!
- Choice of line gap $\Leftrightarrow$ choice of contour
  $\rightsquigarrow$ Line gap part of an infinite contour
- Q, P not unique (choice of scalar product)

Definition of the Relevant Projection P_{rel}





Our approach

- **Pick physically relevant states** $\rightsquigarrow \sigma_{\mathrm{rel}}$
- Define projection P_{rel} onto relevant states
- Symmetries of H and σ_{rel}

  $\Longrightarrow$ symmetries and constraints of P_{rel}
- No homotopy argument, no extended hamiltonian
- Homotopy definition of topological class of P_{rel} relies on continuity of spectrum

Kawabata et al. (real line gap)

- H homotopically deformed to spectrally flattened hamiltonian Q
- $Q = Q^* = \mathbb{1} - 2P \;\longleftrightarrow\; P = P^2 = P^*$
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Jordan Blocks Do Not Enter the Line Gap Classification

$$P_{\mathrm{rel}} := \frac{\mathrm{i}}{2\pi} \int_{\Gamma(\sigma_{\mathrm{rel}})} \mathrm{d}z \, (H - z)^{-1}$$

$$\overset{\star}{=} 1_{\sigma_{\mathrm{rel}}}(H) = 1_{\sigma_{\mathrm{rel}}}(S) = P_{\mathrm{rel}}^{*w}$$

- Equality marked with $\star$ only works when H is spectral
- P_{rel} is *independent* of the quasi-nilpotent part (Jordan block)!
  $\Longrightarrow$ Topological phase determined solely by scalar part
  $\Longrightarrow$ **Jordan blocks do not enter topological classification**
- Relevant projection is always orthogonal
  (with respect to a suitably chosen scalar product)
- Becomes classification problem of orthogonal projections

$\Longrightarrow$ Assume $H = S$ is a scalar operator
(simplifies presentation)

Symmetries of Diagonalizable Operators

Example

Time-reversal symmetry

$$T\,H\,T^{-1} \overset{!}{=} +H$$

$$T\,H_{\mathrm{Re}}\,T^{-1} - \mathrm{i}\,T\,H_{\mathrm{Im}}\,T^{-1} \overset{!}{=} H_{\mathrm{Re}} + \mathrm{i}H_{\mathrm{Im}}$$

Eqiuvalent to

$$T\,H\,T^{-1} = +H \quad \Longleftrightarrow \quad \begin{cases} T\,H_{\mathrm{Re}}\,T^{-1} & = +H_{\mathrm{Re}} \\ T\,H_{\mathrm{Im}}\,T^{-1} & = -H_{\mathrm{Im}} \end{cases}$$

Symmetries of Diagonalizable Operators

Example

Pseudohermiticity

$$V_* \, H \, V_*^{-1} \overset{!}{=} +H^* = +W \, H^{*_W} \, W^{-1}$$

$$V_* \, H_{\mathrm{Re}} \, V_*^{-1} + \mathrm{i} \, V_* \, H_{\mathrm{Im}} \, V_*^{-1} \overset{!}{=} W \, \big(H_{\mathrm{Re}} - \mathrm{i} H_{\mathrm{Im}} \big) \, W^{-1}$$

Eqiuvalent to

$$V_* \, H \, V_*^{-1} = +H^* \quad \Longleftrightarrow \quad \begin{cases} V_* \, H_{\mathrm{Re}} \, V_*^{-1} & = +W \, H_{\mathrm{Re}} \, W^{-1} \\ V_* \, H_{\mathrm{Im}} \, V_*^{-1} & = -W \, H_{\mathrm{Im}} \, W^{-1} \end{cases}$$

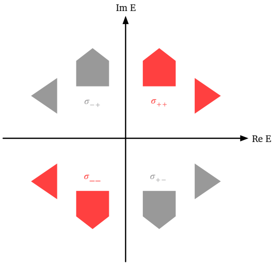
Symmetries of Diagonalizable Operators

General symmetries

$$U\,H\,U^{-1} = \pm H \quad \Longleftrightarrow \quad \begin{cases} U\,H_{\mathrm{Re}}\,U^{-1} & = \pm_{\mathrm{Re}}\,H_{\mathrm{Re}} \\ U\,H_{\mathrm{Im}}\,U^{-1} & = \pm_{\mathrm{Im}}\,H_{\mathrm{Im}} \end{cases}$$

$$U_*\,H\,U_*^{-1} = \pm H^* \quad \Longleftrightarrow \quad \begin{cases} U_*\,H_{\mathrm{Re}}\,U_*^{-1} & = \pm_{\mathrm{Re}}\,W^{-1}\,H_{\mathrm{Re}}\,W \\ U_*\,H_{\mathrm{Im}}\,U_*^{-1} & = \pm_{\mathrm{Im}}\,W^{-1}\,H_{\mathrm{Im}}\,W \end{cases}$$

Example: Pseudoselfadjoint Operator with Odd TR and Odd TR*



Symmetries of H

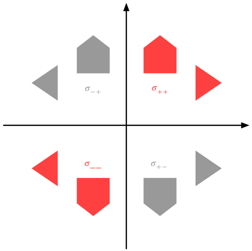
1. Pseudo: $U_* \, H \, U_*^{-1} = +H^*$
   (simplifying assumption: H normal $\Leftrightarrow W = \mathbb{1}_{\mathcal{H}}$)
2. $-\mathrm{TR}$: $T \, H \, T^{-1} = +H, T^2 = -\mathbb{1}_{\mathcal{H}}$
3. $-\mathrm{TR}^*$: $T_* = U_* \, T$: $[T, U_*] = 0$
   $\Longrightarrow \; T_* \, H \, T_*^{-1} = +H^*, T_*^2 = -\mathbb{1}_{\mathcal{H}}$

Symmetries of the relevant spectrum

$$\sigma_{\mathrm{rel}} = \sigma_{++} \cup \sigma_{--} = -\sigma_{\mathrm{rel}}$$

$\Longrightarrow$ incompatible with two symmetries!

Example: Pseudoselfadjoint Operator with Odd TR and Odd TR*



Symmetries of H

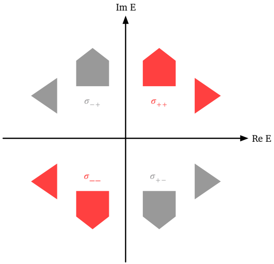
1. Pseudo: $\sigma(H) = +\overline{\sigma(H)}$
2. $-$TR: $\sigma(H) = +\overline{\sigma(H)}$
3. $-\mathrm{TR}^*$: $\sigma(H) = +\sigma(H)$

Symmetries of the relevant spectrum

$$\sigma_{\mathrm{rel}} = \sigma_{++} \cup \sigma_{--} = -\sigma_{\mathrm{rel}}$$

$\Longrightarrow$ incompatible with two symmetries!

Example: Pseudoselfadjoint Operator with Odd TR and Odd TR*



Symmetries of H

1. Pseudo: $U_* \, H \, U_*^{-1} = +H^* \iff \begin{cases} U_* \, H_{\mathrm{Re}} \, U_*^{-1} & = +H_{\mathrm{Re}} \\ U_* \, H_{\mathrm{Im}} \, U_*^{-1} & = -H_{\mathrm{Im}} \end{cases}$

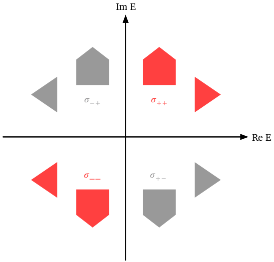
2. $-$TR: $T \, H \, T^{-1} = +H \iff \begin{cases} T \, H_{\mathrm{Re}} \, T^{-1} & = +H_{\mathrm{Re}} \\ T \, H_{\mathrm{Im}} \, T^{-1} & = -H_{\mathrm{Im}} \end{cases}$

3. $-$TR*: $T_* \, H \, T_*^{-1} = +H^* \iff \begin{cases} T_* \, H_{\mathrm{Re}} \, T_*^{-1} & = +H_{\mathrm{Re}} \\ T_* \, H_{\mathrm{Im}} \, T_*^{-1} & = +H_{\mathrm{Im}} \end{cases}$

Projection onto relevant states

$$P_{\mathrm{rel}} = 1_{\sigma_{++}}(H) + 1_{\sigma_{--}}(H)$$

Example: Pseudoselfadjoint Operator with Odd TR and Odd TR*



Projection onto relevant states

$$P_{\mathrm{rel}} = 1_{\sigma_{++}}(H) + 1_{\sigma_{--}}(H)$$

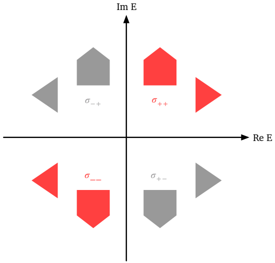
Symmetries of P_{rel}

1. Chiral: $U_* \, P_{\mathrm{rel}} \, U_*^{-1} = \mathbb{1}_{\mathcal{H}} - P_{\mathrm{rel}}$
2. $-$PH: $T \, P_{\mathrm{rel}} \, T^{-1} = \mathbb{1}_{\mathcal{H}} - P_{\mathrm{rel}}, T^2 = -\mathbb{1}_{\mathcal{H}}$
3. $-$TR: $T_* \, P_{\mathrm{rel}} \, T_*^{-1} = P_{\mathrm{rel}}, T_*^2 = -\mathbb{1}_{\mathcal{H}}$

$\Longrightarrow$ Class CII + Index

Coincides with class AII + imaginary line gap classification by Kawabata et al. even though σ_{rel} does not fit the gap type!

Example: Pseudoselfadjoint Operator with Odd TR and Odd TR*



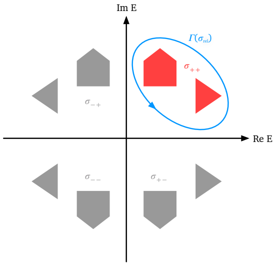
Choices for σ_{rel}

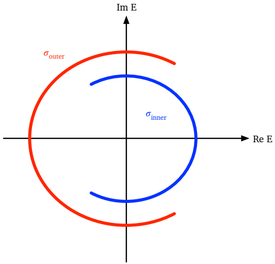
$\sigma_{\mathrm{rel}} =$	*Our classification*	*Kawabata et al.*	*Agreement?*
σ_{++}	class AII + index	AII, η_+, P	Accidental?!*
$\sigma_{++} \cup \sigma_{+-}$	2 × class AII	AII, η_+, $\mathrm{L_r}$	Yes
$\sigma_{++} \cup \sigma_{-+}$	class CII + index	AII, η_+, $\mathrm{L_i}$	Yes*
$\sigma_{++} \cup \sigma_{--}$	class CII + index	Not covered	No

For other examples: comparison of two classifications more subtle

* In principle, a relative index between two projections can be defined, although I do not know whether it can be non-zero.

Symmetries and Constraints of P_{rel}





Presence of symmetries of

- the operator H and
- the relevant spectrum $\sigma_{\mathrm{rel}} \overset{?}{=} -\sigma_{\mathrm{rel}}, \pm\overline{\sigma_{\mathrm{rel}}}$

leads to

1. $(*)$**-Symmetries**

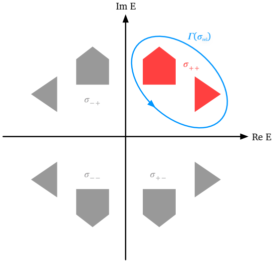
$$U \, P_{\mathrm{rel}} \, U^{-1} = P_{\mathrm{rel}}$$
$$U_* \, P_{\mathrm{rel}} \, U_*^{-1} = W \, P_{*,\mathrm{rel}} \, W^{-1}$$

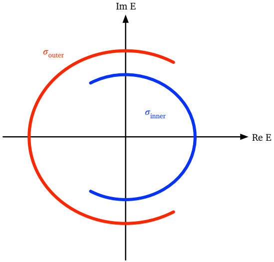
2. $(*)$**-Constraints**

$$U \, P_{\mathrm{rel}} \, U^{-1} = \mathbb{1}_{\mathcal{H}} - P_{\mathrm{rel}}$$
$$U_* \, P_{\mathrm{rel}} \, U_*^{-1} = W \, \big(\mathbb{1}_{\mathcal{H}} - P_{*,\mathrm{rel}}\big) \, W^{-1}$$

between of $P_{\mathrm{rel}} := 1_{\sigma_{\mathrm{rel}}}(H)$ and $P_{*,\mathrm{rel}} := 1_{\overline{\sigma_{\mathrm{rel}}}}(H)$

Symmetries and Constraints of P_{rel}





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leads to

1. $(*)$-**Symmetries**

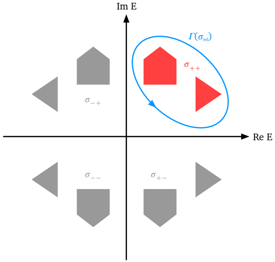
$$U\, 1_{\sigma_{\mathrm{rel}}}(H)\, U^{-1} = 1_{\sigma_{\mathrm{rel}}}(H)$$
$$U_*\, 1_{\sigma_{\mathrm{rel}}}(H)\, U_*^{-1} = 1_{\sigma_{\mathrm{rel}}}(H^*)$$

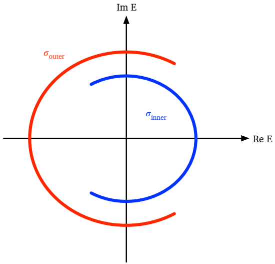
2. $(*)$-**Constraints**

$$U\, 1_{\sigma_{\mathrm{rel}}}(H)\, U^{-1} = \mathbb{1}_{\mathcal{H}} - 1_{\sigma_{\mathrm{rel}}}(H)$$
$$U_*\, 1_{\sigma_{\mathrm{rel}}}(H)\, U_*^{-1} = \mathbb{1}_{\mathcal{H}} - 1_{\sigma_{\mathrm{rel}}}(H^*)$$

$$1_{\sigma_{\mathrm{rel}}}(H^*) = W\, 1_{\sigma_{\mathrm{rel}}}(H^{*_W})\, W^{-1} = W\, 1_{\overline{\sigma_{\mathrm{rel}}}}(H)\, W^{-1}$$

Symmetries and Constraints of P_{rel}





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between of $P_{\mathrm{rel}} := 1_{\sigma_{\mathrm{rel}}}(H)$ and $P_{*,\mathrm{rel}} := 1_{\overline{\sigma_{\mathrm{rel}}}}(H)$

Classification of (Pairs of) Projections with Symmetries and Constraints

1. $(*)$-**Symmetries**

$$U\, P_{\mathrm{rel}}\, U^{-1} = P_{\mathrm{rel}}$$
$$U_*\, P_{\mathrm{rel}}\, U_*^{-1} = W\, P_{*,\mathrm{rel}}\, W^{-1}$$

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between of $P_{\mathrm{rel}} := 1_{\sigma_{\mathrm{rel}}}(H)$
and $P_{*,\mathrm{rel}} := 1_{\overline{\sigma_{\mathrm{rel}}}}(H)$

- Our approach does not rely on any particular classification technique
  ⇝ suitable K-theories (e. g. à la Freed & Moore)
  ⇝ vector bundles with symmetries (De Nittis & Gomi)
- Our approach clearly delineates two classes of topological phenomena:
  1. Analogs of topological phenomena in selfadjoint systems
     ⇝ classification only involves P_{rel}
  2. True non-selfadjoint topological phenomena
     ⇝ classification involves P_{rel} *and* $P_{*,\mathrm{rel}}$
- Presence of similarity transform W does not impact classification in at least the periodic case (under very mild conditions on W)

Classification of (Pairs of) Projections with Symmetries and Constraints

1. $(*)$-**Symmetries**

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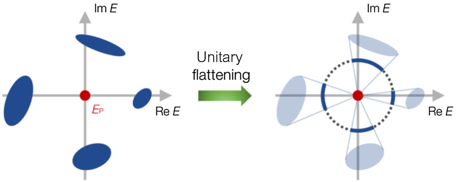
Mathematical Point Gap Classification via Unitary Phase Operator



Our approach

- Define modulus as $|H|$ via analytic functional calculus ($E \mapsto |E|$ analytic away from $E = 0$)
- Phase operator $U := H\,|H|^{-1}$
- $[U, |H|] = 0$
- U unique
- No homotopy argument, extended hamiltonian
- At present I only fully understand the diagonalizable case

Kawabata et al.

- H homotopically deformed to unitary V_H
- $\sigma(V_H)$ is "angular part" of spectrum
- Unless H is normal $[V_H, M] \neq 0$
- V_H not unique! $\rightsquigarrow$ one per scalar product

What Physical Data Choose the Point Gap Over the Line Gap Classification

I do not know yet.

Today's Take-Away

- General classification theory for **spectral operators**
- *Extends 38-fold classification* from periodic tight-binding to generic spectral operators
  $\rightsquigarrow$ Inclusion of effects of disorder
  $\rightsquigarrow$ More/less symmetry classes? Unclear.
- Eliminates superfluous steps from Kawabata et al.
  $\rightsquigarrow$ No homotopy arguments, no extended operator
- *Point gap classification*
  Mathematics well-understood, but I do not understand physics
- *"Line gap" classification*
  Mechanism of identifying symmetries of the relevant operator and their nature more explicit

Future Developments

- Proof of continuity of spectra for spectral operators
  ⇝ Gap in my current work that needs to be filled
- Classification of spectral operators on **Banach** spaces?
  Derivation seems to only depend on algebra, not geometry
- Classification beyond spectral operators? (I am skeptical …)
- Understanding of “truly non-selfadjoint” topological classes
  1. Analogs of topological phenomena in selfadjoint systems
     ⇝ classification only involves P_{rel}
  2. True non-selfadjoint topological phenomena
     ⇝ classification involves P_{rel} *and* $P_{*,\mathrm{rel}}$
- *Bulk-boundary correspondences* for “truly non-selfadjoint” topological classes
- Other topological phenomena in non-selfadjoint systems not covered by existing theory
  ⇝ Yes, they exist! I know of at least one example.

Thank you!

Q&A

Diagonalizable Operators are Normal WRT a Weighted Scalar Product

$$G\,H\,G^{-1} = \int_{\mathbb{C}} E\,\mathrm{d}P(E)$$

implies H is normal with respect to the adjoint $H^{*_W} = W^{-1}\,H^*\,W$ where the weight $W = G^*\,G$ enters the scalar product

$$\langle \varphi, \psi \rangle_W := \langle G\varphi, G\psi \rangle = \langle \varphi, G^*\,G\psi \rangle.$$

$\langle \cdot \, , \cdot \rangle_W$ is a scalar product

- $G \in \mathcal{B}(\mathcal{H})^{-1}$ implies $W \in \mathcal{B}(\mathcal{H})^{-1}$
- $0 < c \leq W \leq C$
- Scalar product not unique!
- $G \rightsquigarrow U\,G\,V$ where U is unitary and $V \in \mathcal{B}(\mathcal{H})^{-1}$ commutes with H

H is normal with respect to ^{*_W}

- $\big[H \, , \, H^{*_W}\big] = 0 \iff \big[G\,H\,G^{-1} \, , \, (G\,H\,G^{-1})^*\big] = 0$
- H admits functional calculus
- $H = H_{\mathrm{Re}} + \mathrm{i} H_{\mathrm{Im}}$, $\big[H_{\mathrm{Re}} \, , \, H_{\mathrm{Im}}\big] = 0$

Characterizations of Diagonalizability

Theorem

The following are equivalent:

1. H *is diagonalizable.*
2. $G\,H\,G^{-1}$ *is normal for some* $G \in \mathcal{B}(\mathcal{H})$.
3. $\exists\, G \in \mathcal{B}(\mathcal{H})^{-1}$: $G\,H\,G^{-1}$ *admits a functional calculus* $f \mapsto f(G\,H\,G^{-1})$.
4. H *is normal with respect to* ^{*w} *for some* $G \in \mathcal{B}(\mathcal{H})$.
5. H *admits a functional calculus* $f \mapsto f(H)$.
6. $H = H_{\mathrm{Re}} + \mathrm{i} H_{\mathrm{Im}}$ *has a cartesian decomposition,* $[H_{\mathrm{Re}}, H_{\mathrm{Im}}] = 0$, $H_{\mathrm{Re},\mathrm{Im}} = H_{\mathrm{Re},\mathrm{Im}}^{*w}$ *for some* $G \in \mathcal{B}(\mathcal{H})^{-1}$.

Characterizations of Diagonalizability

Theorem

1. *$H \in \mathcal{B}(\mathcal{H})$ is diagonalizable exactly when it has a unique* ***cartesian decomposition***

$$H = H_{\mathrm{Re}} + \mathrm{i} H_{\mathrm{Im}} , \qquad [H_{\mathrm{Re}} , H_{\mathrm{Im}}] = 0,$$

and $H_{\mathrm{Re}\,,\mathrm{Im}} = H_{\mathrm{Re}\,,\mathrm{Im}}^{*_W}$ for some $G \in \mathcal{B}(\mathcal{H})^{-1}$.

2. *$H \in \mathcal{B}(\mathcal{H})^{-1}$ is diagonalizable with bounded inverse exactly when it has a unique* ***polar decomposition***

$$H = V_H \, |H|, \qquad [V_H, |H|] = 0,$$

where V_H is $*_W$-unitary and $|H| = |H|^{*_W}$ for some $G \in \mathcal{B}(\mathcal{H})^{-1}$.

Characterizations of Diagonalizability

Theorem

Let $H \in \mathcal{B}(\mathcal{H})$ be diagonalizable and $V \in \mathcal{B}(\mathcal{H})^{-1}$ a similarity transform. Then we have:

1. $V\,H\,V^{-1}$ *is diagonalizable.*
2. $C\,H\,C$ *is diagonalizable where* C *is a complex conjugation.*
3. H^* *is diagonalizable where* * *is any adjoint on* $\mathcal{H}$*.*
4. $(V\,H\,V^{-1})_{\mathrm{Re},\mathrm{Im}} = V\,H_{\mathrm{Re},\mathrm{Im}}\,V^{-1}$
5. $f(V\,H\,V^{-1}) = V\,f(H)\,V^{-1}$