Classification of approximately inner automorphisms of subfactors

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Dedicated to Professor Hikosaburo Komatsu on his sixtieth birthday

Abstract

For classification of approximately inner automorphisms of subfactors, we introduce a new invariant, a *higher obstruction*. From an algebraic viewpoint, this can be regarded as a generalization of the Connes obstruction, and from an analytic viewpoint, this can be regarded as a generalization of the Jones invariant κ. We have two classification theorems for approximately inner automorphisms of strongly amenable subfactors with known invariants and this new one. In particular, our theorems give a complete classification of automorphisms, up to outer conjugacy, of AFD subfactors of type II_1 with index less than four except for one special case for A_{4n-1} and E_6.

1 Introduction

For single factors, classification problems of automorphisms and group actions on injective factors have caught much attention since the fundamental work of A. Connes [7], and many successful classification theories have been available. In subfactor theory initiated by V. F. R. Jones [19], a systematic study of automorphisms of subfactors was initiated by P. H. Loi in [27]. Loi's main motivation was a study of subfactors of type III_λ $(0 < \lambda < 1)$, as one of the main applications of the Connes classification of automorphisms of type II factors was a classification of factors of type III_λ.

In subfactor theory, S. Popa has obtained a powerful classification result of automorphisms of subfactors in [33] in the following form: Properly outer actions of a discrete amenable group on a strongly amenable subfactors of type II_1 are classified by Loi's invariants up to cocycle conjugacy. (This is, of course, based on Popa's "ultimate" analytic classification theorem in [31] that a strongly amenable subfactors of type II_1 has the generating property. — The terminologies in these theorems will be explained below.) This deep theorem solves the original problem of Loi of classification of strongly amenable subfactors of type III_λ completely. Subfactors, however, often have interesting automorphisms with trivial Loi invariant, as shown in our previous work [11], [12], [21], [22]. Thus we expect an interesting classification theory for automorphisms of subfactors which are not classified by Loi's invariant. Our aim in this paper is to get certain classification results for these automorphisms with our new invariants. By the results in [27, Theorem 5.4] and [31], we know that the class of automorphisms of strongly amenable subfactors with trivial Loi invariant is exactly that of approximately inner automorphisms. So our classification results here are for approximately inner automorphisms. (Precise definitions of these notions will be given below.)

Before going into our theory, we first point out that a "complete classification" of automorphisms of subfactors in a sense can be obtained rather easily for strongly amenable subfactors with our previous results in [23]. (This approach was first suggested by S. Popa. T. Ceccherini [4] got the first classification result on this approach.)

That is, for a strongly amenable subfactor $N \subset M$ and an automorphism α of M with $\alpha(N) = N$, we study the following commuting squares of II_1 factors.

$$
\begin{array}{ccc}
\left\{ \left(\begin{array}{ccc} x & 0 & 0 \\ 0 & \alpha(x) & 0 \\ 0 & 0 & \alpha(x) \end{array} \right) \middle| \; x \in N \right\} & \subset & \left\{ \left(\begin{array}{ccc} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{array} \right) \middle| \; x_{ij} \in N \right\} \\
\cap & & \cap \\
\left\{ \left(\begin{array}{ccc} x & 0 & 0 \\ 0 & \alpha(x) & 0 \\ 0 & 0 & \alpha(x) \end{array} \right) \middle| \; x \in M \right\} & \subset & \left\{ \left(\begin{array}{ccc} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{array} \right) \middle| \; x_{ij} \in M \right\}
\end{array}
$$

Then it is easy to see that an outer conjugacy classification of α is equivalent to classification of this commuting squares. This type of commuting squares has been classified completely by us with the standard invariant as in [23, Theorem 3.4]. Furthermore, we are often interested in subfactors with finite depth, and in such a case, any aperiodic automorphism gives a centrally free action of $\mathbf{Z}$ by [20, Proposition 3.1], [27, Proposition 4.4]. Then they are classified by Loi's invariant by Popa's classification theorem. Thus the remaining case involves only automorphisms of finite order. Then the above classification based on [23] is really reduced to a problem of finite combinatorics.

The above "complete solution" is, however, not satisfactory from the viewpoint of classical classification results in [7], [9], [10], and so on. This is because it is very hard to write down a list of all the outer conjugacy classes for a given subfactor in the above approach. That is, although we have a complete invariant, its computation is very hard and we practically cannot determine the range of the invariant. (Even in the case of the Jones subfactors of type A_n in [19], this problem of listing is already hard enough.) So we hope to get a more concrete classification result being parallel to the classical classification results along the line of Connes [7], [9], [10].

We will introduce such concrete invariants here and show that they give a complete classification for a nice class of subfactors such as the Jones subfactors of type A_{4n-3} and the Wenzl subfactors with indices converging to 9. (See Examples 6.2 and 6.7.) These subfactors give the first examples of a complete classification of automorphisms of subfactors which is different from the Connes classification of automorphisms of single factors in [7].

In Section 2, we introduce our new invariants from an algebraic viewpoint. We will show a first classification theorem for approximately inner automorphisms of subfactors in Section 3. In Section 4, we define the invariants from an analytic viewpoint and prove that the algebraic and analytic definitions give essentially same invariants for strongly amenable subfactors. This coincidence generalizes a theorem in [24, Theorem 3.4] where we identified an obstruction to flatness in the quantum $SU(n)_k$ orbifold construction with the relative Jones invariant κ. We then get another classification theorem in Section 5 for the case not covered by the theorem in Section 3. In the last Section 6, we work out the classification on concrete examples of subfactors and give a philosophical reason our higher obstruction should be the *last missing invariant* for classification of automorphisms of subfactors, based on an analogy to classification of automorphisms of injective factors of type III. We also show that our Theorems give a complete classification of (approximately inner) automorphisms, up to outer conjugacy, of a large subclass of the Hecke algebra subfactors of Wenzl [37].

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2 New invariants — an algebraic approach —

We now describe our new invariants. Let $N \subset M$ be a subfactor of type II_1. For simplicity, we also assume in this paper that M [resp. M_1] has no non-trivial normalizer of N [resp. M]. (If we have non-trivial normalizers, then the subfactor has a "classical" intermediate subfactor coming from an action of a finite group. Our philosophy is that an interesting "quantum" subfactor does not have such an intermediate subfactor.) Then $\alpha \in \mathrm{Aut}(M,N)$ is inner on M if and only if it is inner on N, so it makes sense to have an outer period $p_o(\alpha)$.

Let $\mathrm{Aut}(M,N) = \{\alpha \in \mathrm{Aut}(M,N) \mid \alpha(N) = N\}$. We say that $\alpha, \beta \in \mathrm{Aut}(M,N)$ are *outer conjugate* if there exist a unitary $u \in N$ and $\theta \in \mathrm{Aut}(M,N)$ such that $(\mathrm{Ad}\, u) \cdot \alpha = \theta \cdot \beta \cdot \theta^{-1}$.

Consider any $\alpha \in \mathrm{Aut}(M,N)$. Choda–Kosaki gave a definition of strong outerness of automorphisms of subfactors in [6, Definition 1], which is the same as proper outerness of Popa in [33, 1.5]. That is, α is *not* strongly outer if there exists a non-zero $a \in M_k$ in the Jones tower such that we have $\alpha(x)a = ax$ for all $x \in M$. We define the *strongly outer period* $p_s(\alpha) \in \mathbf{N}$ of α by

$$p_s(\alpha)\mathbf{Z} = \{n \in \mathbf{Z} \mid \alpha^n \text{ is not strongly outer.}\}.$$

The characterization of not strongly outer automorphisms by Choda–Kosaki [6, Theorem 2], [26, Theorem 3] is as follows. An automorphism α of $N \subset M$ is not strongly outer if and only if it appears as a vertex of the dual principal graph (as an M-M bimodule). (Also see [13, Theorem 2.2].)

Suppose now $p_s(\alpha) > 0$ and set $p = p_s(\alpha)$. The power α^p gives an element in

$$\{\alpha \in \mathrm{Aut}(M, N) \mid \alpha \text{ is not strongly outer.}\}/\mathrm{Int}(M, N),$$

where $\mathrm{Int}(M, N) = \{\mathrm{Ad}\ u \mid u \in \mathcal{U}(N)\}$. We write $\nu_{\mathrm{alg}}(\alpha)$ for the orbit of the image of α^p under the action of $\mathrm{Aut}(M, N)$ given by $\alpha^p \mapsto \theta \cdot \alpha^p \cdot \theta^{-1}$ for $\theta \in \mathrm{Aut}(M, N)$ and call it an *algebraic ν invariant*. This is clearly an outer conjugacy invariant. (See [15, Example 2.9] for an example where we have to take this orbit to get an outer conjugacy invariant.) In many interesting cases, this orbit turns out to consist of a single element as we will see in Section 6. This invariant can be regarded as an analogue of the modular invariant of Sutherland–Takesaki [35] and the symbol ν comes from this analogy. See Section 6 for more on this analogy.

We know that there exist $k \geq 0$ and $a \in M_k \setminus \{0\}$ such that $\alpha^p(x)a = ax$ for all $x \in M$. We fix a minimal k with this property here. Let

$$I_\alpha = \{a \in M_k \mid \alpha^p(x)a = ax, \text{ for all } x \in M\}.$$

This is a Hilbert space with respect to the inner product arising from the trace. It is clear that α gives a unitary action U_α on I_α. On the space of unitary matrices in a fixed dimension, the unitary matrices themselves acts by adjoint. We define $\gamma_h(\alpha)$ to be the orbit of U_α under this action. We call $\gamma_h(\alpha)$ a *higher obstruction* of α.

Theorem 2.1 *The higher obstruction $\gamma_h(\alpha)$ is an outer conjugacy invariant of α.*

Proof It is clear that $\gamma_h(\alpha)$ is a conjugacy invariant, by definition.

Let u be a unitary in N, and set $\tilde{\alpha} = (\mathrm{Ad}\ u)\cdot\alpha$. Then it is trivial that the strongly outer period of $\tilde{\alpha}$ is also p, and we have

$$\tilde{\alpha}^p = \mathrm{Ad}(u \cdot \alpha(u) \cdots \alpha^{p-1}(u)) \cdot \alpha^p.$$

Set $v = u \cdot \alpha(u) \cdots \alpha^{p-1}(u)$. Then we have $u\alpha(v) = v\alpha^p(u)$. We also have that $\alpha^p(x)a = ax$ if and only of $\tilde{\alpha}^p(x)va = vax$ for all $x \in M$. This gives a natural isomorphism between I_α and $I_{\tilde{\alpha}}$. To prove $\gamma_h(\alpha) = \gamma_h(\tilde{\alpha})$, we need to prove that $\tilde{\alpha}(va) = v\alpha(a)$ for all $a \in I_\alpha$. We see this identity by

$$\tilde{\alpha}(va) = u\alpha(v)\alpha(a)u^* = v\alpha^p(u)\alpha(a)u^* = v\alpha(a)uu^* = v\alpha(a),$$

where we used the property $\alpha(a) \in I_\alpha$. Q.E.D.

We next recall the definition of the Loi invariant $\Phi(\alpha)$ of $\alpha \in \mathrm{Aut}(M, N)$. As in [27, §5], we can extend α to the Jones tower $\{M_k\}$ canonically by setting $\alpha(e_k) = e_k$. Then $\Phi(\alpha)$ is defined as an action of this extension on the commuting squares of the higher relative commutant $\{M' \cap M_k \subset N' \cap M_k\}_k$. The higher obstruction takes a simple form of a scalar if the automorphism has a trivial Loi invariant. This will play a key role in the classification theorem later.

Theorem 2.2 *If $\Phi(\alpha)$ is trivial, then $\gamma_h(\alpha) \in \mathbf{C}$. Furthermore, for any l, if $a \in M_l$ satisfies $\alpha^p(x)a = ax$ for all $x \in M$, then we get $\alpha(a) = \gamma_h(\alpha)a$.*

Proof We use notations and arguments of [13, Theorem 2.2], which is based on [6], [26]. We know that I_α is given by the images of $1 \in M_{\alpha^p}$ by ξ^* for all $\xi \in \mathrm{Hom}({}_M M_{kM}, {}_M M_{\alpha^p M})$. Because U_α is unitary on I_α, we can choose an orthonormal system $\{a_j\}_j \subset I_\alpha$ with $U_\alpha a_j = c_j a_j$ for some scalars c_j. Then we may assume that each a_j is represented as $\xi_j^*(1)$ for an orthonormal family $\{\xi_j\}_j \subset \mathrm{Hom}({}_M M_{kM}, {}_M M_{\alpha^p M})$.

Then $\{\xi_j^*\xi_k\}_{jk}$ gives a system of matrix units in $M' \cap M_{2k}$ (for the summand corresponding to α^p). Because the Loi invariant of α is trivial, we have $\alpha(\xi_j^*\xi_k) = \xi_j^*\xi_k$. It is also easy to see $a_j = \xi_j^*\xi_k a_k$, so by applying α to the both hand sides, we get $c_j a_j = c_k \xi_j^* \xi_k a_k$. This implies $c_j = c_k$ for all j, k, and hence U_α is a scalar.

For $l > k$, we can embed I_α into M_l, and repeat the above argument for M_l to get the conclusion in the second half of the Theorem. Q.E.D.

We also prepare a lemma we will need in the next section.

Lemma 2.3 *Suppose that $p_s(\alpha) > 0$ and that the order n of $\nu_{\mathrm{alg}}(\alpha)$ in the group $\mathrm{Aut}(M,N)/\mathrm{Int}(M,N)$ is finite. (The finiteness of n automatically holds if $N \subset M$ has finite depth by [6, Theorem 2].) We also assume that $\Phi(\alpha)$ is trivial. Then the outer period of α is np and the Connes obstruction of α is $\gamma_h(\alpha)^n$, where $p = p_s(\alpha)$.*

Proof It is trivial that the outer period of α is np.

Let u be a unitary in N with $\alpha^{np} = \mathrm{Ad}\, u$. By Theorem 2.2, the higher obstruction $\gamma_h(\alpha)$ is scalar, and we have $a \in M_k \setminus \{0\}$ with $\alpha^p(x)a = ax$ for all $x \in M$ and $\alpha(a) = \gamma_h(\alpha)a$, where k is chosen to be minimal.

The argument in [13, Theorem 2.2] shows that $a \otimes_M a \otimes_M \cdots \otimes_M a$ is given by $\xi^*(1)$ for some non-zero intertwiner ξ, and thus $a \otimes_M a \otimes_M \cdots \otimes_M a \neq 0$. Let $\tilde{a} \in M_{nk}$ be the corresponding element in the canonical isomorphism between $M_k \otimes_M M_k \otimes_M \cdots \otimes_M M_k$ and M_{nk} as M-M bimodules. That is, we have $\tilde{a} = a f_1 a f_2 a \cdots a f_{n-1} a$, where f_m is a product of a positive scalar and the Jones projections e_j's as in [30]. Then we have $\tilde{a} \neq 0$, $\alpha^{np}(x)\tilde{a} = \tilde{a}x$ for all $x \in M$, and $\alpha(\tilde{a}) = \gamma_h^n(\alpha)\tilde{a}$. We then have $uxu^*\tilde{a} = \alpha^{np}(x)\tilde{a} = \tilde{a}x$ for all $x \in M$. This means $u^*\tilde{a} \in M' \cap M_{nk}$. Because the Loi invariant of α is trivial, we get $\alpha(u^*\tilde{a}) = u^*\tilde{a}$. It then follows that $\alpha(u) = \gamma_h(\alpha)^n u$, as desired. Q.E.D.

The above lemma implies $\gamma_h(\alpha)^{pn^2} = 1$, in particular.

Note that if $N = M$, then $p_s(\alpha)$ is the outer period $p_o(\alpha)$ of α and the higher obstruction $\gamma_h(a)$ is same as the Connes obstruction of α in [10].

Also note that if the $*$-vertex of the dual principal graph of $N \subset M$ is the only vertex with Perron–Frobenius weight 1, then p_s and γ_h of $\alpha \in \mathrm{Aut}(M,N)$ are same as p_o and γ of $\alpha \in \mathrm{Aut}(N)$ (or $\alpha \in \mathrm{Aut}(M)$) by [6, Theorem 2].

3 The first classification theorem

We recall some notations from [27] as follows.

$$\begin{aligned}\overline{\mathrm{Int}}(M,N) &= \text{closure of } \mathrm{Int}(M,N),\\ \mathrm{Ct}(M,N) &= \{\alpha\in\mathrm{Aut}(M,N)\mid \alpha=\mathrm{id} \text{ on } N^\omega\cap M'\},\end{aligned}$$

where ω is any free ultrafilter over $\mathbf{N}$. (The definition of $\mathrm{Ct}(M,N)$ does not depend on a choice of ω.) Automorphisms in $\overline{\mathrm{Int}}(M,N)$ and $\mathrm{Ct}(M,N)$ are called *approximately inner* and *centrally trivial*, respectively.

We here give the first classification theorem for strongly amenable subfactors $N\subset M$. Recall that $\alpha\in\mathrm{Aut}(M,N)$ is approximately inner if and only if $\Phi(\alpha)$ is trivial by [27, Theorem 5.4] and [31] and that $\alpha\in\mathrm{Aut}(M,N)$ is centrally trivial if and only if it is not strongly outer by [32, section 4].

For approximately inner automorphisms, we have the following classification theorem first.

Theorem 3.1 *Let $N\subset M$ be a strongly amenable subfactor of type II_1. Suppose that $\alpha,\beta\in\mathrm{Aut}(M,N)$ have trivial Loi invariants and we have the following properties.*

1. $p_s(\alpha)=p_s(\beta)=p$.
2. $\nu_{\mathrm{alg}}(\alpha)=\nu_{\mathrm{alg}}(\beta)$.
3. $\gamma_h(\alpha)=\gamma_h(\beta)$.
4. $p_o(\sigma)>0$ *for* $\sigma=\alpha^p$.
5. $\gamma_h(\sigma)=1$.

Then α and β are outer conjugate.

Proof The case $p=0$ follows exactly from Popa's classification theorem in [33], so we may assume $p>0$ here. We may also assume that $\beta^p=\sigma$ modulo $\mathrm{Int}(M,N)$.

Set $p=p_s$ and $\gamma=\gamma_h$. Consider an automorphism $s_p^{\bar\gamma}\otimes\alpha$ on $R\otimes N\subset R\otimes M$, where R is an approximately finite dimensional (AFD) II_1 factor and $s_p^{\bar\gamma}$ is the automorphism defined by Connes in [10]. This can be regarded as an automorphism of $N\subset M$ because $R\otimes N\subset R\otimes M$ is isomorphic to $N\subset M$. (Note that these two subfactors have the same higher relative commutants, so we can appeal to the general classification theorem of Popa [31]. This is the property called a *relative McDuff splitting* — see [2].)

Then it is easy to see $p_s(s_p^{\bar\gamma}\otimes\alpha)=p$ and $\gamma_h(s_p^{\bar\gamma}\otimes\alpha)=1$. (See [22, Proposition 4.6] or [26, Corollary 7] for the first equality.) Then α is outer conjugate to $s_p^{\gamma}\otimes s_p^{\bar\gamma}\otimes\alpha$ as in [7, Theorem 2.3.1], so it is enough to prove that $s_p^{\bar\gamma}\otimes\alpha$ and $s_p^{\bar\gamma}\otimes\beta$ are outer conjugate. That is, we may assume that $\gamma=1$ from the beginning. Then the Connes obstructions of α and β are both 1 by Lemma 2.3. Thus we may assume that both α and β give outer actions of $\mathbf{Z}_{np}$. Then the simultaneous crossed products

$N \rtimes_\alpha \mathbf{Z}_{np} \subset M \rtimes_\alpha \mathbf{Z}_{np}$ give an orbifold subfactor in the sense of [15, Definition 2.10]. (See [11], [13], [14], [15], [21], [39], [40] for the orbifold subfactors.)

We claim that the dual actions $\hat\alpha$ and $\hat\beta$ of $\mathbf{Z}_{np}$ are both strongly outer. This is proved for α as follows. Suppose $a = \sum_{j=0}^{np-1} a_j u^j \in M_k \rtimes_\alpha \mathbf{Z}_{np}$ satisfies

$$\begin{aligned} \sum_{j=0}^{np-1} \hat\alpha^l(x) a_j u^j &= \sum_{j=0}^{np-1} a_j u^j x, \quad \text{for all } x \in M, \\ \sum_{j=0}^{np-1} \hat\alpha^l(u) a_j u^j &= \sum_{j=0}^{np-1} a_j u^j u, \end{aligned}$$

for some l with $0 < l < np$, where u is the implementing unitary for the crossed product by the action α. The first identity implies that $x a_j = a_j \alpha^j(x)$ for all $x \in M$, which means $a_j = 0$ unless j is a multiple of p. The second identity implies $e^{2\pi i l/np}\alpha(a_j) = a_j$. Then the assumption $\gamma_h(a) = 1$ gives $a_j = 0$ for all j, and thus the dual action $\hat\alpha$ is strongly outer. This, together with Takesaki duality and Theorem 6.1 in [38], gives that the subfactor $N \rtimes_\alpha \mathbf{Z}_{np} \subset M \rtimes_\alpha \mathbf{Z}_{np}$ is strongly amenable. We have the same result for $\hat\beta$. Because the higher relative commutans of $N \rtimes_\alpha \mathbf{Z}_{np} \subset M \rtimes_\alpha \mathbf{Z}_{np}$ and $N \rtimes_\beta \mathbf{Z}_{np} \subset M \rtimes_\beta \mathbf{Z}_{np}$ are described by p_s and γ_h, they are the same, and hence these two subfactors are isomorphic by Popa's classification theorem in [31].

It is easy to see that both $\hat\alpha$ and $\hat\beta$ have the same Loi invariant, so Popa's theorem in [33] gives outer conjugacy of $\hat\alpha$ and $\hat\beta$. With Takesaki duality [36], this gives the conclusion. Q.E.D.

Take any non-strongly-outer $\alpha \in \mathrm{Aut}(M, N)$ with trivial Loi invariant. We call $\gamma_h(\alpha)$ an *obstruction to flatness in the orbifold construction*. (See [15] for a reason of this name.) If $\gamma_h(\alpha) = 1$ for any such $\alpha \in \mathrm{Aut}(M, N)$, we then say that *the obstruction for flatness in the orbifold construction vanishes for $N \subset M$*. As a corollary to the above theorem, we get the following. We will see in Section 6 that we have several interesting subfactors satisfying the assumption in this Corollary.

Corollary 3.2 *Let $N \subset M$ be a subfactor of the AFD type II_1 factor with finite index and finite depth. Suppose that the obstruction for flatness in the orbifold construction vanishes for $N \subset M$. Then the triple $(p_s(\alpha), \nu_{\mathrm{alg}}(\alpha), \gamma_h(\alpha))$ is a complete invariant of $\alpha \in \overline{\mathrm{Int}}(M, N)$ up to outer conjugacy.*

For the range of the invariant, we have the following theorem.

Theorem 3.3 *Let $N \subset M$ be a strongly amenable subfactor of type II_1. Suppose that a non-strongly-outer $\sigma \in \mathrm{Aut}(M, N)$ with $\Phi(\sigma) = 1$ and $\gamma_h(\sigma) = 1$ is given and that the outer period $p_o(\sigma)$ of σ is finite. Then for any positive integer p and a pth root γ of unity, there exists $\alpha \in \mathrm{Aut}(M, N)$ such that*

$$(p_s(\alpha), \nu_{\mathrm{alg}}(\alpha), \gamma_h(\alpha)) = (p, \sigma, \gamma).$$

Proof We may assume $\gamma = 1$, because then the general case follows from a tensoring the subfactor automorphism with s_p^γ of Connes as above.

By Lemma 2.3, we may assume that σ gives an outer action of $\mathbf{Z}_n$ with $n = p_o(\sigma)$. Let $P \subset Q$ be the orbifold subfactor $N \rtimes_\sigma \mathbf{Z}_n \subset M \rtimes_\sigma \mathbf{Z}_n$. We take a (unique) outer action θ of $\mathbf{Z}_{np}$ on the AFD II_1 factor R. (See [10].) Consider $(R \otimes P) \rtimes_{\theta\otimes\hat\sigma} \mathbf{Z}_{np} \subset (R \otimes Q) \rtimes_{\theta\otimes\hat\sigma} \mathbf{Z}_{np}$. It is easy to see that this subfactor is isomorphic to $N \subset M$ by comparing the higher relative commutants, because the action $\theta \otimes \hat\sigma$ is strongly outer by [26, Corollary 7]. (Here we use an argument in the proof of Theorem 3.1.) We write α for the dual action $\widehat{\theta \otimes \hat\sigma}$. It is easy to see that α is an outer action of $\mathbf{Z}_{np}$ and that $p_s(\alpha) = p$, $\nu_{\mathrm{alg}}(\alpha) = \sigma$, $\gamma_h(\alpha) = 1$. Q.E.D.

4 New invariants — an analytic approach —

We next introduce another set of invariants defined analytically and identify them with those in Section 2 for strongly amenable subfactors.

Let $N \subset M$ be a subfactor of type II_1. (We do *not* have to assume that the Jones index $[M : N]$ is finite here.) As in [22], we set

$$\chi(M,N) = \frac{\mathrm{Ct}(M,N) \cap \overline{\mathrm{Int}}(M,N)}{\mathrm{Int}(M,N)},$$

which is a subfactor analogue of the Connes invariant χ in [8].

Take any $\alpha \in \mathrm{Aut}(M,N)$. We define a *relative asymptotic period* $p_a(\alpha) \in \mathbf{N}$ by

$$p_a(\alpha)\mathbf{Z} = \{n \in \mathbf{Z} \mid \alpha^n \in \mathrm{Ct}(M,N)\}.$$

Suppose now $p_a(\alpha) > 0$ and set $p = p_a(\alpha)$. The power α^p gives an element in $\mathrm{Ct}(M,N)/\mathrm{Int}(M,N)$. We write $\nu_{\mathrm{ana}}(\alpha)$ for the orbit of the image of α^p under the action of $\mathrm{Aut}(M,N)$ given by $\alpha^p \mapsto \theta \cdot \alpha^p \cdot \theta^{-1}$ for $\theta \in \mathrm{Aut}(M,N)$ and call it an *analytic ν invariant*. This is an outer conjugacy invariant.

Suppose $\alpha \in \overline{\mathrm{Int}}(M,N)$ and $p = p_a(\alpha) > 0$. Writing $\alpha = \mathrm{Ad}\, U$ with a unitary $U \in N^\omega$, we get $\alpha^p(U) = \lambda U$ with some $\lambda \in \mathbf{C}$ by the same argument as in [24, Section 2], which is a relative version of [5], [18]. (As usual, his λ does not depend on choices of U or a free ultrafilter ω.) We set $\tilde\kappa(\alpha) = \lambda$ and call it a *generalized relative Jones invariant*. (The case in [24, Section 2] corresponds to the case $p = 1$.) Note that in the original context of Jones [18], his κ was a finer invariant of a *factor* than the Connes invariant χ in [8], but in our context, the invariant $\tilde\kappa$ is an invariant of an *automorphism*.

For a strongly amenable subfactor $N \subset M$ and $\alpha \in \overline{\mathrm{Int}}(M,N)$, we have two triplets of the invariants $(p_s(\alpha), \nu_{\mathrm{alg}}(\alpha), \gamma_h(\alpha))$ and $(p_a(\alpha), \nu_{\mathrm{ana}}(\alpha), \tilde\kappa(\alpha))$, and we now will compare these. We already know that $p_s(\alpha) = p_a(\alpha)$ and $\nu_{\mathrm{alg}}(\alpha) = \nu_{\mathrm{ana}}(\alpha)$. In our current framework, we get the following theorem easily for γ_h and $\tilde\kappa$.

Theorem 4.1 *For strongly amenable subfactors $N \subset M$ of type II_1, we have $\overline{\gamma_h(\alpha)} = \tilde\kappa(\alpha)$ for $\alpha \in \overline{\mathrm{Int}}(M,N)$.*

Proof Let $a \in M_k$ be a non-zero element with $\alpha^p(x)a = ax$ for all $x \in M$. Writing $\alpha = \mathrm{Ad}\, U$ with a unitary $U \in N^\omega$, we get $UaU^* = \gamma_h(\alpha)a$. (This is because the extension of α to the Jones tower is also given by Ad U by Loi's definition.) We also have $\alpha^p(U) = \tilde{\kappa}(\alpha)U$. These give

$$\tilde{\kappa}(\alpha)Ua = \alpha^p(U)a = aU = \overline{\gamma_h(\alpha)}Ua,$$

which gives the conclusion, because $\tilde{\kappa}(\alpha), \gamma_h(\alpha) \in \mathbf{C}$. Q.E.D.

The above theorem was proved for the Hecke algebra subfactors of Wenzl [37] in the case $p_a(\alpha) = 1$ in [24, Theorem 3.4] by a more complicated argument. V. F. R. Jones asked the author whether [24, Theorem 3.4] is valid for a general case or not, and the above theorem gives a positive answer. (Note that here we need a complex conjugate, which was unnecessary in [24] because the value of κ there was ± 1.)

5 The second classification theorem

We next give another classification theorem of approximately inner automorphisms, which covers a different situation from the one in Theorem 3.1. This is based on the Connes technique given in [9, page 466]. (Also see [25, Theorem 20] and a remark at the end of [22].) Note that the case $p_a = 0$ is covered by the relative version of the Connes classification theorem [7, Theorem 1] as in [27]. So we assume that $p_a > 0$.

Theorem 5.1 *Let $N \subset M$ be a subfactor of type II_1 with the relative McDuff splitting property. Suppose that $\alpha, \beta \in \overline{\mathrm{Int}}(M, N)$ satisfying the following properties.*

1. $p_a(\alpha) = p_a(\beta) = p > 0$.
2. $\nu_{\mathrm{ana}}(\alpha) = \nu_{\mathrm{ana}}(\beta)$.
3. $\tilde{\kappa}(\alpha) = \tilde{\kappa}(\beta)$.
4. *The outer period n of α^p is finite.*
5. $(n, p) = 1$.

Then we have a unitary $u \in N$ and $\theta \in \overline{\mathrm{Int}}(M, N)$ such that $(\mathrm{Ad}\, u) \cdot \alpha = \theta \cdot \beta \cdot \theta^{-1}$.

Proof We may and do assume $\alpha^p = \beta^p$ modulo $\mathrm{Int}(M, N)$. Let $\sigma = \alpha^p$.

By 5, there exist integers k, m with $kn + mp = -1$. Then it is easy to see that

$$p_o(\alpha^{pm+1}) = p_a(\alpha^{pm+1}) = p_o(\beta^{pm+1}) = p_a(\beta^{pm+1}) = p$$

and $\alpha^{pm+1}, \beta^{pm+1} \in \overline{\mathrm{Int}}(M, N)$. As in the proof of lemma 2.3, we also get

$$\gamma(\alpha^{pm+1}) = \gamma_h(\alpha)^{(pm+1)^2} = \gamma(\beta^{pm+1}).$$

This shows that there exist $\theta \in \overline{\mathrm{Int}}(M,N)$ and a unitary $u \in N$ with

$$\theta \cdot \alpha \cdot \sigma^m \cdot \theta^{-1} = (\mathrm{Ad}\ u) \cdot \beta \cdot \sigma^m,$$

by the relative "translation" of [7, Theorem 2.3.1]. Because $\sigma^m \in \mathrm{Ct}(M,N)$ and $\theta^{-1} \in \overline{\mathrm{Int}}(M,N)$ commute modulo $\mathrm{Int}(M,N)$ as in [7, Lemma 2.2.2], we can conclude that there exists a unitary $v \in N$ with $\theta \cdot \alpha \cdot \theta^{-1} = (\mathrm{Ad}\ v) \cdot \beta$, which is the desired conclusion. Q.E.D.

In this theorem, the finiteness condition n is not so restrictive, because it holds automatically for subfactors of finite depth again. The condition $(n,p) = 1$ is certainly more restrictive, and we do not know even whether the case $(n,p) > 1$ can actually happens or not.

6 Examples and remarks

In this last section, we study the case $[M : N] < 4$ as examples first. We then study Hecke algebra subfactors of Wenzl constructed in [37]. Finally, we discuss analogy between our classification here and the classification of automorphisms of injective type III factors in [9], [25], [35].

The subfactors with index less than 4 are completely classified by paragroups as in [28], [29], and we label them by the Dynkin diagrams A_n, D_{2n}, E_6, and E_8. (See [1], [16], [17], [20], [34] for more details.) Our classification of automorphisms of these subfactors is complete except for the case of subfactors of type A_{4n-1} or E_6 with non-trivial $\nu(\alpha)$ and even $p_a(\alpha)$, as we shall see.

First recall that $\mathrm{Aut}(M,N) = \overline{\mathrm{Int}}(M,N)$ for AFD type II_1 subfactors with principal graph A_n, E_6 or E_8 by [27]. The first example has already been treated in [22, Corollary 5.1].

Example 6.1 The following is a complete list of outer conjugacy classes automorphisms of AFD type II_1 subfactors $N \subset M$ with principal graph A_{2n} of E_8.

$$\begin{aligned} &s_0 \otimes \mathrm{id}, \\ &s_p^\gamma \otimes \mathrm{id}, \quad p > 0, \gamma^p = 1, \end{aligned}$$

where the tensor product factorization is with respect to the relative McDuff splitting $R \otimes N \subset R \otimes M$. The subfactor with principal graph $E_8^{(1)}$ also has this classification.

Example 6.2 Let $N \subset M$ be an AFD type II_1 subfactor with principal graph A_{4n-3}, $n > 1$. Then as in [16] and [6, Theorem 2] (or [22, Proposition 4.4]), there exists an outer but not strongly outer automorphism $\sigma \in \mathrm{Aut}(M,N)$ of order 2, which is unique up to $\mathrm{Int}(M,N)$. The following is a complete list of outer conjugacy classes automorphisms of $N \subset M$.

$$\begin{aligned} &s_0 \otimes \mathrm{id}, \\ &s_p^\gamma \otimes \mathrm{id}, \quad p > 0, \gamma^p = 1, \\ &s_p^\gamma \otimes \theta_p, \quad p > 0, \gamma^p = 1, \end{aligned}$$

where the tensor product factorization is with respect to the relative McDuff splitting $R \otimes N \subset R \otimes M$, and θ_p is a "pth root" of σ constructed as in the proof of Theorem 3.3. The subfactor with principal graph $E_7^{(1)}$ also has this classification.

Example 6.3 Let $N \subset M$ be an AFD type II_1 subfactor with principal graph A_{4n-1}, $n > 1$, or E_6. Then as in [16] and [6] (or [22]), there exists an outer but not strongly outer automorphism $\sigma \in \mathrm{Aut}(M, N)$ of order 2, which is unique up to $\mathrm{Int}(M, N)$. The following is a complete list of outer conjugacy classes automorphisms of $N \subset M$, except for the case of even p_a and $p_o = 2p_a$.

$$\begin{aligned}&s_0 \otimes \mathrm{id},\\ &s_p^\gamma \otimes \mathrm{id}, \quad p > 0, \gamma^p = 1,\\ &s_p^{-\gamma} \otimes \sigma, \quad p \text{ is odd}, \gamma^p = -1,\end{aligned}$$

where the tensor product factorization is with respect to the relative McDuff splitting $R \otimes N \subset R \otimes M$.

Example 6.4 In order to study the case excluded in the above Example more concretely, we study an outer action of $\mathbf{Z}_4$ on $N \subset M$ with principal graph A_{2m+1}.

On one hand, Example 6.2 says that we have three different outer actions on $N \subset M$ with principal graph A_{4n-3} and these are all. They are listed as follows, where the graph G_1 denotes the co-standard graph of the action in the sense of Popa [32] and the graph G_2 the principal graph of the crossed product subfactor $N \rtimes \mathbf{Z}_4 \subset M \rtimes \mathbf{Z}_4$.

invariants	the graph G_1	the graph G_2
$p_a = p_o = 4$	A_{4n-3}	A_{4n-3}
$p_a = 2$, $p_o = 4$, $\gamma_h = -1$	D_{2n}	A_{4n-3}
$p_a = 2$, $p_o = 4$, $\gamma_h = 1$	D_{2n}	D_{2n}

On the other hand, if $p_a = 2$ and $p_o = 4$ on $N \subset M$ with principal graph A_{4n-1}, then $\gamma_h^2 = -1 = \gamma$ by lemma 2.3. That is, we do not have an action of $\mathbf{Z}_4$. That is, we have only one outer action of $\mathbf{Z}_4$, $s_4^1 \otimes \mathrm{id}$. (The same holds for any $\mathbf{Z}_{2m}$ by the same reason.)

Next we also use the Loi invariant $\Phi(\alpha)$ for $\alpha \in \mathrm{Aut}(M, N)$ as in [27]. The following is a rather easy observation based on Popa's classification theorem [33].

Theorem 6.5 *Suppose that $N \subset M$ is a strongly amenable subfactor of type II_1 with $\mathrm{Ct}(M, N) = \mathrm{Int}(M, N)$. Then the triple $(p_o(\alpha), \gamma(\alpha), \Phi(\alpha))$ is a complete invariant for outer conjugacy for $\alpha \in \mathrm{Aut}(M, N)$, where p_o is an outer period.*

All the combinations of (p_o, γ, Φ) with $\gamma^p = 1$ ($\gamma = 1$ if $p = 0$) and $\Phi^p = 1$ are realized.

Proof The case $p_o = 0$ is exactly follows from Popa's classification theorem in [33], so we assume $p_o > 0$.

As in the proof of Theorem 3.1, we may assume that $\gamma = 1$. Then α gives an outer action of $\mathbf{Z}_{p_o}$, thus Popa's classification theorem in [33] applies.

For the second half, let θ be the standard automorphism with given Loi invariant as in [27], [33]. Note that we have $\theta^p = 1$. Then $s_p^\gamma \otimes \theta$ on the relative McDuff splitting gives the desired automorphism. Q.E.D.

With this theorem, we have the following.

Example 6.6 Let $N \subset M$ be an AFD type II_1 subfactor with principal graph D_{2n}, $n > 2$. Then as in [27] and [33], there exists a strongly outer automorphism $\theta \in \mathrm{Aut}(M, N)$ of order 2 with non-trivial Loi invariant, which is unique up to $\mathrm{Int}(M, N)$. The following is a complete list of outer conjugacy classes automorphisms of $N \subset M$.

$$\begin{aligned}
&s_0 \otimes \mathrm{id},\\
&s_0 \otimes \theta,\\
&s_p^\gamma \otimes \mathrm{id}, \quad p > 0, \gamma^p = 1,\\
&s_p^\gamma \otimes \theta, \quad p \text{ is even}, \gamma^p = 1,
\end{aligned}$$

where the tensor product factorization is with respect to the relative McDuff splitting $R \otimes N \subset R \otimes M$.

We next study the Hecke algebra subfactors of Wenzl [37]. (We use a description of them as quantum $SU(n)_k$ subfactors, which arise from the Wess–Zumino–Witten models $SU(n)_k$ as in [3].)

Example 6.7 First note that the ν invariant $\nu_{\mathrm{alg}}(\alpha)$ (or $\nu_{\mathrm{ana}}(\alpha)$) consists of a single element for the quantum $SU(n)_k$ subfactors for $n \geq 3$ and $k > n$. (Here we have to exclude the case $SU(3)_3$, for example. See [15].) Note that the Jones subfactors of type A_n corresponds to $SU(2)_{n-1}$.

By [39], we have a large subclass of these subfactors where the obstruction to flatness in the orbifold construction vanishes. Because we always have finite depth for these cases, Corollary 3.2 applies, and we get a complete classification of $\overline{\mathrm{Int}}(M, N)$ up to outer conjugacy. In particular, we have this complete classification of approximately inner automorphisms for all the quantum subfactors $SU(2n+1)_k$ with $n \geq 1$, $k > 2n + 1$, because of Xu's result in [39], which has identified the obstruction to flatness in the orbifold construction with a conformal dimension in rational conformal field theory.

Furthermore, with a decsription of a paragroup in [11, Theorem 3.5] and a little bit of work, we can show that all the automorphisms of the quantum $SU(3)_k$ subfactors with $k > 3$ are approximately inner. (It seems that this kind of argument would work for general quantum $SU(n)_k$ subfactors with $n > k$, but the graphs appearing in the paragroups are too complicated for the author to verify it in general.) Thus we have a complete classification of automorphisms of the quantum $SU(3)_k$ subfactors with $k > 3$, up to outer conjugacy.

At the end, we discuss analogy between our classification and a classification of automorphisms of injective type III factors. Because we are interested in approximately inner automorphisms, take the (unique) injective type III_1 factor M, where any automorphism is approximately inner by [9, section 3.8], [25, Theorem 1]. We also have

$$\mathrm{Ct}(M) = \{(\mathrm{Ad}\ u) \cdot \sigma_t^\varphi \mid u \text{ is a unitary in } M, t \in \mathbf{R}\},$$

where σ_t^φ denotes the modular automorphism group (with respect to a weight φ), again by [9, section 3.8], [25, Theorem 1]. Then the classification of automorphisms of this M is given as follows. Let $p_a(\alpha)$ be the asymptotic period of α. If $p_a(\alpha) > 0$, then $\alpha^p = (\mathrm{Ad}\ u) \cdot \sigma_t^\varphi$ for some u, t. We define $\nu(\alpha) = t$ and call it the *modular invariant*. We can choose φ to be a dominant weight with $\varphi \cdot \alpha = \varphi$, then $u \in M_\varphi$, the centralizer. The *modular obstruction* $\gamma_m(\alpha)$ is defined to be the scalar λ with $\alpha(u) = \lambda u$. This is a well-defined outer conjugacy invariant of α. We then have that the triple $(p_a(\alpha), \nu(\alpha), \gamma_m(\alpha))$ is a complete invariant of $\mathrm{Aut}(M)$ up to outer conjugacy. This classification is essentially due to Connes [9], where he sketched only an outline. See [25] for the details.

It is clear that our classification here is conceptually parallel to this classification of automorphisms of injective type III factors. We, however, have something different for subfactors from the case of injective type III factors, that is, the obstruction to flatness in the orbifold construction may not vanish — as in [21], [39]. This causes combinatorial difficulty and gives a reason our classification here is not complete in all the cases. We, however, still believe that our higher obstruction is the last missing invariant in the previous studies based on the above analogy.

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