Paragroups and their actions on subfactors

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1 Introduction

Our aim in this work is to make a "double quantization" of studies of group actions on von Neumann algebras in the setting of Ocneanu's paragroup theory. Detailed proofs will be given in [33].

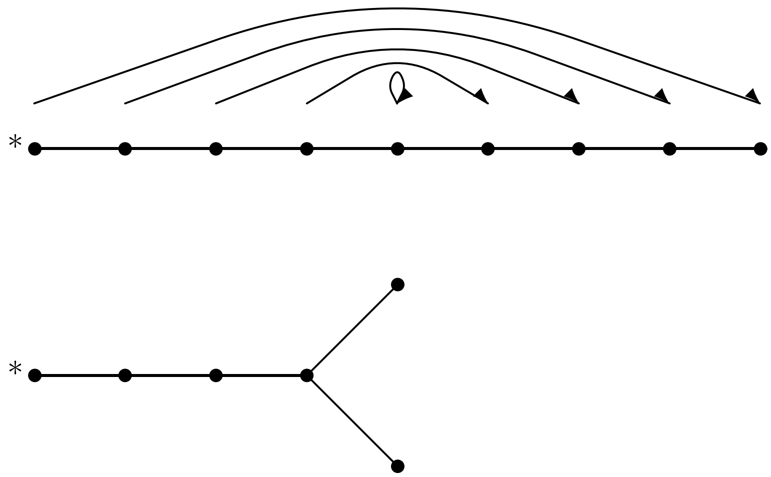
Since the pioneering work [27] of V. F. R. Jones, the theory of subfactors has made much progress both in the internal theory in operator algebras and in relations to low-dimensional (quantum) topology, quantum groups, and theoretical physics. In these interactions, a new and unexpected combinatorial structure has emerged in the subfactor theory. So far, the most powerful machinery to study this structure is Ocneanu's paragroup theory [44], [46], [47].

One one hand, Ocneanu's basic idea was to regard a general subfactor $N \subset M$ as a crossed product by an action of a "paragroup" on a factor N. In this sense, a paragroup is a "quantized" object of an ordinary (finite) group. On the other hand, several studies [9], [29], [32], [36], [38], [56], [57], [70], [71], [72] have been made on group actions on subfactors. These are regarded as studies of actions of ordinary groups on quantum objects. Thus we have the following table, and we will study the missing entry in this work.

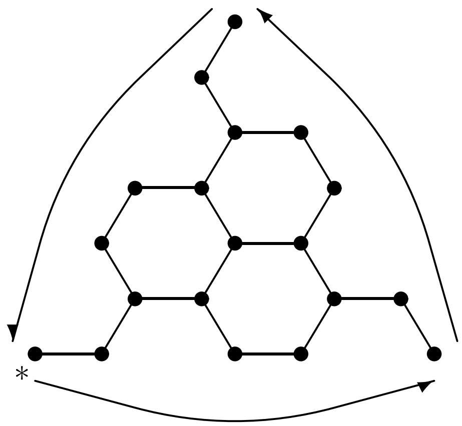
	object	action
subfactor	classical	quantum
$\mathrm{Aut}(M, N)$	quantum	classical
?	quantum	quantum

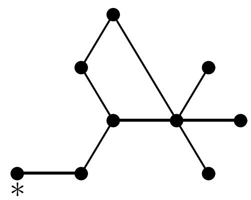
Passing from a function algebra to a non-commutative operator algebra is often called *quantization*, but note that the words quantum/quantization above have a different meaning. In our setting here, a single factor and an ordinary group are classical mathematical objects, and a paragroup represents the quantization.

We give a more concrete motivation for our study here. Orbifold construction in the subfactor theory [13], [30], [75], [76] has been well-studied. Its basic idea, coming from physics [11], [17], [18], [37], [59], is to make a "quotient" of a paragroup with a certain symmetry. The most fundamental example is the $\mathbf{Z}_2$-symmetry of the A_{4n-3} subfactors.



As in the above picture, the principal graph A_{4n-3} has a symmetry of order 2, and this is really a symmetry of the paragroup. Then the quotient by the symmetry, given by a certain fixed point algebra, produces a paragroup of type D_{2n}. The fixed point of the symmetry splits into two vertices by the "orbifold bifurcation". (In the case of A_{4n-1}, the same construction gives back A_{4n-1}, because the obstruction for flatness kills D_{2n+1} as in [30]. This obstruction was further studied in [13] and later identified with a conformal dimension in rational conformal field theory by F. Xu [75].) In this construction, we use an automorphism of order 2 in $\mathrm{Aut}(M,N)$ and make $N^{\mathbf{Z}_2} \subset M^{\mathbf{Z}_2}$, or $N \times \mathbf{Z}_2 \subset M \times \mathbf{Z}_2$. The automorphism appearing in this way is actually the same automorphism appearing in the irreducible decompositions of the descendent sectors or M-M bimodules, as shown in [9], [32], (or [16] more generally). These automorphisms also played an important role in the work of M. Izumi [22], [23]. A more complicated example of the orbifold construction is given below.





The first paragroup, for a subfactor of H. Wenzl [69] with index value $\dfrac{\sin^2 3\pi/9}{\sin^2 \pi/9}$, has a $\mathbf{Z}_3$-symmetry and the quotient is given by the next graph with the same index.

With this kind of construction, we get the following commuting square of type II_1 factors with a finite group G, and the paragroups for $N \subset M$ and $N \times G \subset M \times G$ are different.

$$\begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ N \times G & \subset & M \times G \end{array} \tag{1}$$

This difference suggests that the above commuting square itself has some interesting combinatorial structure and we should study more general commuting squares. For example, the notion of the co-standard graph of S. Popa [56], [57] is determined by the tower $N' \cap (M_k \times G)$ and this graph is an invariant of the above type of commuting squares. So we are led to study of commuting squares of II_1 factors. Our assumption for the study is as follows.

Assumption 1.1 The four algebras $M_{00}, M_{01}, M_{10}, M_{11}$ are type II_1 factors with the following properties.

1. The square

$$\begin{array}{ccc} M_{00} & \subset & M_{01} \\ \cap & & \cap \\ M_{10} & \subset & M_{11} \end{array}$$

of AFD II_1 factors is commuting and co-commuting in the sense of [61, Definition 3.4].

2. $[M_{11} : M_{00}] < \infty$.

3. The subfactor $M_{00} \subset M_{11}$ has finite depth.

Condition (1) means a non-degenerate commuting square in the sense of S. Popa [55].

We could generalize the third condition so that the subfactor $M_{00} \subset M_{11}$ is extremal and strongly amenable in the sense of [55], but we use the above assumption for simplicity. See [33] for the full generality.

In the above setting, we regard the above commuting square in Assumption 1.1 as a quantization of a commuting square (1). Thus the subfactor $M_{10} \subset M_{11}$ is regarded as a "crossed product by a paragroup action on $M_{00} \subset M_{01}$". Note that the situation is now symmetric and we can also regard the subfactor $M_{01} \subset M_{11}$ as a "crossed product by a paragroup action on $M_{00} \subset M_{10}$".

We will show that such a commuting square is completely classified by combinatorial objects generalizing Ocneanu's paragroups and will give axiomatization of such objects.

S. Okamoto tried to determine when the above commuting square is of the following type:

$$\begin{array}{ccc} N \otimes P & \subset & M \otimes P \\ \cap & & \cap \\ N \otimes Q & \subset & M \otimes Q \end{array} \tag{2}$$

From our viewpoint, this problem is regarded as a generalization of the Connes-Jones-Ocneanu type splitting [10], [26], [43] of a "paragroup action" on a subfactor, and we can get a characterization of the above type of commuting squares. But such a commuting square is of rather trivial type, and it is not clear whether we really have many different types of commuting squares. We will give several examples and discuss their relations to Rational Conformal Field Theory (RCFT) and 3-dimensional Topological Quantum Field Theory (TQFT).

At the end of the Introduction, we mention studies of intermediate subfactors [7], [46, II.7], [66]. In our settings, the two factors M_{01} and M_{10} are intermediate subfactors of M_{00} and M_{11}, so the these studies look applicable, but our settings are rather different from theirs. In the study of intermediate subfactors, one subfactor is given, and we look for intermediate subfactors, but in our setting our initial object is usually $M_{00} \subset M_{01}$ and we are interested in what kind of commuting square we can have for the fixed $M_{00} \subset M_{01}$. Also, it is practically impossible to list all the intermediate subfactors of $M_{00} \subset M_{11}$ and determine which pair makes a commuting square.

We finally note that the study of commuting squares of II_1 factors was initiated by Y. Watatani and his students [60], [61], [67], [68]. Our method gives classification and construction of the objects they study.

2 Survey of the paragroup theory

Before going into details of our new set of axioms, we start with the axiomatization of Ocneanu's paragroup theory, because we think that it will be helpful for understanding.

In this section, we mean by a *subfactor* an inclusion $N \subset M$ of approximately finite dimensional (AFD) factors of type II_1 with finite Jones index and finite depth. (Of course, we do not need the AFD condition for getting a paragroup, and the finite depth condition could be weakened, but we just work in simpler cases.)

Ocneanu's fundamental idea [44] to study a subfactor $N \subset M$ was to look at the bimodules naturally given by the inclusion, and have an analogue of a representation theory (of compact groups). As emphasized by A. Connes, a bimodule over von Neumann algebras are a correct analogue of a group representation. (See [52] for a general theory of bimodules.) Then we have the following table of analogies. (We also include entries for an essentially same method based on Longo's sector theory [22], [39], [40] for the subfactor theory of properly infinite factors [35].)

group representation	bimodule	sector
direct sum	direct sum	direct sum
tensor product	relative tensor product	composition
dimension	(Jones index)$^{1/2}$	statistical dimension
contragredient representation	conjugate bimodule	conjugate sector
Frobenius reciprocity	Frobenius reciprocity	Frobenius reciprocity
fundamental representation	${}_N M_M$	$\rho : M \to N$

With these in mind, we can give two (equivalent) axiomatizations [44], [47] of higher relative commutants $\{N_j' \cap M_k\}_{j,k}$. Because the higher relative commutants completely recover the original subfactor in strongly amenable cases, which include finite depth cases, by [55], it is enough to characterize the higher relative commutants in combinatorial terms for classification of subfactors.

One axiomatization [44] is based on Ocneanu's notion of flat connection and the other is based on certain type of (finite) tensor category with (quantum) $6j$-symbols. The both were claimed Ocneanu without details. See [14], [30], [31], [77]. for details. We list the corresponding table of the two axiomatizations.

	flat connection	tensor category
multiplication table	(dual) principal graph	fusion algebra
numerical data	connection	$6j$-symbol
$*$-structure	unitarity	unitarity
symmetry	renormalization rule	tetrahedral symmetry
algebraic compatibility	flatness	pentagon relation
similarity	IRF model	RCFT/TQFT
Jones' A_n subfactor	ABF model	WZW $SU(2)_k$ model KR $\mathcal{U}_q(sl_2)$ $6j$-symbol

In the above, the entries "similarity" mean the two sets of axioms are similar to those of interaction-round-faces (IRF) models in the theory of exactly solvable models [4], and those of rational conformal field theory (RCFT) [8], [41] and topological quantum field theory (TQFT) [3], [65]. The entries "Jones' A_n subfactor" mean that the Jones subfactors correspond to the Andrews-Baxter-Forrester model [2] in the flat connection approach and to the Wess-Zumino-Witten $SU(2)_k$ models [74] and the Kirillov-Reshetikhin $6j$-symbols for $\mathcal{U}_q(sl_2)$ [34] in the tensor category approach.

Note that if we replace the finite depth condition of the subfactor by strong amenability [55], we need two more axioms; ergodicity and amenability. Ergodicity [55] is expressed as factoriality of the model algebras given by the string algebra construction [44], [45], [46], and amenability is expressed by an extremality [50], [51] of the model inclusion.

An outline of the method of getting the combinatorial data in the both ways of axiomatization out of a subfactor is as follows. First we take a bimodule ${}_N L^2(M)_M$, which will be simply denoted by ${}_N M_M$, as a basic object. Next we make a (finite) relative tensor product $\cdots \otimes_N M \otimes_M M \otimes_N M \otimes_M \cdots$ of ${}_N M_M$ and ${}_M M_N$ and make an irreducible decomposition. The finite index assumption implies that we have finitely many irreducible bimodules for each tensor product and the finite depth assumption means that we have finitely many irreducible bimodules for all the possible tensor products. We represent each irreducible bimodule (of one of four kinds, N-N, N-M, M-N, M-M) by a vertex. Next we draw edges between vertices. For an N-N bimodule X and an N-M bimodule Y, the number of edges is given by the multiplicity of Y in $X \otimes_N M$, which is also equal to the dimension of the intertwiner space $\mathrm{Hom}({}_N X \otimes_N M_M, {}_N Y_M)$. This number of edges is also equal to the multiplicity of X in $Y \otimes_M M$ and the dimension of $\mathrm{Hom}({}_N Y \otimes_M M_N, {}_N X_N)$ by the Frobenius reciprocity [47], [77] of bimodules. We also draw edges between N-N bimodules and M-N bimodules, N-M bimodules and M-M bimodules, and M-N bimodules and M-M bimodules. We assign a co-isometric intertwiner to each edge so that the intertwiners make an orthonormal basis and the Frobenius dual to the reversed edge.

Each edge of the Bratteli diagram of $N' \cap M_k$ corresponds to an intertwiner as above. A path in the Bratteli diagram then corresponds to composition of intertwiners. For example, suppose we have a path of length 4 represented by four edges $\xi_1 \in \mathrm{Hom}({}_N N \otimes_N M_M, {}_N M_M)$, $\xi_2 \in \mathrm{Hom}({}_N M \otimes M M_N, {}_N X_N)$, $\xi_3 \in \mathrm{Hom}({}_N X \otimes_N M_M, {}_N Y_M)$, $\xi_4 \in \mathrm{Hom}({}_N Y \otimes M M_N, {}_N Z_N)$, respectively. Then the composite intertwiner represented by this path is

$$\xi_4 \cdot (\xi_3 \otimes id) \cdot (\xi_2 \otimes id) \cdot (\xi_1 \otimes id) \in \mathrm{Hom}({}_N M \otimes_N M \otimes_M M \otimes_N M \otimes_M M_N, {}_N Z_N).$$

We take another path from ${}_N N_N$ to ${}_N Z_N$ with the same length 4 represented by $\eta_1, \eta_2, \eta_3, \eta_4$. We assign to the pair

$$(\xi_1 \cdot \xi_2 \cdot \xi_3 \cdot \xi_4, \eta_1 \cdot \eta_2 \cdot \eta_3 \cdot \eta_4)$$

the composition

$$\begin{aligned} & (\eta_1 \otimes id)^* \cdot (\eta_2 \otimes id)^* \cdot (\eta_3 \otimes id)^* \cdot \eta_4^* \cdot \xi_4 \cdot (\xi_3 \otimes id) \cdot (\xi_2 \otimes id) \cdot (\xi_1 \otimes id) \\ \in \ & \mathrm{End}({}_N M \otimes_N M \otimes_M M \otimes_N M \otimes_M M_N). \end{aligned}$$

Such a pair of paths is called a *string*. In this way, we assign a matrix unit in the higher relative commutant to a string. We summarize this method as follows.

graphical object	meaning
vertex	bimodule
edge	intertwiner
reversed edge	Frobenius dual
path	composition of intertwiners
string	matrix unit in the higher relative commutant

In this way, we get a graphical description of the higher relative commutants $\{N' \cap M_k\}$. For a full description of the higher relative commutants, we have to deal with the double sequence as follows.

$$
\begin{array}{cccccccc}
 & & \overset{\cdot \otimes_N M_M}{} & & \overset{\cdot \otimes_M M_N}{} & & \overset{\cdot \otimes_N M_M}{} & \\
 & \mathrm{End}({}_N N_N) & \subset & \mathrm{End}({}_N M_M) & \subset & \mathrm{End}({}_N M_N) & \subset & \cdots \\
{}_M M_N \otimes \cdot & \cap & & \cap & & \cap & & \\
 & \mathrm{End}({}_M M_N) & \subset & \mathrm{End}({}_M M \otimes_N M_M) & \subset & \mathrm{End}({}_M M \otimes_N M_N) & \subset & \cdots \\
{}_N M_M \otimes \cdot & \cap & & \cap & & \cap & & \\
 & \mathrm{End}({}_N M_N) & \subset & \mathrm{End}({}_N M \otimes_N M_M) & \subset & \mathrm{End}({}_N M \otimes_N M_N) & \subset & \cdots \\
{}_M M_N \otimes \cdot & \cap & & \cap & & \cap & & \\
 & \vdots & & \vdots & & \vdots & &
\end{array}
$$

By the above method, we get the Bratteli diagram for the above double sequence, and we have several systems of matrix units described graphically. Because we have different systems of the matrix units of the same algebra $N_j \cap M_k$, we need to give identification among several bases. This is done “locally” as follows. Make a square with four vertices A, B, C, D and four edges $\xi_1, \xi_2, \xi_3, \xi_4$ from our graphical system.

$$
\begin{array}{ccc}
A & \xrightarrow{\xi_3} & B \\
{\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_4} \\
C & \xrightarrow[\xi_2]{} & D
\end{array}
$$

We will get a complex number from the above picture and the number will be also denoted by this picture. Suppose that A, B, C, D are N-N, N-M, M-N, M-M bimodules, respectively. Then we get the following diagram.

$$
\begin{array}{ccc}
{}_M M \otimes_N A \otimes_N M_M & \xrightarrow{id \otimes \xi_3} & {}_M M \otimes_N B_M \\
{\scriptstyle \xi_1 \otimes id}\downarrow & & \downarrow{\scriptstyle \xi_4} \\
{}_M C \otimes_N M_M & \xrightarrow[\xi_2]{} & {}_M D_M
\end{array}
$$

With this diagram, we can make a composition $\xi_2 \cdot (\xi_1 \otimes id) \cdot (id \otimes \xi_3)^* \cdot \xi_4^* \in \mathrm{End}({}_M D_M) = \mathbf{C}$, which gives the desired number. The other kinds of bimodules are treated similarly. We call this assignment of a number to each square a *connection*. This is a map depending on specific choices of bimodules and intertwiners, and the equivalence class of a connection is a well-defined invariant of a subfactor. This map is, in a sense, a quantized version of the characteristic invariant [26]. The connection is characterized by three axioms. The first one is easy and called *unitarity*.

$$\sum_{C,\xi_3,\xi_4} \begin{array}{ccc} A & \xrightarrow{\xi_1} & B \\ {\scriptstyle \xi_4}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ D & \xrightarrow[\xi_3]{} & C \end{array} \; \overline{\begin{array}{ccc} A & \xrightarrow{\xi_1'} & B' \\ {\scriptstyle \xi_4}\downarrow & & \downarrow{\scriptstyle \xi_2'} \\ D & \xrightarrow[\xi_3]{} & C \end{array}} = \delta_{B,B'} \delta_{\xi_1,\xi_1'} \delta_{\xi_2,\xi_2'}.$$

This just means that our co-isometries are chosen so that they make an orthonormal basis.

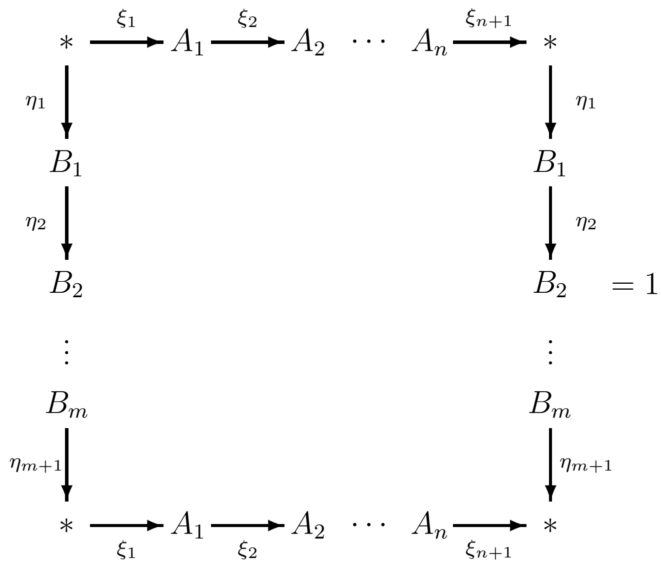
Another axiom is the following, and this is a direct consequence of the Frobenius reciprocity. In the operator algebraic framework, this condition corresponds to the commuting square condition [53], [19] as in [45], [62]. This axiom is called *renormalization rule*.

$$\begin{array}{ccc} A & \xrightarrow{\xi_3} & B \\ {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_4} \\ C & \xrightarrow[\xi_2]{} & D \end{array} = \left(\frac{[B][C]}{[A][D]} \right)^{1/4} \overline{\begin{array}{ccc} B & \xrightarrow{\tilde{\xi}_3} & A \\ {\scriptstyle \xi_4}\downarrow & & \downarrow{\scriptstyle \xi_1} \\ D & \xrightarrow[\tilde{\xi}_2]{} & C \end{array}}$$

$$\begin{array}{ccc} A & \xrightarrow{\xi_3} & B \\ {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_4} \\ C & \xrightarrow[\xi_2]{} & D \end{array} = \left(\frac{[B][C]}{[A][D]} \right)^{1/4} \overline{\begin{array}{ccc} C & \xrightarrow{\xi_2} & D \\ {\scriptstyle \tilde{\xi}_1}\downarrow & & \downarrow{\scriptstyle \tilde{\xi}_4} \\ A & \xrightarrow[\xi_3]{} & B \end{array}}$$

Here $[A]$ means the square of the Jones index of the subfactor corresponding to the bimodule. The vector $[A]^{1/2}$ is the Perron-Frobenius eigenvector of the graphs with the eigenvalue equal to the square root of the Jones index of the original subfactor.

The last main axiom is called *flatness*.



Here a large diagram means a sum of the product of the connection values over all the possible configurations for a fixed boundary as in [44], [46], [29]. See [29] for the linear algebraic meaning of this axiom and [47] for a proof. This represents associativity of the bimodule tensor product in a certain sense.

We also have some minor axioms such as the initialization axiom [44], but we omit them here.

In the tensor category approach, we replace ${}_N M_M$ and ${}_M M_N$ in the above definition of the connection by general bimodules in the system. Then we have a complex number for six bimodules and four intertwiners. This assignment of numbers is called a (quantum) *6j-symbol*. The $6j$-symbols also have to satisfy three axioms. The first two axioms, unitarity and the tetrahedral symmetry, are basically the same as the above unitarity and renormalization rule. The other axiom expresses associativity of the tensor products of bimodules in a more explicit way than the flatness axiom, and is called the *pentagon relation*. This is essentially same as the pentagon relation in ordinary representation theory. See [47], [14] for more details on a relation between the flatness axiom and the pentagon relation.

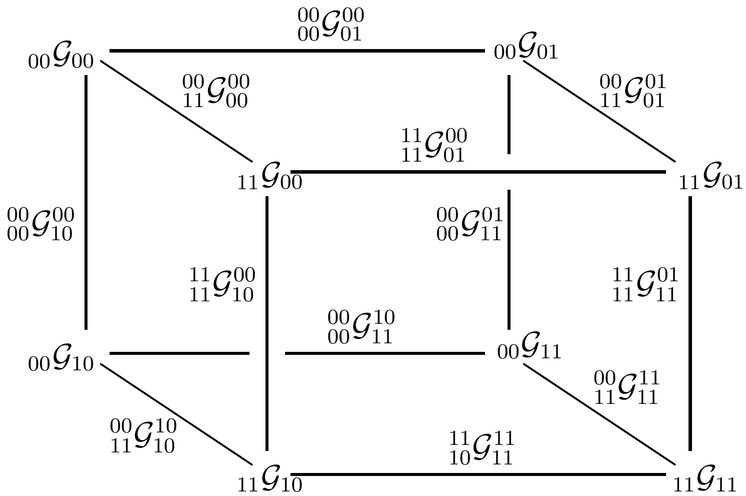
At the end of this section, we discuss a relation between the two axiomatizations of paragroups. The main difference of the two lies in the flatness axiom and the pentagon relations. These are the most important conditions of our objects and also are the most difficult conditions to verify in concrete cases. From the conceptual viewpoint, the tensor category approach has a more transparent meaning, and the pentagon relations look relatively easier to verify than the flatness axiom which involves a huge number of terms. (The number of terms in the summation of the flatness axiom can be easily over billions even for fairly simple examples.) The tensor category approach, however, has a serious drawback if we really want to make a concrete and explicit computations of the connections or $6j$-symbols. That is, we need much more numerical data in the tensor category approach than in the flat connection approach. We have easier equation, the pentagon relation, only because we have more data at the beginning. Take more concrete examples. As announced by A. Ocneanu [44] and verified by [22], [24], [30], [63], we have an A-D-E classification for subfactors with index less than four, and we have A_n, D_{2n}, E_6, E_8 as a possible principal graph. (The E_6 was first constructed by [5] earlier. Also see [15], [21], [25], [31, Appendix], [46] for related results.) For the case A_n, we have explicit formulae for the $6j$-symbols of q-integers of Kirillov-Reshetikhin [34], but the explicit formulae for the case D_{2n}, E_6, E_8 have not been given, and it would be quite difficult to prove a realization of the principal graph E_8 by direct computations based on the tensor category approach. In the flat connection approach, an advantage is that we can often explicitly write down the connection formula for given graphs, if the graphs are relatively simple.

3 Paragroup actions on subfactors

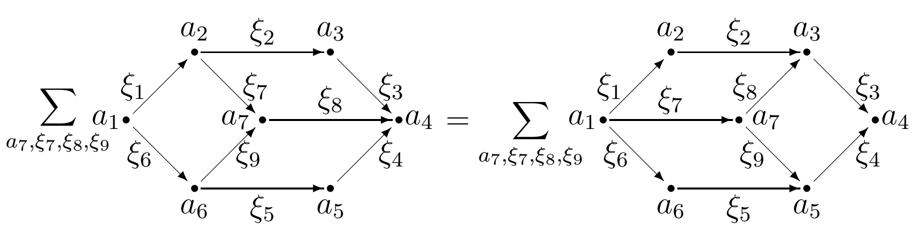
Our aim is to generalize the construction of a paragroup out of a subfactor in the previous section to the commuting squares of II_1 factors. We assume Assumption 1.1. A first observation is that we can make a basic construction of the commuting square as in [61, Section 7] so that we have a double sequence $\{M_{kl}\}_{k,l=1,2,3,\dots}$ of commuting squares of II_1 factors. Next, if we choose a generating tunnel $\{M_{-j,-j}\}_{j=1,2,3,\dots}$ for the subfactor $M_{00} \subset M_{11}$ by [54], [55], then it is easy to see $\bigvee_{j=0}^{\infty}(M'_{-j,-j} \cap M_{kl} = M_{kl}$ for all $k, l \geq 0$. Thus we only have to characterize the triple sequence $\{M'_{-j,-j} \cap M_{kl}\}_{j,k,l=0,1,2,\dots}$.

Again we use a method based on bimodules. A basic observation is that we have natural isomorphisms $M_{01} \otimes_{00} M_{10} \cong M_{11}$ as M_{01}-M_{10} bimodules, and $M_{10} \otimes_{00} M_{01} \cong M_{11}$ as M_{10}-M_{01} bimodules. (This easily follows from [61, Corollary 7.1], and this condition actually characterizes the commuting and co-commuting squares in the sense of Sano-Watatani [61].) With these isomorphisms, we can start with M_{00} as an M_{00}-M_{00} bimodule and make a left tensor product by M_{11} as an M_{00}-M_{11} bimodule or an M_{11}-M_{00} bimodule, and make a right tensor product by M_{01} an M_{00}-M_{01} bimodule, M_{11} an M_{10}-M_{11} bimodule, M_{10} an M_{00}-M_{10} bimodule, or M_{11} an M_{01}-M_{11} bimodule. In this way, we get eight kinds of bimodules. (We have four factors, so we can get $4^2 = 16$ kinds of bimodules and they make a finite system of bimodules over four II_1 factors in the sense of Ocneanu [47]. For our purpose here, however, it is enough to handle eight kinds.)

We can then define a connection in a similar way as in the previous section. Now we have a three-dimensional Bratteli diagram, so we have a graph connected like a cube as in the following picture.



Note that some graphs, e.g., $^{00}_{00}\mathcal{G}^{00}_{10}$, may not be connected. Basically, we have the same kind of axioms as in the ordinary paragroup case in the previous section, but we have a new additional axiom. That is, we have a triple sequence of string algebras, so we have several different kind of identification of bases, and the identifications have to be compatible. This is expressed by the intertwining Yang-Baxter equation as follows.



The both hand sides are sums of products of three connection values over all the possible configurations for a fixed boundary of the hexagon. This equation first appeared in [30] as a compatibility condition of embeddings of string algebras.

Thus we have one-dimension higher objects here than in the ordinary paragroup case. We can summarize the correspondences as follows.

paragroup	paragroup action on a subfactor
(dual) principal graph	(dual) principal connection
(canonical) commuting square	(canonical) commuting cube
double sequence of string algebras	triple sequence of string algebras

The Yang-Baxter equation was also used to prove certain flatness in[13], but in general neither of the flatness axiom and the Yang-Baxter equation implies the other.

Summing up, we have the following theorem.

Theorem 3.1 *The isomorphism classes of commuting squares of type II_1 factors satisfying Assumption 1.1 are in a bijective correspondence to the isomorphism classes of combinatorial systems satisfying the axioms described above.*

We call our system of connections with the above axioms the *standard invariant*. See [33] for the exact forms of the axioms and a proof of the theorem.

4 Examples

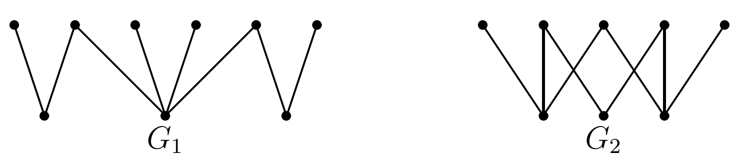
We have established the general theory in the previous section, but we still do not know what kind of "non-trivial" commuting squares we have; if all the commuting squares satisfying Assumption 1.1 were like (1) or (2) in Section 1, then our general machinery would be empty.

First recall that rational conformal field theory (RCFT) in the sense of Moore-Seiberg [41] is a vast source of paragroups [8]. Our set of axioms is essentially the set of the paragroup axioms plus the intertwining Yang-Baxter equation, and the Yang-Baxter equation is in RCFT, so the method of [8] gives families of commuting squares of II_1 factors in the "canonical" form and it is easy to see that they are not of form (1) or (2). The correspondence table between our axioms and those of RCFT is summarized as follows.

paragroup action on a subfactor	RCFT
unitarity	unitarity
renormalization	tetrahedral symmetry
flatness	braiding-fusion relation
intertwining Yang-Baxter equation	Yang-Baxter equation

The orbifold construction in the form of F. Xu [75] also works in our settings.

A more interesting example is given by the Goodman-de la Harpe-Jones subfactors in [19, Section 4.5]. The principal graphs of these subfactors were computed by S. Okamoto [49], and the graph for the smallest index case, $3+\sqrt{3}$, is given by G_1 in the following.



From the connection viewpoint, this construction gives a flat connection without the initialization axiom, and the above G_2 is a part of the connection data, but G_2

cannot be a principal graph of any subfactor. (Also see [31, Remark 2.2].) From this connection and ordinary connections on A_{11} and E_6 [44], we get a system of connections with the intertwining Yang-Baxter equation and thus get a commuting square of II_1 factors. (The intertwining Yang-Baxter equation in this case follows from the flatness of the Jones projections, as pointed out by Jones.) In the notations of Assumption 1.1, the subfactor $M_{00} \subset M_{01}$ is of type A_{11}, the subfactor $M_{10} \subset M_{11}$ is of type E_6, and both subfactors $M_{00} \subset M_{10}$ and $M_{01} \subset M_{11}$ are the Goodman-de la Harpe-Jones subfactor with index $3+\sqrt{3}$. In this case, the above graph G_2 does appear as a part of the "standard invariant". We note that the graphs D_{2n+1} and E_7 also appear as a part of standard invariant of some commuting squares of II_1 factors.

The same construction works for E_7 and E_8 instead of E_6. Also, a similar construction gives a commuting square such that the subfactors $M_{00} \subset M_{01}$ and $M_{10} \subset M_{11}$ is of type E_8 and the subfactors $M_{00} \subset M_{10}$ and $M_{01} \subset M_{11}$ are of A_4, but the commuting square is not of form (2). This gives a concrete interpretation of the fact that "E_8 has an A_4 symmetry" stated in [22, page 969].

Another interesting construction is given out of the above example. In de Boer-Goeree [8], they gave a correspondence table between paragroups and RCFT. The orbifold construction is one of them, and it has been well-studied. Another interesting entry is the coset construction. From a purely operator algebraic viewpoint, this construction gives a subfactor $S' \cap N \subset S' \cap M$ from a subfactor $N \subset M$ and a subalgebra S of N. We show that the above commuting square involving the Goodman-de la Harpe Jones subfactor with index $3+\sqrt{3}$ gives an interesting example of the coset construction in this sense. We make basic constructions vertically, and take limits vertically (as the GNS-completion of the unions) as follows.

$$
\begin{array}{ccccc}
 & & A_{11} & & \\
 & M_{00} & \subset & M_{01} & \\
3+\sqrt{3} & \cap & E_6 & \cap & 3+\sqrt{3} \\
 & M_{10} & \subset & M_{11} & \\
 & \cap & A_{11} & \cap & \\
 & M_{20} & \subset & M_{21} & \\
 & \cap & E_6 & \cap & \\
 & M_{30} & \subset & M_{31} & \\
 & \cap & & \cap & \\
 & \vdots & A_{11} & \vdots & \\
N = & M_{\infty,0} & \subset & M_{\infty,1} & = M
\end{array}
$$

Set $S = \bigvee_{k\geq 1}(M'_{10} \cap M_{k0}$. Then it is easy to see $S' \cap N = M_{10}$ and $S' \cap M = M_{11}$. Thus we get a subfactor $S' \cap N \subset S' \cap M$ of type E_6 out of a subfactor $N \subset M$ of type A_{11}. In short, we can say that the E_6 subfactors are constructed from the A_{11} subfactor with the coset construction. The same method works for E_8, too.

At the end of this section, we discuss commuting squares arising from a group action on a subfactor. As in ordinary subfactor cases where many problems of group actions on single factors are reduced problems of subfactors, our theory gives a method to classify certain group actions on subfactors. For example, (not-necessarily-outer) actions of finite groups on subfactors (with finite depth) are among them. (Recall that an aperiodic automorphism, in an appropriate sense, of a subfactor with finite depth is automatically properly outer [57] [strongly outer [9]], so they are classified by Loi's invariant [38] by a deep classification theorem of S. Popa [57]. So we are now mainly interested in finite group actions.) But unfortunately, it is quite complicated to regard non-outer actions of finite groups as paragroup actions via locally trivial subfactors. It is the same situation as in the single factor case. The connection of a locally trivial subfactor is determined by the characteristic invariant [26] of the action, but it is not easy to list all the possible invariant arising from a fixed group in the connection approach. In our settings, a main difficulty in listing all the possible standard invariants for a fixed group lies in the fact that we have several disconnected graphs.

5 Topological quantum field theory

After the astonishing discovery of the Jones polynomial as an invariant of links [28], "quantum" aspects of three dimensional topology and their relations have been extensively studied. Witten's program [73] to realize topological quantum field theories in the sense of Atiyah [3] based on physical ideas has been very influential. Reshetikhin and Turaev [58] gave a mathematically rigorous version of Witten's topological quantum field theory based on surgery and the Jones polynomial, and Turaev and Viro gave another formulation of topological quantum field theory based on triangulations and the Kirillov-Reshetikhin quantum $6j$-symbols by refining a classical theorem of Alexander [1]. (See [64] for a relation between the two approaches. Also see Ocneanu's article in this volume [48].) Ocneanu [47] has realized that the Turaev-Viro machinery works for general subfactors with finite depth and that the axioms of quantum $6j$-symbols for the Turaev-Viro TQFT and those for the paragroups are essentially same. (See [14] for the exact statement of the equivalence and a proof.)

In this section, we discuss TQFT's arising from subfactors as an application of our method here.

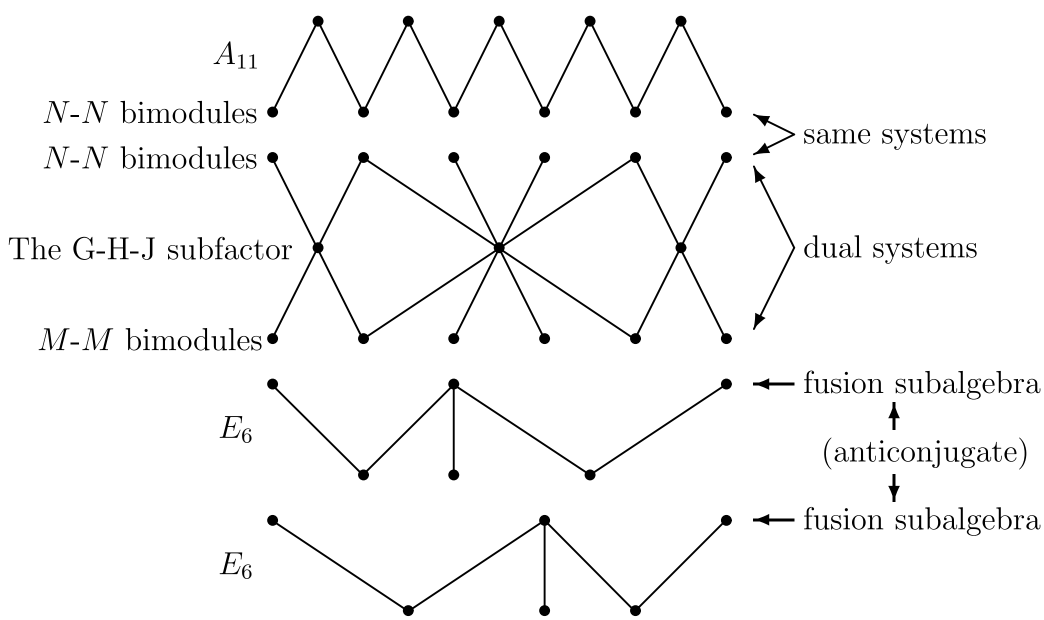
The known constructions of finite depth subfactors are listed as follows.

1. RCFT (Wess-Zumino-Witten models)
2. orbifold construction
3. group actions (or Hopf algebra actions)
4. others

We have E_6, E_8, the Goodman-de la Harpe-Jones subfactors, the Haagerup subfactor [20] among the others. In the case of RCFT, the TQFT has been extensively studied by several people without working on subfactors, so we do not deal with them here. (But strictly speaking, our TQFT is slightly different from the one usually considered by topologists because we have "grading" of bimodules. See [14] for this difference.) The TQFT for the orbifold subfactors is not known explicitly, but it is hard to imagine that they are dramatically different from the ones arising from RCFT. The case of group actions is less exotic. (See [12] for example.) Thus we are led to study "the other" subfactors. We will discuss the E_6 subfactors and the Goodman-de la Harpe-Jones subfactor with index $3+\sqrt{3}$ here.

First, a purely combinatorial argument [31] shows that the dual principal graph of the Goodman-de la Harpe-Jones subfactor with index $3+\sqrt{3}$ is the same as the principal graph Okamoto [49] computed. With this, we can work on the axioms on the standard invariant for paragroup actions on subfactors, and we can determine several other graphs by combinatorial arguments.

Then we find that the system of N-N bimodules of the Goodman-de la Harpe-Jones subfactor with index $3+\sqrt{3}$ is exactly same as the one for the A_{11} subfactor and that the system of M-M bimodules of the Goodman-de la Harpe-Jones subfactor with index $3+\sqrt{3}$ contains that of the E_6 subfactor as a "fusion subalgebra". (We have the two subfactors for E_6, and the corresponding two fusion algebras are embedded into a single fusion algebra differently.) It is expressed graphically as follows.



D. Bisch [7] has tried to compute the fusion algebra of the Goodman-de la Harpe-Jones subfactor with index $3+\sqrt{3}$ just from the principal graph, but had five possibilities and could not determine which the right multiplication table is. Our method here shows that the fusion algebra of the N-N bimodules gives the fifth table in [7] and that of the M-M bimodules gives the first table in [7]. In particular, this subfactor $N \subset M$ is not conjugate to its dual $M \subset M_1$ although these two subfactors have the same principal graphs. (M. Izumi computed the flat connection of this subfactor, and his method also shows this non-conjugacy result, but it is difficult to see the difference of the two fusion algebras directly from the flat connection.) Furthermore, a recent result of Haagerup [20] shows that this subfactor has the smallest index among subfactors such that the principal and the dual principal graphs are the same but the two fusion algebras are different.

This shows that our TQFT for the Goodman-de la Harpe-Jones subfactor with index $3+\sqrt{3}$ is exactly same as that for the A_{11} subfactor, which is certainly disappointing. The same method works for the other Goodman-de la Harpe-Jones subfactors, and we have no new TQFT. The two systems of N-N bimodules and M-M bimodules of a single subfactor give the same TQFT as in [14], so we can say that the E_6 TQFT's are based on partial systems of the A_{11} TQFT. But it does not necessarily mean that the E_6 TQFT's are less interesting than the A_{11} TQFT. Computations of Niţică and Török [42] show that the E_6 TQFT's do detect orientations of certain lens spaces while it is easy to see that the A_{11} TQFT does not detect an orientation of any 3-manifold.

The case of the Haagerup subfactor with index $(5+\sqrt{13})/2$ [20] gives a candidate of a really interesting new TQFT, but unfortunately, almost nothing is known on this TQFT.

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