# Paragroups as quantized Galois groups for subfactors

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V. F. R. Jones initiated the theory of subfactors in 1982 in [27] and it has revolutionized the theory of operator algebras. Furthermore, he found the Jones polynomial, an invariant of links, based on the theory of subfactors in 1984 in [28], and it has created a stream of new mathematics involving quantum groups, conformal field theory, solvable lattice models, and low dimensional topology. This discovery gave him a Fields medal in 1990 as well known. From an operator algebraic viewpoint, the best machinery to understand algebraic and combinatorial structures of subfactors is the *paragroup* theory which was introduced by A. Ocneanu [43] in 1987. As an analogue of the classical Galois theory in which Galois groups describe relations between fields and their subfields, paragroups describe relations between certain kinds of algebras of (bounded linear) operators (on Hilbert spaces) and their subalgebras. Passing from function algebras to non-commutative operator algebras is often called "quantization", and in this sense, a paragroup is regarded as a "quantized Galois group". It is very similar to a fusion algebra with braiding/fusion matrices in rational conformal field theory (RCFT), and it is also related to a representation theory of quantum groups at roots of unity, an

Interaction-Round-Faces (IRF) model in exactly solvable lattice model theory, and a quantum $6j$-symbol producing Turaev-Viro type 3-dimensional topological quantum field theory (TQFT). Unfortunately, Ocneanu has not published details of his paragroup theory, a unified account of the paragroup theory still does not exist in literature, and many arguments have been scattered in several papers without full details. Here we will give a survey of the theory with an emphasis on the basic parts so that non-experts will be able to grasp the ideas of the theory. Also see an exposition [35] of H. Kosaki on the basics of the Jones index theory. We refer readers to [57, 58] as general references of the theory of operator algebras.

## §1 $\mathrm{II}_1$ factors and the Jones index

We will give a brief exposition on operator algebras first. Let $H$ be a Hilbert space and $B(H)$ be the Banach algebra of all the bounded linear operators on $H$. A $*$-subalgebra of $B(H)$ containing the identity operator is called a von Neumann algebra if it is closed with respect to the strong (or weak) operator topology. If a von Neumann algebra $M$ has a unique linear functional tr with the following three properties, we say that $M$ is a $\mathrm{II}_1$ factor.

$$(1) \quad \mathrm{tr}(xy) = \mathrm{tr}(yx), \quad x, y \in M, \qquad (2) \quad \mathrm{tr}(x^*x) \geq 0, \quad x \in M, \qquad (3) \quad \mathrm{tr}(1) = 1.$$

A simple way to construct such a $\mathrm{II}_1$ factor is as follows. Consider an increasing sequence of finite dimensional algebras

$$M_2(\mathbf{C}) \subset M_2(\mathbf{C}) \otimes M_2(\mathbf{C}) \subset M_2(\mathbf{C}) \otimes M_2(\mathbf{C}) \otimes M_2(\mathbf{C}) \subset \cdots$$

with the embedding $x \mapsto x \otimes 1$, and let the union $A$ of these algebras act on itself by the left multiplication. Next we define tr to be the ordinary trace in linear algebra divided by the size of the matrix, which is compatible with the above embedding. Define an inner product of $A$ by $(x, y) = \mathrm{tr}(y^* x)$ and regard $A$ as a set of operators acting on the completion of $A$ with respect to this inner product. Taking a closure $M$ of $A$ with respect to the weak operator topology, we get a $\mathrm{II}_1$ factor with an extension of tr on it. This $\mathrm{II}_1$ factor is especially important among all the $\mathrm{II}_1$ factors, and it is called approximately finite dimensional (AFD), because it is approximated by an increasing sequence of finite dimensional algebras. It has been known that all the AFD $\mathrm{II}_1$ factors are mutually isomorphic.

The original idea of Jones was how a $\mathrm{II}_1$ factor $N$ is embedded into another $\mathrm{II}_1$ factor $M$. We call such an $N$ a *subfactor* of $M$. From today's viewpoint, it is known that many classification problems in the theory of classical operator algebras are reduced to classification problems of subfactors. Jones [27] initiated the study of the most fundamental invariant of a subfactor $N \subset M$, the Jones index $[M : N]$. This is a kind of rank of $M$ regarded as a left $N$-module by the left multiplication. We can also think this number as a relative size of an infinite dimensional space $M$ with respect to $N$, or an analogue of the degree of an extension in the Galois theory. (In general, when a $\mathrm{II}_1$ factor $M$ acts on a Hilbert space $H$, the "rank" of $H$ as a left or right $M$-module has been defined by Murray-von Neumann with the name "coupling constant". This can take any positive number, and we write $\dim {}_M H$ or $\dim H_M$.) Jones [27] studied the possible values of the index $[M : N]$, and found that the range of the Jones index values is $\{4\cos^2(\pi/n) \mid n = 3, 4, 5, \ldots\} \cup [4, \infty]$.

For this study of the range, Jones made a use of a basic construction, which had been introduced by C. Skau for a different purpose. In the rest of this article, we assume $[M:N]<\infty$. The basic construction is given as follows. Let $L^2(N), L^2(M)$ be the Hilbert spaces given as completions of $N, M$ with respect to the inner products induced by the trace. We can naturally regard $L^2(N)$ as a subspace of the Hilbert space $L^2(M)$, so we have an orthogonal projection $e_1$ onto $L^2(N)$, which is called the Jones projection. Because $M$ acts on $L^2(M)$ by the left multiplications, we make a von Neumann algebra $M_1$ generated by $M$ and $e_1$. An important fact is that $M_1$ is again a $\mathrm{II}_1$ factor with a trace satisfying $\mathrm{tr}(e_1)=[M:N]^{-1}$ and we have $[M_1:M]=[M:N]$. This construction is called the *basic construction.* By repeating this procedure, we get $e_2, M_2$ on $L^2(M_1)$, and similarly we get a sequence of the Jones projections $e_1, e_2, e_3, \dots$ and an increasing sequence of $\mathrm{II}_1$ factors $N\subset M\subset M_1\subset M_2\subset\cdots$ by changing the Hilbert spaces at each step. We have the following celebrated relation among the Jones projections.

$$e_i e_{i\pm1} e_i = [M:N]^{-1} e_i, \qquad e_i e_j = e_j e_i, \quad |i-j|\neq 1.$$

This relation has given a representation of the braid group, and produced the Jones polynomial [28], a link invariant, via a theorem of Markov on braids and knots.

Jones further introduced the "higher relative commutants" $N'\cap M_k=\{x\in M_k \mid xy=yx, \quad \forall y\in N\}$ as a finer invariant of subfactors than the Jones index.

If $[M : N] < \infty$, then this algebra is finite dimensional for all $k$. Next look at the Bratteli diagram of the increasing sequence of the higher relative commutant

$$N' \cap N = \mathbf{C} \subset N' \cap M \subset N' \cap M_1 \subset N' \cap M_2 \subset \cdots .$$

Here the Bratteli diagram is defined as follows. In general, a finite dimensional operator algebra is isomorphic to a direct sum of matrix algebras. For a pair of finite dimensional operator algebras $A \subset B$, we decompose $A$, $B$ into direct summands, and study how a summand $A_0$ of $A$ is embedded into a summand $B_0$ of $B$. The algebra $A_0$ is isomorphic to $M_k(\mathbf{C})$ for some $k$, and let $p$ be the identity of $A_0$. Then $A_0$ is embedded into $pB_0p$ with multiplicity $l$ for some integer $l$. Then we draw $l$ edges from a vertex representing a summand $A_0$ to a vertex representing $B_0$.

Figure 1.

It turns out that the graph indicating the embedding at each step consists of the reflection of the graph in the previous step and a new part, as illustrated in the above example. The graph consisting of the new parts in all the steps (the bold part in the above example) is called the *principal graph* of the subfactor. We label the first vertex of the principal graph with the symbol $*$. This important invariant of subfactors was introduced by Jones. In general, there is no reason the principal graph is finite, but if it is finite, we say that the subfactor is of finite depth. We can also define a similar graph for another sequence of the higher relative commutants

$$M' \cap M = \mathbf{C} \subset M' \cap M_1 \subset M' \cap M_2 \subset M' \cap M_3 \subset \cdots ,$$

and we call it the *dual principal graph*. If the principal graph is finite, then the dual principal graph is automatically finite. This finite depth condition is similar to the finite level condition for the Wess-Zumino-Witten models and the roots-of-unity condition for quantum groups. In the rest of this exposition, we assume the finite depth condition.

§2 Bimodules, intertwiners, and the Frobenius reciprocity

The above-mentioned Jones index and the (dual) principal graphs are certainly a part of the data describing a subfactor $N \subset M$, but they do not look line analogues of the Galois groups. It was Ocneanu [43, 44, 45] who found a conceptually clearer formulation of the structure in the form of paragroups. We note that the direct analogue of the classical Galois group, $\{\alpha \in \mathrm{Aut}(M) \mid \alpha(x) = x, \quad \forall x \in N\}$, is almost always $\{1\}$ for subfactors related to quantum groups or conformal field theory and hence useless.

Ocneanu's fundamental idea came from a viewpoint based on bimodules over $\mathrm{II}_1$ factors. Importance of bimodules in the theory of operator algebras was emphasized by Connes and described in detail by Popa [51]. We regard $M$ itself as four kinds of bimodules, $M$-$M$, $M$-$N$, $N$-$M$, $N$-$N$ by the left and right multiplications by $N$ and $M$. (Strictly speaking, we should use the completion of $M$ when we regard $M$ as a bimodule, but we write just $M$ for simplicity.) The first important observation is that we can define relative tensor products of bimodules and $M \otimes_N M$ is naturally identified with $M_1$ as bimodules with a map $x \otimes y \mapsto xe_1y$. (See [17].) Furthermore, we identify $M_k$ with $M \otimes_N M \otimes_N \cdots \otimes_N M$ ($k+1$ copies) as bimodules naturally by induction. Ocneanu, based on these facts, found that we can get the higher relative commutants and the (dual) principal graphs in terms of bimodules as follows. We denote by $\mathrm{End}({}_N X_N)$ the set of bimodule endomorphisms, i.e., the set of bounded linear operators on the Hilbert space $X$ which commute with the left and right actions of $N$, for an $N$-$N$ bimodule $X$. We use similar notations for the other three kinds of bimodules, and we also use the endomorphism spaces of left (right) modules. Because we can identify $\mathrm{End}(M_{k\,N})$ with $M_{2k+1}$ as the basic construction of $N \subset M_k$ naturally by [49], we can identify the tower of the higher relative commutants

$$N' \cap N = \mathbf{C} \subset N' \cap M \subset N' \cap M_1 \subset N' \cap M_2 \subset N' \cap M_3 \subset \cdots$$

with

$$\mathrm{End}({}_N N_N) \subset \mathrm{End}({}_N M_M) \subset \mathrm{End}({}_N M_N) \subset \mathrm{End}({}_N M \otimes_N M_M) \subset \mathrm{End}({}_N M \otimes_N M_N) \subset \cdots .$$

Here the embeddings in the latter increasing sequence are given by the trivial inclusion $\mathrm{End}({}_N M_{k\,M}) \subset \mathrm{End}({}_N M_{k\,N})$ and the embedding $\mathrm{End}({}_N M_{k\,N}) \subset \mathrm{End}({}_N M_k \otimes_N M_M)$ given by the map $x \mapsto x \otimes_N \mathrm{id}_M$. It is shown by simple calculations that these two types of embeddings are identified naturally with those of the tower of the higher relative commutants.

With the above observations, the Bratteli diagram of the higher relative commutants and the principal graph are expressed in terms of bimodules. First, we make an irreducible decomposition of ${}_N M_{k N}$ or ${}_N M_{k M}$ as a bimodule at each step, and write a vertex for each irreducible component. (We have only finitely many components.) In the former case, we draw edges from ${}_N X_N$ to ${}_N Y_M$ with the multiplicity of ${}_N Y_M$ in ${}_N X \otimes_N M_M$, and in the latter case draw edges from ${}_N Y_M$ to ${}_N X_N$ with the multiplicity of ${}_N X_N$ in ${}_N Y \otimes_M M_N = {}_N Y_N$. Repeating this procedure, we get the desired Bratteli diagram. The fact that each step of the Bratteli diagram consists of the reflection of the previous step and a new part means that the multiplicity of ${}_N Y_M$ in ${}_N X \otimes_N M_M$ is equal to the multiplicity of ${}_N X_N$ in ${}_N Y \otimes_M M_N = {}_N Y_N$. This is an analogue of the Frobenius reciprocity in the representation theory, and we also call this fact the *Frobenius reciprocity* for bimodules. We can interpret the principal graph as a graph describing the branching rule of the relative tensor products of ${}_N N_N$ with ${}_N M_M$, ${}_M M_N$, ... , from the right. In this interpretation, we also use the name induction-restriction graph for the principal graph.

This way to interpret the (dual) principal graph, however, does not give enough information. As a next step, we make a correspondence of intertwiners to downward paths with finite length from $*$ on the Bratteli diagram. For this purpose, we need a finer version of the Frobenius reciprocity. Let ${}_N X_N$, ${}_N Y_M$ be irreducible bimodules. If the multiplicity of ${}_N Y_M$ in ${}_N X \otimes_N M_M$ is $m$, then the dimension of $\mathrm{Hom}({}_N X \otimes_N M_M, {}_N Y_M)$, which is defined in a similar way to the above End, is also $m$. Also note that for elements $\xi, \eta$ in this Hom, we have an inner product $(\xi, \eta) =$

$\xi\eta^* \in \mathrm{End}({}_N Y_M) = \mathbf{C}$. (We have $\mathbf{C}$ in this formula by the irreducibility of ${}_N Y_M$.) Next we take an orthonormal basis $\{\xi_1, \xi_2, \ldots, \xi_m\}$ of $\mathrm{Hom}({}_N X \otimes_N M_M, {}_N Y_M)$ with respect to the above inner product. Note that each $\xi_j$ is a co-isometry by the definition of the inner product. By the Frobenius reciprocity, the dimension of $\mathrm{Hom}({}_N Y \otimes_M M_N, {}_N X_N)$ is also $m$, and our aim is to express an orthonormal basis in this space in terms of the above $\xi_j$. For $\xi_j \in \mathrm{Hom}({}_N X \otimes_N M_M, {}_N Y_M)$, define $\xi'_j \in \mathrm{Hom}({}_N Y \otimes_M M_N, {}_N X_N)$ by $\xi'_j(y \otimes_M a) = \sqrt{\dim X_N / \dim Y_M}\pi_r(a^*)^* \xi^*(y)$. Here $y \in {}_N Y_M$, $a \in {}_M M_N$, $a^* \in {}_N M_M$, and $\pi_r(a^*) : {}_N X_N \to {}_N X \otimes_N M_M$ is defined by $\pi_r(a^*) : x \mapsto x \otimes_N a^*$ for $x \in {}_N X_N$. Then the coefficients $\sqrt{\dim X_N / \dim Y_M}$ make $\{\xi'_h\}$ an orthonormal basis. (See [45, 65].) Similarly, an orthonormal basis $\{\eta_j\}$ of $\mathrm{Hom}({}_N Y \otimes_M M_N, {}_N X_N)$ produces an orthonormal basis $\{\eta'_j\}$ of $\mathrm{Hom}({}_N X \otimes_N M_M, {}_N Y_M)$, and we can show that $\xi''_j = \xi_j$. In the following procedure, we assign an intertwiner to each edge in the Bratteli diagram.

(0) Let $k = 0$, and start with a bimodule ${}_N N_N$ at level 0.

(1) Depending on the parity of $k$, we make a tensor product of the bimodules at level $k$ with ${}_N M_M$ or ${}_M M_N$ from the right, and make the bimodules at level $k+1$ by irreducible decompositions.

(2) To an edge appearing in the Bratteli diagram from the level $k$ to $k+1$ which is a reflection of an edge in the previous level, we assign $\xi'$, where $\xi$ is the intertwiner assigned to the edge in the previous step. To the other edges, we assign new intertwiners so that they make an orthonormal basis.

(3) Increase $k$ by 1 and go to (1).

Note that because we assume the finite depth condition, we will have no new bimodules any more at some level in step (1). Next we assign intertwiners to downward paths of finite length on the Bratteli diagram from $*$. For example, take a path with length 3, and suppose that the three edges are labeled by the intertwiners $\xi_1 : {}_N N \otimes_N M_M \to {}_N X_{1M}$, $\xi_2 : {}_N X_1 \otimes_M M_N \to {}_N X_{2N}$, $\xi_3 : {}_N X_2 \otimes_N M_M \to {}_N X_{3M}$ from the top to the bottom. We assign to this path the composition $\xi_3(\xi_2 \otimes \mathrm{id}_{{}_N M_M})(\xi_1 \otimes \mathrm{id}_{{}_M M_N} \otimes \mathrm{id}_{{}_N M_M}) : {}_N N \otimes_N M \otimes_M M \otimes_N M_M \to {}_N X_{3M}$ of these three intertwiners. A general path is dealt with similarly.

We then take a pair of downward paths $\xi_+, \xi_-$ on the Bratteli diagram from $*$ so that the paths have the same length and same endpoint. Identifying the paths with the compositions of intertwiners, we assign the composition $\xi_+^* \xi_-$ of the intertwiners to the pair $(\xi_+, \xi_-)$. If $\xi_+, \xi_-$ have length 3 as in the above example, this composition is in $\mathrm{End}({}_N N \otimes_N M \otimes_M M \otimes_N M_M) = \mathrm{End}({}_N M_{1M}) = N' \cap M_2$. In general, if a pair $(\xi_+, \xi_-)$ has length $k$, the composition $\xi_+^* \xi_-$ of the intertwiners gives a partial isometry in a matrix unit of $N' \cap M_{k-1}$, and we know that $N' \cap M_{k-1}$ is spanned by such pairs $(\xi_+, \xi_-)$ by counting the dimension. We next study the embedding $N' \cap M_{k-1} \subset N' \cap M_k$ in this setting, and get that a pair $(\xi_+, \xi_-)$ with length $k$ is embedded from $N' \cap M_{k-1}$ into $N' \cap M_k$ as $\sum_\eta (\xi_+ \cdot \eta, \xi_- \cdot \eta)$ of length $k+1$. Here $\eta$ means a path with length 1 from the endpoint of $\xi_\pm$, and $\xi_\pm \cdot \eta$ means a concatenation of the paths. (Note that in the concatenation of the paths, we write a path below after a path above, which gives a reversed order of the compositions of the corresponding intertwiners.) In this way, we get a series of specific bases for the increasing sequence of the finite dimensional algebras $\{N' \cap M_k\}_k$. Ocneanu called a pair $(\xi_+, \xi_-)$ and such a sequence of finite dimensional algebras a *string* and *string algebras* respectively.

The increasing sequence $\{N' \cap M_k\}_k$ is not enough to recover the original subfactor $N \subset M$. We also take the following increasing sequences.

$$\begin{array}{ccccccc}
\mathrm{End}({}_N N_N) & \subset & \mathrm{End}({}_N M_M) & \subset & \mathrm{End}({}_N M_N) & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\mathrm{End}({}_M M_N) & \subset & \mathrm{End}({}_M M \otimes_N M_M) & \subset & \mathrm{End}({}_M M \otimes_N M_N) & \subset & \cdots
\end{array}$$

Here the horizontal embeddings are as above, and the vertical embeddings are given by $\mathrm{End}({}_N M_N) \ni \xi \mapsto \mathrm{id}_{{}_M M_N} \otimes \xi \in \mathrm{End}({}_M M \otimes_N M_N)$, for example. By an argument similar to the above, we can identify the second line in the above sequence with the increasing sequence $M' \cap M_1 \subset M' \cap M_2 \subset M' \cap M_3 \subset \cdots$. The first line was identified with $N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset \cdots$, and the vertical embeddings shows that the algebra $N' \cap M_k$ here is embedded into $M' \cap M_{k+2}$ in the second line as $M_1' \cap M_{k+2}$. Thus the above sequence is identified with the following.

$$\begin{array}{ccccccc}
M_1' \cap M_1 & \subset & M_1' \cap M_2 & \subset & M_1' \cap M_3 & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
M' \cap M_1 & \subset & M' \cap M_2 & \subset & M' \cap M_3 & \subset & \cdots
\end{array}$$

The above-mentioned string algebra method produces a specific basis for each finite dimensional algebra here. (That is, in the Bratteli diagram of the above sequence, downward or right-bound path with finite length from $*$ corresponds to a composition of intertwiners. Note that the Bratteli diagram of the increasing sequence $\{N' \cap M_k\}_k$ is now written horizontally, though it was written vertically above.) Now we have several natural bases for the same algebra. For example, $\mathrm{End}({}_M M \otimes_N M_N)$ in the above sequence has three paths from $\mathrm{End}({}_N N_N) = \mathbf{C}$;

$$\mathrm{End}({}_N N_N) \subset \mathrm{End}({}_N M_M) \subset \mathrm{End}({}_N M_N) \subset \mathrm{End}({}_M M \otimes_N M_N),$$

$$\mathrm{End}({}_N N_N) \subset \mathrm{End}({}_N M_M) \subset \mathrm{End}({}_M M \otimes_N M_M) \subset \mathrm{End}({}_M M \otimes_N M_N),$$

$$\mathrm{End}({}_N N_N) \subset \mathrm{End}({}_M M_N) \subset \mathrm{End}({}_M M \otimes_N M_M) \subset \mathrm{End}({}_M M \otimes_N M_N).$$

These three give different bases. The base changes among them are given by a *connection*. Take the first two of the above three. For paths with length three, the first parts $\mathrm{End}({}_N N_N) \subset \mathrm{End}({}_N M_M)$ are the same, so the difference arises from whether the path through the upper-right corner or the lower-left corner in the next block. Take a path $\xi_1$ from an upper-left vertex ($N$-$M$ bimodule) $A$ to an upper-right vertex ($N$-$N$ bimodule) $B$, a path $\xi_2$ from $B$ to an lower-right vertex ($M$-$N$ bimodule) $D$, a path $\xi_3$ from an upper-left vertex ($N$-$M$ bimodule) $A$ to a lower-left vertex ($M$-$M$ bimodule) $C$, and a path $\xi_4$ from $C$ to a lower-right vertex ($M$-$N$ bimodule) $D$. Then the compositions $\xi_2(\mathrm{id}_{{}_M M_N} \otimes \xi_1)$ and $\xi_4(\xi_3 \otimes \mathrm{id}_{{}_M M_N})$ are both in $\mathrm{Hom}({}_M M \otimes_N A \otimes_M M_N, D)$, and if we let each $\xi_j$ vary, the two ways of compositions both give orthonormal bases of this Hom space. Assign a complex number $\xi_4(\xi_3 \otimes \mathrm{id}_{{}_M M_N})(\mathrm{id}_{{}_M M_N} \otimes \xi_1)^* \xi_2^*$ to the square determined by $A, B, C, D, \xi_1, \xi_2, \xi_3, \xi_4$, and denote this number by the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{\xi_1} & B \\ {\scriptstyle \xi_3}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ C & \xrightarrow[\xi_4]{} & D \end{array}$$

We have two kinds of diagrams because the upper-left vertex is an $N$-$N$ bimodule or an $N$-$M$ bimodule. In both cases, we get the following identity, which is called the *unitarity*.

$$\sum_{B,\xi_1,\xi_2} \begin{array}{ccc} A & \xrightarrow{\xi_1} & B \\ {\scriptstyle \xi_3}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ C & \xrightarrow[\xi_4]{} & D \end{array} \cdot \overline{\begin{array}{ccc} A & \xrightarrow{\xi_1} & B \\ {\scriptstyle \eta_3}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ C' & \xrightarrow[\eta_4]{} & D \end{array}} = \delta_{\xi_3,\eta_3}\delta_{\xi_4,\eta_4}\delta_{C,C'},$$

(The number of paths from $A$ to $D$ with length 2 is the same for paths through the upper-right corner and those through the lower-left corner.) Furthermore, the coefficients appearing in the Frobenius reciprocity give

$$\begin{array}{ccc} A & \xrightarrow{\xi_1} & B \\ {\scriptstyle \xi_3}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ C & \xrightarrow[\xi_4]{} & D \end{array} = \sqrt{\frac{\mu(B)\mu(C)}{\mu(A)\mu(D)}} \overline{\begin{array}{ccc} B & \xrightarrow{\tilde{\xi}_1} & A \\ {\scriptstyle \xi_2}\downarrow & & \downarrow{\scriptstyle \xi_3} \\ D & \xrightarrow[\tilde{\xi}_4]{} & C \end{array}}$$

(See [46, 65].) Here the symbol $\tilde{\xi}_j$ means the edge $\xi_j$ with the reversed orientation and $\mu(A)$ means $\dim_N A$, $[M:N]^{-1/2}\dim_N A$, $[M:N]^{1/2}\dim_M A$, or $\dim_M A$ if $A$ is an $N$-$N$, $N$-$M$, $M$-$N$, or $M$-$M$ bimodule respectively. This identity is called *renormalization rule* or *crossing symmetry*. The renormalization rule together with the unitarity is called *bi-unitarity*.

Looking at the Bratteli diagram of the above sequence from the left to the right, we notice that the graphs at each step consist of the reflection (with respect to a vertical axis) of the previous step and a new part. The finite depth assumption means that the new parts disappear at some step, so finally we have a combination of four kinds of graphs as follows.

$$\begin{array}{ccc} \bullet & \xrightarrow{\mathcal{G}} & \bullet \\ {\scriptstyle \mathcal{G}'}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{H}} \\ \bullet & \xrightarrow[\mathcal{H}']{} & \bullet \end{array}$$

Here we have $N$-$N$ bimodules appearing in the irreducible decompositions of $M \otimes_N \cdots \otimes_N M$ at the upper left corner, and $N$-$M$ bimodules at the upper-right corner, $M$-$N$ bimodules at the lower-left corner, and $M$-$M$ bimodules at the lower-right corner. Here two graphs $\mathcal{G}$, $\mathcal{G}'$ are the same as abstract graphs and equal to the principal graph as shown below. That is, the graph $\mathcal{G}$ is determined by how an $N$-$N$ bimodule $X$ is decomposed when ${}_N M_M$ is tensored from the right as in ${}_N X \otimes_N M_M = \oplus_j n_j {}_N(Y_j)_M$, but taking the conjugate bimodules of the both hand sides, we get ${}_M M \otimes_N \bar{X}_N = \oplus_j n_j {}_M(\bar{Y}_j)_N$, which gives the decomposition rule for the left tensor multiplication by ${}_M M_N$. (Here the conjugate bimodule $\bar{X}$ of a bimodule $X$ is the conjugate Hilbert space $\bar{X}$ with the actions defined by $x \cdot \bar{\xi} \cdot y = \overline{y^* \cdot \xi \cdot x^*}$.) Similarly, both $\mathcal{H}$ and $\mathcal{H}'$ are equal to the dual principal graph.

It turns out that the above quantity $\mu(\cdot)$ gives a Perron-Frobenius eigenvector of the incidence matrices of these four graphs with the eigenvalue $[M : N]^{1/2}$. For example, if $X$ is an $N$-$N$ bimodule $X$, then we get $[M : N]^{1/2}\mu(X) = \sum_{{}_N Y_M} \mu(Y)$. Here the summation $\sum_{{}_N Y_M}$ is over $Y$ which is connected to $X$ with the principal graph and $\mu(Y)$ is summed $n$ times if the number of edges connecting $X$ and $Y$ is $n$. In an abstract setting with four graphs related as above and a rule of assigning complex number to squares arising from the graphs as above, we say that the assignment gives a bi-unitary connection (or simply, a connection) if it satisfies the bi-unitarity.

We come back to the problem of the base changes of the same End space. Take a partial isometry $(\xi_{1,+}\cdot\xi_{2,+}\cdot\xi_{3,+},\xi_{1,-}\cdot\xi_{2,-}\cdot\xi_{3,-})$ in a matrix unit of $\mathrm{End}({}_M M\otimes_N M_N)$ corresponding to the embeddings $\mathrm{End}({}_N N_N)\subset \mathrm{End}({}_N M_M)\subset \mathrm{End}({}_N M_N)\subset \mathrm{End}({}_M M\otimes_N M_N)$. We express this operator in terms of a basis corresponding to the embeddings $\mathrm{End}({}_N N_N)\subset \mathrm{End}({}_M M_N)\subset \mathrm{End}({}_M M\otimes_N M_M)\subset \mathrm{End}({}_M M\otimes_N M_N)$. We need a coefficient in the expansion

$$\sum c_{\xi_{1,+}\cdot\xi_{2,+}\cdot\xi_{3,+},\eta_{1,+}\cdot\eta_{2,+}\cdot\eta_{3,+}}\bar{c}_{\xi_{1,-}\cdot\xi_{2,-}\cdot\xi_{3,-},\eta_{1,-}\cdot\eta_{2,-}\cdot\eta_{3,-}}(\eta_{1,+}\cdot\eta_{2,+}\cdot\eta_{3,+},\eta_{1,-}\cdot\eta_{2,-}\cdot\eta_{3,-})$$

with respect to the latter basis $\{(\eta_{1,+}\cdot\eta_{2,+}\cdot\eta_{3,+},\eta_{1,-}\cdot\eta_{2,-}\cdot\eta_{3,-})\}$, and this coefficient $c_{\xi_{1,+}\cdot\xi_{2,+}\cdot\xi_{3,+},\eta_{1,+}\cdot\eta_{2,+}\cdot\eta_{3,+}}$ is given as follows.

$$\sum_{\zeta}\begin{array}{ccc} * & \xrightarrow{\xi_1} & \cdot \\ {\scriptstyle\eta_1}\downarrow & & \downarrow{\scriptstyle\zeta} \\ \cdot & \xrightarrow[\eta_2]{} & \cdot \end{array}\quad \begin{array}{ccc} \cdot & \xrightarrow{\xi_2} & \cdot \\ {\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle\xi_3} \\ \cdot & \xrightarrow[\eta_3]{} & \cdot \end{array} = \begin{array}{ccccc} * & \xrightarrow{\xi_1} & \cdot & \xrightarrow{\xi_2} & \cdot \\ {\scriptstyle\eta_1}\downarrow & & & & \downarrow{\scriptstyle\xi_3} \\ \cdot & \xrightarrow[\eta_2]{} & \cdot & \xrightarrow[\eta_3]{} & \cdot \end{array}$$

Here the summation in the left hand side is over all $\zeta$ connecting the endpoint of $\xi_1$ and the endpoint of $\eta_2$, and the right hand side is defined by the left hand side. (That is, we make a product of the connections for each configuration of the inside of the square and a summation of the products over all the configurations.) We have a similar formula for $c_{\xi_{1,-}\cdot\xi_{2,-}\cdot\xi_{3,-},\eta_{1,-}\cdot\eta_{2,-}\cdot\eta_{3,-}}$ and longer paths are dealt with similarly. Note that these coefficients are similar to the partition functions in statistical mechanics.

Next we take the "limits" $M_1'\cap M_\infty$, $M'\cap M_\infty$ of the increasing sequences $\{M_1'\cap M_k\}_k$ and $\{M'\cap M_k\}_k$. (Strictly speaking, the limit means the weak closure of the increasing sequence in the GNS-representation with respect to the trace.) It has been known that if $N\subset M$ is AFD and has finite depth, then $M_1'\cap M_\infty\subset M'\cap M_\infty$ is anti-isomorphic to $N\subset M$. This was first claimed by Ocneanu [43], but his proof has not been published, and Popa [52] gave a proof. Popa [53] further gave necessary and sufficient conditions for this anti-isomorphism, which are weaker than the finite depth condition, in the ultimate form. Thus the knowledge of the connection is enough to recover the original subfactor. Note that choices of orthonormal bases of the intertwiners are not unique, so we have a cohomological class of connections as an invariant of subfactors. See Ocneanu [43, page 154] for the exact equivalent relations of connections.

## §3 Flatness of bi-unitary connections

By the results in the previous section, it is enough to classify (equivalence classes of ) connections for classifying (AFD type $\mathrm{II}_1$) subfactors (with finite index and finite depth). But it turns out that the bi-unitarity axiom is not enough to characterize the connections arising from subfactors. The flatness axiom gives a missing property in the characterization.

Two bimodules ${}_N N_N$, ${}_M M_M$ play a role of identity in the tensor product operations. We write $*$ for one of the two. It corresponds to the inial vertex $*$ of the (dual) principal graph. In the above, we had an analogue of a partition function for rectangles with size $1 \times n$. Here we have more general rectangles. For this purpose, we take the following double sequence of the End spaces.

$$\begin{array}{ccccccc}
\mathrm{End}({}_N N_N) & \subset & \mathrm{End}({}_N M_M) & \subset & \mathrm{End}({}_N M_N) & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\mathrm{End}({}_M M_N) & \subset & \mathrm{End}({}_M M \otimes_N M_M) & \subset & \mathrm{End}({}_M M \otimes_N M_N) & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\mathrm{End}({}_N M_N) & \subset & \mathrm{End}({}_N M \otimes_N M_M) & \subset & \mathrm{End}({}_N M \otimes_N M_N) & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Here the vertical embedding is given by tensoring ${}_M M_N$ or ${}_N M_M$ from the left. For example, the first entry $\mathrm{End}({}_N M_N)$ in the third line and the third entry $\mathrm{End}({}_N M_N)$ in the first line are of course isomorphic, but they are not identified, because they are embedded into the third entry $\mathrm{End}({}_N M \otimes_N M_N)$ of the third line as different subalgebras. By generalizing the argument identifying the End spaces with the higher relative commutants in the first two lines, we can identify this double sequence with $\{N_k' \cap M_{l-1}\}$. Here the seqeunce $\{N_k\}$ is chosen so that

$$\cdots \subset N_2 \subset N_1 \subset N_0 = N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots$$

gives a series of basic constructions. Such a choice is essentially unique by [48].

For each algebra in the double sequence, we have several matrix unit systems. In order to get base changes among different systems, we apply the same method as above, and here we need the vertical Frobenius reciprocity which changes the orientations of the arrows $A \to C$, $B \to D$. An analogue of a partition function for a general rectangle with size $m \times n$ gives a coefficient in the base changes.

An important fact is that special types of diagrams have value 0 or 1. Take the following diagram.

$$
\begin{array}{ccccccccc}
* & \xrightarrow{\xi_1} & \cdot & \xrightarrow{\xi_2} & \cdot & \cdots & \cdot & \xrightarrow{\xi_m} & * \\
\eta_1 \big\downarrow & & & & & & & & \big\downarrow \eta_1 \\
\cdot & & & & & & & & \cdot \\
\eta_2 \big\downarrow & & & & & & & & \big\downarrow \eta_2 \\
\vdots & & & & & & & & \vdots \\
\eta_n \big\downarrow & & & & & & & & \big\downarrow \eta_n \\
* & \xrightarrow[\xi_1]{} & \cdot & \xrightarrow[\xi_2]{} & \cdot & \cdots & \cdot & \xrightarrow[\xi_m]{} & *
\end{array}
$$

Because all the four corners are $*$, the definition of compositions of intertwiners implies that the intertwiner through the upper-right corner and that through the lower-left corner both give

$$(\eta_n \cdot (\mathrm{id} \otimes \eta_{n-1}) \cdot \cdots \cdot (\mathrm{id} \otimes \cdots \otimes \mathrm{id}) \otimes \eta_1) \otimes \mathrm{id}_N \otimes (\xi_m \cdot (\xi_{m-1} \otimes \mathrm{id}) \cdot \cdots \cdot \xi_1 \otimes (\mathrm{id} \otimes \cdots \otimes \mathrm{id})).$$

That is, the above diagram has value 1. If we change a part of a left vertical path or a bottom right-bound path, then the unitarity implies that the new diagram has value 0 ([46]). This property is called *flatness*. The name *flat connection* comes from an interpretation that this condition is a discrete analogue of the flatness of connection in differential geometry in the sense that "parallel transport" along a loop from $*$ does not change the form of a loop. ([31,43].)

Now suppose that we have a flat bi-unitary connection on an arbitrary graph in an abstract sense. The string algebra construction gives a double sequence $A_{k,l}$

of string algebras from $*$. Different systems of matrix units of the same $A_{k,l}$ are identified by the bi-unitary connection. This system of string algebras has a unique trace and the weak closures of $\cup_l A_{k,l}$ in the GNS-representation with respect to the trace give $\mathrm{II}_1$ factors $A_{k,\infty}$. The renormalization rule implies the commuting square condition in the theory of operator algebras [50, 17], hence $A_{0,\infty} \subset A_{1,\infty} \subset A_{2,\infty} \subset \cdots$ is the Jones tower. ([45, 31, 10]) A linear algebra argument [31, Theorem 2.1] shows that the flatness axiom implies $A_{k,0} \subset A'_{0,\infty} \cap A_{k,\infty}$ and the dimension estimate of Wenzl [60, Theorem 1.6] implies the equality here. Repeating this argument, we know that the series $A_{k,l}$ appears as the higher relative commutants of some subfactor. ([32, §2]) By this kind of argument (with the initialization axiom in [43], strictly speaking), we get a bijective correspondence between equivalence classes of flat bi-unitary connections and (AFD) subfactors (with finite index and finite depth). We call the (dual) principal graphs with a flat bi-unitarity connection a *paragroup*. This name was given as a "quantization" of ordinary finite groups, as explained below.

So far we have assumed the finite depth condition, but in the infinite depth cases, we need two more axioms. They are called "amenability" by Ocneanu in [45, II.6], but they correspond to strong amenability of Popa [53] and they are equivalent to extremality [48, 49] of von Neumann subalgebras given by the higher relative commutants. According to [53], the axioms are given as ergodicity and amenability.

Longo [39, 40] found based on ideas in quantum field theory that replacing bimodules by sectors of type III factors simplifies Ocneanu type arguments. The paragroup theory based on sectors of type III factors would give an essentially same machinery.

At the end of this section, we explain how to regard an ordinary finite group $G$ as a paragroup. Any finite group $G$ acts on the AFD $\mathrm{II}_1$ factor $R$ freely. Then the crossed product construction, which is an analogue of a semi-direct product of groups, gives a bigger von Neumann algebra $R \times G$. This $R \times G$ is also a $\mathrm{II}_1$ factor, and we get $[R \times G : R] = |G| < \infty$. By constructing a paragroup with $N = R$, $M = R \times G$, we get $N$-$N$ bimodules parametrized by the elements of $G$, $M$-$M$ bimodules parametrized by the elements of $\hat{G}$, a single $M$-$N$ bimodule, and a single $N$-$M$ bimodule. The principal graph has one edge from the single $N$-$M$ bimodule to each element of $G$, and the dual principal graph has edges from the unique $M$-$N$ bimodule to each element of $\hat{G}$ with multiplicity equal to the dimension of the representation. The flat connection of this paragroup is given by

$$\begin{array}{ccc} g & \longrightarrow & \cdot \\ \Big\downarrow & & \Big\downarrow{\scriptstyle j} \\ \cdot & \underset{i}{\longrightarrow} & \sigma \end{array} = \sigma_{ij}(g), \qquad g \in G, \sigma \in \hat{G}.$$

In short, this fact is expressed as any finite group can appear as a "quantized Galois group" over the AFD $\mathrm{II}_1$ factor.

## §4 RCFT, TQFT, and paragroups

At the end, we explain relations among paragroups, solvable lattice models, (R)CFT, TQFT, and quantum groups.

First, the connection is similar to the Boltzmann weight [2] of exactly solvable models, especially IRF models. (But we have no spectral parameters for connections.) The bi-unitarity is clearly similar to the first/second inversion relations in [2], so we expect some relation between the flatness, the other axiom for paragroups, and the Yang-Baxter equation in IRF models. Unfortunately, neither of the two implies the other in general, but in some good cases, we can show that the Yang-Baxter equation implies flatness. The subfactors of type $A_n$ constructed by Jones [27, 60] (with principal graph $A_n$) correspond to the Andrews-Baxter-Forrester models [1], and the Wenzl subfactors arising from Hecke algebras of type $A$ [60] has paragroups given by the solution by Jimbo-Miwa-Okado [24, 25] to the Yang-Baxter equation ([10]). In the polynomial invariants of links, the former corresponds to the Jones polynomial [28], and the latter to the HOMFLY polynomial [16].

Next we explain a relation to the $A$-$D$-$E$ classification in CFT [5, 29]. It had been known [17] that if index is less than 4, the principal graph must be one of the Dynkin diagrams of type $A$-$D$-$E$, but the problem to determine the possible paragroup structure for each graph is more difficult. Ocneanu [43] announced a solution to this problem without a proof, and a proof has been given by [3, 19, 21, 31, 56]. (Also see [18].) A more exact comparison of this classification with the $A$-$D$-$E$ classification in CFT was given in [12].

Next we take a combinatorial approach to RCFT by Moore-Seiberg [42]. This is also similar to a paragroup.

Indeed, de Boer-Goeree [4] found that combinatorial data of RCFT in the sense of Moore-Seiberg produce paragroups. In particular, the $SU(N)_k$ Wess-Zumino-Witten models give the above-mentioned subfactors of Jones and Wenzl.

In [31], an analogue of the orbifold construction in CFT and solvable lattice model theory [8, 9, 14, 15,37] was initiated in a rather primitive form in order to solve the flatness problem of the paragroups corresponding the Dynkin diagrams $D_n$. This method was established as a general method in [10]. Furthermore, Xu [64] found a relation between the orbifold construction and the work [4] and that the conformal dimension in RCFT plays an important role in the orbifold construction.

Turaev-Viro [59] has shown that generalized $6j$-symbols give 3-dimensional TQFT based on triangulations, and had the quantum $6j$-symbol corresponding to the quantum group $\mathcal{U}_q(sl_2)$ of Kirillov-Reshetikhin [34] as an example of such a generalized $6j$-symbol. For closed 3-manifolds, this produces the square of the absolute value of the quantum $SU(2)$ invariant which was predicted by Witten [63] and established by Reshetikhin-Turaev [55]. Ocneanu [46] claimed without a full proof that this kind of generalized $6j$-symbol arises from a paragroup and produces a paragroup conversely. (See [11] for an exact statement and a proof.) We first note that the four kinds of bimodules arising from a paragroup give an algebraic system closed under tensor products. This is often called a *fusion algebra*, but note that the tensor product here is non-commutative in general. As a generalization of a connection, we get a number as a composition $\xi_4(\xi_3 \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \xi_1)^* \xi_2^*$ of the four intertwiners in the diagram

$$\begin{array}{ccc} X \otimes A \otimes Y & \xrightarrow{\mathrm{id}_X \otimes \xi_1} & X \otimes B \\ {\scriptstyle \xi_3 \otimes \mathrm{id}_Y} \big\downarrow & & \big\downarrow {\scriptstyle \xi_2} \\ C \otimes Y & \xrightarrow[\xi_4]{} & D \end{array}$$

for 6 bimodules $A, B, C, D, X, Y$. This assignment of complex numbers is called a *(quantum) $6j$-symbol*. That is, $X$, $Y$ are now general bimodules in the system.

Then we get the so-called tetrahedral symmetry of the $6j$-symbol by the Frobenius reciprocity. The unitarity of the $6j$-symbol is obtained in the same way as in the paragroup case. We use an improved version of Alexander's theorem [41, 47] to show the TQFT does not depend on triangulations, and the condition of $6j$-symbols for this property [59] is a so-called *pentagon relation*. This is actually equivalent to the flatness. This is not trivial at all, but a proof was given in [11].

We have seen relations of paragroups to "trendy" topics, but we note at the end that classical viewpoints in the theory of operator algebras are also useful for paragroups. Studies of automorphism groups have played important roles in the theory of operator algebras, and we have many work [6, 30, 33, 36, 38, 54, 61, 62] in the subfactor theory in this direction. It has turned out that the automorphisms arising from a paragroup symmetry, which was used in the orbifold construction [10, 31, 32, 64], the automorphisms in the fusion algebra appearing Izumi's work [19, 20, 22, 23], automorphisms giving a subfactor version [33] of the Connes invariant $\chi(M)$ of $\mathrm{II}_1$ factors [7, 26], and the automorphisms appearing in the Tomita-Takesaki theory [58] have much similarity in common. In short, we can say that studies of subfactors are discrete analogues of studies of type III von Neumann algebras. See [13, 33] for details of this viewpoint.

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