Subfactors and Quantum Symmetry

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§0 Introduction

Since the pioneering work of V. F. R. Jones [28, 29], the subfactor theory has experienced more and more unexpected deep interactions with other fields such as low dimensional topology, quantum group theory, solvable lattice model theory, and conformal field theory.

Within the theory of operator algebras, the subfactor technique has produced solutions to old and new problems. The most important problem in the analytic theory of subfactors is to determine when the higher relative commutants $\bigvee_k(M'\cap M_k)\subset\bigvee_k(N'\cap M_k)$ gives a subfactor anti-conjugate to the original subfactor $N\subset M$, where M_k is given by the Jones basic construction. S. Popa [55, 56, 58, 59] has given an answer to this problem in the strongest form. That is, if a subfactor $N\subset M$ is extremal and strongly amenable [51, 52, 56], the higher relative commutants $\{M_j'\cap M_k\}_{jk}$ contains the complete information about the subfactor. The double sequence $\{M_j'\cap M_k\}_{jk}$ is a double sequence of commuting squares [53] of finite dimensional algebras, but not all the double sequences of commuting squares of finite dimensional algebras come from subfactors as higher relative commutants, so we want to get a system of axioms characterizing the double sequences $\{M_j'\cap M_k\}_{jk}$ in an abstract way. Ocneanu's paragroup [44, 46] was proposed as such a machinery, and it has been very successful. With this general method, we have a complete classification of all the subfactors (of the hyperfinite II_1 factor) with index less than or equal to 4 [4, 19, 20, 22, 24, 32, 33, 44, 46, 56, 62]. The paragroup machinery has been very powerful not only within the operator algebra theory, but also in connections with solvable lattice model theory [2, 3], rational conformal field theory (RCFT) [43, 68], and 3-dimensional topological quantum field theory (TQFT) [60, 63, 69]. We will start with a review of the paragroup theory and then explain its relations to RCFT/TQFT, and applications in operator algebra theory. For basics of the subfactor theory, see [18, 28].

§1 Paragroup theory

Ocneanu's basic idea was to start with a special type of subfactors $N\subset N\rtimes G=M$, where we have an outer action of a finite group G on N, and regard the general subfactor $N\subset M$ as a "quantization" of the above type of subfactors. Then a "paragroup" appears as a "quantized Galois group". The classical Galois theory studies inclusions of fields, and our "quantized" Galois theory studies inclusions of (non-commutative and infinite dimensional) algebras. Paragroups also have surprising similarities to several objects in other fields of mathematics and physics. That is, a paragroup can be roughly regarded as "discrete" differential geometry [44, 45, 46], IRF (Interaction-Round-Faces) models without a spectral parameter

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[2, 3, 9, 11, 14, 19, 25, 26, 32, 33, 61], Rational Conformal Field Theory in an abstract combinatorial sense [5, 12, 14, 36, 43, 70, 71] , generalized $6j$-symbols for representations of quantum groups at roots of unity [12, 37], and generalized $6j$-symbols giving 3-dimensional topological quantum field theory [12, 14, 60, 63, 69]. We note that all the finite groups can also appear as paragroups.

There are two equivalent formulations of paragroups and the both were claimed by A. Ocneanu in [44] and [47] (without details of the proof). We start with a comparison table of the two formulations.

	flat connection	tensor category
multiplication	(dual) principal graph	fusion algebra
numerical data	connection	$6j$-symbol
$*$-structure	unitarity	unitarity
symmetry	renormalization rule	tetrahedral symmetry
algebraic compatibility	flatness	pentagon relation
similarity	IRF model	RCFT/TQFT
Jones A_n subfactors	ABF	WZW $SU(2)_k$, KR $\mathcal{U}_q(sl_2)$

Here ABF, WZW, and KR stand for Andrews-Baxter-Forrester [2], Wess-Zumino-Witten [68], and Kirillov-Reshetikhin [37] respectively. (Also see an exposition [35].)

The both formulations are based on bimodule/intertwiner theory of II_1 factors. See [47, 54, 72] for basics of the bimodule theory. Longo's sector theory [41, 42] also gives essentially equivalent framework.

We start with a hyperfinite type II_1 subfactor $N \subset M$ with finite index and finite depth. (Or more generally, we assume that the subfactor $N \subset M$ is extremal and strongly amenable.) We define call ${}_N M_M$ the *standard* bimodule. (Strictly speaking, we have to look at the $L^2(M)$, the L^2-completion of M with respect to the trace, but we omit the notation $L^2(\cdot)$ for simplicity. A basic idea, due to A. Connes, is that we can regard bimodules as a "quantum" version of (unitary) representations of (compact) groups. So we regard ${}_N M_M$ as an analogue of the fundamental representation. A relative tensor product of bimodules was introduced by Connes. Using this, we make tensor products, ${}_N N_N$, ${}_N M_M$, ${}_N M_N$, ${}_N M \otimes_N M_M$, ${}_N M \otimes_N M_N, \ldots$. We decompose each bimodule into irreducible pieces, and draw a diagram of "branching" rule. Then it turns out that the diagram is same as the Bratteli diagram of $N' \cap M_k$. By the Frobenius reciprocity of bimodules, we know that each step of this diagram is a reflection of the previous step and a new extra part, which could be empty. Thus this diagram can be given by a single bipartite graph and a distinguished vertex $*$ of the graph. We call the graph *principal graph*. The finite depth condition means the principal graph is finite. Similarly, we can start with ${}_M M_M$, and then we get another graph, which is called the dual principal graph. Each vertex of the graphs denotes a bimodule (one of four kinds), and each edge denotes an intertwiner. (We choose intertwiners so that the choice makes orthonormal basis with respect to the natural inner product of intertwiners.) The distinguished vertex $*$ denotes the identity bimodule ${}_N N_N$ or ${}_M M_M$. Next we look at a (finite) path on the Bratteli diagram starting at $*$. Because each edge means an intertwiner, the path means a composition of intertwiners. Furthermore, we look at a pair of paths (ξ,η) with the same length and same endpoints. Such a pair is called a string. We identify ξ,η with the composite intertwiners they represent, then the string is identified with the composition $\xi^*\eta$. Then it is easy to see that strings make a system of matrix units in the higher relative commutants. We summarize the graphical meaning as follows.

graphical objects	meaning
(dual) principal graph	multiplication by the generator
vertex	bimodule
edge	intertwiner
path	composition of intertwiners
string	matrix unit

With these, the tower of the higher relative commutants is naturally identified with

$$\mathrm{End}({}_N N_N) \subset \mathrm{End}({}_N M_M) \subset \mathrm{End}({}_N M_N) \subset \mathrm{End}({}_N M \otimes_N M_M) \subset \mathrm{End}({}_N M \otimes_N M_N) \subset \cdots .$$

Next, we make a double sequence of endomorphism spaces as follows.

$$\begin{array}{ccccccc} \mathrm{End}({}_N N_N) & \subset & \mathrm{End}({}_N M_M) & \subset & \mathrm{End}({}_N M_N) & \subset & \cdots \\ \cap & & \cap & & \cap & & \\ \mathrm{End}({}_M M_N) & \subset & \mathrm{End}({}_M M \otimes_N M_M) & \subset & \mathrm{End}({}_M M \otimes_N M_N) & \subset & \cdots \end{array}$$

Then we can identify the double sequence with $\{M'_{-j} \cap M_k\}_{j,k}$, where $\{M_{-j}\}_j$ is a choice of a tunnel.

Next we discuss the connection. For example, take four bimodules, N-M bimodule A, N-N bimodule B, M-M bimodule C, and M-N bimodule D. We look at the diagram

$$\begin{array}{ccc} {}_M M \otimes_N A \otimes_M M_N & \xrightarrow{\xi_1} & {}_M M \otimes_N B_N \\ {\scriptstyle \xi_3}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ {}_M C \otimes_M M_N & \xrightarrow[\xi_4]{} & {}_M D_N \end{array}$$

and make the composition of the four intertwiners $\xi_4(\xi_3 \otimes \mathrm{id}_{{}_M M_N})(\mathrm{id}_{{}_M M_N} \otimes \xi_1)^* \xi_2^*$, which is an endomorphism from D to D, so it must be a scalar. We denote this number simply by

$$\begin{array}{ccc} A & \xrightarrow{\xi_1} & B \\ {\scriptstyle \xi_3}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ C & \xrightarrow[\xi_4]{} & D \end{array}$$

This is called a connection. If we replace ${}_M M_N$, ${}_M M_N$ by more general bimodules in the system, we have 6 bimodules and 4 intertwiners to get a number. Such an assignment of the number is called a $6j$-symbol. We want to get a system of abstract axioms for connections and $6j$-symbols.

The first axiom is *Unitarity*, which is given as follows.

$$\sum_{B,\xi_1,\xi_2} \begin{array}{ccc} A & \xrightarrow{\xi_1} & B \\ {\scriptstyle \xi_3}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ C & \xrightarrow[\xi_4]{} & D \end{array} \cdot \overline{\begin{array}{ccc} A & \xrightarrow{\xi_1} & B \\ {\scriptstyle \eta_3}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ C' & \xrightarrow[\eta_4]{} & D \end{array}} = \delta_{\xi_3,\eta_3}\delta_{\xi_4,\eta_4}\delta_{C,C'}.$$

We have a similar formula for $6j$-symbols.

The second axiom is *Renormalization*, which is as follows.

$$\begin{array}{ccc} A & \xrightarrow{\xi_1} & B \\ {\scriptstyle \xi_3}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ C & \xrightarrow[\xi_4]{} & D \end{array} = \sqrt{\frac{\mu(B)\mu(C)}{\mu(A)\mu(D)}} \overline{\begin{array}{ccc} B & \xrightarrow{\tilde{\xi}_1} & A \\ {\scriptstyle \xi_2}\downarrow & & \downarrow{\scriptstyle \xi_3} \\ D & \xrightarrow[\tilde{\xi}_4]{} & C \end{array}}$$

Here the coefficient comes from the Frobenius reciprocity. The number $\mu(\cdot)$ is given by $\mu({}_AX_B) = \sqrt{\dim_A X \dim X_B}$, where A, B is N or M. From the operator algebraic viewpoint, this is really a commuting square condition [53, 18, 45, 46]. For $6j$-symbols, we have a slightly more general form of symmetry, which is called the *tetrahedral symmetry*. This again comes from the Frobenius reciprocity.

The last and most important axiom is *Flatness*. The flatness of the connection means that the following partition function has a value 1.

$$\begin{array}{ccccccc} * & \xrightarrow{\xi_1} & \cdot & \xrightarrow{\xi_2} & \cdot \;\cdots\; \cdot & \xrightarrow{\xi_m} & * \\ {\scriptstyle \eta_1}\downarrow & & & & & & \downarrow{\scriptstyle \eta_1} \\ \cdot & & & & & & \cdot \\ {\scriptstyle \eta_2}\downarrow & & & & & & \downarrow{\scriptstyle \eta_2} \\ \vdots & & & & & & \vdots \\ {\scriptstyle \eta_n}\downarrow & & & & & & \downarrow{\scriptstyle \eta_n} \\ * & \xrightarrow[\xi_1]{} & \cdot & \xrightarrow[\xi_2]{} & \cdot \;\cdots\; \cdot & \xrightarrow[\xi_m]{} & * \end{array}$$

Here $*$ denotes the identity bimodule ${}_NN_N$ or ${}_MM_M$. The above partition function is defined exactly same as the partition function in statistical mechanics. The meaning of this formula is as follows. From the definition of the connection, it describes the following double sequence of the endomorphism spaces.

$$\begin{array}{ccccccc}
\mathrm{End}({}_N N_N) & \subset & \mathrm{End}({}_N M_M) & \subset & \mathrm{End}({}_N M_N) & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\mathrm{End}({}_M M_N) & \subset & \mathrm{End}({}_M M \otimes_N M_M) & \subset & \mathrm{End}({}_M M \otimes_N M_N) & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\mathrm{End}({}_N M_N) & \subset & \mathrm{End}({}_N M \otimes_N M_M) & \subset & \mathrm{End}({}_N M \otimes_N M_N) & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

The flatness axiom means that the vertical "strings" commute with the horizontal "strings" [32].

For $6j$-symbols, the flatness axiom transforms into another form, which is called the pentagon relation [47, 12].

These axioms are enough to characterize higher relative commutants as abstract sequences of finite dimensional algebras.

§2 Relation between paragroups and RCFT

A combinatorial axiomatization of RCFT was given by Moore and Seiberg in [43]. Their axioms are similar to those for flat connections/$6j$-symbols. Indeed, de Boer-Goeree [5] constructed a paragroup from a given RCFT. By applying this procedure to the Wess-Zumino-Witten models [68], we get a large family of paragroups including the subfactors of Jones [28] and Wenzl [64].

The converse direction, from a paragroup to an RCFT, is not automatic. A direct approach often fails as in [5], but Ocneanu [50] claims that the asymptotic inclusion [46], which can be regarded as an analogue of Drinfeld's quantum double construction, produces a system satisfying the Moore-Seiberg axioms.

Another relation between RCFT and paragroups is the orbifold construction, which has a physical origin [9, 10, 15, 16, 39]. In the subfactor theory, D. E. Evans and the author [11, 32, 33] initiated the orbifold construction, and F. Xu [70, 71] clarified its relation to RCFT. That is, in the orbifold procedure, we have an obstruction for flatness in general, but Xu identified it with the conformal weights in RCFT.

Another similarity between CFT and paragroups is the A-D-E classification. In CFT, Cappeli-Itzykson-Zuber [6] and Kato [30] got an A-D-E classification, and it resembles the A-D-E classification in the paragroup theory [4, 19, 20, 22, 32, 33, 44, 46, 56, 62]. The last missing piece of this similarity was given by Evans and the author [13, 14] by computing the flat parts of the non-flat connection on E_7. This answered a conjecture given by Zuber.

In CFT, the Yang-Baxter equation plays an important role, so it is expected that the Yang-Baxter equation is related to axioms in the paragroup theory, especially flatness. But neither of The Yang-Baxter equation and flatness implies the other, and the first clear relation between the two was given in [11] based on an idea of Roche [61]. With this technique, one can compute the paragroup of the subfactors of Wenzl [64] arising from the Hecke algebras of type A. This uses solutions of the Yang-Baxter equation by Jimbo-Miwa-Okado [25, 26].

§3 Relations between paragroups and TQFT

From the original discovery of the Jones polynomial [29] for knots and links, the theory of operator algebras has had deep interactions with low dimensional topology.

In 3-dimensional topology, E. Witten [69] proposed a general topological invariant based on physical ideas, and Turaev-Reshetikhin [60] and Turaev-Viro [63] gave mathematical rigorous formulations of 3-dimensional topological quantum field theory in the sense of Atiyah. The latter is based on triangulations of compact 3-manifolds, and uses a new version of an old theorem of Alexander [1]. As initial data, the latter uses Kirillov-Reshetikhin quantum $6j$-symbols [37].

Ocneanu [47, 48, 49] claimed that the $6j$-symbols arising from subfactors give a generalized version of the Turaev-Viro TQFT, and Evans and the author [12, 14] gave details of the proof. Roughly speaking, the three axioms for $6j$-symbols arising from the subfactors are equivalent to the three axioms for abstract $6j$-symbols for TQFT. (See [12] for the exact statements.) Recently, Ocneanu [50] discussed chirality in the subfactor theory, and generalized the Reshetikhin-Turaev TQFT in subfactor setting with an extra assumption "full braiding". Furthermore, he claims a general version of Turaev's theorem which states that the Turaev-Viro TQFT splits as a tensor product of the Reshetikhin-Turaev TQFT and its complex conjugate.

§4 Automorphisms of subfactors and paragroup actions on subfactors

As in classical von Neumann algebra theory, the automorphism groups $\mathrm{Aut}(M, N)$, which are the automorphisms of M fixing N globally, has been extensively studied [7, 31, 38, 40, 57, 65, 66, 67].

The author introduced $\chi(M, N)$ in [34] as a relative version of Connes' $\chi(M)$ [8, 27], and found that the finite group action used in the orbifold construction [11, 32, 70] give the entire $\chi(M, N)$. This class also coincides with the automorphisms Izumi studied in several examples [21, 23]. Choda-Kosaki [7, 38] and Popa [57] also studied the same class from a different viewpoint.

From the study [34], it has turned out that strongly amenable subfactors of type II_1 are rather similar to injective type III factors. Roughly speaking, the paragroup is a discrete analogue of the flow of weights of Connes-Takesaki, and Loi's invariant [40] is a discrete analogue of the Connes-Takesaki module, and the above class of automorphisms is an analogue of modular automorphism groups.

We can make a further "quantization". On one hand, because paragroups are regarded as quantization of ordinary (finite) groups, we think of a subfactor problem as a quantized version of a group action problem. On the other hand, the other problems on $\mathrm{Aut}(M, N)$ are problems of classical actions on quantized objects. So as a "double quantization", we can think of paragroup actions [36], which are really certain types of commuting squares of type II_1 factors. Under the strong amenability assumption, we can classify them in terms of combinatorial objects generalizing paragroups. Essentially, we have similar axioms to those for the paragroups, but we have a new axiom, which is a kind of the Yang-Baxter equation. This idea is also useful for ordinary subfactor problem. For example, we can determine the fusion algebras of the Goodman-de la Harpe-Jones subfactor with index $3 + \sqrt{3}$ [18].

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