Cohomology of actions of discrete groups on factors of type $\mathbf{II}_1$

Yasuyuki Kawahigashi

Department of Mathematics, Faculty of Science

University of Tokyo, Hongo, Tokyo, 113, JAPAN

(email: d33844%tansei.cc.u-tokyo.ac.jp@RELAY.CS.NET)

Abstract. We study 1-cohomology of discrete group actions on factors of type II_1. Characterizations of Kazhdan's property T and amenability for discrete groups in terms of cocycles and coboundaries are given, and we show that each of $SL(n, \mathbf{Z})$, $n \geq 3$, and $Sp(n, \mathbf{Z})$, $n \geq 2$, has a continuous family of mutually non-cocycle conjugate free actions on the AFD factor of type II_1 as an application. We also introduce and compute entropy for discrete amenable group action on factors of type II_1.

§0 Introduction

In this paper, we study 1-cohomology of discrete group actions on factors of type II_1. We give characterizations of Kazhdan's property T and amenability for discrete groups in terms of 1-cocycles and coboundaries for actions on factors of type II_1. As an application, we also show that each of $SL(n, \mathbf{Z})$, $n \geq 3$, and $Sp(n, \mathbf{Z})$, $n \geq 2$, has a continuous family of mutually non-cocycle conjugate ergodic free actions on the approximately finite dimensional (AFD) factor of type II_1. These are typical groups with Kazhdan's property T. We introduce and compute entropy of discrete amenable group actions on the AFD factor of type II_1.

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Complete classification of actions of discrete amenable groups on the AFD factor $\mathcal{R}$ of type II_1 up to cocycle conjugacy was given in Ocneanu [O]. In particular, he showed that any two free actions of a discrete amenable group on $\mathcal{R}$ are cocycle conjugate. Then Jones [J2] showed that this statement is no longer valid for any discrete non-amenable group. He constructed two free actions and used the ergodicity at infinity to distinguish the two. This shows that non-amenable discrete groups are quite different from amenable ones in the theory of group actions on factors. In order to understand cocycle conjugacy of non-amenable group actions, we start to study 1-cohomology of the actions and get several von Neumann algebra analogues of Schmidt's work [S] on ergodic actions on probability spaces. Major difference between the cohomology theory on probability spaces and one on von Neumann algebras is that we do not have the group structure on the space of cocycles in the latter case, which causes technical difficulty.

First we work on groups with Kazhdan's property T, which are far from being amenable. (See §1 for the definition.) Since Connes [C3], several authors have shown that the discrete groups with Kazhdan's property T are the opposite extreme of amenable groups with respect to representations and actions on probability spaces and von Neumann algebras. Here a characterization of Kazhdan's property T in terms of cocycles and coboundaries is given in §1. Jones [J1] showed two mutually non-cocycle conjugate free actions of discrete groups with Kazhdan's property T on the AFD factors of type II_1, and M. Choda [Ch2] showed four of such actions. We exhibit a continuous family of mutually non-cocycle conjugate free actions on the AFD factor of type II_1 for certain groups with Kazhdan's property T like $SL(n, \mathbf{Z})$, $n \geq 3$ and $Sp(n, \mathbf{Z})$, $n \geq 2$, using her construction in [Ch2]. These cannot be distinguished by ergodicity at infinity as in Jones [J2]. Instead, we make use of rigidity argument of cocycles to show that "almost all" pair in the family is not mutually cocycle conjugate. This shows another aspect of rigidity in operator algebras. (See Connes [C3], Connes-Jones [CJ1, CJ2] Popa [P1].)

In §2, we work on amenable groups. We introduce and compute Connes-Størmer entropy for discrete amenable group actions on the AFD factor of type II_1 to distinguish continuously many non-commutative Bernoulli shifts. Then we get a characterization of amenability in terms of cocycles and coboundaries, based on Ocneanu's work [O]. This shows another remarkable difference between the discrete groups with Kazhdan's property T and discrete amenable groups.

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§1 Kazhdan's property T and cohomology classes

Let G be a discrete (countable) group and α an action of G on a factor $\mathcal{M}$ of type II_1. Let

$$Z_\alpha^1 = \{u : G \to \mathcal{U}(\mathcal{M}) \mid u_g \alpha_g(u_h) = u_{gh}, g, h \in G\},$$

$$B_\alpha^1 = \{u : G \to \mathcal{U}(\mathcal{M}) \mid \text{There exists } v \in \mathcal{U}(\mathcal{M}) \text{ such that } u_g = v\alpha_g(v^*)\}.$$

Topology of Z_α^1 is given by the strong convergence at each $g \in G$. This topology is given by the following metric:

$$d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|u_{g_n} - v_{g_n}\|_2, \qquad \text{where } G = \{g_n \mid n \geq 1\}.$$

Consider the induced metric on B^1_α.

We say two cocycles u_g, v_g are cohomologous and write $u_g \sim v_g$ if there exists a unitary $u \in \mathcal{M}$ such that $u_g = uv_g\alpha_g(u^*)$. Let $H^1_\alpha = Z^1_\alpha/\sim$ be the set of cohomology classes. Araki-Choda [ACh] called an action α strong if any asymptotically α-fixed sequence in $\mathcal{M}$ is equivalent to a bounded sequence in $\mathcal{M}^\alpha$ and showed that a discrete group has Kazhdan's property T if and only if every action of it on a factor of type II_1 is strong. This is a non-commutative analogue of Connes-Weiss [CW] to the effect that a discrete group has Kazhdan's property T if and only if every ergodic measure-preserving action of it on a probability space is strongly ergodic.

We show that a strong actions have a certain good property about cocycles. The following technique is based on that of Connes [C1, Theorem 3.1], and we note an analogy between central sequences and asymptotically α-fixed sequences.

Proposition 1. *If α is strong in the above context, then the space of coboundaries B^1_α is closed.*

Proof. We define a bijective map Φ from the coset space $\mathcal{U}(\mathcal{M})/\mathcal{U}(\mathcal{M}^\alpha)$ to Z^1_α by $\Phi(v) = \{v\alpha_g(v^*)\}$. Because the metric defined by L^2-norm is translation invariant, $\mathcal{U}(\mathcal{M})/\mathcal{U}(\mathcal{M}^\alpha)$ is a Polish space. We show that this Φ is a homeomorphism.

If $v_n\mathcal{U}(\mathcal{M}^\alpha) \to v\mathcal{U}(\mathcal{M}^\alpha)$ in the topology of the coset space, then there exists a sequence of unitaries (w_n) in $\mathcal{U}(\mathcal{M}^\alpha)$ such that $\|v_n w_n - v\| \to 0$. This implies $\|v_n\alpha_g(v_n^*) - v\alpha_g(v^*)\|_2 \to 0$, hence Φ is continuous.

On the other hand, suppose $v_n\alpha_g(v_n^*) \to v\alpha(v^*)$. Then the sequence (v^*v_n) is asymptotically α-fixed, hence there exists a sequence (w_n) in $\mathcal{M}^\alpha$ such that $\|v^*v_n - w_n\|_2 \to 0$. We may assume each w_n is a unitary by changing w_n by a small operator if necessary, by Proposition 1.1.3 in Connes [C2]. Then the cosets $v_n\mathcal{U}(\mathcal{M}^\alpha)$ converges to $v\mathcal{U}(\mathcal{M}^\alpha)$, hence Φ^{-1} is also continuous. Thus there exists a positive number ε_n such that $d(u\alpha_g(u^*),1) < \varepsilon_n$ implies existence of $w \in \mathcal{U}(\mathcal{M}^\alpha)$ with $\|uw-1\|_2 < \frac{1}{2^n}$. Suppose we have a sequence $(v_n\alpha_g(v_n^*))$ converging to u_g in Z^1_α. Choosing a subsequence if necessary, we may assume $d(v_n\alpha_g(v_n^*), v_{n+1}\alpha_g(v_{n+1}^*)) < \varepsilon_n$. We choose a sequence of unitaries (u_n) in $\mathcal{U}(\mathcal{M}^\alpha)$ by induction so that $\|v_nu_n - v_{n+1}u_{n+1}\|_2 < \frac{1}{2^n}$. Suppose $u_1,\dots,u_n$ are chosen. Then there exists a unitary $w \in \mathcal{U}(\mathcal{M}^\alpha)$ such that $\|v_{n+1}^*v_nw_{n+1}-1\|_2 < \frac{1}{2^n}$ by the definition of ε_n. Now set $u_{n+1}=u_nw_{n+1}$. The sequence (v_nu_n) converges to some unitary $u \in \mathcal{M}$. We then have $u_g = v\alpha_g(v^*)$. Q.E.D.

Here we recall the definition of Kazhdan's property T for later use. A discrete group G is said to have Kazhdan's property T if it has the following condition: There exists a finite subset $F \subset G$ and a positive number ε such that for any unitary representation U_g of G on H, if there exists a vector $\xi \in \mathcal{H}$ such that $\|\xi\|=1$ and $\|U_g\xi-\xi\|<\varepsilon$ for all $g \in F$, then there exists a non-zero vector $\eta \in \mathcal{H}$ such that $U_g\eta=\eta$ for all $g \in G$. (See Kazhdan [K] or Zimmer [Z2] for more about property T.)

We have the following characterization of Kazhdan's property T. Equivalence of (1), (2), and (3) are a non-commutative analogue of Theorem 3.2 in Schmidt [S]. Because H^1_α is not a group here, we consider all the cohomology classes in (3). (See Example 8.)

Theorem 2. *Let G ba a discrete group. Then the following conditions are equivalent.*

(1) *G has Kazhdan's property T.*

(2) *Any action of G on a factor of type II_1 is strong.*

(3) *For any action α of G on a factor of type II_1, each cohomology class is closed in Z^1_α.*

(4) *For any action α of G on a factor of type II_1, the cohomology space H^1_α with the quotient topology is Hausdorff.*

Proof. The equivalence of (1) and (2) was proved in Araki-Choda [ACh].

(2) $\Rightarrow$ (3) : Fix a unitary cocycle u_g for α. Because $\mathrm{Ad}(u_g) \cdot \alpha_g$ is strong, we know that $\{vu_g\alpha_g(v^*)u_g^* \mid v \in \mathcal{U}(\mathcal{M})\}$ is closed by Proposition 1. Then it follows that $\{vu_g\alpha_g(v^*) \mid v \in \mathcal{U}(\mathcal{M})\}$ is closed.

(3) $\Rightarrow$ (4) : Because the metric in Z^1_α is invariant under the action of $\mathcal{U}(\mathcal{M})$, we have a metric on H^1_α and get the conclusion. (This is not just a pseudo-metric by closedness of each class.)

(4) $\Rightarrow$ (2) : If we define an action of $\mathcal{U}(\mathcal{M})$ on Z^1_α by $v \cdot u_g = vu_g\alpha_g(v^*)$, then $(\mathcal{U}(\mathcal{M}), Z^1_\alpha)$ is a Polish transformation group in the sense of Effros [E]. The orbit space H^1_α is Hausdorff, hence T_0, thus Theorem 2.1 in [E] implies that B^α_1 is homeomorphic to $\mathcal{U}(\mathcal{M})/\mathcal{U}(\mathcal{M}^\alpha)$. Suppose a sequence (x_n) in $\mathcal{M}$ is asymptotically α-fixed. We show that the sequence is equivalent to another sequence in $\mathcal{M}^\alpha$. We may assume all x_n's are unitaries by a standard argument. Then the sequence $(x_n\alpha_g(x_n^*))$ in B^1_α converges to 1, thus the sequence $(x_n\mathcal{U}(\mathcal{M}^\alpha))$ converges to $\mathcal{U}(\mathcal{M}^\alpha)$ in $\mathcal{U}(\mathcal{M})/\mathcal{U}(\mathcal{M}^\alpha)$. This shows (2) and complete the proof. Q.E.D.

Next we deal with openness of B^1_α. The following technique is taken from Connes [C3]. This is also a non-commutative analogue of Theorem 3.4 (3) of Schmidt [S] and Theorem 2.11 of Zimmer [Z1].

Proposition 3. *If α is an ergodic action of a discrete group G with Kazhdan's property T on a factor $\mathcal{M}$ of type II_1, then the set of coboundaries B^1_α is open in Z^1_α.*

Proof. Take a coboundary $u\alpha_g(u^*) \in Z^1_\alpha$. Let F and ε be as in the definition of Kazhdan's property T. Choose ε_0 so that $d(u\alpha_g(u^*), v_g) < \varepsilon_0$ implies $\|u\alpha_g(u^*) - v_g\|_2 < \varepsilon$ for all $g \in F$, and let

$$\mathcal{V} = \{v_g \in Z^1_\alpha \mid d(u\alpha_g(u^*), v_g) < \varepsilon_0\}.$$

It is enough this neighborhood $\mathcal{V}$ of $u\alpha_g(u^*)$ is contained in the set of coboundaries. We assume $\mathcal{M}$ acts on the L^2-completion $\mathcal{H}$ of $\mathcal{M}$ by the left multiplication. Define a unitary representation U_g of G on $\mathcal{H}$ by $U_g(x\xi_0) = u\alpha_g(u^*)\alpha_g(x)v_g^*\xi_0$, where ξ_0 is the vector in $\mathcal{H}$ corresponding to $1 \in \mathcal{M}$. (The equality $U_{gh} = U_g U_h$ follows from the cocycle conditions of v_g.) We now have

$$\|U_g(\xi_0) - \xi_0\| = \|u\alpha_g(u^*) - v_g\|_2 < \varepsilon, \qquad \text{for all } g \in F.$$

Thus there exists a non-zero vector $\eta \in \mathcal{H}$ such that $U_g(\eta) = \eta$ for all $g \in G$. The operator $\eta\eta^*$ is well-defined and belongs to $L^1(\mathcal{M})_+$. This operator is fixed by $\mathrm{Ad}(u) \cdot \alpha \cdot \mathrm{Ad}(u^*)$, which is ergodic. Thus η is a unitary up to scalar, and we get a unitary v such that $v = u\alpha_g(u^*)\alpha_g(v)v_g^*$, which means that the cocycle v_g is a coboundary. Q.E.D.

In Theorem 2 (3), we did not need an assumption on the action α, but we assumed ergodicity of α in Proposition 3. We show that we cannot drop this assumption in general by the following example. The difference arised from the lack of the group structure of Z^1_α.

Example 4. Let G be a discrete group with infinite conjugacy classes and Kazhdan's property T. (Take $G = SL(3, \mathbf{Z})$, for instance.) Then consider the left regular representation u_g of G, and let $\mathcal{R}(G)$ be the factor of type II_1 generated by u_g's. Let $\mathcal{M} = \mathcal{R}(G) \bar{\otimes} \mathcal{R}(G)$, and α be the trivial action of G on $\mathcal{M}$. Define

$$v_g^{(n)} = u_g \otimes e_n + 1 \otimes (1 - e_n), \qquad \text{for } n \in \mathbf{N}, g \in G,$$

where e_n is a projection in $\mathcal{R}(G)$ with the trace $1/n$. Then these $v_g^{(n)}$ are cocycles for the trivial action α, and $\|1 - v_g^{(n)}\|_2 \leq 2/\sqrt{n} \to 0$ as $n \to \infty$. But the set of coboundaries of α is just $\{1\}$. This shows that B^1_α is not open.

Thus, we concentrate on ergodic actions. We would like to show openness of each cohomology class for ergodic actions, but Proposition 3 does not imply it immediately because of lack of group structure in Z^1_α. Indeed, cohomology classes are not open in general as Example 8 shows. For this reason, we consider only cocycles connecting ergodic actions. For this purpose, we show the following continuity first.

Proposition 5. *Let α be an action of a discrete group G with Kazhdan's property T on a factor $\mathcal{M}$ of type II_1. Then the correspondence $u_g \in Z^1_\alpha \mapsto \mathcal{M}^{Ad(u_g)\cdot\alpha_g}$ is uniformly continuous in the following sense: For any $\varepsilon_0 > 0$, there exists $\delta > 0$ such that if $d(u_g, v_g) < \delta$, then $\|\mathcal{M}^{Ad(u_g)\cdot\alpha_g} - \mathcal{M}^{Ad(v_g)\cdot\alpha_g}\|_2 < \varepsilon_0$. (See Definition in p. 21 of Christensen* [**Chr**] *for notation.)*

Proof. Let $F \subset G$ and $\varepsilon > 0$ be as in the definition of Kazhdan's property T. Choose δ so that $d(u_g, v_g) < \delta$ implies $\|u_g - v_g\| < \varepsilon\varepsilon_0/2$ for all $g \in F$, and let x

be an element in $\mathcal{M}^{\mathrm{Ad}(v_g)\cdot\alpha_g}$ with $\|x\|_\infty \leq 1$. Set $\mathcal{N} = \mathcal{M}^{\mathrm{Ad}(u_g)\cdot\alpha_g}$, $y = \mathcal{E}_{\mathcal{N}}(x)$, and $z = x - y$, where $\mathcal{E}_{\mathcal{N}}$ is a conditional expectation onto $\mathcal{N}$.

By the GNS representation with respect to the trace τ, we may assume $\mathcal{M}$ acts on the L^2-completion of $\mathcal{M}$ by the left multiplication. Let ξ_0 be the vector corresponding to $1 \in \mathcal{M}$. We get a unitary representation U_g of G on this Hilbert space defined by $U_g(x\xi_0) = (\mathrm{Ad}(u_g)\cdot\alpha_g(x))\xi_0$, $x \in \mathcal{M}$. Then this U_g restricts onto the orthogonal complement $\mathcal{H}$ of the L^2-completion of $\mathcal{N}$. Because we do not have a non-trivial invariant vector in $\mathcal{H}$ for U_g, we have an inequality

$$\|\mathrm{Ad}(u_{g'})\cdot\alpha_{g'}(x) - x\|_2 = \|\mathrm{Ad}(u_{g'})\cdot\alpha_{g'}(z) - z\|_2 \geq \varepsilon\|z\|_2,$$

for some $g' \in F$. Because

$$\|\mathrm{Ad}(u_{g'})\cdot\alpha_{g'}(x) - x\|_2 = \|\mathrm{Ad}(u_{g'}v_{g'}^*)(x) - x\|_2 \leq 2\|u_{g'} - v_{g'}\|_2 \leq \varepsilon\varepsilon_0,$$

we get $\mathcal{M}^{\mathrm{Ad}(v_g)\cdot\alpha_g} \overset{\varepsilon_0}{\subset} \mathcal{N}$. By symmetry, we get the conclusion. Q.E.D.

Now we work on ergodic actions of discrete groups. Let α be an ergodic action of a discrete group G on a factor $\mathcal{M}$ of type II_1. Define

$$Z^1_{\alpha,erg} = \{u : G \to \mathcal{U}(\mathcal{M}) \mid u_g \in Z^1_\alpha, \mathrm{Ad}(u_g)\cdot\alpha_g \text{ is also ergodic.}\} \supset B^1_\alpha,$$

and $H^1_{\alpha,erg} = Z^1_{\alpha,erg}/\sim$. Consider the induced metric on $Z^1_{\alpha,erg}$. We assume now G has Kazhdan's property T and α is ergodic in the rest of this section. Then we get the following immediately from Proposition 5.

Corollary 6. *In the above context, $Z^1_{\alpha,erg}$ is closed in Z^1_α.*

Proof. The inequality $\|\mathcal{N} - \mathbf{C}\|_2 < \delta$ for any $\delta > 0$ implies $\mathcal{N} = \mathbf{C}$. Q.E.D.

We show that property T implies the discreteness of the space $H^1_{\alpha,erg}$. The following proof is similar to that of Theorem 3. It implies the number of "different" cocycles connecting ergodic actions is "small". This is a rigidity result for cocycles.

Theorem 7. *Under the above context, $H^1_{\alpha,erg}$ is at most countable.*

Proof. Since $H^1_{\alpha,erg}$ is separable, it is enough to show that each cohomology class in $Z^1_{\alpha,erg}$ is open and closed. Then, since $Z^1_{\alpha,erg}$ is a disjoint union of classes, it is sufficient to show that each class is open.

Take a cocycle $u_g \in Z^1_{\alpha,erg}$. Let F and ε be as in the definition of Kazhdan's property T. Choose ε_0 so that $d(u_g, v_g) < \varepsilon_0$ implies $\|u_g - v_g\|_2 < \varepsilon$ for all $g \in F$, and let

$$\mathcal{V} = \{v_g \in Z^1_{\alpha,erg} \mid d(u_g, v_g) < \varepsilon_0\}.$$

It is enough this neighborhood $\mathcal{V}$ of u_g is contained in the class of u_g. We assume $\mathcal{M}$ acts on the L^2-completion $\mathcal{H}$ of $\mathcal{M}$ by the left multiplication. Define a unitary representation U_g of G on $\mathcal{H}$ by $U_g(x\xi_0) = u_g\alpha_g(x)v_g^*\xi_0$, where ξ_0 is the vector in $\mathcal{H}$ corresponding to $1 \in \mathcal{M}$. (The equality $U_{gh} = U_g U_h$ follows from the cocycle conditions of u_g and v_g.) We now have $\|U_g(\xi_0) - \xi_0\| = \|u_g - v_g\|_2 < \varepsilon$ for all $g \in F$. Thus there exists a non-zero vector $\eta \in \mathcal{H}$ such that $U_g(\eta) = \eta$ for all $g \in G$. The operator $\eta^*\eta$ is well-defined and belongs to $L^1(\mathcal{M})_+$. This operator is fixed by $\mathrm{Ad}(v_g) \cdot \alpha_g$, which is ergodic. Thus η is a unitary up to scalar, and we get a unitary v such that $v = u_g\alpha_g(v)v_g^*$, which means $u_g \sim v_g$. Q.E.D.

A direct analogue of Theorem 3.4 (3) of Schmidt [S] and Theorem 2.11 of Zimmer [Z1] would be that each cohomology class of an ergodic action of a discrete group with Kazhdan's property T would be open. But this statement is invalid in general as the following example, similar to Example 4, shows. This justifies we considered only $Z^1_{\alpha,erg}$ in the above Theorem.

Example 8. Let $G, u_g, \mathcal{M}$ be as in Example 4. Let α be the action of $G\times G$ on $\mathcal{M}$ defined by $\alpha_{(g,h)} = \mathrm{Ad}(u_g \otimes u_h)$ for $g, h \in G$. Note that $G\times G$ also has Kazhdan's property T (see Connes [C3]), and this action α is ergodic. Define $v_{(g,h)} = u_g^* \otimes u_h^*$ and

$$v^{(n)}_{(g,h)} = 1\otimes(e_n u_h^*) + u_g^* \otimes (1-e_n)u_h^*, \quad n\in\mathbf{N}, g, h\in G,$$

where e_n is a projection in $\mathcal{R}(G)$ with the trace $1/n$. An easy computation shows that these are α-cocycles. Then $\|v_{(g,h)} - v^{(n)}_{(g,h)}\|_2 \le 2/\sqrt{n} \to 0$ as $n\to\infty$. If we have $v_{(g,h)} \sim v^{(n)}_{(g,h)}$ for some n, then it implies $v_{(g,h)} = v^{(n)}_{(g,h)}$, which is a contradiction. This show that the cohomology class of $v_{(g,h)}$ is not open.

If we have "too many" ergodic actions compared to cocycles, then it means the number of cocycle conjugacy classes is large. Jones [J1] first showed that a discrte group with Kazhdan's property T has two mutually non-cocycle conjugate free actions on the AFD factor of type II_1. M. Choda [Ch1, Ch2] constructed a continuous family of mutually non-conjugate ergodic free actions of $SL(n,\mathbf{Z})$, $n\ge 3$, and $Sp(n,\mathbf{Z})$, $n\ge 2$, on the AFD factor $\mathcal{R}$ of type II_1, and she asked in the first question of page 534 of [Ch2] whether a discrete group with Kazhdan's property T has two non-cocycle conjugate ergodic free actions on the AFD factor of type II_1, and obtained an affirmative answer in it by constructing two different crossed product algebras. We get the following Corollary about this question. It shows that these groups have the totally opposite property of amenable groups with respect to free actions on the AFD factor of type II_1. (Theorem 2.7 in Ocneanu [O] asserts that every discrete amenable group has the unique free action, up to cocycle conjugacy, on the AFD factor $\mathcal{R}$ of type II_1.)

Corollary 9. *Each of the groups $SL(n, \mathbf{Z})$, $n \geq 3$, and $Sp(n, \mathbf{Z})$, $n \geq 2$, has a continuous family of ergodic free actions on the AFD factor of type II_1 such that any two of it are not cocycle conjugate.*

Proof. Suppose the number of different cocycle conjugacy classes of actions in the family M. Choda constructed is countable. Then at least one class contains continuously many actions. Let α be an action in this class. Since the other actions in this class are ergodic and cocycle conjugate to α, there exist continuously many cocycles such that any two of them are not cohomologous, hence $H^1_{\alpha,erg}$ for this α is uncountable, which is a contradiction. Q.E.D.

Noe that in Corollary to Theorem 4 of Popa [P2], he shows that a countable group G with Kazhdan's property T have uncountably many mutually nonconjugate properly outer cocycle-actions on the AFD factor $\mathcal{R}$ of type II_1.

Remark 10. All the above actions are ergodic, hence ergodic at infinity in the sense of Jones [J2]. Thus any two of them cannot be distinguished by the method of Jones [J2]. All the crossed product algebras of the above actions have property T as shown in Choda [Ch2]

§2 Entropy and cohomology for amenable group actions

In this section, we study what follows from Ocneanu's work [O] about cohomological properties of free actions of discrete amenable groups on the AFD factor $\mathcal{R}$ of type II_1. In Proposition 7.2 in Ocneanu [O], he obtained 1-cohomology vanishing in the ultraproduct algebra. His method appeals to Shapiro's lemma type argument based on his non-commutative Rohlin theorem. If we go back to the original algebra, we get the following by his method.

Proposition 11. *Let α be a free action of a discrete amenable group G on the AFD factor $\mathcal{R}$ of type II_1. Then B^1_α is dense in Z^1_α.*

Proof. Apply the proof of Proposition 7.2 in [O] to a given cocycle v_g. Though v_g is not in the ultraproduct, the proof works until line 11 of page 63 if we think w in the proof is an element of $\mathcal{M}^\omega$, because we have a non-commutative Rohlin Theorem for our α. Then v_g is equal to a coboundary $w\alpha_g(w^*)$ in the ultraproduct algebra with a small error. Then choosing a unitary u_n from the sequence representing w in the proof, we get a coboundary $u_n\alpha_g(u_n^*)$ such that $\|u_n\alpha_g(u_n^*) - v_g\|_1 < 1/n$ for all $g \in F_n$, where F_n is an increasing sequence of finite subsets of G with $\cup_n F_n = G$. This shows the desired density. Q.E.D.

We consider the number of unitary cocycles next. As in Choda's result used in §1, we would like to obtain a large number of conjugacy classes of ergodic free actions. In order to distinguish general discrete amenable group actions, we extend Connes-Størmer entropy in [CS] to countable amenable group actions. It will give us continuous conjugacy classes of non-commutative Bernoulli shifts. (See Ornstein-Weiss [OW] for entropy of group actions on probability spaces.)

For reference, we list basic definitions and properties of Connes-Størmer entropy from [CS].

[Definition 1 of [CS]]**.** For finite dimensional von Neumann subalgebras $N_1, \dots, N_k$ of $\mathcal{R}$, define

$$H(N_1, \dots, N_k) = \sup_{x \in S_k} (\sum \eta(\tau(x_{i_1,\dots,i_k})) - \sum \tau(\eta(E_{N_l}(x^l_{i_l})))),$$

where $\eta(x) = -x \log x$, S_k is the set of all families $(x_{i_1,\dots,i_k})$, $i_j \in \mathbf{N}$, of positive elements of $\mathcal{R}$, zero except for a finite number of indices, and with the sum equal to 1, and

$$x^l_{i_l} = \sum_{i_1,\dots,i_{l-1},i_{l+1},\dots,i_k} x_{i_1,\dots,i_k}.$$

[Properties of H**].**

(A) $H(N_1, \dots, N_k) \le H(P_1, \dots, P_k)$ when $N_j \subset P_j$,

(B) $H(N_1, \dots, N_p) \le H(N_1, \dots, N_k) + H(N_{k+1}, \dots, N_p)$,

(C) $P_1, \dots, P_n \subset P \Rightarrow H(P_1, \dots, P_n, P_{n+1}, \dots, P_m) \le H(P, P_{n+1}, \dots, P_m)$,

(D) $H(N) = \sum \eta(\tau(e_\alpha))$, when $\sum e_\alpha = 1, e_\alpha$: minimal,

(E) $H(N_1, \dots, N_k) = H((N_1 \cup \dots \cup \mathbf{N}_k)'')$,

when $(N_1 \cup \dots \cup N_k)''$ is generated by pairwise commuting subalgebras $P_j \subset N_j$,

(F) $H(N_1, \dots, N_k) \le H(P_1, \dots, P_k) + \sum H(N_j | P_j)$,

where $H(N|P) = \sup_{x \in S_1} \sum (\tau(\eta(E_P(x_i))) - \tau(\eta(E_N(x_i))))$. ∎

[Theorem 1 of [CS]**].** For each $n \in \mathbf{N}$ and $\varepsilon > 0$, there is a $\delta > 0$ such that for any pair of von Neumann subalgebras $N, P \subset \mathcal{R}$, the conditions $\dim N = n$ and $N \overset{\delta}{\subset} P$ imply $H(N|P) < \varepsilon$.

Let G be a countable amenable group. Choose an increasing sequence of finite subsets $\{A_n\}$ by amenability such that

$$\lim_{n\to\infty} \frac{|gA_n \triangle A_n|}{|A_n|} = 0, \qquad \text{for all } g \in G.$$

(It is well-known that existence of such a sequence is equivalent to amenability. See Greenleaf [G] for instance.)

For an action α of G on a factor $\mathcal{M}$ of type II_1, we define

$$\begin{aligned} H(N, \alpha, A) &= H(\alpha_{g_1}(N), \ldots, \alpha_{g_m}(N)), \qquad \text{where } A = \{g_1, \ldots, g_m\} \subset G, \\ H(N, \alpha) &= \limsup_{n\to\infty} \frac{1}{|A_n|} H(N, \alpha, A_n), \\ H(\alpha) &= \sup_{N \subset \mathcal{M}, \text{finite dimensional}} H(N, \alpha). \end{aligned}$$

Note that $H(N, \alpha) \leq H(N)$ by property (B) in Connes-Størmer [CS]. The sup in the third definition may be infinity.

The following is the non-commutative Kolmogorov-Sinai theorem and an analogue of Theorem 2 in [CS].

Proposition 12. *Let P_q, $q \geq 1$, be an increasing sequence of finite dimensional subalgebras of the AFD factor $\mathcal{R}$ of type II_1 with $\overline{\cup_{q=1}^{\infty} P_q} = \mathcal{R}$. Then*

$$H(\alpha) = \lim_{q\to\infty} H(P_q, \alpha).$$

Proof. The same proof as that of Theorem 2 in [CS] works. (Use property (F) and Theorem 1 in [CS] to get $H(N, \alpha) \leq H(P_q, \alpha) + \varepsilon$ for given ε.) Q.E.D.

The following is a computation of entropy for Bernoulli shifts and corresponds to Theorem 3 in [CS].

Proposition 13. *Represent the AFD factor $\mathcal{R}$ of type II_1 as the infinite tensor product of k-dimensional matrix algebra $M_k(\mathbf{C})$ with respect to the trace over a countable amenable group G. Define an action α_g of G on $\mathcal{R}$ by the Bernoulli shift. Then we get $H(\alpha) = \log k$.*

Proof. For $q \geq 1$, define

$$P_q = \bigotimes_{g \in A_q} M_k(\mathbf{C}) \otimes \bigotimes_{g \notin A_q} \mathbf{C}.$$

It is clear that P_q is an increasing sequence of finite dimensional algebras in $\mathcal{R}$ and $\cup_{q=1}^{\infty} P_q$ is weakly dense in $\mathcal{R}$. Thus we can apply Proposition 12, and get

$$\begin{aligned} H(\alpha) &= \lim_{q \to \infty} \lim_{n \to \infty} \frac{1}{|A_n|} H(P_q, \alpha, A_n) \\ &= \lim_{q \to \infty} \lim_{n \to \infty} \frac{|A_q A_n|}{|A_n|} \log k \\ &= \log k, \end{aligned}$$

by the definition of A_n. Q.E.D.

The following shows that this entropy for group actions is more poweful than entropy of single automorphisms.

Example 14. Let $G = \mathbf{Z}^2$ and apply the above construction for $k = 2, 3$ to get actions α and β of $\mathbf{Z}^2$ on $\mathcal{R}$. Then for any $g \in \mathbf{Z}^2$, $g \neq 0$, α_g and β_g have the entropy infinity as single automorphisms. Actually, they are both conjugate to the shift on $\bigotimes_{\mathbf{Z}} \mathcal{R} \cong \mathcal{R}$. But as group actions, they have different entropy, and thus they are non-conjugate.

Let $G = \mathbf{Z}[1/2]/\mathbf{Z}$ and apply the above construction for $k = 2, 3$ to get actions α and β of $\mathbf{Z}[1/2]/\mathbf{Z}$ on $\mathcal{R}$. Then for any $g \in \mathbf{Z}[1/2]/\mathbf{Z}$, α_g and β_g have the entropy zero as single automorphisms because they are both periodic, by Remark 6 of Connes-Størmer [CS]. But as group actions, they have different entropy, and thus they are non-conjugate.

We would like to get continuously many values of the entropy, so we introduce the following as in Theorem 4 in [CS].

Let $\mathcal{M}$ be the infinite tensor product of $M_k(\mathbf{C})$ with respect to the product state $\psi^\lambda = \otimes \varphi^\lambda$, $\lambda = (\lambda_1, \ldots, \lambda_k)$, where φ^λ on $M_k(\mathbf{C})$ is defined by

$$\varphi^\lambda(x) = \mathrm{Tr}\left(x \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \right), \qquad \lambda_1 + \cdots + \lambda_k = 1,\ \lambda_j > 0.$$

Let β_g^λ be the ergodic action of a countable amenable group G on $\mathcal{M}$ given by the Bernoulli shift. The centralizer $\mathcal{M}_{\psi^\lambda}$ is isomorphic to $\mathcal{R}$ as in the proof of Theorem 4 of Connes-Størmer [CS]. Define the action α_g^λ to be the restriction of β_g^λ on $\mathcal{M}_{\psi^\lambda}$. This is also ergodic, and this is a free action. We use entropy as a conjugacy invariant to distinguish these actions.

Proposition 15. *The entropy* $H(\alpha^\lambda)$ *of the above action* α_g^λ *is given by*

$$H(\alpha^\lambda) = \sum_{j=1}^{k} -\lambda_j \log \lambda_j.$$

Proof. Define

$$F_p = (\otimes_{g\in A_p}(M_k(\mathbf{C}), \varphi^\lambda))_{\otimes\varphi^\lambda}; \qquad D_p = \otimes_{g\in A_p} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Then using Properties (C), (D), (E) and (C) of Connes-Størmer [CS], we get

$$\begin{aligned} H(F_p, \alpha, A_n) &\leq H((\otimes_{g\in A_pA_n}(M_k(\mathbf{C}), \varphi^\lambda))_{\otimes\varphi^\lambda}) \\ &= |A_pA_n| \sum_{j=1}^{k} -\lambda_j \log \lambda_j \\ &= H(D_p, \alpha, A_n) \\ &\leq H(F_p, \alpha, A_n), \end{aligned}$$

as in the computation in p. 304 of [CS]. Thus we have

$$\begin{aligned} H(\alpha^\lambda) &= \lim_p \lim_n \frac{1}{|A_n|} H(F_p, \alpha, A_n) \\ &= \sum_{j=1}^{k} -\lambda_j \log \lambda_j. \end{aligned}$$

as in the proof of Proposition 13. Q.E.D.

Theorem 16. *For a free action α of a discrete amenable group G on the AFD factor $\mathcal{R}$ of type II_1, H^1_α is uncountable. For an ergodic free action α of G on $\mathcal{R}$, $H^1_{\alpha,erg}$ is uncountable.*

Proof. By Proposition 15, we know that G has a continuous family of mutually non-conjugate ergodic free actions on $\mathcal{R}$. By Ocneanu's result [O, Theorem 2.7], its members are all cocycle conjugate. Thus we get the conclusion. Q.E.D.

Now we get the following. This is a von Neumann algebra analogue of Schmidt [S, Remark 3.5].

Theorem 17. *Let G be a countable group. Then the following conditions are equivalent.*

(1) *G is amenable.*

(2) *No free action α of G on the AFD factor $\mathcal{R}$ of type II_1 is strong.*

(3) *$B^1_\alpha \subsetneq \overline{B^1_\alpha} = Z^1_\alpha$ for all free actions a of G on the AFD factor $\mathcal{R}$ of type II_1.*

Proof. (2) $\Rightarrow$ (1) : If G is not amenable, then G has a strongly ergodic free action α on $\mathcal{R}$ obtained by the Bernoulli shift as in Jones [J2].

(1) $\Rightarrow$ (3) : If G is amenable, we get the conclusion by Proposition 11 and Theorem 16.

(3) $\Rightarrow$ (2) : This follows from Proposition 1. Q.E.D.

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