Local conformal nets arising from framed vertex operator algebras

Yasuyuki Kawahigashi*
Department of Mathematical Sciences
University of Tokyo, Komaba, Tokyo, 153-8914, Japan
e-mail: yasuyuki@ms.u-tokyo.ac.jp

Roberto Longo†
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1, I-00133 Roma, Italy
e-mail: longo@mat.uniroma2.it

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Abstract

We apply an idea of framed vertex operator algebras to a construction of local conformal nets of (injective type III_1) factors on the circle corresponding to various lattice vertex operator algebras and their twisted orbifolds. In particular, we give a local conformal net corresponding to the moonshine vertex operator algebras of Frenkel-Lepowsky-Meurman. Its central charge is 24, it has a trivial representation theory in the sense that the vacuum sector is the only irreducible DHR sector, its vacuum character is the modular invariant J-function and its automorphism group (the *gauge group*) is the Monster group. We use our previous tools such as α-induction and complete rationality to study extensions of local conformal nets.

1 Introduction

We have two mathematically rigorous approaches to study chiral conformal field theory using infinite dimensional algebraic systems. One is algebraic quantum field theory

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where we study local conformal nets of von Neumann algebras (factors) on the circle, and the other is theory of vertex operator algebras. One local conformal net of factors corresponds to one vertex operator algebra, at least conceptually, and each describes one chiral conformal field theory. Since these two mathematical theories are supposed to study the same physical objects, it is natural that the two theories have much in common. For example, both theories have mathematical objects corresponding to the affine Lie algebras and the Virasoro algebra, and also, both have simple current extension, the coset construction, and the orbifold construction as constructions of a new object from a given object. However, the interactions between the two theories have been relatively small, and different people have studied different aspects of the two approaches from different motivations. Comparing the two theories, one easily notices that study of lattice vertex operator algebras and their twisted orbifolds has been extensively pursued, but that the corresponding study in algebraic quantum field theory is relatively small, although we have had some works such as [9], [18], [51], [54]. The most celebrated vertex operator algebra, the moonshine vertex operator algebra [22], belongs to this class, and almost nothing has been studied about its counterpart in algebraic quantum field theory. The automorphism group of this moonshine vertex operator algebra is the Monster group [25], the largest among the 26 sporadic finite simple groups, and Borcherds has solved the celebrated moonshine conjecture of Conway-Norton [11] on the McKay-Thompson series arising from this moonshine vertex operator algebra in [6]. Our aim in this paper is to study such counterparts in algebraic quantum field theory, using an idea of framed vertex operator algebras in [15]. In particular, we construct a local conformal net corresponding to the moonshine vertex operator algebra, the "moonshine net", in Example 3.8. It has central charge 24 and a trivial representation theory, its vacuum character is the modular invariant J-function, and its automorphism group is the Monster group.

Here we briefly explain the relation between local conformal nets and vertex operator algebras. In algebraic quantum field theory, we consider a family of von Neumann algebras of bounded linear operators generated by self-adjoint operators corresponding to the "observables" in a spacetime region. (The book [27] is a standard textbook.) In the case of chiral conformal field theory, the "spacetime" is the one-dimensional circle and the spacetime regions are intervals, that is, non-dense non-empty open connected sets on the circle. We thus have a family of von Neumann algebras $\{\mathcal{A}(I)\}_I$ parameterized by intervals I on a fixed Hilbert space which has a vacuum vector. Other properties such as covariance and locality are imposed upon this family of von Neumann algebras. Such a family of von Neumann algebras is called a local conformal net, or simply a net, of von Neumann algebras, or factors, where a factor means a von Neumann algebra with a trivial center. See [26, 36] for the precise set of axioms. If we start with a vertex operator algebra (with unitarity), then each vertex operator should be an operator-valued distribution on the circle, so we apply test functions supported on an interval I and obtain an algebra of bounded operators generated by these (possibly unbounded) operators arising from such pairing. This should give a local conformal nets $\{\mathcal{A}(I)\}_I$, at least conceptually. Note that locality for a local conformal net $\{\mathcal{A}(I)\}_I$ takes a very simple form, that is, if I and J are disjoint intervals of the circle, then $\mathcal{A}(I)$ and $\mathcal{A}(J)$ mutually commute. The axioms of vertex operator algebras essentially come from Fourier expansions of Wightman fields on the circle. By the so-called state-field correspondence, one obtains a bijection between a certain set of Wightman fields and a dense subspace of the Hilbert space. With this identification, each vector in this dense subspace gives a vertex operator and through the Fourier expansion coefficients, it produces a countable family of (possibly unbounded) operators parameterized by integers. An abstract vertex operator algebras is, roughly speaking, such a vector space each vector of which gives a countable family of operators, subject to various compatibility conditions of these operators such as locality and covariance. The diffeomorphism covariance is represented as existence of a special vector called the Virasoro element. See [22] for the precise definition. Fredenhagen and Jörß[24] have given a rigorous correspondence between a local conformal nets of factors and a family of Wightman fields, but still, the exact relations between local conformal nets and vertex operator algebras are not clear. (Note that unitarity is always assumed from the beginning in algebraic quantum field theory.)

Dong, Griess and Höhn [15] have established a general theory of framed vertex operator algebras and in particular shown how the moonshine vertex operator algebra of Frenkel-Lepowsky-Meurman decomposes into irreducible modules as a module of the 48th tensor power of the Virasoro vertex operator algebra of central charge 1/2. (Also see predating works [14, 16].) A framed vertex operator algebra is an extension of a tensor power of the Virasoro vertex operator algebra having central charge 1/2. We "translate" this work into the framework of algebraic quantum field theory in this paper, which is a nontrivial task. We hope that the operator algebraic viewpoint gives a new light on the structure of the Monster group, and in particular, the Monstrous Moonshine [11].

A framed vertex operator algebra is a particular example of an extension of a vertex operator algebra. A general extension problem of a vertex operator algebra is formulated as follows. Take a vertex operator algebra V and let $\{M_i\}_i$ be a set of representatives from equivalence classes of irreducible V-modules. We take M_0 to be V itself regarded as a V-module. Then we make a direct sum $\bigoplus_i n_i M_i$, where n_i is a multiplicity and $n_0 = 1$, and we would like to know when this direct sum has a structure of a vertex operator algebra with the same Virasoro element in M_0 (and how many different vertex operator algebra structures we have on this). The corresponding extension problem has been studied well in algebraic quantum field theory. That is, for a local conformal net $\mathcal{A}$ with irreducible Doplicher-Haag-Roberts (DHR) sectors $\{\lambda_i\}_i$ (as in [19] and [21]), we would like to determine when $\bigoplus_i n_i \lambda_i$ gives an extension of a local conformal net, where n_i is a multiplicity, λ_0 is the vacuum sector, and $n_0 = 1$. (A DHR sector of a local conformal net is a unitary equivalence class of representations of a local conformal net. A local conformal net acts on some Hilbert space from the beginning, and this is called a vacuum representation when regarded as a representation. The representation category of a local conformal net is a braided tensor category.) A complete characterization when this gives an extension of a local conformal net was given in [44, Theorem 4.9] using a notion of a Q-system [42]. (An essentially same characterization was given in [40, Definition 1.1] in the framework of abstract tensor categories. Conditions 1 and 3 there correspond to the axioms of the Q-system in [42], Condition 4 corresponds to irreducibility, and Condition 2 corresponds to chiral locality in [44, Theorem 4.9] in the sense of [3, page 454].) When we do have such an extension, we can compare representation theories of the original local conformal net and its extension by a method of α-induction, which gives a (soliton) representation of a larger local conformal net from a representation of a smaller one, using a braiding. This induction machinery was defined in [44], many interesting properties and examples, particularly ones related to conformal embedding, were found in [52], and it has been further studied in [1, 2, 3, 4, 5], particularly on its relation to modular invariants. Classification of all possible structures on a given $\bigoplus_i n_i\lambda_i$ was studied as a certain 2-cohomology problem in [29] and finiteness of the number of possible structures has been proved in general there. Further studies on such 2-cohomology were made in [37] and various vanishing results on this 2-cohomology were obtained. Using these tools, a complete classification of such extensions for the Virasoro nets with central charge less than 1, arising from the usual Virasoro algebra, has been obtained in [36]. The classification list consists of the Virasoro nets themselves, their simple current extensions of index 2, and four exceptionals at the central charges $21/22, 25/26, 144/145, 154/155$. By the above identification of the two extension problems, this result also gives a complete classification of extensions of the Virasoro vertex operator algebras with $c < 1$.

We refer to [36] and its references for a general definition and properties of local conformal nets (with diffeomorphism covariance) and α-induction. Also see [34, 35] for reviews on [36]. Subfactor theory initiated by Jones [31] also plays an important role. See [20] for general theory of subfactors.

2 Framed vertex operator algebras and extension of local conformal nets

Staszkiewicz [51] gave a construction of a local conformal net from a lattice L and constructed representations of the net corresponding to the elements of L^*/L, where L^* is the dual lattice of L. Since he also computed the μ-index of the net, the result in [39, Proposition 24] implies that the DHR sectors he constructed exhaust all. Furthermore, after the first version of this paper was posted on arXiv, a paper of Dong and Xu [18] appeared on arXiv studying the operator algebraic counterparts of lattice vertex operator algebras and their twisted orbifolds. Here we use a different approach based on an idea of framed vertex operator algebras in [15].

We start with a Virasoro net with $c = 1/2$ constructed as a coset in [53] and studied in [36]. In [36], we have shown that this net is completely rational in the sense of [39, Definition 8], and that it has three irreducible DHR sectors of statistical dimensions $1, \sqrt{2}, 1$, respectively. So the μ-index, the index of the two-interval subfactor as in [39, Proposition 5], of this net is 4 and their conformal weights are $0, 1/16, 1/2$, respectively. (The μ-index of a local conformal net is equal to the square sum of the statistical dimensions of all irreducible DHR sectors of the net by [39, Proposition 24]. This quantity of a braided tensor category also plays an important role in studies of quantum invariants in three-dimensional topology.) We denote this net by $\mathrm{Vir}_{1/2}$. We next consider the net $\mathrm{Vir}_{1/2} \otimes \mathrm{Vir}_{1/2}$. This net is also completely rational and has μ-index 16. It has 9 irreducible DHR sectors, since each such sector is a tensor product of two irreducible DHR sectors of $\mathrm{Vir}_{1/2}$. Using the conformal weights of the two such irreducible DHR sectors of $\mathrm{Vir}_{1/2}$, we label the 9 irreducible DHR sectors of $\mathrm{Vir}_{1/2} \otimes \mathrm{Vir}_{1/2}$ as

$$\lambda_{0,0}, \lambda_{0,1/16}, \lambda_{0,1/2}, \lambda_{1/16,0}, \lambda_{1/16,1/16}, \lambda_{1/16,1/2}, \lambda_{1/2,0}, \lambda_{1/2,1/16}, \lambda_{1/2,1/2}.$$

We denote the conformal weight, the lowest eigenvalue of the image of L_0, and the conformal spin of an irreducible sector λ by h_λ and ω_λ, respectively. By the spin-statistics theorem of Guido-Longo [26], we have $\exp(2\pi i h_\lambda) = \omega_\lambda$.

Lemma 2.1. *Let $\mathcal{A}$ be a local conformal net on S^1. Suppose we have a finite system $\{\lambda_j\}_j$ of irreducible DHR endomorphisms of $\mathcal{A}$ and each λ_j has a statistical dimension 1. If all λ_j have a conformal spin 1, then the crossed product of $\mathcal{A}$ by the finite abelian group G given by $\{\lambda_j\}_j$ produces a local extension of the net $\mathcal{A}$.*

Proof First by [50, Lemma 4.4], we can change the representatives $\{\lambda_j\}_j$ within their unitary equivalence classes so that the system $\{\lambda_j\}_j$ gives an action of G on a local algebra $A(I)$ for a fixed interval I. That is, we decompose $G = \prod_i G_i$, where each G_i is a cyclic group, and localize generators of G_i's on mutually disjoint intervals within I. By the assumption on the conformal spins, we can adjust each generator so that it gives an action of G_i. Then as in [1, Part II, Section 3], we can make an extension of the local conformal net $\mathcal{A}$ to the crossed product by the G-action. The proof of [1, Part II, Lemma 3.6] gives the desired locality, because the conformal spins are now all 1. (Lemma 3.6 in [1, Part II] deals with only actions of a cyclic group, but the same argument works in our current setting.) $\square$

This extended local conformal net is called a *simple current extension* of $\mathcal{A}$ by G. We now apply Lemma 2.1 to the net $\mathrm{Vir}_{1/2} \otimes \mathrm{Vir}_{1/2}$ and its system of irreducible DHR sectors consisting of $\lambda_{0,0}, \lambda_{1/2,1/2}$. Note that the conformal weight of the sector $\lambda_{1/2,1/2}$ is 1, so its conformal spin is 1 and we can apply Lemma 2.1 with $G = \mathbb{Z}_2$.

Proposition 2.2. *This net $\mathcal{A}$ is completely rational and it has four irreducible DHR sectors and all have statistical dimensions 1. The fusion rules of these four sectors are given by the group $\mathbb{Z}_4$ and the conformal weights are $0, 1/16, 1/2, 1/16$.*

Proof The net $\mathcal{A}$ is completely rational by [43]. By [39, Proposition 24], the μ-index is 4. Since $\alpha^\pm$-induction of $\lambda_{0,0}$ and $\lambda_{0,1/2}$ give two irreducible DHR sectors of the net $\mathcal{A}$, we obtain two inequivalent irreducible DHR sectors of $\mathcal{A}$ having statistical dimensions 1. The $\alpha^{\pm}$-induction of $\lambda_{1/16,1/16}$ gives either one irreducible DHR sector of statistical dimension 2 or two inequivalent irreducible DHR sectors of statistical dimensions 1. Since the square sum of the statistical dimensions of all the irreducible DHR sectors is 4, the latter case in the above occurs. In this way, we obtain four inequivalent irreducible DHR sectors of statistical dimensions 1, and these exhaust all the irreducible DHR sectors of $\mathcal{A}$. Note that two of the irreducible DHR sectors have conformal weights 1/8, and thus conformal spins $\exp(\pi i/4)$.

The fusion rules are given by either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If we have the fusion rules of $\mathbb{Z}_2 \times \mathbb{Z}_2$, then the two irreducible DHR sectors of conformal weights 1/8 must have order 2, but this would violate [49, Corollary on page 343]. Thus we obtain the former fusion rules. □

Remark 2.3. This net $\mathcal{A}$ has a central charge 1. In [56], Xu classified all local conformal nets on S^1 under an additional assumption called a spectrum condition. It is conjectured that this spectrum condition always holds. Thus, our net $\mathcal{A}$ should be in the classification list in [56]. Since we have only one net having the μ-indexes equal to 1 in the list in [56], our net $\mathcal{A}$ should be isomorphic to the net $U(1)_4$. (See [55, Section 3.5] for the nets $U(1)_{2k}$.)

We now use the framework of Dong-Griess-Höhn [15]. As [15, Section 4], let C be a doubly-even linear binary code of length $d \in 8\mathbb{Z}$ containing the vector $(1, 1, \ldots, 1)$. (A doubly-even code is sometimes called type II.) This C is a subset of $\mathbb{F}_2^d$, where $\mathbb{F}_2$ is a field consisting two elements, and any element in C is naturally regarded as an element in $\mathbb{Z}^d$. As in [12], we associate two even lattices with such a code C as follows.

$$\begin{aligned} L_C &= \{(c+x)/\sqrt{2} \mid c \in C, x \in (2\mathbb{Z})^d\}, \\ \tilde{L}_C &= \{(c+y)/\sqrt{2} \mid c \in C, y \in (2\mathbb{Z})^d, \textstyle\sum y_i \in 4\mathbb{Z}\} \cup \\ &\quad \{(c+y+(1/2,1/2,\ldots,1/2))/\sqrt{2} \mid c \in C, y \in (2\mathbb{Z})^d, 1-(-1)^{d/8} + \textstyle\sum y_i \in 4\mathbb{Z}\}. \end{aligned}$$

We then have the corresponding vertex operator algebras $V_{L_C}, V_{\tilde{L}_C}$ as in [22], but the construction directly involves vertex operators and it seems very difficult to "translate" this construction into the operator algebraic framework. So we will take a different approach based on an idea of a framed vertex operator algebras. (If the lattice is D_1, we do have a counterpart of the lattice vertex operator algebra V_{D_1} and it is the above net $\mathcal{A}$. This has been already noted in [54, page 14072] and also is a basis of [51].)

Let L be a positive definite even lattice of rank d containing D_1^d as a sublattice, where $D_1 = 2\mathbb{Z}$ and $\langle \alpha, \beta \rangle = \alpha\beta$ for $\alpha, \beta \in 2\mathbb{Z}$. Such a sublattice is called a D_1-frame. Recall that we have a non-degenerate symmetric $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$ on L and $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in L$. Note that $(D_1^*/D_1)^d$ is isomorphic to $\mathbb{Z}_4^d$, thus we have a set $\Delta(L) = L/D_1^d \subset (D_1^*/D_1)^d$ and this set is a code over $\mathbb{Z}_4$, that is, $\Delta(L)$ is a subgroup of $\mathbb{Z}_4^d$. An element of a code is called a codeword. It is known that $\Delta(L)$ is self-dual if and only if L is self-dual. (See [15, page 426].) For the above lattices

$L_C, \tilde{L}_C$, the corresponding $\mathbb{Z}_4$-codes are explicitly known as follows. (Also see [15, page 426].)

Let $\hat{}$ be a map from $\mathbb{F}_2^2$ to $\mathbb{Z}_4^2$ given by $00 \mapsto 00$, $11 \mapsto 20$, $10 \mapsto 11$, and $01 \mapsto 31$. By regarding $\mathbb{F}_2^d = (\mathbb{F}_2^2)^{d/2}$ and $\mathbb{Z}_4^d = (\mathbb{Z}_4^2)^{d/2}$, and applying the map $\hat{}$ componentwise, we obtain the map, still denoted by $\hat{}$, from $\mathbb{F}_2^d$ to $\mathbb{Z}_4^d$. We also define $\Sigma_2^n = \{(00), (22)\}^n$ and let $(\Sigma_2^n)_0$ be the subcode of the $\mathbb{Z}_4$-code Σ_2^n consisting of codewords of Hamming weights divisible by 4. (The Hamming weight of a codeword is the number of nonzero entries of the codeword.) The we have

$$\begin{aligned}
\Delta(L_C) &= \hat{C} + \Sigma_2^{d/2}, \\
\Delta(\tilde{L}_C) &= \hat{C} + (\Sigma_2^{d/2})_0 \cup \hat{C} + (\Sigma_2^{d/2})_0 + \begin{cases} (1,0,\cdots,1,0,1,0), & \text{if } d \equiv 0 \mod 16, \\ (1,0,\cdots,1,0,3,2), & \text{if } d \equiv 8 \mod 16. \end{cases}
\end{aligned}$$

Let G be the abelian group given by $\Delta(L_C)$ or $\Delta(\tilde{L}_C)$, regarded as a subgroup of $\mathbb{Z}_4^d$. We consider a completely rational local conformal net $\mathcal{A}^{\otimes d}$. As above, each irreducible DHR sector of this net is labeled with an element of $\mathbb{Z}_4^d$ naturally.

Lemma 2.4. *Each irreducible DHR sector corresponding to an element in G has a conformal spin* 1*.*

Proof We consider a conformal weight of each DHR sector. For the four elements $00, 01, 10, 11 \in \mathbb{F}_2^2$, the images by the map $\hat{}$ are $00, 31, 11, 20$, respectively. For a net $\mathcal{A} \otimes \mathcal{A}$, we have the corresponding irreducible DHR sectors, and their conformal weights are $0, 1/4, 1/4, 1/2$, respectively. This means that $1 \in \mathbb{F}_2$ has a contribution $1/4$ to the conformal weight and $0 \in F_2$ has a contribution 0. Since the code C is doubly-even, the number of nonzero entries in any element c in C is a multiple of 4. This means that for any element $c \in C$, the irreducible DHR sector corresponding to $\hat{c}$ has an integer conformal weight.

Since the irreducible DHR sector corresponding to the element $2 \in \mathbb{Z}_4$ has a conformal weight $1/2$, the irreducible DHR sector corresponding to any element in $\Sigma_2^{d/2}$ also has an integer conformal weight. The irreducible DHR sector corresponding to the elements $1, 3 \in \mathbb{Z}_4$ have conformal weights $1/8$. So, if $d \equiv 0 \mod 16$, then the irreducible DHR sector corresponding to the element $(1,0,\cdots,1,0,1,0) \in \mathbb{Z}_4^d$ has a conformal weight $d/16$, which is an integer. If $d \equiv 8 \mod 16$, then the irreducible DHR sector corresponding to the element $(1,0,\cdots,1,0,3,2) \in \mathbb{Z}_4^d$ has a conformal weight $(d+8)/16$, which is again an integer.

We have thus proved that for any element in $\Delta(L_C)$ or $\Delta(\tilde{L}_C)$, the corresponding irreducible DHR sector has an integer conformal weight, hence a conformal spin 1 by the spin-statistics theorem [26]. □

We then have the following theorem.

Theorem 2.5. *We can extend $\mathcal{A}^{\otimes d}$ to a local conformal net as a simple current extension by the group $G = \Delta(L_C), \Delta(\tilde{L}_C)$ as above. If the code C is self-dual, the extended local conformal net has μ-index* 1*.*

Proof The first claim is clear by Lemmas 2.1, 2.4. If the code C is self-dual, its cardinality is $2^{d/2}$, so the cardinality of G is 2^d. (See [22, (10.1.8)].) Since the μ-index of $\mathcal{A}^{\otimes d}$ is 4^d, we obtain the conclusion by [39, Proposition 24]. □

The condition that the μ-index is 1 means that the vacuum representation is the only irreducible representation of the net. This property is well-studied in theory of vertex operator algebras and such a vertex operator algebra is called holomorphic (or also self-dual). So we give the following definition.

Definition 2.6. A local conformal net is said to be *holomorphic* if it is completely rational and has μ-index 1.

Recall that the vacuum character of a local conformal net is defined as

$$\mathrm{Tr}(e^{2\pi i\tau(L_0-c/24)})$$

on the vacuum Hilbert space. The vacuum character of a vertex operator algebra has been defined in a similar way.

Proposition 2.7. *Let L be an even lattice arising from a doubly-even linear binary code C of length $d \in 8\mathbb{Z}$ containing the vector $(1, 1, \dots, 1)$ as L_C or $\tilde{L}_C$ as above. Denote the local conformal net arising from L as above by $\mathcal{A}_L$ and the vertex operator algebra arising from L by V_L. If we start with a self-dual code C, both $\mathcal{A}_L$ and V_L are holomorphic. The central charges of $\mathcal{A}_L$ and V_L are both equal to d. Furthermore, the vacuum characters of $\mathcal{A}_L$ and V_L are equal.*

Proof The statements on holomorphy and the central charge are now obvious. The Virasoro net $\mathrm{Vir}_{1/2}$ and the Virasoro vertex operator algebra $L(1/2, 0)$ have the same representation theory consisting of three irreducible objects, and their corresponding characters are also equal. So both of their $2d$-th tensor powers also have the same representation theory and the same characters. When we decompose V_L as a module over $L(1/2, 0)^{\otimes 2d}$ and decompose the canonical endomorphism of the inclusion of $\mathrm{Vir}_{1/2}^{\otimes 2d} \subset \mathcal{A}_L$ into irreducible DHR sectors of $\mathrm{Vir}_{1/2}^{\otimes 2d}$, we can identify the two decompositions from the above construction. This implies, in particular, that the vacuum characters are equal. □

We study the characters for local conformal net $\mathcal{A}$. For an irreducible DHR sector λ of $\mathcal{A}$, we define the specialized character $\chi_\lambda(\tau)$ for complex numbers τ with $\mathrm{Im}\,\tau > 0$ as follows.

$$\chi_\lambda(\tau) = \mathrm{Tr}(e^{2\pi i\tau(L_{0,\lambda}-c/24)}),$$

where the operator $L_{0,\lambda}$ is conformal Hamiltonian in the representation λ and c is the central charge. For a module λ of vertex operator algebra V, we define the specialized character by the same formula. Zhu [57] has proved under some general condition that the group $SL(2, \mathbb{Z})$ acts on the linear span of these specialized characters through the usual change of variables τ and this result applies to lattice vertex operator algebras. In particular, if a lattice vertex operator algebra is holomorphic, then the only specialized character is the vacuum one, and this vacuum character must be invariant under $SL(2,\mathbb{Z})$, that is, a modular function. We thus know that our local conformal nets $\mathcal{A}_L$ also have modular functions as the vacuum characters. This can be proved also as follows. For the Virasoro net $\mathrm{Vir}_{1/2}$, the specialized characters for the irreducible DHR sectors coincide with the usual characters for the modules of the Virasoro vertex operator algebra $L(1/2,0)$. Then by Proposition 2 in [38], the vacuum character of a holomorphic net $\mathcal{A}_L$ is a modular invariant function.

We show some concrete examples, following [15, Section 5].

Example 2.8. The Hamming code H_8 is a self-dual double-even binary code of length 8. This gives the E_8 lattice as both L_{H_8} and $\tilde{L}_{H_8}$, and the above construction then produces a holomorphic local conformal net with central charge 8. In theory of vertex operator algebras, the corresponding lattice vertex operator algebra V_{E_8} is isomorphic to the one arising from the affine Lie algebra $E_8^{(1)}$ at level 1, thus we also expect that the above local conformal net is isomorphic to the one arising from the loop group representation for $E_8^{(1)}$ at level 1.

Example 2.9. Consider the Golay Code $\mathcal{G}_{24}$ as in [22, Chapter 10], which is a self-dual double-even binary code of length 24. Then the lattice $\tilde{L}_{\mathcal{G}_{24}}$ is the Leech lattice, which is the unique positive definite even self-dual lattice of rank 24 having no vectors of square length 2 as in [22]. Its vacuum character is the following, which is equal to $J(q)+24$.

$$q^{-1}+24+196884q+21493760q^2+8642909970q^3+\cdots.$$

3 Twisted orbifolds and α-induction

We now perform the twisted orbifold construction for the net arising from a lattice as above. Let C be self-dual double-even binary code of length d and L be L_C or $\tilde{L}_C$, the lattice arising from C as in the above section. We study the net $\mathcal{A}_L$, which is holomorphic by Theorem 2.5.

The basic idea of the twisted orbifold construction can be formulated in the framework of algebraic quantum field theory quite easily. We follow Huang's idea [28] in theory of vertex operator algebras. Take a holomorphic local conformal net $\mathcal{A}$ and its automorphism σ of order 2. Then the fixed point net $\mathcal{A}^\sigma$ has μ-index 4, and it is easy to see that this fixed point net has four irreducible DHR sectors of statistical dimensions 1, and the fusion rules are given by $\mathbb{Z}_2\times\mathbb{Z}_2$. Besides the vacuum sector, two of them have conformal spins 1, so we can make a simple current extension of $\mathcal{A}^\sigma$ with each of these two sectors. One of them obviously gives back $\mathcal{A}$, and the other is the twisted orbifold of $\mathcal{A}$. (Here we have presented only an outline. The actual details are given below.) In this way, we can obtain the counterpart of the moonshine vertex operator algebra from $\mathcal{A}_\Lambda$, where Λ is the Leech lattice, but it is hard to see the structure of the twisted orbifold in this abstract approach. For example, an important property of the moonshine vertex operator algebras is that its vacuum character is the modular invariant J-function, but it is hard to see this property for the corresponding net in the above approach. Moreover, we are interested in the automorphism group of the twisted orbifold, but it is essentially impossible to determine the automorphism group through this abstract construction of a local conformal net. So we need a more concrete study of the twisted orbifold construction for local conformal nets, and we will make such a study with technique of α-induction below.

Recall that the net $\mathcal{A}$ is constructed as a crossed product by a $\mathbb{Z}_2$-action in Proposition 2.2, so it has a natural dual action of $\mathbb{Z}_2$. We take its d-th tensor power on $\mathcal{A}^{\otimes d}$ and denote it by σ. This is an automorphism of the net $\mathcal{A}^{\otimes d}$ of order 2. We extend $\mathcal{A}^{\otimes d}$ to a local conformal net $\mathcal{A}_L$ as in Theorem 2.5. Then σ extends to an automorphism of $\mathcal{A}_L$ of order 2 by Lemma 2.4 and [8, Proposition 2.1], and we denote the extension again by σ. Note that the fusion rules of $\mathcal{A}^{\otimes d}$ are naturally given by $\mathbb{Z}_4^d$, and the action of σ on the fusion rules $\mathbb{Z}_4$ is given by $j \mapsto -j \in \mathbb{Z}_4$.

Lemma 3.1. *The fixed point net $\mathcal{A}_L^\sigma$ has four irreducible DHR sectors of statistical dimensions 1 and they exhaust all the irreducible DHR sectors.*

Proof The μ-index of the fixed point net is 4 by [39, Proposition 24]. So either we have four irreducible DHR sectors of statistical dimensions 1 or we have three irreducible DHR sectors of statistical dimensions 1, 1, $\sqrt{2}$. Suppose we had the latter case and draw the induction-restriction graph for the inclusion $\mathcal{A}_L^\sigma \subset \mathcal{A}_L$. Then the index of this inclusion is 2 and the Dynkin diagram A_3 is the only graph having the Perron-Frobenius eigenvalue $\sqrt{2}$, and statistical dimensions are always larger than or equal to 1, thus we would have an irreducible DHR sector of statistical dimension $\sqrt{2}$ for the holomorphic net $\mathcal{A}_L$, which is impossible. We thus know that we have four irreducible DHR sectors of statistical dimensions 1 and these give all the irreducible DHR sectors of the net $\mathcal{A}_L^\sigma$. □

We study the following inclusions of nets.

$$\begin{array}{ccccc} & & \mathcal{A}^{\otimes d} & \subset & \mathcal{A}_L \\ & & \cup & & \cup \\ \mathrm{Vir}_{1/2}^{\otimes 2d} & \subset & (\mathcal{A}^{\otimes d})^\sigma & \subset & \mathcal{A}_L^\sigma. \end{array}$$

Our aim is to study decompositions of irreducible DHR sectors of the net $\mathcal{A}_L^\sigma$ restricted to $\mathrm{Vir}_{1/2}^{\otimes 2d}$. We describe such decompositions through studies of soliton sectors of $\mathcal{A}^{\otimes d}$ and $\mathcal{A}_L$. This is because the inclusions $\mathrm{Vir}_{1/2}^{\otimes 2d} \subset \mathcal{A}^{\otimes d} \subset \mathcal{A}_L$ are easier to study.

Lemma 3.2. *The irreducible DHR sectors of the net $(\mathcal{A}^{\otimes d})^\sigma$ consist of 4^{d-1} sectors of statistical dimensions 2, 4^d sectors of statistical dimensions 1, and 2^{d+1} sectors of statistical dimensions $2^{d/2}$.*

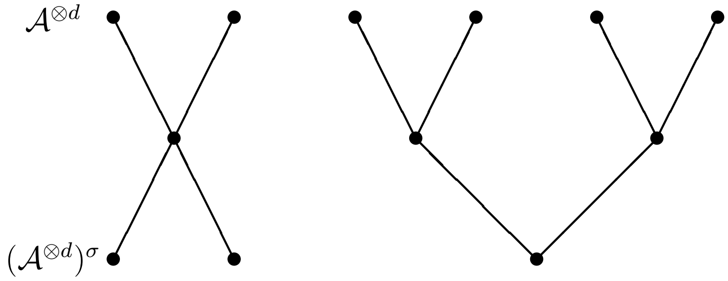

Figure 1: Induction-restriction graphs

After applying the α^+-induction for the inclusion $(\mathcal{A}^{\otimes d})^\sigma \subset \mathcal{A}^{\otimes d}$ and the irreducible DHR sectors of $(\mathcal{A}^{\otimes d})^\sigma$ as above, we obtain irreducible soliton sectors consisting of 4^d sectors of statistical dimensions 1 and 2^d sectors of statistical dimensions $2^{d/2}$. The 4^d sectors of statistical dimensions 1 precisely give the irreducible DHR sectors of the net $\mathcal{A}^{\otimes d}$.

Proof Let

$$K = \{(a_j) \in \mathbb{Z}_2^d \mid \sum_j a_j = 0 \in \mathbb{Z}_2\},$$

which is a subgroup of $\mathbb{Z}_2^d$ of order 2^{d-1}. Then the inclusion of nets $\mathrm{Vir}_{1/2}^{\otimes 2d} \subset (\mathcal{A}^{\otimes d})^\sigma \subset \mathcal{A}^{\otimes d}$ is identified with $\mathrm{Vir}_{1/2}^{\otimes 2d} \subset (\mathrm{Vir}_{1/2}^{\otimes 2d}) \rtimes K \subset \mathrm{Vir}_{1/2}^{\otimes 2d} \rtimes \mathbb{Z}_2^d$. From this description of the inclusion and information of the representation category of $\mathrm{Vir}_{1/2}^{\otimes 2d}$, we obtain the description of the irreducible DHR sectors for the net $(\mathcal{A}^{\otimes d})^\sigma$ as follows.

Take an irreducible DHR sector $\lambda = \bigotimes_{k=1}^d \lambda_k$ of $\mathrm{Vir}_{1/2}^{\otimes 2d}$, where each λ_k is one of the irreducible DHR sectors $\lambda_{0,1/16}, \lambda_{1/2,1/16}, \lambda_{1/16,0}, \lambda_{1/16,1/2}$ of $\mathrm{Vir}_{1/2} \otimes \mathrm{Vir}_{1/2}$. Let θ be the dual canonical endomorphism for the extension $\mathrm{Vir}_{1/2}^{\otimes 2d} \subset (\mathrm{Vir}_{1/2}^{\otimes 2d}) \rtimes K$. Then we see that the monodromy of θ and λ is trivial because each automorphism appearing in θ has a trivial monodromy with λ by the description of the S-matrix given by [13, (10.138)]. Then [1, Part I, Proposition 3.23] implies that α^+- and α^--inductions of this λ give the same sector, thus they produce a DHR sector of the net $(\mathrm{Vir}_{1/2}^{\otimes 2d}) \rtimes K$. It is irreducible by [52, Theorem 3.3] and has a statistical dimension $2^{d/2}$. Again by [52, Theorem 3.3], we conclude that we have 2^{d+1} mutually distinct irreducible DHR sectors of this form.

We now determine the induction-restriction graph for the inclusion $(\mathcal{A}^{\otimes d})^\sigma \subset \mathcal{A}^{\otimes d}$. By chiral locality, each connected component of this graph for all the irreducible DHR sectors of $(\mathcal{A}^{\otimes d})^\sigma$ must be one of the two components in Figure 1.

Since the irreducible DHR sectors of the net $\mathcal{A}^{\otimes d}$ is labeled with elements of $\mathbb{Z}_4^d$ and the action of σ on this fusion rule algebra is given by -1 on each $\mathbb{Z}_4$, the induction-restriction graph for the irreducible DHR sectors of the net $\mathcal{A}^{\otimes d}$ contains 4^{d-1} copies

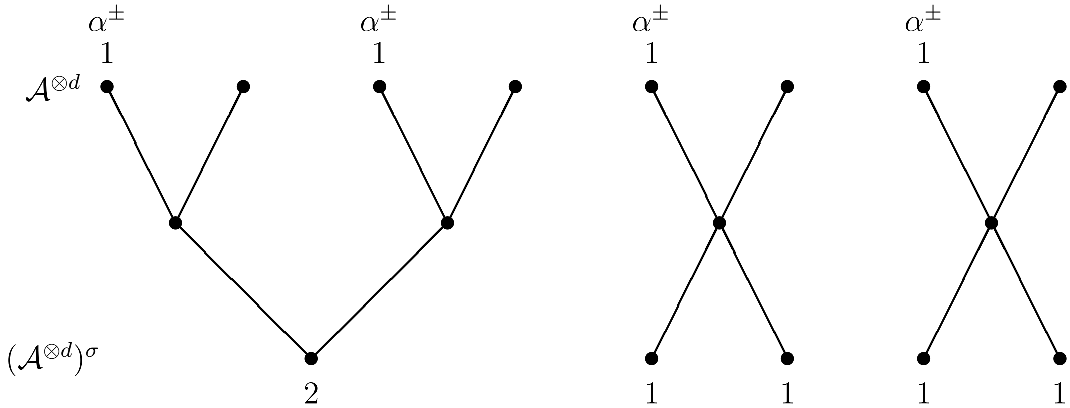

Figure 2: DHR sectors and α-induction (1)

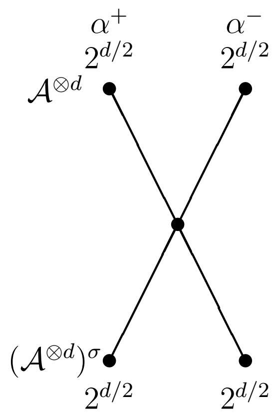

Figure 3: DHR sectors and α-induction (2)

of the graphs in Figure 2. The labels $\alpha^\pm$ mean that these irreducible DHR sectors arise from both α^+- and α^--inductions, and the numbers represent the statistical dimensions.

By the above description of the irreducible DHR sectors of the net $(\mathcal{A}^{\otimes d})^\sigma$, we know that the induction-restriction graph for the inclusion $(\mathcal{A}^{\otimes d})^\sigma \subset \mathcal{A}^{\otimes d}$ and all these irreducible DHR sectors of the net $(\mathcal{A}^{\otimes d})^\sigma$ contains 2^d copies of the graph in Figure 3, besides 4^{d-1} copies of the graphs in Figure 2 described above. The labels α^+ and α^- mean that these two irreducible DHR sectors result from α^+- and α^--inductions, respectively. (Note that 4^{d-1} copies of the graphs in Figure 2 already give all the irreducible DHR sectors of the net $(\mathcal{A}^{\otimes d})$, thus α^+- and α^--inductions in Figure 3 must produce distinct soliton sectors.)

The description of these graphs now gives the desired conclusion of the Lemma, because the total μ-index of the sectors already described gives the correct μ-index. □

We fix an interval $I \subset S^1$ and consider the following commuting square of factors.

$$\begin{array}{ccc} \mathcal{A}^{\otimes d}(I) & \subset & \mathcal{A}_L(I) \\ \cup & & \cup \\ (\mathcal{A}^{\otimes d})^\sigma(I) & \subset & \mathcal{A}_L^\sigma(I). \end{array}$$

As $(\mathcal{A}^{\otimes d})^\sigma(I)$-$(\mathcal{A}^{\otimes d})^\sigma(I)$ sectors, we consider those arising from the irreducible DHR sectors of the net $(\mathcal{A}^{\otimes d})^\sigma$ and draw the induction-restriction graphs for the above commuting square.

Two of the four irreducible DHR sectors of the net $\mathcal{A}_L^\sigma$ appear in the decomposition of the vacuum sector of $\mathcal{A}_L$. Denote the other two irreducible DHR sectors of the net $\mathcal{A}_L^\sigma$ by β_1, β_2. Apply the α^+-induction for the inclusion $\mathcal{A}_L^\sigma \subset \mathcal{A}_L$ then both β_1, β_2 give the same irreducible soliton sector of statistical dimension 1. Let $\tilde{\beta}_1$ be this soliton sector. We consider the decomposition of $\tilde{\beta}_1$ restricted to $\mathcal{A}^{\otimes d}$. This gives a direct sum of irreducible soliton sectors of $\mathcal{A}^{\otimes d}$ arising from the α^+-induction for the inclusion $(\mathcal{A}^{\otimes d})^\sigma \subset \mathcal{A}^{\otimes d}$. Note that the automorphism group G as in Lemma 2.1 gives a fusion rule subalgebra of the irreducible DHR sectors of the net $\mathcal{A}^{\otimes d}$. This group acts on the system of irreducible soliton sectors arising from α^+-induction for the inclusion $(\mathcal{A}^{\otimes d})^\sigma \subset \mathcal{A}^{\otimes d}$ by multiplication. Then this system of such soliton sectors decomposes into G-orbits.

Lemma 3.3. *The decomposition of $\tilde{\beta}_1$ into such soliton sectors give one G-orbit with a common multiplicity for all the soliton sectors.*

Proof The subnet $(\mathcal{A}^{\otimes d})^\sigma \subset \mathcal{A}_L$ is given by a fixed point net of a finite group, so all the soliton sectors of $\mathcal{A}_L$ arising from $\alpha^\pm$-inductions have statistical dimension 1 by [48, Corollary 3.20]. Consider the inclusion $Q/D_1^d \subset L/D_1^d = G \subset \mathbb{Z}_4^d$ as in [15, page 428]. Let H be the kernel of the action of G on the set of above irreducible soliton sectors of statistical dimensions $2^{d/2}$. We then have $Q/D_1^d \subset H \subset G$. Then each connected component of the induction-restriction graph for the inclusion $\mathcal{A}^{\otimes d} \subset \mathcal{A}_L$ produces a G-orbit of such soliton sectors because the dual canonical endomorphism for the inclusion $\mathcal{A}^{\otimes d} \subset \mathcal{A}_L$ gives a group G. Since all the soliton sectors of $\mathcal{A}_L$ now have dimensions 1, a simple Perron-Frobenius argument gives that one connected component of the induction-restriction graph involving the irreducible soliton sectors of statistical dimensions $2^{d/2}$ of $\mathcal{A}^{\otimes d}$ has one G-orbit. Then a decomposition of $\tilde{\beta}_1$ into soliton sectors gives such a G-orbit with a common multiplicity $[H : Q/D_1^d]$. □

Lemma 3.4. *We decompose $\beta_1 \oplus \beta_2$ for the restriction of $\mathcal{A}_L^\sigma$ to $\mathrm{Vir}_{1/2}^{\otimes 2d}$. Then this decomposition exactly corresponds to the decomposition of V_L^T in [15, Proposition 4.8].*

Proof Recall that the irreducible DHR sectors of the net $\mathrm{Vir}_{1/2}^{\otimes 2d}$ are labeled with $J \in \{0, 1/2, 1/16\}^{2d}$. So the decomposition of $\beta_1 \oplus \beta_2$ for the restriction to $\mathrm{Vir}_{1/2}^{\otimes 2d}$ is given as $\bigoplus_J n_J \lambda_J$, where n_J is the multiplicity of λ_J given by the label J as above. The irreducible modules of the vertex operator algebra $L(1/2, 0)^{\otimes 2d}$ are also labeled with the same J and we denote these irreducible modules as M_J. Then V_L^T is decomposed as $\bigoplus_J m_J M_J$ as a $L(1/2,0)^{\otimes 2d}$-module as in [15, Proposition 4.8]. By comparing this decomposition and Lemma 3.3, we conclude that $n_J = m_J$. □

Lemma 3.5. *Among the two irreducible DHR sectors β_1, β_2 of the net $\mathcal{A}_L^\sigma$, one has a conformal spin 1 and the other has -1. The fusion rules of all the four irreducible DHR sectors of the net $(\mathcal{A}_L^\sigma)$ are given by the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof The weights of $\beta_1 \oplus \beta_2$ consist of integers and half-integers by Lemma 3.4 and the description in [15, Proposition 4.8], since weights are the same in both setting of the Virasoro algebra as local conformal nets and vertex operator algebras. We know that this DHR-endomorphism decomposes into a sum of two irreducible DHR-endomorphisms β_1 and β_2, thus one of them must have all the vectors having integer weights and the other has all the vectors having half-integer weights. The former DHR-endomorphism then has a conformal spin 1 and the other has -1, as desired.

We have two possible fusion rules here, that is, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$. By Rehren [49], we have $\omega_{\alpha^n} = (\omega_\alpha)^{n^2}$, which shows that the fusion rules must be given by $\mathbb{Z}_2 \times \mathbb{Z}_2$, not $\mathbb{Z}_4$. □

We may and do assume that the DHR sector β_1 has a conformal spin 1 and β_2 has -1. By Lemma 2.1, we can make an extension of $\mathcal{A}_L^\sigma$ of index 2 using the vacuum sector and the sector β_1. We denote this net by $\tilde{\mathcal{A}}_L$. This corresponds to the twisted orbifold vertex operator algebra $\tilde{V}_L$. We now have the following theorem.

Theorem 3.6. *The local conformal net $\tilde{\mathcal{A}}_L$ constructed above is holomorphic and its vacuum character is the same as the one for the vertex operator algebra $\tilde{V}_L$.*

Proof By [39, Proposition 24], we know that the μ-index of $\tilde{\mathcal{A}}_L$ is 1. By the description of the weights in the proof of Lemma 3.5 above, the decomposition of $\tilde{\mathcal{A}}_L$ for the restriction to $\mathrm{Vir}_{1/2}^{\otimes 2d}$ exactly corresponds to the decomposition in [15, Theorem 4.10], in a similar way to the decomposition in Lemma 3.5. Since the characters for the irreducible DHR endomorphisms of the Virasoro net $\mathrm{Vir}_{1/2}$ and the characters for the modules of the Virasoro vertex operator algebra $L(1/2,0)$ coincide, the vacuum characters for $\tilde{A}_L$ and $\tilde{V}_L$ also coincide. □

In [23, page 625], Fröhlich and Gabbiani have conjectured that the S-matrix arising from a braiding on a system of irreducible DHR sectors as in [49] coincides with the S-matrix arising from the change of variables for the characters. (We have called a completely rational net satisfying this property *modular* in [38].) If a local conformal net is holomorphic, this conjecture would imply that the vacuum character is invariant under the change of variables given by $SL(2,\mathbb{Z})$, that is, a modular invariant function. The next corollary shows that this modular invariance is true for our current examples.

Corollary 3.7. *Let C be self-dual double-even binary code of length d and L be L_C or $\tilde{L}_C$, the lattice arising from C as above. Let $\mathcal{A}$ be the local conformal net $\mathcal{A}_L$ or its twisted orbifold as above. Then the vacuum character of the net $\mathcal{A}$ is invariant under $SL(2,\mathbb{Z})$.*

Proof Such modular invariance for the vacuum characters of the vertex operator algebras V_L and $\tilde{V}_L$ have been proved by Zhu [57]. Thus Theorem 3.6 gives the conclusion.

The conclusion also follows from Proposition 2 in [38]. □

Example 3.8. As in Example 2.9, we consider the Leech lattice Λ as L above. Then the corresponding net $\tilde{A}_\Lambda$ has the vacuum character as follows, which is the modular invariant J-function.

$$q^{-1} + 196884q + 21493760q^2 + 8642909970q^3 + \cdots .$$

This net $\tilde{\mathcal{A}}_\Lambda$ is the counterpart of the moonshine vertex operator algebra, and has the same vacuum character as the moonshine vertex operator algebra. We call this the *moonshine net* and denote it by $\mathcal{A}^\natural$. The near-coincidence of the coefficient 195884 in this modular function and the dimension 196883 of the smallest non-trivial irreducible representation of the Monster group, noticed by J. McKay, was the starting point of the Moonshine conjecture [11]. Note that the nets $\mathcal{A}_\Lambda$ and $\tilde{\mathcal{A}}_\Lambda$ both have central charge 24 and are holomorphic, but these nets are not isomorphic because the vacuum characters are different. To the best knowledge of the authors, this pair is the first such example.

4 Preliminaries on automorphism of framed nets

For a local conformal net $\mathcal{A}$ on a Hilbert space H with a vacuum vector Ω, its automorphism group $\mathrm{Aut}(\mathcal{A})$ is defined to be set of unitary operators U on H satisfying $U\Omega = \Omega$ and $U\mathcal{A}(I)U^* = \mathcal{A}(I)$ for all intervals I on the circle. This group is also called the *gauge group* of a net. (See [54, Definition 3.1], where Xu studies an orbifold construction with a finite subgroup of $\mathrm{Aut}(\mathcal{A})$.)

Definition 4.1. If a local conformal net $\mathcal{A}$ is an irreducible extension of $\mathrm{Vir}_{1/2}^{\otimes d}$ for some d, we say that $\mathcal{A}$ is a *framed net*.

We prepare general statements on the automorphism groups of framed nets here. We first recall some facts on automorphisms of a vertex operator algebra. Our net $\tilde{\mathcal{A}}_L$ is defined as a simple current extension by the group action of $\mathbb{Z}_2$, so it has an obvious dual action of $\mathbb{Z}_2$, and it gives an order 2 automorphism of the net $\tilde{\mathcal{A}}_L$, whose counterpart for the moonshine vertex operator algebra has been studied by Huang [28]. It is known that the Monster group has exactly two conjugacy classes of order 2, and they are called 2A and 2B. The above order 2 automorphism of the moonshine vertex operator algebra belongs to the 2B-conjugacy class. In theory of vertex operator algebras, a certain construction of an automorphism of order 2 of a vertex operator algebra has been well studied by Miyamoto [46]. In the case of the moonshine vertex operator algebra, it is known that this gives an automorphism belonging to the 2A-conjugacy class. We construct its counterpart for our framed net as follows.

We start with a general situation where we have three completely rational nets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ with an irreducible inclusion $\mathcal{B}\otimes\mathcal{C}\subset\mathcal{A}$ and $\mathcal{B}$ is isomorphic to $\mathrm{Vir}_{1/2}$. (Later we will take $\mathcal{A}$ to be a framed net.) Then the inclusion $\mathcal{B}\otimes\mathcal{C}\subset\mathcal{A}$ automatically has a finite index as in [36, Proposition 2.3]. Let θ be the dual canonical endomorphism for the inclusion $\mathcal{B}\otimes\mathcal{C}\subset\mathcal{A}$. Then we have a decomposition $\theta=\bigoplus_{h,j} n_{h,j}\lambda_h\otimes\tilde\lambda_j$, where $\{\lambda_0=\mathrm{id},\lambda_{1/16},\lambda_{1/2}\}$ is the system of irreducible DHR sectors of $\mathcal{B}\cong\mathrm{Vir}_{1/2}$, $\{\tilde\lambda_0=\mathrm{id},\tilde\lambda_1,\cdots,\tilde\lambda_m\}$ is the system of irreducible DHR sectors of $\mathcal{C}$, and $n_{h,j}$ is a multiplicity. Set $\hat\theta=\bigoplus_{h=0,1/2} n_{h,j}\lambda_h\otimes\tilde\lambda_j$. We assume that $\hat\theta$ is different from θ, that is, we have some j for which the multiplicity $n_{1/16,j}$ is nonzero. Then by [30, Corollary 3.10], this $\hat\theta$ is a dual canonical endomorphism of some intermediate net $\hat{\mathcal{A}}$ between $\mathcal{B}\otimes\mathcal{C}$ and $\mathcal{A}$. (Note that $\hat{\mathcal{A}}$ is local because $\mathcal{A}$ is local.) We have the following lemma.

Lemma 4.2. *The index* $[\mathcal{A}:\hat{\mathcal{A}}]$ *is* 2.

Proof Localize θ on an interval and let $N=\mathcal{B}(I)\otimes\mathcal{C}(I)$, $\hat M=\hat{\mathcal{A}}(I)$, $M=\mathcal{A}(I)$. The following map π gives an automorphism of order 2 of the fusion rule algebra.

$$\begin{aligned}\pi(\lambda_h\otimes\tilde\lambda_j) &= \lambda_h\otimes\tilde\lambda_j, \quad \text{if } h=0,1/2,\\ \pi(\lambda_h\otimes\tilde\lambda_j) &= -\lambda_h\otimes\tilde\lambda_j, \quad \text{if } h=1/16.\end{aligned}$$

By the crossed product type description of the algebra M in [30, Section 3], this π induces an automorphism of order 2 ot the factor M. (By the assumption on $\hat\theta$, this map π is not identity.) We denote this automorphism again by π. It is easy to see that $\hat M$ is the fixed point algebra of π on $\mathcal{M}$. Since this fixed point algebra is a factor, π is outer and thus the index $[\mathcal{A}:\hat{\mathcal{A}}]$ is 2. □

Thus the extension $\hat{\mathcal{A}}\subset\mathcal{A}$ is a simple current extension of index 2, that is, given as the crossed product by an action of $\mathbb{Z}_2$, and thus, we have the dual action on $\mathcal{A}$. Note that the vacuum Hilbert space of $\mathcal{A}$ decomposes into a direct sum according to the decomposition of θ, and on each Hilbert subspace, this automorphism acts as a multiplication by 1 [resp. -1], if the Hilbert subspace corresponds to a sector containing $\lambda_0,\lambda_{1/2}$ [resp. $\lambda_{1/16}$] as a tensoring factor. So this automorphism of order 2 corresponds to the involution of a vertex operator algebra studied by Miyamoto [46, Theorem 4.6]. We denote this automorphism by τ_B. When $\mathcal{A}$ is a framed net, an irreducible extension of $\mathrm{Vir}_{1/2}^{\otimes d}$, then we can take the k-th component of $\mathrm{Vir}_{1/2}^{\otimes d}$ as above $\mathcal{B}$, and we obtain an automorphism τ_k for this choice of $\mathcal{B}$.

Suppose now that the above θ' is equal to θ. That is, no subsector of θ contains $\lambda_{1/16}$ as a tensoring factor. Then we can define another map π' as follows.

$$\pi'(\lambda_h \otimes \tilde{\lambda}_j) = \lambda_h \otimes \tilde{\lambda}_j, \quad \text{if } h = 0,$$
$$\pi'(\lambda_h \otimes \tilde{\lambda}_j) = -\lambda_h \otimes \tilde{\lambda}_j, \quad \text{if } h = 1/2.$$

In a very similar way to the above, this map also induces an automorphism of order 2 of the net $\mathcal{A}$. This is a counterpart of another involution given in [46, Theorem 4.8].

We now have the following general description of a framed net.

Theorem 4.3. *Let $\mathcal{A}$ be a framed net, an irreducible extension of $\mathrm{Vir}_{1/2}^{\otimes d}$. Then there exist integers k, l and actions of $\mathbb{Z}_2^k$, $\mathbb{Z}_2^l$ such that $\mathcal{A}$ is isomorphic to a simple current extension of a simple current extension of $\mathrm{Vir}_{1/2}^{\otimes d}$ as follows.*

$$\mathcal{A} \cong (\mathrm{Vir}_{1/2}^{\otimes d} \rtimes \mathbb{Z}_2^k) \rtimes \mathbb{Z}_2^l.$$

Proof For the fixed subnet $\mathcal{B}$, consider the group of automorphisms of $\mathcal{A}$ generated by τ_j, $j = 1, 2, \ldots, d$. This is an abelian group and any element in this group has order at most 2, thus this group is isomorphic to $\mathbb{Z}_2^l$ for some l. Note that the net $\mathcal{A}$ is realized as an extension of this fixed point net by the group $\mathbb{Z}_2^l$ and the dual action of this group action recovers the original action of $\mathbb{Z}_2^l$ on $\mathcal{A}$.

Consider the fixed point net of $\mathcal{A}$ under this group $\mathbb{Z}_2^l$. This net contains $\mathcal{B}$ as a subnet, so decompose the vacuum representation of this net as a representation of $\mathcal{B}$, then we have only tensor products of λ_0, $\lambda_{1/2}$ and each tensor product has multiplicity 1 since each has dimension 1. This is then a crossed product extension of the net $\mathcal{B}$ and the group involved is an abelian group whose elements have order at most 2, so the group is isomorphic to $\mathbb{Z}_2^k$. □

Remark 4.4. When we make a simple current extension of a local conformal net $\mathcal{A}$ by a group G, the extended local net is unique, as long as the group G is fixed as a subcategory of the representation category of $\mathcal{A}$. That is, we have a choice of a representative, up to unitary, of an automorphism for each element of G, but such choices produce the unique extension. (In general, we may have two Q-systems depending on the choices, and they may differ by a 2-cocycle $(c_{gh})_{gh}$ on G as in [29], but locality forces $c_{gh} = c_{hg}$, which in turn gives triviality of the cocycle. So all the local Q-systems realizing G are unitarily equivalent.) We have a two-step extension of $\mathrm{Vir}_{1/2}^{\otimes d}$, thus the Q-system of this extension and the resulting extended net are both unique as long as the two groups $\mathbb{Z}_2^l$, $\mathbb{Z}_2^k$ are fixed in the representation categories.

5 The automorphism group of the moonshine net

In this section, we make a more detailed study of the automorphism group and show that it is indeed the monster group.

We begin with some comments.

Starting with a Möbius covariant local family of Wightman fields $\{\varphi_a(f)\}$, one can (obviously) canonically construct a Möbius covariant local net $\mathcal{A}$ if each operator $\varphi_a(f)$ is essentially selfadjoint on the common invariant domain and bounded continuous functions of $\varphi_a(f_1)$ and $\varphi_b(f_2)$ commute when the real test functions f_1, f_2 have support in disjoint intervals of S^1. In this case we shall say that the family $\{\varphi_a(f)\}$ is *strongly local.*

With V a vertex algebras equipped with a natural scalar product, we shall denote by $\mathrm{Aut}(V)$ the automorphism group of V that we define as follows: $g \in \mathrm{Aut}(V)$ if g is a linear invertible map from V onto V preserving the inner product with $gL = Lg$ for any Möbius group generator L, $g\Omega = \Omega$, and such that $gY(a,z)g^{-1}$ is a vertex operator of V if $Y(a,z)$ is a vertex operator of V. (One does not have a positive definite inner product for a general vertex operator algebra, but we are to consider the moonshine vertex operator algebra, and this has a natural scalar product preserved by the action of the Monster group by [22, Section 12.5], so we can use this definition of the automorphism group for a vertex operator algebra.) By the state-field correspondence it then follows that $gY(a,z)g^{-1} = Y(ga,z)$. To simplify the notation we shall denote by g also the closure of g on the Hilbert space completion of V, which is a unitary operator.

Let now $L(1/2,0)^{\otimes 48} \subset V^\natural$ be the realization of the moonshine vertex operator algebra as a framed vertex operator algebra. Let $\mathrm{Vir}_{1/2}^{\otimes 48} \subset \mathcal{A}^\natural$ be the corresponding realization of the moonshine net as a framed net as in Example 3.8. (By the "correspondence", we mean that the two Q-systems are identified.) Note in particular that $V^\natural$ is equipped with a natural scalar product and the completion of $V^\natural$ under with respect to this inner product is denoted by H.

Lemma 5.1. *The vertex operator sub-algebras V of $V^\natural$ generated by $g(L(1/2,0)^{\otimes 48})$ for all $g \in \mathrm{Aut}(V^\natural)$ is equal to $V^\natural$.*

Proof The subspace V_2 contains the Virasoro element of $V^\natural$ and some other vector which is not a scalar multiple of the Virasoro element, and it is a subspace of $V_2^\natural$, and has a representation of $\mathrm{Aut}(V^\natural)$, the monster group. The representation of the monster group on the space $V_2^\natural$ is a direct sum of the trivial representation and the one having the smallest dimension, 196883, among the nontrivial irreducible representations. From these, we conclude $V_2 = V_2^\natural$. Then by [22, Theorem 12.3.1 (g)], we know that $V = V^\natural$. □

Let H be the Hilbert space arising as the completion of $V^\natural$ with respect to the natural inner product. Because of the identification of the Q-systems, we may regard that the net $\mathcal{A}^\natural$ acts on this H.

Denote $\mathcal{B}$ the local conformal net associated with $L(1/2,0)^{\otimes 48}$. Clearly $\mathcal{B} = \bigotimes_{k=1}^{48} \mathcal{B}_k$ where the $\mathcal{B}_k$'s are all equal to the local conformal net $\mathrm{Vir}_{1/2}$.

Let $T_k(z)$ and $T(z)$ denote the stress-energy tensor of $\mathcal{B}_k$ and $\mathcal{B}$. We consider $T_k(z)$ as acting on the Hilbert space H of $\mathcal{B}$ (correspondingly to the obvious identification of of the Hilbert space of $\mathcal{B}_k$ as a subspace of H). As is well known each T_k is a strongly local Wightman field on the domain of finite-energy vectors for T_k (which is dense on the Hilbert space of $\mathcal{B}_k$) [10]. Therefore, by tensor independence, the full family $\{T_k\}_{k=1}^{48}$ (and T) is a strongly local family of Wightman fields on H. Furthermore we have:

Lemma 5.2. *The family of Wightman fields* $\{T_{k,g} \equiv gT_kg^{-1} : g \in \mathrm{Aut}(V^\natural), k = 1, 2, \dots 48\}$ *is strongly local.*

Proof Fix $g, g' \in \mathrm{Aut}(V^\natural)$ and $k, k' \in \{1, 2, \dots 48\}$. As vertex operators (formal distributions) we have

$$T_{g,k}(z)T_{g',k'}(w) = T_{g',k'}(w)T_{g,k}(z), \quad z \neq w$$

because $T_{g,k}$ and $T_{g',k'}$ belong to $V^\natural$. Therefore

$$T_{g,k}(f)T_{g',k'}(f')v = T_{g',k'}(f')T_{g,k}(f)v,$$

if f, f' are C^∞ functions with support in disjoint intervals of S^1 and v is a vector of $V^\natural$.

Recall that the vectors of $V^\natural$ are the finite energy vectors of L_0, the conformal Hamiltonian of T. Denoting by $L_{0;k,g}$ the conformal Hamitonian of $T_{g,k}$ we have the energy bounds in [10]

$$||T_{g,k}(f)v|| \leq c_f||(L_{0;k,g} + 1)v|| \leq c_f||(L_0 + 1)v||$$

where v is a finite energy vector, c_f is a constant, and in the last inequality we have used that $gTg^{-1} = T$ and

$$L_0 = \sum_k L_{0;k,\iota} = \sum_k L_{0;k,g}$$

(on finite-energy vectors) thus $L_{0;k,g} < L_0$.

Analogously, as in [10], one can verify the other bound necessary to apply a theorem in [17]

$$||[T_{g,k}(f), L_0]v|| \leq c_f||(L_0 + 1)v|| \;,$$

check the required core conditions, and conclude that the $T_{g,k}$'s form a strongly local family of Wightman fields. □

Corollary 5.3. *Let $\tilde{\mathcal{A}}$ be the conformal net on H generated by all these fields $T_{g,k}$, namely by all the nets $g\mathcal{B}_kg^{-1}$, $g \in \mathrm{Aut}(V^\natural)$ and $k \in \{1, 2, \dots 48\}$. Then $\tilde{\mathcal{A}}$ is* local *and we may identify $\tilde{\mathcal{A}}$ and $\mathcal{A}^\natural$.*

Proof The locality of $\tilde{\mathcal{A}}$ is is immediate from the above lemma. We will only need to check the cyclicity of the vacuum vector Ω.

Let E be the selfadjoint projection of H onto the closure of $\tilde{\mathcal{A}}\Omega$. Then E commutes with $\tilde{\mathcal{A}}$ and with the unitary Möbius group action. The unitary $U \equiv 2E-1$ thus fixes the vacuum, maps finite energy vectors to finite energy vectors and $UT_{g,k}U^* = T_{g,k}$. By Lemma 5.1 $UY(a,z)U^* = Y(a,z)$ for every vertex operator $Y(a,z)$ of $V^\natural$. Thus $Ua = UY(0,a)\Omega = Y(0,a)U\Omega = Y(0,a)\Omega = a$ for every vector $a \in V^\natural$, namely U and E are the identity, so Ω is cyclic.

Last, the identification of $\tilde{\mathcal{A}}$ and $\mathcal{A}^\natural$ follows by Remark 4.4. □

Theorem 5.4. *There is a natural identification between* $\mathrm{Aut}(V^\natural)$*, the automorphism group of the moonshine vertex operator algebra, and* $\mathrm{Aut}(\mathcal{A}^\natural)$*, the automorphism group of the moonshine local conformal net. Thus* $\mathrm{Aut}(\mathcal{A}^\natural)$ *is the Monster group by [22].*

Proof Since $\mathcal{A}^\natural(I)$ is generated by $g\mathcal{B}(I)g^{-1}$ as g varies in $\mathrm{Aut}(V^\natural)$, clearly every $g \in \mathrm{Aut}(V^\natural)$ gives rise to an automorphism of $\mathcal{A}^\natural$.

It remains to prove the converse. So let $g \in \mathrm{Aut}(\mathcal{A}^\natural)$ be given; we shall prove that g corresponds to an automorphism of $V^\natural$.

By definition g is a unitary operator on H that implements an automorphism of each $\mathcal{A}^\natural(I)$ and fixes the vacuum. Then g commutes with the Möbius group unitary action [7], in particular with the conformal Hamiltonian, (the Möbius group action is determined by the modular structure); therefore g maps finite energy vectors to finite energy vectors, i.e. $gV^\natural = V^\natural$.

Since $V^\natural$ is generated by the vertex operators $T_{k,g'}$ (as g' varies in $\mathrm{Aut}(V^\natural)$), and $\mathrm{Aut}(\mathcal{A}^\natural) \supset \mathrm{Aut}(V^\natural)$, it is sufficient to prove that gT_kg^{-1} is a vertex operator of $V^\natural$.

The proof is based on the state-field correspondence. Set $W(z) = gT_k(z)g^{-1}$ and recall that W is a Wightman field, in particular $W(z)\Omega$ is an analytic vector valued function for $|z| < 1$.

With f a real test function with support in an interval I, the operator $W(f)$ is affiliated to the von Neumann algebra $g\mathcal{A}^\natural(I)g^{-1} = \mathcal{A}^\natural(I)$ and it thus follows that W is strongly local with respect to $\{T_{k,g'}\}$, $g' \in \mathrm{Aut}(V^\natural)$ and $k \in \{1,2,\dots 48\}$. Since these fields generate $V^\natural$, by Dong's lemma [32] W is local with respect to all vertex operators of $V^\natural$.

Set $a \equiv W(0)\Omega$ and let $Y(a,z)$ be the vertex operator of $V^\natural$ corresponding to a in the state field correspondence, thus $Y(a,0)\Omega = a$.

We have

$$W(z)\Omega = e^{zL_{-1}}W(0)\Omega = e^{zL_{-1}}a = e^{zL_{-1}}Y(a,0)\Omega = Y(a,z)\Omega, \quad |z| < 1,$$

where L_{-1} is the infinitesimal translation operator of $V^\natural$. Now we have for $z \neq w$ and $b \in V^\natural$ (cf. [32, Sect. 4.4])

$$W(z)e^{wL_{-1}}b\Omega = W(z)Y(b,w)\Omega = Y(b,w)W(z)\Omega = Y(b,w)e^{zL_{-1}}a\Omega;$$

here we have made use of the mutual locality of $W(z)$ and $Y(a,w)$ and that the infinitesimal translation operator of W is L_{-1} because $gL_{-1}g^{-1} = L_{-1}$. Letting $w = 0$ a we get

$$W(z)b = Y(b,0)e^{zL_{-1}}a \ .$$

Analogously $Y(a,z)b = Y(b,0)e^{zL_{-1}}a$, so $W(z) = Y(a,z)$ as desired.

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