Compact Abelian Group Actions on Injective Factors

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Abstract. We classify compact abelian group actions on injective type III factors up to conjugacy, which completes the final step of classification of compact abelian group actions on injective factors.

§0 Introduction

The purpose of this paper is to provide a classification, up to conjugacy, of actions of a (separable) compact abelian group on injective factors of type III (Theorem 3.1).

Studying automorphism groups has been a powerful and fruitful approach to deepening our understanding of the structure of operator algebras, and the class of injective factors, which are approximately finite dimensional, (AFD), by [C5], has been the most important and well-studied. Group and groupoid actions on AFD factors have been extensively studied in recent years by Jones [J] (for finite groups),

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Ocneanu [O] (for discrete amenable groups), Jones-Takesaki [JT] (for compact abelian groups), and Sutherland-Takesaki [ST1] (for orbitally discrete amenable groupoids), since Connes' breakthrough [C2, C4] for integer actions. Sutherland-Takesaki started a classification of amenable discrete group actions on AFD factors of type III_λ, $0 \le \lambda < 1$, in [ST2], while [ST1] gave a complete classification only on semifinite algebras, and Kawahigashi-Sutherland-Takesaki extended the classification of discrete abelian group actions to the case of type III_1 in [KST]. Here we classify (separable) compact abelian group actions on AFD factors of type III. This completes the classification of compact abelian group actions on AFD factors as the natural continuation of Jones-Takesaki [JT].

In §1, we prepare some technical results on automorphisms of AFD factors of type III. The key is Theorem 1.2, by which we obtain a special type of approximation of approximately inner automorphisms by inner automorphisms. Because we already have a complete list of automorphisms of AFD factors of type III by [ST2, KST], we check the property for each automorphism in the list.

Section 2 handles centrally ergodic actions of discrete abelian groups on AFD von Neumann algebras of type III which are not necessarily factors. Such actions arise as dual actions of compact abelian group actions. We heavily rely on the method of [JT, ST1], but a new difficulty arises from the fact that the isomorphism class of each fibre of the central decomposition of the algebra is not unique in general, while it is unique in the semifinite case.

We apply the result of §2 to compact abelian group actions in §3 by Takesaki duality. With the aid of inner invariant, we get a classification up to conjugacy. We also give an example. For prime actions with properly infinite fixed point algebras, we obtain simpler classification result, which extends Thomsen's result [Th].

Section 4 is devoted to detailed study of the 1-dimensional torus. We determine all the possible combinations of types of the original factors and the crossed product algebras for prime actions.

The basic references are Connes [C1] and Connes-Takesaki [CT] for type III von Neumann algebras, and Jones-Takesaki [JT] and Sutherland-Takesaki [ST1,2] for group actions. We use notations and results from these sources freely.

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§1 Preliminaries on automorphism groups of AFD factors of type III.

Let $\mathcal{M}$ be an AFD factor of type III and $\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbf{R}$ be the continuous decomposition of $\mathcal{M}$. Define maps

$$\mathcal{U}(\mathcal{Z}(\mathcal{N}))/\mathbf{T} \to \mathcal{U}(\mathcal{M})/\mathbf{T} \rtimes \mathrm{Aut}(\mathcal{M} \mid \mathcal{N}),$$

$$\mathcal{U}(\mathcal{M})/\mathbf{T} \rtimes \mathrm{Aut}(\mathcal{M} \mid \mathcal{N}) \to \mathrm{Cnt}(\mathcal{M}),$$

by $v \mapsto (v, \sigma^v)$ and $(v, \sigma) \mapsto \mathrm{Ad}(v) \cdot \sigma$ respectively, where σ^v is defined by $\sigma^v(x) = x$ for $x \in \mathcal{N}$ and $\sigma^v(u(s)) = v\theta_s(v^*)u(s)$ for the unitaries $u(s)$ implementing θ_s in $\mathcal{M}$. Here $\mathrm{Aut}(\mathcal{M} \mid \mathcal{N})$ denotes the group of automorphisms which fix the subalgebra $\mathcal{N}$ elementwise.

Proposition 1.1. *In the above context, the following sequence is exact:*

$$1 \longrightarrow \mathcal{U}(\mathcal{Z}(\mathcal{N}))/\mathbf{T} \longrightarrow \mathcal{U}(\mathcal{M})/\mathbf{T} \rtimes Aut(\mathcal{M} \mid \mathcal{N}) \longrightarrow Cnt(\mathcal{M}) \longrightarrow 1.$$

Hence $Cnt(\mathcal{M})$ is a Polish group under the quotient group topology. In particular, $Cnt(\mathcal{M})$ is Borel in $Aut(\mathcal{M})$.

Proof. We get the exactness at $\mathrm{Cnt}(\mathcal{M})$ by Theorem 1(ii) in [KST]. Exactness at the other points is clear. Since $\mathcal{U}(\mathcal{Z}(\mathcal{N}))$, $\mathcal{U}(\mathcal{M})$, and $\mathrm{Aut}(\mathcal{M} \mid \mathcal{N})$ are Polish groups and $\mathcal{U}(\mathcal{Z}(\mathcal{N}))/\mathbf{T}$ is the quotient group by the compact group $\mathbf{T}$, $\mathrm{Cnt}(\mathcal{M})$ is also a Polish group, being the image of a continuous homomorphism. Q.E.D.

We need a special type of approximation for approximately inner automorphisms of AFD factors of type III. This will be used when we apply the generalized cohomology lemma in [Su2] to our problem.

Theorem 1.2. *Let $\mathcal{M}$ be an AFD factor of type III, and φ be a dominant weight on $\mathcal{M}$. If an automorphism α of $\mathcal{M}$ is approximately inner, then there exist a unitary $u \in \mathcal{M}$ and a sequence of unitaries $\{v_n\}$ in $\mathcal{M}_\varphi$ such that $\alpha = Ad(u) \cdot \lim_{n\to\infty} Ad(v_n)$.*

We need some lemmas for the proof.

Lemma 1.3. *Let $\mathcal{M}$ be a factor of type III_0 and φ a dominant weight on $\mathcal{M}$. Let $\mathcal{M} = \mathcal{M}_\varphi \rtimes_\theta \mathbf{R}$ be the continuous decomposition of $\mathcal{M}$ and c be a θ-cocycle in $\mathcal{Z}(\mathcal{M}_\varphi)$. Then for a positive integer n, there exist a θ-cocycle d on $\mathcal{Z}(\mathcal{M}_\varphi)$ and a unitary u in $\mathcal{M}$ such that $\bar{\sigma}^\varphi_c = Ad(u) \cdot (\bar{\sigma}^\varphi_d)^n$.*

Proof. We have an isomorphism $\Phi : H^1_\theta(\mathbf{R},\mathcal{U}(\mathcal{C}_\varphi)) \to H^1(\mathbf{Z},\mathcal{U}(\mathcal{C}_\psi))$ by Appendix in [CT], where ψ is a faithful normal semifinite lacunary weight of infinite multiplicity and $\mathcal{M} = \mathcal{M}_\psi \rtimes \mathbf{Z}$ is a discrete decomposition of $\mathcal{M}$. (See also Theorem 3.1 in [ST2] and Corollary 5.6 in [HS].) Let $[c'] = \Phi([c])$, where c' is a 1-cocycle on $\mathcal{U}(\mathcal{C}_\psi)$. Then there exists a 1-cocycle d' on $\mathcal{U}(\mathcal{C}_\psi)$ such that $(d')^n = c'$ by taking an n-th root in $\mathcal{U}(\mathcal{C}_\varphi)$. Setting $[d] = \Phi^{-1}([d'])$, we get $\bar{\sigma}^\varphi_c \equiv (\bar{\sigma}^\varphi_d)^n$ modulo $\mathrm{Int}(\mathcal{M})$.

Q.E.D.

Regard elements of $Z^1_\theta(\mathbf{R},\mathcal{U}(\mathcal{C}_\varphi))$ as maps from $\mathbf{R}$ to a Polish group $\mathcal{U}(\mathcal{C}_\varphi)$, and topologize $Z^1_\theta(\mathbf{R},\mathcal{Z}(\mathcal{M}_\varphi))$ by uniform convergence on compact sets.

Lemma 1.4. *Let $\mathcal{M}$ be a factor of type III_0, φ a dominant weight on $\mathcal{M}$, and $\mathcal{M} = \mathcal{M}_\varphi \rtimes_\theta \mathbf{R}$ the continuous decomposition of $\mathcal{M}$. The map $\Sigma : c \in Z^1_\theta(\mathbf{R},\mathcal{U}(\mathcal{C}_\varphi)) \mapsto \bar{\sigma}^\varphi_c \in Aut(\mathcal{M})$ is continuous.*

Proof. Both $Z^1_\theta(\mathbf{R},\mathcal{U}(\mathcal{C}_\varphi))$ and $\mathrm{Aut}(\mathcal{M})$ are Polish groups. Since the map $\Sigma : c \mapsto \bar{\sigma}^\varphi_c$ is a group homomorphism, it is enough to show that Σ is closed. Suppose $c_n \to c$ and $\bar{\sigma}^\varphi_{c_n} \to \sigma$ as $n \to \infty$. Then σ is identity on $\mathcal{M}_\varphi$, so σ is of the form $\bar{\sigma}^\varphi_d$ by the relative commutant theorem in [CT]. Comparing $\bar{\sigma}^\varphi_d(u(s))$ and $\bar{\sigma}^\varphi_{c_n}(u(s))$ for the unitaries $u(s)$ implementing θ_s, we get $\lim_{n\to\infty} c_n(s) = d(s)$ in the strong topology for all $s \in \mathbf{R}$. Thus we have $c = d$, which shows the closedness of Σ. Q.E.D.

Next we approximate extended modular automorphisms by certain inner automorphisms.

Lemma 1.5. *Let $\mathcal{M}$ be an AFD factor of type III and φ be a dominant weight on $\mathcal{M}$. For an extended modular automorphism $\bar{\sigma}^\varphi_c$, $c \in Z^1_\theta(\mathbf{R},\mathcal{U}(\mathcal{C}_\varphi))$, there exist a* *unitary u in $\mathcal{M}$ and a sequence of unitaries $\{v_n\}$ in $\mathcal{M}_\varphi$ such that $\bar{\sigma}_c^\varphi = Ad(u) \cdot \lim_{n\to\infty} Ad(v_n)$.*

Proof. Case 1 ($\mathcal{M}$ is of type III_λ, $0 < \lambda < 1$): Let ψ be the Powers state on $\otimes_{j=1}^\infty M_2(\mathbf{C})$ corresponding to λ. Then we may assume $\varphi = \psi \otimes \omega$, where ω is a weight on $\mathcal{L}(L^2(\mathbf{R}))$ such that $\sigma_t^\omega = \mathrm{Ad}(\rho_t)$. (Here ρ_t denotes the translation on $\mathbf{R}$ by t.) Then $\sigma_t^{\psi\otimes \mathrm{Tr}} = \left(\otimes_{j=1}^\infty \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{it}\end{pmatrix}\right) \otimes id|_{\mathcal{L}(L^2(\mathbf{R}))}$, and setting $v_n = \left(\otimes_{j=1}^n \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{it}\end{pmatrix}\right) \otimes 1 \in \mathcal{M}_\varphi$, we get $\lim_{n\to\infty} \mathrm{Ad}(v_n) = \sigma_t^{\psi\otimes \mathrm{Tr}}$. This completes the proof in this case because $\sigma_t^\varphi \equiv \sigma_t^{\psi\otimes\mathrm{Tr}}$ mod $\mathrm{Int}(\mathcal{M})$ by the Connes Radon-Nikodym cocycle theorem.

Case 2 ($\mathcal{M}$ is of type III_1): This case is proved in a way similar to the above, by regarding the AFD factor of type III_1 as a tensor product of the AFD factors of type III_λ and III_μ, $\log\lambda/\log\mu \notin \mathbf{Q}$, by the uniqueness of the AFD factor of type III_1 [C6], [H].

Case 3 ($\mathcal{M}$ is of type III_0): Let $\mathcal{C}_\varphi \cong L^\infty(X)$, and regard the flow θ of weights on X as the flow built over the base $\{B, m, Q\}$ under the ceiling function $f : B \to \mathbf{R}_+$ as in §3 of [ST2]. Let $v \in Z^1(B \rtimes_Q \mathbf{Z})$ be a cocycle such that $L^\infty \cdot p^*([v]) = [c] \in H_\theta^1(\mathbf{R}, \mathcal{U}(L^\infty(X)))$ in the notation of Theorem 3.1 in [ST2]. Then we have a θ-cocycle $c' = p^*(v)$. It is enough to show that $\bar{\sigma}_{c'}^\varphi$ can be approximated by $\mathrm{Ad}(v_n)$ because $\bar{\sigma}_c^\varphi \equiv \bar{\sigma}_{c'}^\varphi$ mod $\mathrm{Int}(\mathcal{M})$. As in the proof of Theorem 4.4 in [HS], an argument based on the Rohlin lemma shows that v can be approximated by coboundaries w_n in the strong topology. Then c' is approximated by coboundaries $d_n = p^*(w_n)$ in our topology defined above by the definition of p^*. Let $\mathrm{Ad}(v_n)$,

$v_n \in \mathcal{M}_\varphi$, be the inner automorphisms corresponding to $\bar{\sigma}^{\varphi}_{d_n}$. Then by Lemma 4, we get $\bar{\sigma}^{\varphi}_{c'} = \lim_{n\to\infty} \mathrm{Ad}(v_n)$. Q.E.D.

Proof of Theorem 1.2. We may assume $\varphi \cdot \alpha = \varphi$ replacing α by an inner perturbation of α if necessary.

First note that if the conclusion holds for some $\alpha \in \mathrm{Aut}(\mathcal{M})$, then it also holds for $\rho \cdot \alpha \cdot \rho^{-1}$ for any $\rho \in \mathrm{Aut}(\mathcal{M})$. The reason is as follows. Suppose $\alpha = \mathrm{Ad}(u) \cdot \lim_{n\to\infty} \mathrm{Ad}(v_n)$, $v_n \in \mathcal{U}(\mathcal{M}_\varphi)$. Since $\rho(\mathcal{M}_\varphi) = v\mathcal{M}_\varphi v^*$ for some $v \in \mathcal{U}(\mathcal{M})$, φ being dominant, there exists a sequence of unitaries w_n in $\mathcal{M}_\varphi$ such that $\rho(v_n) = v w_n v^*$. Then

$$\rho \cdot \alpha \cdot \rho^{-1} = \lim_{n\to\infty} \mathrm{Ad}(\rho(u)\rho(v_n)) = \mathrm{Ad}(\rho(u)v) \cdot \mathrm{Ad}(\lim_{n\to\infty} w_n v^* w_n^*) \cdot \lim_{n\to\infty} \mathrm{Ad}(w_n).$$

We regard $\mathcal{M} = \mathcal{M}_0 \bar{\otimes} \mathcal{R}$, $\varphi = \varphi_0 \otimes \mathrm{tr}$, and $\mathcal{M}_{0,\varphi_0} \bar{\otimes} \mathcal{R} \subset \mathcal{M}_\varphi$, where $\mathcal{M} \cong \mathcal{M}_0$, $\mathcal{R}$ is the AFD factor of type II_1, and φ_0 is a dominant weight on $\mathcal{M}_0$.

Assume first the asymptotic period $p_a(\alpha)$ to be 0. Then by Theorem 2 in [C2], we may assume α is of the form $id_{\mathcal{M}_0} \otimes \sigma$, where σ is a free automorphism on $\mathcal{R}$. Since $\overline{\mathrm{Int}}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})$, we are done.

Next assume $p_a(\alpha) = p \neq 0$. Then by Theorem 1(ii) in [KST], α^p is of the form $\mathrm{Ad}(w) \cdot \bar{\sigma}^{\varphi}_{c}$. By Lemma 1.3, take a cocycle d such that $\bar{\sigma}^{\varphi}_{c} \equiv (\bar{\sigma}^{\varphi}_{d})^p \bmod \mathrm{Int}(\mathcal{M})$. Setting $\beta = \alpha \cdot (\bar{\sigma}^{\varphi}_{d})^{-1}$, we get $p_o(\beta) = p_a(\beta) = p$ because $\varphi \cdot \alpha = \varphi$ and $\mathrm{mod}(\alpha) = 1$. Thus by Theorem 2.7 of [O] (see also Theorem 1.5 of [C4]), we may assume β is of the form $id_{\mathcal{M}_0} \otimes \sigma_{p,\gamma}$, where γ is an obstruction of β and $\sigma_{p,\gamma}$ is the model automorphism of $\mathcal{R}$ with obstruction γ constructed in [C4]. Now α is of the form $\bar{\sigma}^{\varphi_0}_{d'} \otimes \sigma_{p,\gamma}$ modulo inner automorphisms for some $d' \in Z^1_\theta(\mathbf{R}, \mathcal{Z}(\mathcal{M}_{0,\varphi_0}))$ by the

Connes Radon-Nikodym cocycle theorem. Thus Lemma 1.5 completes the proof.

Q.E.D.

§2 Centrally ergodic actions of discrete abelian groups on AFD von Neumann algebras of type III.

We will classify centrally ergodic actions of (countable) discrete abelian groups on AFD von Neumann algebras of type III up to cocycle conjugacy.

Let $\mathcal{M}$ be an AFD von Neumann algebra of type III, and α be an action α of a discrete abelian group G on $\mathcal{M}$. The case where $\mathcal{M}$ is a factor was handled in [ST2] and §2 of [KST]. Here we do not assume that $\mathcal{M}$ is a factor. Instead we assume that α is centrally ergodic, i.e., $\mathcal{Z}(\mathcal{M})^\alpha = \mathbf{C}$. Let $\mathcal{M} = \int_X^\oplus \mathcal{M}(x)\, d\mu(x)$ be the central decomposition of $\mathcal{M}$. Then each $\mathcal{M}(x)$ is an AFD factor of type III by Proposition 6.5 in [C5]. (We ignore measure-zero sets here and later unless specified otherwise.) Note that the isomorphism class of $\mathcal{M}(x)$ is not necessarily unique. (See p. 405 of [T3] for instance.) This is different from the case where $\mathcal{M}$ is semifinite and causes technical difficulty.

Lemma 2.1. *In the above context, the map $x \in X \mapsto Cnt(\mathcal{M}(x))$ is Borel in the sense of* [**Su2**].

Proof. This follows immediately from Theorem 3.1 (ii) in [Su2] and Proposition 1.1. Q.E.D.

Let $H = \{h \in G : \alpha_h|_{\mathcal{Z}(\mathcal{M})} = id\}$. For $h \in H$ and $x \in X$, let α_h^x be the automorphism of $\mathcal{M}(x)$ corresponding to α_h in the central decomposition of $\mathcal{M}$.

Define $N(x) = \{h \in H : \alpha_h^x \in \mathrm{Cnt}(\mathcal{M}(x))\}$. Then $N(x)$ is constant on each G/H-orbit, thus by Lemma 2.1 and the central ergodicity, $N(x)$ is independent of x. We denote it by $N(\alpha) \equiv N$. Let $\mathcal{M} = \mathcal{M}_\varphi \rtimes_\theta \mathbf{R}$ be the continuous decomposition of $\mathcal{M}$ for a dominant weight φ. We can define $\chi_\alpha = [\lambda_\alpha, \mu_\alpha] \in \Lambda(G, N(\alpha), \mathcal{U}(\mathcal{F}(\mathcal{M})))$, $\nu_\alpha = [c_\alpha] \in H^1(\mathcal{F}(\mathcal{M}))$, and $\mathrm{mod}(\alpha) : G \to \mathrm{Aut}(\mathcal{F}(\mathcal{M}))$ as in p. 437 of [ST2] even though $\mathcal{M}$ is not necessarily a factor here. (Here $\mathcal{F}(\mathcal{M})$ denotes the flow of weights of $\mathcal{M}$.) The center $\mathcal{Z}(\mathcal{M})$ is then identified with the fixed point subalgebra of $\mathcal{F}(\mathcal{M})$ under the flow. As in [ST2], $\rho \in \mathrm{Aut}(\mathcal{F}(\mathcal{M}))$ can act on these invariants.

Theorem 2.2. *If α and β are centrally ergodic actions of a discrete abelian group G on $\mathcal{M}$, then these are cocycle conjugate if and only if $N(\alpha) = N(\beta)$ and there exists a $\rho \in Aut(\mathcal{F}(\mathcal{M}))$ such that*

$$\rho|_{\mathcal{Z}(\mathcal{M})} \cdot \alpha|_{\mathcal{Z}(\mathcal{M})} \cdot \rho|_{\mathcal{Z}(\mathcal{M})}^{-1} = \beta|_{\mathcal{Z}(\mathcal{M})},$$

$$\rho(mod(\alpha), \chi_\alpha, \nu_\alpha) = (mod(\beta), \chi_\beta, \nu_\beta).$$

It is clear that the conditions are necessary as in [ST2]. We will prove that these conditions are also sufficient in this section. In the rest of this section, assume α and β satisfy the conditions in the theorem. We will prove that α and β are cocycle conjugate. We will follow the lines of [JT] and [ST1].

Identify $\mathcal{Z}(\mathcal{M})$ with $L^\infty(X)$. Define a groupoid $\mathcal{G}$ by $\mathcal{G} = X \rtimes G$. Setting $K = G/H$ and $\mathcal{K} = X \rtimes K$, we can identify $\mathcal{G}$ with $H \times \mathcal{K}$ as in Lemma 2.2.11 in [JT].

We extend ρ to an automorphism of $\mathcal{M}$ and replace α by $\rho\cdot\alpha\cdot\rho^{-1}$ so that $\alpha=\beta$ on $\mathcal{Z}(\mathcal{M})$. We may and do assume that α and β have an invariant dominant weight φ, and satisfy

$$(*) \qquad \alpha_g(u(s)) = u(s) = \beta_g(u(s)), \quad g\in G,\ s\in\mathbf{R},$$

where $\{u(s)\}$ is the one parameter unitary group corresponding to the continuous decomposition $\mathcal{M}=\mathcal{M}_\varphi \rtimes_\theta \mathbf{R}$, by Lemmas 5.10, 5.11 and 5.12 in [ST2]. (The fact that $\mathcal{M}$ is a factor was not used in [ST2].)

Lemma 2.3. *There exists an action κ of a groupoid $H\times\mathcal{K}$ on the unique AFD factor $\mathcal{R}$ of type II_1 with $\chi_\kappa=1\in\Lambda(H\times\mathcal{K},N,\mathbf{T})$ such that for each homomorphism q of $\mathcal{K}$ into $\hat{H}$, there exists a Borel map $\rho : x\mapsto\rho_x\in Aut(\mathcal{R})$ with the following properties:*

$$\rho_y\cdot\kappa_{1,k}\cdot\rho_x^{-1} = Ad(u_k)\cdot\kappa_{1,k}, \quad \textit{for a cocycle } u_k;$$

$$\rho_x\cdot\kappa_{h,x}\cdot\rho_x^{-1} = Ad(v_{h,x})\cdot\kappa_{h,x}, \quad \textit{for a cocycle } v_{h,x};$$

$$u_\gamma\kappa_{1,k}(v_{h,x}) = \langle h,q(k)\rangle v_{h,y}\kappa_{h,y}(u_k), \quad \textit{where } k\in\mathcal{K}: x\mapsto y.$$

Proof. This is nothing but Theorem 3.1 (b) in [ST1]. Q.E.D.

We regard this κ as the action of G on $\mathcal{R}\mathbin{\bar{\otimes}} L^\infty(X)$.

According to the central decomposition $\mathcal{M}=\int_X^\oplus \mathcal{M}(x)\,d\mu(x)$, we get a decomposition $\alpha_{g,x}:\mathcal{M}(x)\to\mathcal{M}(gx)$ of α_g, where gx stands for the image of x by the transformation induced on X by α_g. Based on the fact: $\mathcal{M} \cong \int_X^{\oplus} \mathcal{M}(x) \bar{\otimes} \mathcal{R}\, d\mu(x)$, we define the model action m of G on $\mathcal{M}$ by

$$m_{g,x} = \alpha_{g,x} \otimes \kappa_{g,x} : \mathcal{M}(x) \bar{\otimes} \mathcal{R} \to \mathcal{M}(gx) \bar{\otimes} \mathcal{R}.$$

This m also satisfies the above condition $(*)$. We will prove that m and β are cocycle conjugate, which is enough for the proof of Theorem 2.2.

For each $x \in X$, we have actions m^x and β^x of H on $\mathcal{M}(x)$. Now m and β have the same invariants by construction. Hence by [ST2, Theorem 5.9] and [KST, Theorem 19], m^x and β^x are cocycle conjugate for each x as actions of H, thus there exist $\tau^x \in \mathrm{Aut}(\mathcal{M}(x))$ and an m_h^x-cocycle v_h^x for each x so that $\tau^x \cdot \beta_h^x \cdot (\tau^x)^{-1} = \mathrm{Ad}(v_h^x) \cdot m_h^x$. Integrating τ^x and v_h^x respectively based on the von Neumann measurable cross-section theorem, we obtain $\tau \in \mathrm{Aut}(\mathcal{M})$ and an m-cocycle $\{v_h\}$ in $\mathcal{M}$, so that $\tau \cdot \beta_h \cdot \tau^{-1} = \mathrm{Ad}(v_h) \cdot m_h, h \in H$. Now we replace $\beta_{h,k} : \mathcal{M}(x) \to \mathcal{M}(y)$ by $\tau^y \cdot \beta_{h,k} \cdot (\tau^x)^{-1}$, where $(h,k) \in H \times \mathcal{K}$ and $k : x \mapsto y$. It is enough to prove that this new β is cocycle conjugate to m. We will write $\beta_k = \beta_{1,k}$ and $m_k = m_{1,k}$. Summarizing the above change, we come to the situation:

$$\beta_h = \mathrm{Ad}(v_h) \cdot m_h, \qquad h \in H\,; \quad \text{equivalently} \quad \beta_h^x = \mathrm{Ad}(v_h^x) \cdot m_h^x, \qquad h \in H.$$

Set

$$\mathcal{A}_0(x) = \{(\rho^x, w_h^x) : \rho^x \in \mathrm{Aut}(\mathcal{M}(x)), \rho^x \cdot m_h^x \cdot (\rho^x)^{-1} = \mathrm{Ad}(w_h^x) \cdot m_h^x,$$
$$w_h^x \text{ is an } m_h^x\text{-cocycle}, h \in H\},$$

$$\mathcal{B}_0(x) = \{(\mathrm{Ad}(u^x), u^x m_h^x((u^x)^*)) : u^x \in \mathcal{U}(\mathcal{M}(x))\}.$$

The group structure of $\mathcal{A}_0(x)$ is given by

$$(\rho_1^x, w_h^x) \cdot (\rho_2^x, v_h^x) = (\rho_1^x \rho_2^x, \rho_1^x(v_h^x) w_h^x),$$

and the topology of $\mathcal{A}_0(x)$ is given by the topology of the semi-direct product of $\mathrm{Aut}(\mathcal{M}(x))$ and $\mathcal{U}(\mathcal{M}(x))^H$, the group of all functions of H into $\mathcal{U}(\mathcal{M}(x))$ equipped with the product topology, as in page 242 of [JT]. The above group structure and the topology of $\mathcal{A}_0(x)$ may appear to be artificial. We should note, however, that $\mathcal{A}_0(x)$ can be viewed as the group of automorphisms of $\mathcal{M}(x) \rtimes_{m^x} H$ commuting with the dual action $\widehat{m^x}$ of $\hat{H}$ under the identification:

$$(\rho^x, w^x) \mapsto \bar{\rho}^x \in \mathrm{Aut}(\mathcal{M}(x) \rtimes_{m^x} H)$$

where

$$\bar{\rho}^x(a) = \rho^x(a), \qquad a \in \mathcal{M}(x);$$

$$\bar{\rho}^x(u^x(h)) = w_h^x u^x(h), \qquad h \in H.$$

Here $\{u^x(h)\}$ is of course the unitary representation of H in the crossed product. It then follows that $B_0(x)$ corresponds to the group of inner automorphisms commuting with $\widehat{m^x}$.

Since $\hat{H}$ is commutative, $\hat{H}$, or more precisely its action via $\hat{m}^x$, is a closed subgroup of $\mathcal{A}_0(x)$, which consists of those elements (id, p), $p \in \hat{H}$, regarding each $p \in \hat{H}$ as an m^x-cocycle: $h \to p(h) \in \mathbf{T}$. Define $\mathcal{A}(x)$ by $\mathcal{A}(x) = \mathcal{A}_0(x)/\hat{H}$ and π

to be the quotient map: $\mathcal{A}_0(x) \to \mathcal{A}(x)$. It is a normal Borel subgroup of a Polish group $\mathcal{A}(x)$.

We define $n_k \in \mathrm{Aut}(\mathcal{M}(y))$ by $n_k = \beta_k m_k^{-1}$ for $k \in \mathcal{K} : x \mapsto y$. Since $\beta_h^x = \mathrm{Ad}(v_h^x) \cdot m_h^x$, we have

$$
\begin{aligned}
n_k \cdot m_h^y \cdot n_k^{-1} &= \beta_k \cdot m_h^x \cdot \beta_k^{-1} \\
&= \beta_k \cdot (\mathrm{Ad}((v_h^x)^*) \cdot \beta_h^x) \cdot \beta_k^{-1} \\
&= \mathrm{Ad}(\beta_k((v_h^x)^*)) \cdot \beta_h^y \\
&= \mathrm{Ad}(\beta_k((v_h^x)^*) v_h^y) \cdot m_h^y.
\end{aligned}
$$

Define $w_h^k = \beta_k((v_h^x)^*) v_h^y$, which can be shown to be an m_h^y-cocycle by a routine calculation. This means the class $[n_k, w_h^k]$ of (n_k, w_h^k) belongs to $\mathcal{A}(x)$.

Lemma 2.4. *In the above context,* $[n_k, w_h^k]$ *belongs to* $\overline{\mathcal{B}(y)}$.

Proof. By the conditions $(*)$ and the definition of n, we know that n_γ satisfies $\varphi^y \cdot n_\gamma = \varphi^y$ and $n_k(u^y(s)) = u^y(s)$ in $\mathcal{M}(y)$ for $k \in \mathcal{K} : x \mapsto y$, where φ^x and $u^x(s)$ are given by the central decompositions $\varphi = \int_X^\oplus \varphi^x \, d\mu(x)$ and $u(s) = \int_X^\oplus u^x(s) \, d\mu(x)$. We also know $\mathrm{mod}(n_k) = 1$, because m and β have the same modules. Hence by [KST, Theorem 1. (i)], n_k is approximately inner. By Theorem 1.2, we have a unitary $u \in \mathcal{M}(y)$ and a sequence of unitaries $\{v_n\}$ in $\mathcal{M}(y)_{\varphi^y}$ such that $n_k = \mathrm{Ad}(u) \cdot \lim_{n\to\infty} \mathrm{Ad}(v_n)$. Set $\rho = \lim_{n\to\infty} \mathrm{Ad}(v_n) \in \mathrm{Aut}(\mathcal{M}(y))$. Since n_k leaves φ invariant and $\mathrm{mod}(n_k) = 1$, n_k commutes with the extended modular automorphism $\bar{\sigma}_c$ corresponding to the continuous decomposition: $\mathcal{M} = \mathcal{M}_\varphi \rtimes_\theta \mathbf{R}$. We have also $\bar{\sigma}_c \cdot \rho = \rho \cdot \bar{\sigma}_c$, hence $\bar{\sigma}_c \cdot \mathrm{Ad}(u) = \mathrm{Ad}(u) \cdot \bar{\sigma}_c$. Set

$$(n'_k, {w'}^k_h) = (\mathrm{Ad}(u), um^y_h(u^*))^{-1} \cdot (n_k, w^k_h) = (\mathrm{Ad}(u^*)n_k, u^* w^k_h m^y_h(u)).$$

For $h \in N \subset H$, we have $m^y_h = \mathrm{Ad}(a^y_h) \cdot \bar{\sigma}^{\varphi^y}_{c^y(h)}$, where $a^y_h \in \mathcal{U}(\mathcal{M}(y)_{\varphi^y})$ and $c^y(h)(s) = (a^y_h)^* \theta_s(a^y_h) \in \mathcal{U}(\mathcal{Z}(\mathcal{M}(y)_{\varphi^y}))$, see [ST2, Lemma 5.12. (iii)]. Hence,

$$\mathrm{Ad}({w'}^k_h a^y_h) \cdot \bar{\sigma}^{\varphi^y}_{c^y(h)} = \mathrm{Ad}({w'}^k_h) \cdot m^y_h = n'_k \cdot m^y_h \cdot (n'_k)^{-1} = \mathrm{Ad}(n'_k(a^y_h)) \cdot \bar{\sigma}^{\varphi^y}_{c^y(h)},$$

which implies that (i) there exists $c^k_h \in \mathcal{U}(\mathcal{Z}(\mathcal{M}(y))) = \mathbf{T}$ such that

$$n'_k(a^y_h) = c^k_h {w'}^k_h a^y_h \, ;$$

(ii) ${w'}^k_k$ belongs to $\mathcal{M}(y)_{\varphi^y}$. Using $\mu^y(h,h')a^y_{hh'} = a^y_h a^y_{h'}$, $h, h' \in N$, and $\mu^y(h,h') \in \mathcal{Z}(\mathcal{M}(y)_{\varphi^y})$, we can show that c^k_h is a character of N, by a computation similar to that in p. 243 of [JT]. We extend c^k to a character $\bar{c}^k$ of H. Therefore we may assume $c^k = 1$ by changing $(n'_k, {w'}^k_h)$ within its class in $\mathcal{A}(x)$. For simplicity, we write (n_k, w^k_h) for this new $(n'_k, {w'}^k_h)$. Now $n_k(a^y_h) = w^k_h a^y_h$, $n_k \cdot m^y_h \cdot n_k^{-1} = \mathrm{Ad}(w^k_h) \cdot m^y_h$, and $n_k = \lim_{n\to\infty} \mathrm{Ad}(v_n)$ in our new notation, and it is enough to show that (n_k, w^k_h) can be approximated by elements in $\mathcal{B}_0(x)$. The sequence $\{(w^k_h)^* v_n m^y_h(v^*_n)\}$ is strongly central in $\mathcal{M}(y)$, and we get

$$(w^k_h)^* v_n m^y_h(v^*_n) = (w^k_h)^* v_n a^y_h v^*_n (a^y_h)^* \to (w^k_h)^* n_\gamma(a^y_h)(a^y_h)^* = 1 \quad \text{as } n \to \infty,$$

for $h \in N$. Thus the same argument as in p. 244 of [JT] appealing to the 1-cohomology vanishing theorem in the ultraproduct algebra of [O] completes the proof. Q.E.D.

Proof of Theorem 2.2. We define a covariant Borel functor F from $\mathcal{K}$ to Polish groups in the sense of [Su2, Definition 4.1] by

$$F(x) = \mathrm{Aut}(\mathcal{M}(x)),$$

$$F_k(\rho) = m_k \cdot \rho \cdot m_k^{-1}, \quad k \in \mathcal{K} : x \to y \quad \text{and} \quad \rho \in F(x).$$

Define F-cocycles ρ_1 and ρ_2 by

$$\rho_1(k) = n_k, \qquad \rho_2(k) = id, \qquad k \in \mathcal{K}.$$

Then $\rho_1(k) \equiv \rho_2(k) \mathrm{mod}(\overline{\mathcal{B}(r(k))})$. Thus the Cohomology Reduction Lemma, [Su2, Theorem 5.5], entails the existence of $\theta_x \in \mathrm{Aut}(\mathcal{M}(x))$, an m^x-cocycle $\{w_h^k\}$ and $a_k \in \mathcal{M}(r(k))$ such that for $k \in \mathcal{K} : x \to y$ and $h \in H$,

$$\theta_y \cdot \beta_k \cdot \theta_x^{-1} = \mathrm{Ad}(a_k) \cdot m_k,$$

$$\theta_x \cdot m_h^x \cdot \theta_x^{-1} = \mathrm{Ad}(w_h^x) \cdot m_h.$$

We now follow the line of arguments of [JT, pages 247–248] and [ST1, pages 1111–1112]. Since $\mathcal{K}$ is hyperfinite, $\{a_k\}$ can be chosen to be an m-cocycle. Recalling $\beta_h^x = \mathrm{Ad}(v_h^x) \cdot m_h^x$, we replace β_γ by $\theta_y \cdot \beta_\gamma \cdot \theta_x^{-1}$ where $\gamma = hk \in H \times \mathcal{K} = \mathcal{G}$ and set

$$b_{hk} = \theta_y(v_h^y) w_h^y m_h(a_k),$$

so that we have now

$$(**) \qquad \beta_{hk} = \mathrm{Ad}(b_{hk}) \cdot m_{hk}, \qquad hk \in H \times \mathcal{K}.$$

It then follows easily that the map: $h \in H \mapsto b_{hx} \in \mathcal{U}(\mathcal{M}(x))$ is an m-cocycle, and that $k \in \mathcal{K} \mapsto b_k = a_k \in \mathcal{U}(\mathcal{M}(r(k)))$ is an m-cocycle also. As [ST1, Lemma 4.4], we have

$$b_{gk} m_{gk}(b_{h\ell}) = \langle h, q(k) \rangle b_{gkh\ell}, \qquad g,\, h \in H;\ k, \ell \in \mathcal{K}.$$

for some $q \in \mathrm{Hom}(\mathcal{K}, \hat{H})$. By the construction of m, we can choose $\theta_x \in \mathrm{Aut}(\mathcal{M}(x))$ and unitaries $c_\gamma \in \mathcal{U}(\mathcal{M}(r(\gamma)))$, $\gamma \in \mathcal{G}$, such that

$$\theta_y \cdot m_\gamma \cdot \theta_x^{-1} = \mathrm{Ad}(c_\gamma) \cdot m_\gamma, \gamma \in \mathcal{G}\,;$$

$$c_{gk} m_{gk}(c_{h\ell}) = \langle\, \overline{h, q(k)}\, \rangle\ c_{gkh\ell}, g, h \in H, k, \ell \in \mathcal{K}\,;$$

$$h \in H \mapsto c_{hx} \in \mathcal{U}(\mathcal{M}(x)) \quad \text{is an } m^x\text{-cocycle}\,;$$

$$k \in \mathcal{K} \mapsto c_k \quad \text{is an } m\text{-cocycle.}$$

With β as in $(**)$, we have, for $\gamma \in \mathcal{G} : x \to y$,

$$\theta_y \cdot \beta_\gamma \cdot \theta_x^{-1} = \mathrm{Ad}(\theta_y(b_\gamma) c_\gamma) \cdot m_\gamma.$$

It is now routine to check that $\gamma \mapsto \theta_{r(\gamma)}(b_\gamma) c_\gamma$ is an m-cocycle over $\mathcal{G}$, and that β is cocycle conjugate to m. Q.E.D.

§3 Actions of compact abelian groups.

Let A be a compact, (separable), abelian group. We define the *dual invariant* $\partial(\alpha)$ and the *inner invariant* $\iota(\alpha)$ as in [JT, Definition 3.2.1] for each action α of A on an AFD factor. With these two invariant, we complete the conjugacy classification of actions of A on an AFD factor as follows:

Theorem 3.1. *Let α and β be actions of a compact abelian group on an AFD factor $\mathcal{M}$ of type III. Then we conclude:*

i) α and β are cocycle conjugate if and only if

a) $\mathcal{M} \rtimes_\alpha A \cong \mathcal{M} \rtimes_\beta A$,

b) $N(\hat{\alpha}) = N(\hat{\beta})$,

c) there exists an isomorphism θ of $\mathcal{F}(\mathcal{M} \rtimes_\alpha A)$ onto $\mathcal{F}(\mathcal{M} \rtimes_\beta A)$ which conjugates the restriction of $\hat{\alpha}$ and $\hat{\beta}$ to the center $\mathcal{Z}(\mathcal{M} \rtimes_\alpha A)$ and $\mathcal{Z}(\mathcal{M} \rtimes_\beta A)$ and $\theta(mod(\hat{\alpha}), \chi_{\hat{\alpha}}, \nu_{\hat{\alpha}}) = (mod(\hat{\beta}), \chi_{\hat{\beta}}, \nu_{\hat{\beta}})$.

ii) α and β are conjugate if $\partial(\alpha) = \partial(\beta)$ and $\iota(\alpha) = \iota(\beta)$.

Proof. i) Since $\mathcal{M}$ is properly infinite, stable conjugacy implies cocycle conjugacy. Each fibre of the central decomposition of the crossed product algebra is semifinite or of type III. Thus we get the theorem by Theorem 1.2 in [ST1] and Theorem 2.2 in each case, respectively.

ii) This follows from the proof of [JT, Proposition 3.2.2] without change.

Q.E.D.

We will construct an example of a $\mathbf{T}^2$-action α on an AFD factor $\mathcal{M}$ of type III_1 in Example 3.4 such that all the fibres of the central decomposition of $\mathcal{M} \rtimes_\alpha \mathbf{T}^2$ are not isomorphic. We prove Theorem 3.3 for computation of flow of weights of crossed product algebras. (At the final stage of the preparation of this paper, we received a preprint of Sekine, who proves the same result as Theorem 3.3 as the main theorem of [Se].)

Let $\mathcal{M}$ be an injective factor of type III and $\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbf{R}$ be the continuous decomposition of $\mathcal{M}$. Thus we have a trace τ on the semifinite von Neumann algebra $\mathcal{N}$ and the one-parameter automorphism group θ of $\mathcal{N}$ satisfying $\tau \cdot \theta_t = e^{-t} \cdot \tau$, and θ on the center $\mathcal{Z}(\mathcal{N})$ induces the flow of weights of $\mathcal{M}$. Let $\mathcal{N} = \int_X^\oplus \mathcal{N}(x)\, d\mu(x)$ be the central decomposition of $\mathcal{N}$. Here each $\mathcal{N}(x)$ is isomorphic to the AFD factor of type II_∞, $\mathcal{R}_{0,1}$.

Lemma 3.2. *Suppose α is an automorphism of $\mathcal{N}$ and commutes with θ in the above context. If there exists a non-zero $a \in \mathcal{N}$ such that $xa = a\alpha(x)$ for all $x \in \mathcal{N}$, then there exists a unitary $u \in \mathcal{N}$ and an element $b \in \mathcal{Z}(\mathcal{N})$ such that $\alpha = Ad(u)$ and $a = u^* b$.*

Proof. Let $a = hu$ be the polar decomposition of a with $h = (aa^*)^{1/2}$. Since $aa^* \in \mathcal{Z}(\mathcal{N})$, we have $h \in \mathcal{Z}(\mathcal{N})$, and we get $xu = u\alpha(x)$ for all x. Let $\mathcal{Z}(\mathcal{N}) = L^\infty(X, \mu)$ and T be the transformation on X determined by $(\alpha f)(x) = f(T^{-1}x)$, $f \in L^\infty(X, \mu)$. Define $Y = \{x \in X : Tx = x\}$, and consider $\alpha^x \in \mathrm{Aut}(\mathcal{N}(x))$ arising from the central decomposition for $x \in Y$. Define $Y' = \{x \in Y : \alpha^x \in \mathrm{Int}(\mathcal{N}(x))\}$. The set Y' is invariant under the flow on X induced from θ because $\theta \cdot \alpha \cdot \theta^{-1} = \alpha$, and has a positive measure because uu^* and u^*u are central projections. Thus we get $Y' = X$, u is a unitary, and $\alpha = \mathrm{Ad}(u)$. By $xa = auxu^*$, a is of the form u^*b for $b \in \mathcal{Z}(\mathcal{N})$. Q.E.D.

Let G be a discrete (countable) group, and α be an action of G on an injective von Neumann algebra $\mathcal{M}$ of type III. For the continuous decomposition $\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbf{R}$, we may assume the action α keeps a dominant weight φ invariant, $\mathcal{N} = \mathcal{M}_\varphi$, α commutes with θ on $\mathcal{N}$, and α keeps the implementing unitaries of θ invariant by Lemma 5.11 in [ST]. We denote also by α the restriction of α on $\mathcal{N}$. Let $N(\alpha) = \{n \in G : \alpha_n \text{ is inner on } \mathcal{N}.\}$, and choose a unitary $v_n \in \mathcal{N}$ with $\alpha_n = \mathrm{Ad}(v_n)$ for $n \in N(\alpha)$. Let

$$\mathcal{R}(\mu_\alpha^*; \mathcal{Z}(\mathcal{N})) = \{ \sum_{n \in N(\alpha)} a_n v_n^* U_n : a_n \in \mathcal{Z}(\mathcal{N})\} \subset \mathcal{N} \rtimes_\alpha G,$$

where U_g denotes the implementing unitary of the crossed product algebra. Now G and $\mathbf{R}$ act on $\mathcal{R}(\mu_\alpha^*; \mathcal{Z}(\mathcal{N}))$ by $\mathrm{Ad}(U_g)$ and θ_t. The product and the G-action on $\mathcal{R}(\mu_\alpha^*; \mathcal{Z}(\mathcal{N}))$ is given by

$$a_n v_n^* U_n \cdot a_m v_m^* U_m = a_n a_m \mu_\alpha(n,m)^* v_{nm}^* U_{nm},$$

$$U_g(a_n v_n^* U_n) U_g^* = \alpha_g(a_n) \lambda_\alpha(g, gng^{-1})^* v_{gng^{-1}}^* U_{gng^{-1}},$$

This is why we use the notation $\mathcal{R}(\mu_\alpha^*; \mathcal{Z}(\mathcal{N}))$ for this algebra. For $n \in N(\alpha)$, the automorphism α_n of $\mathcal{M}$ is centrally trivial, and it is of the form $\alpha_n = \mathrm{Ad}(v_n) \cdot \bar{\sigma}_{c(n,\cdot)} \in \mathrm{Aut}(\mathcal{M})$, where $\bar{\sigma}$ is an extended modular automorphism, by Theorem 1(ii) of [KST]. Since $\alpha_n(u(t)) = u(t)$ for the implementing unitary $u(t)$ of $\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbf{R}$, the $\mathbf{R}$-action on $\mathcal{R}(\mu_\alpha^*; \mathcal{Z}(\mathcal{N}))$ is given by

$$a_n v_n^* U_n \longmapsto c(n,t)^* \theta_t(a_n) v_n^* U_n.$$

Let $\mathcal{A} = \mathcal{R}(\mu_\alpha^*; \mathcal{Z}(\mathcal{N}))^G$, then the above $\mathbf{R}$-action can be restricted to this commutative algebra $\mathcal{A} = \mathcal{Z}(\mathcal{N} \rtimes_\alpha G)$. We denote this flow by $\tilde{F}$.

Theorem 3.3. *In the above context, the flow of weights of $\mathcal{M} \rtimes_\alpha G$ is given by $(\mathcal{A}, \tilde{F}_t)$.*

Proof. Since α and the modular automorphism σ of $\mathcal{M}$ commute, the flow of weights of $\mathcal{M} \rtimes_\alpha G = \mathcal{N} \rtimes_{\alpha'} (G \times \mathbf{R})$ is given by $\bar{\theta}$ on $\mathcal{Z}(\mathcal{N} \rtimes_\alpha G) = (\mathcal{N}' \cap \mathcal{N} \rtimes_\alpha G)^G$, where $\bar{\theta}$ is an extension of θ to $\mathcal{N} \rtimes_\alpha G$.

We compute $\mathcal{N}' \cap \mathcal{N} \rtimes_\alpha G$. Suppose $\sum_{g \in G} x_g U_g$ commute with every $y \in \mathcal{N}$, where $x_g \in \mathcal{N}$, and U_g is the implementing unitary of the crossed product algebra. Then we get $y x_g = x_g \alpha_g(y)$ for every $y \in \mathcal{N}$, thus by Lemma 3.2, we get $g \in N(\alpha)$, and x_g is of the form $v_g^* a_g$, where $\alpha_g = \mathrm{Ad}(v_g)$ on $\mathcal{N}$ and $a_g \in \mathcal{Z}(\mathcal{N})$. Thus the center $\mathcal{Z}(\mathcal{N} \rtimes_\alpha G)$ is given by $\mathcal{A}$ as in the theorem. The flow on $\mathcal{A}$ given by θ is exactly our $\tilde{F}$. Q.E.D.

Now we can construct a $\mathbf{T}^2$-action on the AFD factor of type III_1 such that the isomorphism class of each fibre of the central decomposition of the crossed product algebra is not unique.

Example 3.4. Consider the AFD factor $\mathcal{R}_\infty$ of type III_1, and the modular automorphism group σ_t of $\mathcal{R}_\infty$. For $\lambda \in \mathbf{R} \setminus \mathbf{Q}$, set $\mathcal{M}(\lambda) = \mathcal{R}_\infty \rtimes_\sigma (\mathbf{Q} + \lambda \mathbf{Q})$. Then as in p. 406 of [T3], we get a von Neumann algebra $\mathcal{M} = \int_{\mathbf{R} \setminus \mathbf{Q}}^{\oplus} \mathcal{M}(\lambda)\, d\lambda$. Here $\mathcal{M}(\lambda)$ and $\mathcal{M}(\lambda')$ are isomorphic if and only if $\lambda = a\lambda' + b$ for some $a, b \in \mathbf{Q}$, $a \neq 0$. The $ax + b$-group $(a, b \in \mathbf{Q}, a \neq 0)$ defines the hyperfinite relation on $\mathbf{R} \setminus \mathbf{Q}$, hence this relation is generated by a single transformation T on $\mathbf{R} \setminus \mathbf{Q}$ by [CFW]. Then we have $\mathcal{M}(\lambda) = \mathcal{M}(T^n \lambda)$ for $n \in \mathbf{Z}$ and $\lambda \in \mathbf{R} \setminus \mathbf{Q}$. We define $\alpha_{n,\lambda} = id : \mathcal{M}(\lambda) \to \mathcal{M}(T^n \lambda)$. This defines a $\mathbf{Z}$-action α on $\mathcal{M}$. We also define another $\mathbf{Z}$-action α' on $\mathcal{M}$ with $\alpha'|_{\mathcal{Z}(\mathcal{M})} = id$ by $\alpha'_{n,\lambda}(x) = x$, $x \in \mathcal{R}_\infty \subset \mathcal{M}(\lambda)$, and $\alpha'_{n,\lambda}(u(p)) = \exp(ip)u(p)$, $p \in \mathbf{Q} + \lambda\mathbf{Q}$, where $\alpha'_{n,\lambda} : \mathcal{M}(\lambda) \to \mathcal{M}(\lambda)$ and $u(p)$ is the unitary implementing σ_p in $\mathcal{M}(\lambda)$. These α and α' commute, hence define a $\mathbf{Z}^2$-action. We denote it by α again. Then this action satisfies the assumption of Theorem 3.3, thus we can compute the flow of weights of $\mathcal{M} \rtimes_\alpha \mathbf{Z}^2$. Now $\mathcal{N}' \cap (\mathcal{N} \rtimes \mathbf{Z}^2)$ is $\mathcal{Z}(\mathcal{N}) \cong L^\infty(\hat{\mathbf{Q}} \times \hat{\mathbf{Q}}) \bar{\otimes} L^\infty(\mathbf{R} \setminus \mathbf{Q})$. The fixed point algebra of $\mathcal{Z}(\mathcal{N})$ by the action of the second component of $\mathbf{Z}^2$ is now $L^\infty(\mathbf{R} \setminus \mathbf{Q})$ because the action is ergodic on $\mathcal{N}(\lambda)$ for $\lambda \in \mathbf{R} \setminus \mathbf{Q} \setminus 2\pi\mathbf{Z}$. Considering the fixed point algebra under the action of the other $\mathbf{Z}$, we get the flow of weights is the trivial flow on $\mathbf{C}$. Thus the crossed product algebra is the AFD factor $\mathcal{R}_\infty$ of type III_1. Let β be the dual action of α on $\mathcal{R}_\infty$. Then this β has the desired property.

In the rest of this section, we consider a faithful action α of a compact abelian group A on an AFD factor $\mathcal{M}$ of type III with the condition that the fixed point algebra $\mathcal{M}^\alpha$ is a factor. Such an action is called prime. The fixed point algebra is a factor if and only if the crossed product algebra $\mathcal{M} \rtimes_\alpha A$ is a factor by [P, Corollary 4.7], and thus this condition is also equivalent to $\Gamma(\alpha) = \hat{A}$. Thomsen studied prime actions of compact abelian groups on AFD factors, and classified them on semifinite AFD factors in [Th]. We apply Theorem 3.1 to prime actions to obtain a classification result.

We assume that the fixed point algebra $\mathcal{M}^\alpha$ is properly infinite. Because $\mathcal{M}$ is of type III, $\mathcal{M}^\alpha$ cannot be of type I_n, $n < \infty$, by [St, 3.4]. Thus we assume that $\mathcal{M}^\alpha$ is not of type II_1. (Note that if $\mathcal{M}^\alpha$ is of type II_1, we can change α within its stable conjugacy class so that $\mathcal{M}^\alpha$ is of type II_∞.)

Then we know that α is a dual action by [Th, Theorem 4.3]. That is, we have a full unitary spectrum by maximality argument as in [Th, Lemma 2.5] and adjust U_g, $g \in \hat{A}$, by [Su1, Theorem 4.3.3] so that $\mathrm{Ad}(U_g)$ is an action on $\mathcal{M}_\alpha$. Lemma 5.10 in [ST2] shows that we have a dominant weight φ on $\mathcal{M}^\alpha$ which is invariant under the action $\mathrm{Ad}(U_g)$, by changing U_g if necessary. Let $\mathcal{N} = (\mathcal{M}^\alpha)_\varphi$. Then the action $\mathrm{Ad}(U_g)$ of $\hat{A}$ restricts on $\mathcal{N}$. Define $N(\alpha) = \{g \in \hat{A} : \mathrm{Ad}(U_g) \text{ on } \mathcal{N} \text{ is inner.}\}$. As in [ST2, page 437], we can define $\chi \in \Lambda(\hat{A}, N(\alpha), \mathcal{U}(\mathcal{Z}(\mathcal{N})))$ and $\nu \in H^1(\mathcal{Z}(\mathcal{N}))$ for $\mathrm{Ad}(U_g)$. We denote them by $\Omega(\alpha)$ and $\Phi(\alpha)$ respectively. These are conjugacy invariants of α. We also define $\Delta(\alpha) : \hat{A} \to \mathrm{Aut}(\mathcal{Z}(\mathcal{N}))$ by $\Delta(\alpha)(g) = \mathrm{Ad}(U_g)|_{\mathcal{Z}(\mathcal{N})}$.

Lemma 3.5. *The invariants $N(\alpha)$, $\Omega(\alpha)$, $\Phi(\alpha)$, $\Delta(\alpha)$ are equal to $N(\hat{\alpha})$, $\chi(\hat{\alpha})$, $\nu(\hat{\alpha})$, mod$(\hat{\alpha})$ in Theorem 3.1 i).*

Proof. Because the actions $\mathrm{Ad}(U_g)$ and $\hat{\alpha}$ of $\hat{A}$ are stably conjugate, we get the conclusion. Q.E.D.

Now we get the following.

Theorem 3.6. *Let α, β be prime actions of a compact abelian group A on an AFD factor $\mathcal{M}$ of type III with properly infinite fixed point algebras. Then α and β are conjugate if and only if $N(\alpha) = N(\beta)$ and there is an isomorphism $\theta : \mathcal{M}^\alpha \to \mathcal{M}^\beta$ with $\theta(\Omega(\alpha), \Phi(\alpha), \Delta(\alpha)) = (\Omega(\beta), \Phi(\beta), \Delta(\beta))$.*

Proof. By Theorem 3.1 ii) and Lemma 3.5, it is enough to show $\iota(\alpha) = \iota(\beta)$. Because the fixed point algebras are properly infinite factors, this is proved as in (3.2.3) and (3.2.4) in [JT]. Q.E.D.

Remark 3.7. Thomsen studied prime actions on semifinite AFD factors in [Th], and he also obtained the above theorem in the case $\mathcal{M}^\alpha \cong \mathcal{R}_{0,1}$ [Th, Theorem 8.1]. We extend his result so that the type III fixed point algebra case is included as above.

We now close the section with a brief discussion on the case where $\mathcal{M}^\alpha$ is a factor of type II_1. As seen in the structure analysis of a factor of type III_λ, $0 < \lambda < 1$, [T1], there exists an action of the one dimensional torus group $\mathbf{T}$ whose fixed point algebra is a factor of type II_1, yet it is not dual. On the other hand, if $\mathcal{M}^\alpha$ is a factor, then $\alpha \otimes id$ on $\mathcal{M} \bar{\otimes} \mathcal{L}(\ell^2)$ is dominant, and hence it is dual. Therefore, any prime action α of a compact abelian group A on an AFD factor of type III is conjugate to a reduction of a dominant action $\tilde{\alpha}$ of A by a projection in the fixed point algebra of $\tilde{\alpha}$, which means that the conjugacy comparison of two prime cocycle conjugate actions is reduced to the equivalence analysis of projections in the fixed point algebra under $\tilde{\alpha}$. In the case that $\mathcal{M}^\alpha$ is a finite factor. The comparison of projections in $\mathcal{M}^{\tilde{\alpha}}$ is nothing but the comparison of the relative dimension of the projections. In the next section, we will discuss prime actions more in detail for the group $\mathbf{T}$.

§4 Actions of the 1-dimensional torus T.

In this section, we make a detailed study of actions of the 1-dimensional torus $\mathbf{T}$ on AFD factors $\mathcal{M}$ of type III. We identify $\mathbf{T}$ with $\mathbf{R}/\mathbf{Z}$.

Theorem 4.1. *Let $\mathcal{M}$ be an AFD factor of type III, and α be an action of $\mathbf{T}$ on $\mathcal{M}$ with $\Gamma(\alpha) = \mathbf{Z}$. Then the possible combinations of types of $\mathcal{M}$ and $\mathcal{N} = \mathcal{M} \rtimes_\alpha \mathbf{T}$ are as follows.*

		Type of $\mathcal{N}$			
		III_0	III_λ	III_1	II_∞
Type of $\mathcal{M}$	III_0	$\triangle$	$\triangle$	$\times$	$\times$
	III_λ	$\triangle$	$\bigcirc$	$\bigcirc$	$\bigcirc$
	III_1	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\times$

Here "III_λ" means "$III_\lambda(0 < \lambda < 1)$".

The symbol $\bigcirc$ means all the combinations are possible.

The symbol $\triangle$ means only some combinations are possible.

The symbol $\times$ means no combinations are possible.

Proof. We will give a proof for each case. Now $\mathcal{N}$ is a factor because $\Gamma(\alpha) = \hat{\mathbf{T}}$.

Case 1 ($\mathcal{N}$ is of type II_∞): It is clear.

Case 2 ($\mathcal{N}$ is of type III_1): Let $\beta = \hat{\alpha} \in \mathrm{Aut}(\mathcal{M})$. If $p_a(\beta) = 0$, then β is unique up to outer conjugacy by Theorem 1(i) in [KST] and Theorem 2 in [C2]. In this case, $\mathcal{M}$ is of type III_1.

Let $p_a(\beta) = p \neq 0$. Then we may assume $\beta^p = \mathrm{Ad}(u) \cdot \sigma_T^\varphi$, $u \in \mathcal{U}(\mathcal{N}_\varphi)$, for a dominant weight φ on $\mathcal{N}$ by Theorem 1(2) in [KST]. Because $\mathcal{N} \rtimes_\beta \mathbf{Z}$ is isomorphic to a factor $\mathcal{M}$ by Takesaki duality (Theorem 4.5 in [T2]), $p_o(\beta) = 0$ and $T \neq 0$. Let $\beta(u) = \gamma u$, $\gamma = \exp(2\pi i k/p)$. Then by Theorem 3.3, we know that the flow of weights of $\mathcal{M}$ is periodic with period $\exp(-pT/k)$. Here T can be any non-zero number.

Case 3 ($\mathcal{N}$ is of type $\mathrm{III}_\lambda(0 < \lambda < 1)$): Suppose $p_a(\beta) = p \neq 0$ first. Then β^p may be assumed to be of the form $\mathrm{Ad}(u) \cdot \sigma_T^\varphi$ as above. Then by Theorem 3.3, the flow of weights of $\mathcal{M}$ is given by $(L^\infty(\mathbf{T}) \otimes L^\infty(\mathbf{T}))^{\mathrm{mod}(\beta)\times S}$, where the flow is given by the speed $-\log\lambda$ and T on the first and second components respectively, and S is given by the rotation by k/p, $\beta(u) = \exp(2\pi ik/p)u$. Here we have all the possibilities III_0, $\mathrm{III}_\mu(0 < \mu < 1)$, and III_1. But we cannot get all the AFD factors of type III_0, because we can construct only type III_0 factors with the flow of weights having pure point spectrum. Again, T can be any non-zero number.

If $p_a(\beta) = 0$, then the flow of weights of $\mathcal{M}$ is given by $L^\infty(\mathbf{T})^{\mathrm{mod}\beta}$, where the flow on $L^\infty(\mathbf{T})$ is given by the speed $-\log\lambda$. We get factors of type III_1 and III_μ, where $\log\mu = (\log\lambda)/n$ for some $n \in \mathbf{N}$.

Case 4 ($\mathcal{N}$ is of type III_0): If $p_a(\beta) = 0$, then the flow of weights of $\mathcal{M}$ is given by $\mathcal{F}(\mathcal{N})^{\mathrm{mod}\beta}$. This can be of type III_0, of course. If $\mathcal{F}(\mathcal{N})$ is a flow under the constant ceiling function, then we can make a factor of type $\mathrm{III}_\lambda(0 < \lambda < 1)$. By using θ_T as $\mathrm{mod}(\beta)$ for some T, we can get a factor of type III_1. (If θ_T is not ergodic on $\mathcal{F}(\mathcal{N})$ for some $T \neq 0$, then the original flow is a flow under the constant ceiling function. Then we can use another $\theta_{T'}$ with $T'/T \notin \mathbf{Q}$ as an ergodic transformation.)

If $p_o(\beta) = p \neq 0$, then as in Case 3, we know that the flow of weights of $\mathcal{M}$ is given by $(\mathcal{F}(\mathcal{M}) \otimes L^\infty(\mathbf{T}))^{\mathrm{mod}(\beta)\times S}$, where S is defined as above. Q.E.D.

We consider actions α of $\mathbf{T}$ with $\Gamma(\alpha) \neq \mathbf{Z}$ next. If $\Gamma(\alpha) = p\mathbf{Z}$ for some $p \neq 1, 0$, then we may assume α has a period $1/p$ by changing α within its cocycle conjugacy class if necessary by Corollaire 2.3.1 and Lemma 2.3.14 in [C1]. Then an action α has the full Connes spectrum as an action of $\mathbf{T}/(\mathbf{Z}/p)$. So these are reduced to the above case.

Finally we consider the case $\Gamma(\alpha) = \{0\}$. In this case, $\mathbf{Z} = \hat{\mathbf{T}}$ acts on X freely because of $\Gamma(\alpha) = \{0\}$, where $\mathcal{Z}(\mathcal{M} \rtimes_\alpha \mathbf{T}) = L^\infty(X)$. So we have $H = \{0\}$ for $\hat{\alpha}$

in the notation in §2. Thus we do not have characteristic invariants nor modular invariants. Hence, these are classified by modules.

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