One-parameter automorphism groups of the hyperfinite type II_1 factor

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Abstract. We show the uniqueness, up to cocycle conjugacy, of an action of a separable locally compact abelian group G on the hyperfinite type II_1 factor $\mathcal{R}$, which fixes a Cartan subalgebra of $\mathcal{R}$ elementwise and has the Connes spectrum $\hat{G}$. We also show the uniqueness, up to cocycle conjugacy, of an almost periodic prime action α of a separable locally compact abelian group on the hyperfinite type II_1 factor $\mathcal{R}$ with $(\mathcal{R}^\alpha)' \cap \mathcal{R} = \mathbf{C}I$, and the uniqueness, up to cocycle conjugacy, of a quasi-free one-parameter automorphism group of $\mathcal{R}$ arising from the CAR C^*-algebra, which has the Connes spectrum $\mathbf{R}$.

§0 Introduction

In this paper, we show the uniqueness, up to cocycle conjugacy, of a one-parameter automorphism group of the hyperfinite type II_1 factor $\mathcal{R}$, which fixes a Cartan subalgebra of $\mathcal{R}$ elementwise and has the Connes spectrum $\hat{\mathbf{R}}$. This result is valid for any separable locally compact abelian group G instead of $\mathbf{R}$. As its application, we also show the uniqueness, up to cocycle conjugacy, of an almost periodic prime action α of a separable locally compact abelian group on the hyperfinite type II_1 factor $\mathcal{R}$ with $(\mathcal{R}^\alpha)' \cap \mathcal{R} = \mathbf{C}I$, and the uniqueness, up to cocycle conjugacy, of a quasi-free one-parameter automorphism group of $\mathcal{R}$ arising from the CAR C^*-algebra, which has the Connes spectrum $\mathbf{R}$. Methods of computation of the Connes spectrum in terms of the asymptotic range are also given. Several examples are shown to be identical up to cocycle conjugacy as an application.

In the classification problem of group actions on the hyperfinite type II_1 factor, non-compact continuous groups have not been studied well. Thus we study the classification problem for the real number group $\mathbf{R}$. Note that the uniqueness of the injective type III_1 factor, which was finally solved by Haagerup [8], is equivalent to the uniqueness of the trace scaling one-parameter automorphism group, $\mathrm{tr}\circ\alpha_t = e^{-t}\mathrm{tr}$, of the hyperfinite type II_∞ factor $\mathcal{R}_{0,1}$ by Takesaki [22]. The trace preserving cases are still open.

We solved the classification problem for an action α of $\mathbf{R}$ up to stable conjugacy in the previous paper [13] for the case $\Gamma(\alpha) \neq \mathbf{R}$. In §1, we will deal with the case $\Gamma(\alpha) = \mathbf{R}$ under the condition that an action α fixes a Cartan subalgebra of $\mathcal{R}$. If an action fixes a Cartan subalgebra elementwise, we can write down the explicit form of this type of action by the work of Feldman-Moore [7] and Connes-Feldman-Weiss [5], and we will classify this type of actions. We will use the technique of T-array in Krieger [14], and reduce the general cases to infinite tensor product type actions. The results in this section are stated for a general separable locally compact abelian group G. The method in this section can also be applied to the hyperfinite type II_∞ factor.

In §2, we show uniqueness, up to cocycle conjugacy, of an almost periodic prime action α of a separable locally compact abelian group on the hyperfinite type II_1

factor $\mathcal{R}$ with $(\mathcal{R}^\alpha)' \cap \mathcal{R} = \mathbf{C}I$ as an application of the result in §1. This type of actions were studied by Thomsen [24].

In §3, we will use the construction in §1 and its modification to show that all the ergodic flows actually occur as $\hat{\alpha}$ on $\mathcal{Z}(\mathcal{M} \rtimes_\alpha \mathbf{R})$, which was used as complete invariants together with the type of the crossed product algebra for a classification of an action α of $\mathbf{R}$ on the hyperfinite type II_1 or II_∞ factor $\mathcal{M}$ with $\Gamma(\alpha) \neq \mathbf{R}$ in Kawahigashi [13]. We also show a one-parameter automorphism group α has a trivial relative commutant property $\mathcal{R}' \cap \mathcal{R} \rtimes_\alpha \mathbf{R} = \mathbf{C}I$ if α fixes a Cartan subalgebra of $\mathcal{R}$ and $\Gamma(\alpha) = \mathbf{R}$. We show examples of a one-parameter automorphism with $\Gamma(\alpha) = \mathbf{R}$ at the end of this section.

In §4, we use the result in §1 for quasi-free actions of $\mathbf{R}$ on $\mathcal{R}$, which is a weak extension of a one-parameter automorphism group on the CAR C^*-algebra coming from the Bogoliubov automorphism given by a one-parameter unitary group on a separable Hilbert space. We reduce these actions to the above type of actions by expansionals in Araki [1]. As an example, we can apply this result to "CAR-flow" which is a type II_1 factor automorphism group version of the endomorphism semigroup of $\mathcal{L}(\mathcal{H})$ in Powers-Robinson [20].

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§1 Uniqueness result for locally compact abelian groups

We study actions of a locally compact abelian group G on the hyperfinite type II_1 factor $\mathcal{R}$ defined as follows. (Though our main interest lies in the case $G = \mathbf{R}$, the results in this section are valid for more general cases.) Let T be an ergodic measure preserving transformation on a measure space (X, μ), $\mu(X) = 1$. Then $\sigma \in \mathrm{Aut}(L^\infty(X, \mu))$ is defined by $\sigma(\varphi)(x) = \varphi(T^{-1}x)$ for $\varphi \in L^\infty(X, \mu)$. The group measure space construction $L^\infty(X, \mu) \rtimes_\sigma \mathbf{Z}$ gives us $\mathcal{R}$. Let u be the implementing unitary for this crossed product algebra. We denote $L^\infty(X, \mu)$ by $\mathcal{A}$ in the following. For a separable locally compact abelian group G, take a measurable function h from X to $\hat{G}$, and we define an action α_t of G, $t \in G$ by

$$\begin{cases} \alpha_t(\varphi) = \varphi, & \text{for } \varphi \in L^\infty(X, \mu) \\ \alpha_t(u) = \langle t, h(x) \rangle u, & \end{cases}$$

where $\langle t, h(x) \rangle$ denotes the duality pairing of $t \in G$ and $h(x) \in \hat{G}$ for $x \in X$. Note that if α is an action of G which fixes a Cartan subalgebra of $\mathcal{R}$ elementwise, then α is of this form. (See Definition 3.1, Theorem 1, Theorem 5 in Feldman-Moore [7] and Theorem 10 in Connes-Feldman-Weiss [5].) In this section, α will denote this action. Every set in the following is assumed to be measurable.

We use a groupoid $X \rtimes_T \mathbf{Z}$ for which the multiplication is defined by $(x, n) \cdot (T^{-n}x, m) = (x, n + m)$, where $x \in X$, $n, m \in \mathbf{Z}$.

DEFINITION 1.1. For the above measurable function h and an ergodic transformation T on X, we denote by $r(h, T)$ the asymptotic range $r^*(c)$ (Definition 8.2 in Feldman-Moore [6]) for the following cocycle on a groupoid $X \rtimes_T \mathbf{Z}$.

$$c(x, n) = \begin{cases} h(x) + h(T^{-1}x) + \cdots + h(T^{-n+1}x), & \text{if } n > 0, x \in X, \\ 0, & \text{if } n = 0, x \in X, \\ h(Tx) + h(T^2x) + \cdots + h(T^{-n}x), & \text{if } n < 0, x \in X. \end{cases}$$

The asymptotic range $r^*(c)$ is a closed subgroup of $\hat{G}$ by Proposition 8.5 in Feldman-Moore [6].

For the Connes spectrum $\Gamma(\alpha)$ (see Définition 2.2.1 in Connes [3]), we get the following, as expected. (See Proposition 2.11 in Feldman-Moore [7].)

Proposition 1.2. *For the above action* α, $\Gamma(\alpha) = r(h, T)$.

Proof. By IV.5.4 in Takesaki [23] and $\mathcal{Z}(\mathcal{R}^\alpha) \subseteq \mathcal{A} \subseteq \mathcal{R}^\alpha$ we get

$$\begin{aligned}\Gamma(\alpha) &= \cap\{\mathrm{Sp}(\alpha^e) \mid e \in \mathrm{Proj}(\mathcal{R}^\alpha)\} \subseteq \cap\{\mathrm{Sp}(\alpha^e) \mid e \in \mathrm{Proj}(\mathcal{A})\} \\ &\subseteq \cap\{\mathrm{Sp}(\alpha^e) \mid e \in \mathrm{Proj}(\mathcal{Z}(\mathcal{R}^\alpha))\} = \Gamma(\alpha).\end{aligned}$$

(Here the symbol $\mathcal{Z}$ means the center.) Thus $\lambda \in \Gamma(\alpha)$ if and only if $\lambda \in \mathrm{Sp}(\alpha^e)$ for every $e = \chi_B \in \mathcal{A}$, where $B \subseteq X$, $\mu(B) > 0$.

Suppose $\lambda \in r(h, T)$. Choose an arbitrary $B \subseteq X$, $\mu(B) > 0$, and set $e = \chi_B \in \mathcal{A}$. We have to show $\hat{f}(\lambda) = 0$ if we have an $f \in L^1(G)$ such that

$$(*) \qquad \int_G f(-t)\alpha_t^e(y)\, dt = 0, \qquad \text{for every } y \in \mathcal{R}_e.$$

Assume $\hat{f}(\lambda) \neq 0$, and take an open neighborhood $U \subseteq \hat{G}$ of λ so that $\hat{f} \neq 0$ on U. Now by the definition of the asymptotic range, there exists an integer n such that we have $\mu(B') > 0$ for

$$B' = \{x \in B \mid T^{-n}x \in B, c(x, n) \in U\},$$

where we used the cocycle c as in Definition 1.1. Take $y = eu^n e$ in $(*)$. We have

$$\int_G f(-t)\chi_B \alpha_t(u^n)\chi_B \, dt = \int_G f(t)\chi_B(x)\overline{\langle t, c(x,n)\rangle}\chi_{T^n B}(x)\, dt \cdot u^n = 0.$$

This implies

$$\int_G f(t)\overline{\langle t, c(x,n)\rangle}\, dt = 0, \qquad \text{for almost all } \ x \in B'.$$

This contradicts $c(x,n) \in U$ for $x \in B'$ and $\hat{f} \neq 0$ on U.

Conversely, assume $\lambda \in \mathrm{Sp}(\alpha^e)$ for every $e = \chi_B$, $B \subseteq X$, $\mu(B) > 0$. Suppose $\lambda \notin r(h,T)$. Then there exist $B \subseteq X$, $\mu(B) > 0$, and a neighborhood $U \subseteq \hat{G}$ of λ such that

$$(**) \qquad \mu(\{x \in B \mid T^{-n}x \in B, c(x,n) \in U\}) = 0, \qquad \text{for every integer } n.$$

Take an $f \in L^1(G)$ such that $\mathrm{supp}(\hat{f}) \subseteq U$ and $\hat{f}(\lambda) \neq 0$. Then for every integer n and $\varphi \in L^\infty(X,\mu)$, we have

$$\int_G f(-t)\alpha_t(e\varphi u^n e)\, dt = \int_G f(t)\chi_B \varphi \overline{\langle t, c(x,n)\rangle}\chi_{T^n B}\, dt \cdot u^n.$$

But the right hand side of this is 0 because of $(**)$ and $\mathrm{supp}(\hat{f}) \subseteq U$. Thus by the definition of $\mathrm{Sp}(\alpha^e)$, we get $\hat{f}(\lambda) = 0$, which contradicts the construction of f. Q.E.D.

We get the following for the Poincaré flow. (See Definition 8.1 in Feldman-Moore [6].)

Proposition 1.3. *For the above action α, the flow on $\mathcal{Z}(\mathcal{R} \rtimes_\alpha G)$ given by the dual action $\hat{\alpha}$ of $\hat{G}$ is the Poincaré flow of the cocycle c.*

Proof. By considering the dual action $\hat{\alpha}$ on $\mathcal{Z}(L^\infty(X,\mu) \rtimes_\sigma \mathbf{Z} \rtimes_\alpha G)$, it follows easily from the definition of the Poincaré flow. Q.E.D.

We are interested in the case $r(h,T) = \Gamma(\alpha) = \hat{G}$. Thus in the rest of this section, we assume this equality. In the following we use the notation $[T]$ for the full group of T.

DEFINITION 1.4 For $S \in [T]$ and a measurable function h from X to $\hat{G}$, we define

$$F(h;x,S) = \begin{cases} h(x) + h(T^{-1}x) + \cdots + h(T^{-n+1}x), & \text{if } n > 0, \\ 0, \quad \text{if } n = 0, \\ h(Tx) + h(T^2x) + \cdots + h(T^{-n}x), & \text{if } n < 0, \end{cases}$$

on the set $\{x \in X \mid Sx = T^{-n}x\}$.

Lemma 1.5. *For every $A, B \subseteq X$, $\mu(A) = \mu(B) > 0$, $\lambda \in \hat{G}$ and a neighborhood $W \subseteq \hat{G}$ of λ, there exists $S \in [T]$ such that $S(A) = B$, $F(h;x,S) \in W$ for almost all $x \in A$.*

Proof. Because T is ergodic, there exist $A_1 \subseteq A$, $B_1 \subseteq B$, $\mu(A_1) = \mu(B_1) > 0$ and an integer n such that $T^{-n}A_1 = B_1$. By considering $c(x,n)$ on A_1, there exist $A_2 \subseteq A_1$, $B_2 \subseteq B_1$, $\mu(A_2) = \mu(B_2) > 0$, $\lambda' \in \hat{G}$, an open neighborhood $W' \subseteq \hat{G}$ of λ', and an open neighborhood $W'' \subseteq \hat{G}$ of $\lambda - \lambda'$ such that $c(x,n) \in W'$ for almost all $x \in A_2$ and $W' + W'' \subseteq W$ because $\hat{G}$ is second countable. Then by the definition of the asymptotic range, there exist $B_3, B_4 \subseteq B_2$, $\mu(B_3) = \mu(B_4) > 0$, and an integer m such that $T^{-m}B_3 = B_4$ and $c(x,m) \in W''$ for almost all $x \in B_3$. Now set $A_3 = T^n B_3$. Then we have $c(x, n+m) \in W$ for almost all $x \in A_3$.

Now let $\mathcal{F}$ be the set of family $\{A_i, B_i\}_{i\in I}$, where $\mu(A_i) = \mu(B_i) > 0$, A_i's are mutually disjoint subsets of A, B_i's are mutually disjoint subsets of B, and for each $i \in I$, there exists an integer n_i such that $T^{-n_i}A_i = B_i$ and $c(x, n_i) \in W$ for almost all $x \in A_i$. Consider the usual order on $\mathcal{F}$, then it is inductively ordered. Thus take a maximal $\{A_i, B_i\}_{i\in I}$ in $\mathcal{F}$, then $\mu(\cup A_i) = \mu(\cup B_i)$. If $\mu(A - \cup A_i) = \mu(B - \cup B_i) > 0$, then we can find another A' and B' by applying the above argument to $A - \cup A_i$, $B - \cup B_i$, which contradicts the maximality of $\{A_i, B_i\}_{i\in I}$. Thus $A = \cup A_i$, and $B = \cup B_i$, hence we are done. (The transformation S is defined to be T^{-n_i} on A_i.) Q.E.D.

While this Lemma 1.5 is similar to Lemma 2.7 in Krieger [14], the important difference is that λ is arbitrary here.

Take an action β of G on $\mathcal{R}$ of the form $\beta_t = \bigotimes_{j=1}^{\infty} diag(\langle t, \nu_0^j\rangle, \ldots, \langle t, \nu_{N_j}^j\rangle)$, where $diag(\langle t, \nu_0^j\rangle, \ldots, \langle t, \nu_{N_j}^j\rangle)$ stands for the $(N_j+1)\times(N_j+1)$ diagonal matrix with diagonal entries $\langle t, \nu_0^j\rangle, \ldots, \langle t, \nu_{N_j}^j\rangle$, and ν_i^j's are in $\hat{G}$. We say this action is of the infinite tensor product type. In this expression, we may assume $\nu_0^j = 0$ for all j, thus we assume this in this section, and fix β. We will prove the following theorem on the uniqueness of actions up to cocycle conjugacy. (See p. 215 of Jones-Takesaki [10] for a definition of cocycle conjugacy.)

Theorem 1.6. *If an action α of a locally compact abelian separable group G on the hyperfinite type II_1 factor $\mathcal{R}$ fixes a Cartan subalgebra elementwise and $\Gamma(\alpha) = \hat{G}$, and another action β is of the infinite tensor product type, then α is cocycle* *conjugate to an action of the infinite tensor product type, and $\alpha \otimes \beta$ is cocycle conjugate to α.*

Note that the infinite tensor product type actions are particular cases of the actions in this theorem.

We need some lemmas for the proof of this theorem. We will use the technique of T-array of Krieger. (See p. 166 in Krieger [14] and V.5 in Takesaki [23] for definitions and notations.) In our convention here, we assume $U(a,b)Z(a) = Z(b)$, the index set A is finite, and $\cup_{a\in A} Z(a) = X$ for a T-array $\mathcal{A} = \{Z(a), U(a,b) \mid a, b \in A\}$. We use a notation $\partial k(x) = k(x) - k(T^{-1}x)$ for a measurable function k from X to $\hat{G}$.

Lemma 1.7. *Suppose a T-array $\mathcal{A}_1 = \{Z_1(a), U_1(a,b) \mid a, b \in A_1\}$, $B_1, \dots, B_m \subseteq X$, a measurable function h_1 from X to $\hat{G}$, $\varepsilon > 0$, and an open neighborhood $W \subseteq \hat{G}$ of 0 are given. Moreover, we assume that $F(h_1; x, U_1(a,b))$ is an almost everywhere constant function on $Z_1(b)$. Then there exists an integer n_0 such that for every integer $n \geq n_0$, and $\lambda_0, \dots, \lambda_{n-1} \in \hat{G}$, where $\lambda_0 = 0$, there exist an extension T-array of $\mathcal{A}_1$*

$$\mathcal{A}_2 = \{Z_2(a), U_2(a,b) \mid a, b \in A_2 = A_1 \times \mathbf{Z}_n\}$$

and a measurable function h_2 from X to $\hat{G}$ such that

$$h_2(x) \in W, \qquad \text{for almost all } x \in X,$$

$$B_k \overset{\varepsilon}{\in} \mathcal{B}\{Z_2(a) \mid a \in A_2\}, \qquad \text{for every } 1 \leq k \leq m,$$

$$F(h_1; x, U_1(a,b)) = F(h_1 + \partial h_2; x, U_1(a,b)), \qquad \text{for almost all } x \in X,$$

$$F(h_1 + \partial h_2; x, U_2((a,j),(a,0))) = \lambda_j, \qquad a \in A_1, \text{for almost all } x \in Z_2(a,0). \qquad \blacksquare$$

Proof. Take n_0 as in Lemma V.5.7 in Takesaki [23], and for $n \geq n_0$, take $Z_2(a,j)$ for $(a,j) \in A_1 \times \mathbf{Z}_n$ as in the proof of Lemma V.5.7 in Takesaki [23] so that $B_k \overset{\varepsilon}{\in} \mathcal{B}\{Z_2(a) \mid a \in A_2\}$, where the right hand side means the σ-algebra generated by $Z_2(a)$, $a \in A_2$. Fix $a_0 \in A$. Now take $U_2((a_0,j),(a_0,0)) \in [T]$ by Lemma 1.5 so that

$$F(h_1; x, U_2((a_0,j),(a_0,0))) \in \lambda_j + W, \qquad \text{for almost all } x \in Z_2(a_0,0).$$

(For $j = 0$, take $U_2((a_0,0),(a_0,0)) = id$.) Now define

$$U_2((a_0,i),(a_0,j)) = U_2((a_0,i),(a_0,0))U_2((a_0,j),(a_0,0))^{-1},$$

and extend this as usual. (See V.5.6 in Takesaki [23].) We define

$$h_2(x) = F(h_1; U_2((a,0),(a,j))x, U_2((a,j),(a,0))) - \lambda_j$$

on $Z_2(a,j)$, $(a,j) \in A_1 \times \mathbf{Z}_n$. Then $h_2(x) \in W$ for almost all $x \in X$. For almost all $x \in Z_2(a,0)$, we have

$$\begin{aligned}
&F(h_1 + \partial h_2; x, U_2((a,j),(a,0))) \\
=&h_1(x) + h_1(T^{-1}x) + \cdots + h_1(T^{-n+1}x) + h_2(x) - h_2(T^{-n}x) \\
=&h_1(x) + h_1(T^{-1}x) + \cdots + h_1(T^{-n+1}x) - F(h_1; x, U_2((a,j),(a,0))) + \lambda_j \\
=&\lambda_j,
\end{aligned}$$

where n is given by $U_2((a,j),(a,0))x = T^{-n}x$. Thus by construction of h_2, we have the desired equalities. Q.E.D.

We use a notation $\mathrm{Orb}_U(x) = \{U(a,b)x \mid a,b \in A\}$ for a T-array $\mathcal{A} = \{Z(a), U(a,b) \mid a,b \in A\}$ and $x \in X$.

Lemma 1.8. *For a given T-array $\mathcal{A}_1 = \{Z_1(a), U_1(a,b) \mid a,b \in A_1\}$, $\varepsilon > 0$, an open neighborhood $W \subseteq \hat{G}$ of 0, and a measurable function h_1 from X to $\hat{G}$ such that $F(h_1; U_1(a,b), x)$ is an almost everywhere constant function on $Z_1(b)$, there exist an integer n, an extension T-array $\mathcal{A}_2 = \{Z_2(a), U_2(a,b) \mid a,b \in A_2 = A_1 \times \mathbf{Z}_n\}$, a measurable function h_2 from X to $\hat{G}$, and $\rho_0, \ldots, \rho_{n-1} \in \hat{G}$ such that*

$$\mu(\{x \in X \mid Tx \notin Orb_{U_2}(x)\}) < \varepsilon,$$

$$\mu(\{x \in X \mid h_2(x) \in W\}) > 1 - \varepsilon,$$

$$F(h_1; x, U_1(a,b)) = F(h_1 + \partial h_2; x, U_1(a,b)), \qquad \text{for almost all } x \in Z_1(b),$$

$$F(h_1 + \partial h_2; x, U_2((a,j),(a,0)) = \rho_j, \qquad a \in A_1, \text{for almost all } x \in Z_2(a,0).$$

Proof. First, take an extension T-array $\mathcal{A}_1' = \{Z_1'(a), U_1'(a,b) \mid a,b \in A_1' = A_1 \times \mathbf{Z}_m\}$ for some integer m by Lemma V.5.8 in Takesaki [23] such that

$$\mu(\{x \in X \mid Tx \notin \mathrm{Orb}_{U_1'}(x)\}) < \varepsilon.$$

We make an extension of this by technique on p. 168 in Krieger [14]. Fix $a_0 \in A_1$. Then there exist $E \subseteq Z_1'(a_0, 0)$, an open neighborhood $W' \subseteq \hat{G}$ of 0, and $\rho_0^0, \ldots, \rho_{m-1}^0 \in \hat{G}$ such that $\mu(E) > 0$, $W' + W' \subseteq W$, and

$$F(h_1; x, U_1'((a_0, j), (a_0, 0))) \in \rho_j^0 + W', \qquad 0 \leq j \leq m-1, \text{for almost all } x \in E.$$

By maximality argument, we can find a family of mutually disjoint sets $\{E'_i\}_{i\in\mathbf{N}}$ in $Z'_1(a_0,0)$ and elements $\{\rho'_{i,j}\}_{i\in\mathbf{N},0\le j\le m-1}$ of $\hat{G}$ such that $\rho'_{i,0}=0$, $\mu(E'_i)>0$, $\mu(Z'_1(a_0,0)-\cup_{i\in\mathbf{N}}E'_i)=0$ and

$$F(h_1;x,U'_1((a_0,j),(a_0,0)))\in\rho'_{i,j}+W',\qquad \text{for all } j \text{ and almost all } x\in E'_i.$$

Take $l_0\in\mathbf{N}$ such that $\mu(\cup_{i\ge l_0}E'_i)<\varepsilon\mu(Z'_1(a_0,0))/2$. By approximating $\cup_{i\ge l_0}E'_i$ and E'_i's $(0\le i\le l_0-1)$ by unions of smaller sets, we get integers l_1, l, a family $\{E_i\}_{0\le i\le l-1}$ of mutually disjoint sets in $Z'_1(a_0,0)$ and elements $\{\rho_{i,j}\}_{0\le i\le l-1,0\le j\le m-1}$ of $\hat{G}$ such that

$$\begin{aligned}
\rho_{i,0}&=0,\\
\mu(E_i)&=\mu(E_{i'}),\\
\mu(Z'_1(a_0,0)-\cup_{0\le i\le l-1}E_i)&=0,\\
\mu(\cup_{l_1\le i\le l-1}E_i)&\le\varepsilon\mu(Z'_1(a_0,0)),\\
F(h_1;x,U'_1((a_0,j),(a_0,0)))&\in\rho_{i,j}+W',\qquad 0\le i\le l_1-1,\ \text{for almost all } x\in E_i.
\end{aligned}$$

We define $Z_2(a_0,j,i)=U'_1((a_0,j),(a_0,0))E_i$, choose $U_2((a_0,0,i),(a_0,0,0))\in[T]$ such that

$$\begin{aligned}
U_2((a_0,0,i),(a_0,0,0))Z_2(a_0,0,0)&=Z_2(a_0,0,i),\\
U_2((a_0,0,0),(a_0,0,0))&=id,\\
F(h_1;x,U_2((a_0,0,i),(a_0,0,0)))&\in W',\qquad \text{for almost all } x\in Z_2(a_0,0,0),
\end{aligned}$$

by Lemma 1.5. Now we can define

$$U_2((a_0, j, i), (a_0, 0, 0)) = U_1'((a_0, j), (a_0, 0))U_2((a_0, 0, i), (a_0, 0, 0)),$$

and extend this as usual. Now we define

$$h_2(x) = F(h_1; U_2((a_0, 0, 0), (a_0, j, i))x, U_2((a_0, j, i), (a_0, 0, 0))) - \rho_{i,j}$$

on $Z_2(a_0, j, i)$. Thus this $h_2(x)$ is defined on $Z_1(a_0)$. We extend this to the entire set X by $h_2(x) = h_2(U_1(a_0, a)x)$ on $Z_1(a)$. Then we know that $\mu(\{x \in X \mid h_2(x) \in W\}) > 1 - \varepsilon$. Set $n = lm$. Because two equalities for F are proved as in the proof of Lemma 1.7, $\mathcal{A}_2 = \{Z_2(a), U_2(a, b) \mid a, b \in A_2 = A_1 \times \mathbf{Z}_n\}$, ρ's and h_2 satisfy the desired properties. Q.E.D.

Now we can prove Theorem 1.6.

Proof of Theorem 1.6. Let $\tilde{\beta}$ be the infinite tensor product of copies of β. Because $\tilde{\beta}$ is also of the infinite tensor product type, we represent this as

$$\tilde{\beta} = \bigotimes_{j=0}^{\infty} \mathrm{Ad}(diag(\langle t, \lambda_0^j \rangle, \ldots, \langle t, \lambda_{l_j}^j \rangle)), \qquad \lambda_i^j \in \hat{G}.$$

Let $\{B_n\}_{n \in \mathbf{N}}$ be a sequence of Borel sets which generates the σ-algebra of X. Choose a sequence $\{W_j\}_{j \in \mathbf{N}}$ of open neighborhoods of 0 in $\hat{G}$ such that $\sum_{j=0}^{\infty} \lambda_j$ always converges for an arbitrary sequence $\{\lambda_j\}_{j \in \mathbf{N}}$, $\lambda_j \in W_j$. We can construct a sequence of T-arrays, $\mathcal{A}_0^1$, $\mathcal{A}_0^2$, $\mathcal{A}_1^1$, $\mathcal{A}_1^2, \ldots$, a sequence of measurable functions from X to $\hat{G}$, $\{h_j^1, h_j^2\}_{j \in \mathbf{N}}$, a sequence of integers $\{n_j^1, n_j^2\}_{j \in \mathbf{N}}$, a sequence of elements

$\{\rho^1_{j,k}, \rho^2_{j,k}\}_{j\in\mathbf{N}, 0\le k\le n_j-1}$ of $\hat{G}$, $(\rho^1_{j,0} = \rho^2_{j,0} = 0)$, and a strictly increasing sequence of integers $\{m_j\}_{j\in\mathbf{N}}$, $(m_0 = 0)$, by applying Lemma 1.7 and Lemma 1.8 alternately so that the following conditions are satisfied.

(1) $\mathcal{A}^i_j = \{Z^i_j(a), U^i_j(a,b) \mid a, b \in A^i_j\}, \qquad i = 1, 2,\ j \in \mathbf{N}$,

(2) $\mathcal{A}^2_j$ is an extension of $\mathcal{A}^1_j$,

(3) $\mathcal{A}^1_{j+1}$ is an extension of $\mathcal{A}^2_j$,

(4) $A^1_j = \mathbf{Z}_{n^1_1} \times \mathbf{Z}_{n^2_1} \times \cdots \times \mathbf{Z}_{n^1_j}$,

(5) $A^2_j = A^1_j \times \mathbf{Z}_{n^2_k}$,

(6) $B_k \overset{1/2^j}{\in} \mathcal{B}\{Z^1_j(a) \mid a \in A^1_j\}, \qquad k \le j$,

(7) $\mu(\{x \in X \mid Tx \in \mathrm{Orb}_{U^2_j}(x)\}) > 1 - 1/2^j$,

(8) $h^1_j(x) \in W_j, \qquad$ for almost all $x \in X$,

(9) $\mu(\{x \in X \mid h^2_j(x) \in W_j\}) \ge 1 - 1/2^j$,

(10) $F(h + \partial(h^1_0 + h^2_0 + h^1_1 + h^2_1 + \cdots + h^1_j); x, U^1_j((a,k),(a,0))) = \rho^1_{j,k}$, for $a \in A^2_{j-1}$, $0 \le k \le n^1_j - 1$, and almost all $x \in Z^1_k(a,0)$,

(11) $F(h + \partial(h^1_0 + h^2_0 + h^1_1 + h^2_1 + \cdots + h^2_j); x, U^2_j((a,k),(a,0))) = \rho^2_{j,k}$, for $a \in A^1_j$, $0 \le k \le n^2_j - 1$, and almost all $x \in Z^2_k(a,0)$,

(12) $diag(\langle t, \rho^1_{j,0}\rangle, \ldots, \langle t, \rho^1_{j,n^1_j}\rangle) = \bigotimes_{n=m_j}^{m_{j+1}-1} diag(\langle t, \lambda^n_0\rangle, \ldots, \langle t, \lambda^n_{l_n}\rangle)$.

Note that there exists a measurable function $h' = \sum_{j=0}^\infty (h^1_j + h^2_j)$ on X by (8) and (9). By (2), (3), (4), (5), (6) and (7), $L^\infty(X,\mu) \rtimes_\sigma \mathbf{Z}$ is isomorphic to $L^\infty(\prod_{j=0}^\infty(\mathbf{Z}_{n^1_j} \times \mathbf{Z}_{n^2_j}), \nu) \rtimes \bigoplus_{j=0}^\infty(\mathbf{Z}_{n^1_j} \oplus \mathbf{Z}_{n^2_j})$, where the action is given by the natural addition, and the measure ν is the product measure of ν^1_j on $\mathbf{Z}_{n^1_j}$ and ν^2_j on $\mathbf{Z}_{n^2_j}$, $\nu^1_j(pt) = 1/n^1_j$, $\nu^2_j(pt) = 1/n^2_j$. Under this isomorphism, $\mathrm{Ad}(\langle t, h'\rangle)\alpha_t$ is conjugate to

$$\bigotimes_{j=0}^{\infty}(\mathrm{Ad}(diag(\langle t,\rho^1_{j,0}\rangle,\ldots,\langle t,\rho^1_{j,n^1_j}\rangle))\otimes\mathrm{Ad}(diag(\langle t,\rho^2_{j,0}\rangle,\ldots,\langle t,\rho^2_{j,n^2_j}\rangle))),$$

by (10), and (11). Setting

$$\alpha'_t=\bigotimes_{j=0}^{\infty}\mathrm{Ad}(diag(\langle t,\rho^2_{j,0}\rangle,\ldots,\langle t,\rho^2_{j,n^2_j}\rangle)),$$

we know by (12) that α is cocycle conjugate to $\tilde{\beta}\otimes\alpha'$, which is of the infinite tensor product type. We also know $\alpha\otimes\beta$ is cocycle conjugate to $\beta\otimes\tilde{\beta}\otimes\alpha'\cong\tilde{\beta}\otimes\alpha'$, which is cocycle conjugate to α. Q.E.D.

Corollary 1.9. *If an action α of a separable locally compact abelian group G on the hyperfinite type II_1 factor $\mathcal{R}$ fixes a Cartan subalgebra and $\Gamma(\alpha)=\hat{G}$, then this α is unique up to cocycle conjugacy.*

Proof. Suppose α, β be actions as in the statement. We may assume these are of the above type, and by Theorem 1.6, we may also assume these are of the infinite tensor product type by changing these within their cocycle conjugacy classes if necessary. Now again by Theorem 1.6, both α and β are cocycle conjugate to $\alpha\otimes\beta$, thus α and β are cocycle conjugate. Q.E.D.

We can apply the above technique to the hyperfinite type II_∞ factor $\mathcal{R}_{0,1}$, too.

Proposition 1.10. *If α is an action of a separable locally compact abelian group G on the hyperfinite type II_∞ factor $\mathcal{R}_{0,1}$ which fixes a Cartan subalgebra and $\Gamma(\alpha)=\hat{G}$, then this α is unique up to cocycle conjugacy.*

Proof. We may assume $\mathcal{R}_{0,1} = L^\infty(X,\mu) \rtimes_\sigma \mathbf{Z}$, where T is a measure preserving transformation on a measure space $L^\infty(X,\mu)$, $\mu(X) = \infty$, and α is given by

$$\begin{cases} \alpha_t(\varphi) = \varphi, & \text{for } \varphi \in L^\infty(X,\mu) \\ \alpha_t(u) = \langle t, h(x)\rangle u, \end{cases}$$

Then choose a sequence of mutually disjoint measurable sets $\{X_n\}_{n\in N}$ in X such that $X = \cup_{n\in N} X_n$, $\mu(X_n) = 1$. Choose an open neighborhood $W \subseteq \hat{G}$ of 0. Then by applying Lemma 1.5, we get $S_n \in [T]$ such that $S_0 = id$, $S_n(X_0) = X_n$, and $F(h; x, S_n) \in W$ for almost all $x \in X_0$. Define $h_1(x) = F(h; S_n^{-1}x, x)$ for $x \in X_n$, and consider the induced transformation T_0 on X_0, the reduced cocycle c_{X_0}, and the action α' given by c_{X_0} on the hyperfinite type II_1 factor $L^\infty(X_0,\mu) \rtimes_{T_0} \mathbf{Z}$ as above. Then $\mathrm{Ad}(\langle t, h_1\rangle)\alpha$ is given by $\alpha' \otimes i$ and $\Gamma(\alpha') = \hat{G}$, where i stands for the trivial action on the type I_∞ factor. If β is another action as in the proposition, we get β' similarly. Now α' and β' cocycle conjugate, hence we know that α and β are cocycle conjugate. Q.E.D.

§2 Almost periodic prime actions of locally compact abelian groups

As an application of the result in §1, we consider almost periodic prime actions of separable locally compact abelian groups. We keep denoting a separable locally compact abelian group by G, and consider an action α of G on the hyperfinite type II_1 factor $\mathcal{R}$. We say α is *prime* if the fixed point algebra $\mathcal{R}^\alpha$ is a factor. We define an eigenspace $\mathcal{R}(p)$ for $p \in \hat{G}$ by

$$\mathcal{R}(p) = \{x \in \mathcal{R} | \alpha_g(x) = \langle g, p\rangle x, \text{ for all } g \in G\},$$

and the pure point spectrum $\mathrm{Sp}_d(\alpha)$ by

$$\mathrm{Sp}_d(\alpha) = \{p \in \hat{G} | \mathcal{R}(p) \neq 0\}.$$

We say α is almost periodic if the linear span of the subspaces $\mathcal{R}(p)$, $p \in \hat{G}$, is weakly dense in $\mathcal{R}$. (Definition 7.3 in Olesen-Pedersen-Takesaki [16] and Definition 7.1 in Thomsen [24].) Note the assumption of the existence of a faithful normal α-invariant state in definitions of Olesen-Pedersen-Takesaki [16] and Thomsen [24] is unnecessary here because we consider the type II_1 factor. In this section, we assume α is a faithful, prime, almost periodic action of G on the hyperfinite type II_1 factor $\mathcal{R}$ with $(\mathcal{R}^\alpha)' \cap \mathcal{R} = \mathbf{C}I$, and we will show the uniqueness up to cocycle conjugacy of this type of action.

Lemma 2.1. *Let H be a separable compact abelian group, and let $\hat{H} = \{\lambda_n | n \in \mathbf{N}\}$. Define the infinite tensor product type action σ of H on the hyperfinite type II_1 factor $\mathcal{R}$ by*

$$\sigma_h = \bigotimes_{j,n \in \mathbf{N}} Ad \begin{pmatrix} 1 & 0 \\ 0 & \langle h, \lambda_{n,j} \rangle \end{pmatrix},$$

where $\lambda_{n,j} = \lambda_n$ for all $j \in \mathbf{N}$. Then this action is faithful and we have $(\mathcal{R}^\sigma)' \cap \mathcal{R} = \mathbf{C}I$.

Proof. It is trivial that this σ is faithful.

First we calculate the Connes spectrum $\Gamma(\sigma)$. Although it will follow from $(\mathcal{R}^\sigma)' \cap \mathcal{R} = \mathbf{C}I$ that $\Gamma(\sigma) = \hat{G}$ by Corollary 4.7 in Paschke [18], we need this type of computation later.

Make a sequence $\{\mu_n\}_{n\in\mathbf{N}}$ by renumbering the double sequence $\{\lambda_{n,j}\}_{n,j\in\mathbf{N}}$. Set $X=\prod_{n=1}^{\infty}\{0,1\}$, and let μ be a product measure of a measure ν on $\{0,1\}$, $\nu(\{0\})=\nu(\{1\})=1/2$. We define an equivalence relation $x\sim y$ for $x=(x_n), y=(y_n)\in X$ by

$$x\sim y \Longleftrightarrow x_n=y_n \text{ for all sufficiently large } n\text{'s}.$$

Then this induces a groupoid, and we can define an $\hat{H}$-valued 1-cocycle c by $c(x,y)=\sum_{n=1}^{\infty}(y_n-x_n)\mu_n$ for $x\sim y\in X$. (Note that the sum is actually a finite sum.) Because this groupoid is amenable, this cocycle is of the type considered in §3.2, and the obtained one-parameter automorphism group is exactly σ. Thus it is enough to show $r^*(c)=\hat{H}$ by Proposition 1.2, and it is also enough to show that for every $E\subseteq X$ there exists an integer k such that if $n>k$, then $\mu_n\in r^*(c_E)$. (Here c_E is the restriction of the cocycle c above to E. See Proposition 7.6 in Feldman-Moore [6].)

Fix $E\subseteq X$, $\mu(E)\neq 0$ and take $\varepsilon=\mu(E)/9$. Then there exist an integer k and $F\subseteq\prod_{j=1}^{k}\{0,1\}\subseteq X$ such that $\mu(E\bigtriangleup F)<\varepsilon$. (We identify a set $A\subseteq\prod_{j=1}^{k}\{0,1\}$ with $A\times\prod_{j=k+1}^{\infty}\{0,1\}$.) Let $F=\bigsqcup_{l=1}^{L}F_l$, where each F_l is a singleton in $\prod_{j=1}^{k}\{0,1\}$. Note that each F_l has measure $1/2^k$. We show $\mu(g_n(E\cap G_n^0)\cap E\cap G_n^1)\neq 0$ for $n>k$, where

$$g_n=(0,\dots,0,1,0,\dots)\in\oplus_{j=1}^{\infty}\mathbf{Z}_2,$$

(1 is at the n-th entry),

$$G_n^0=\prod_{j=1}^{n-1}\{0,1\}\times\{0\}\times\prod_{j=n+1}^{\infty}\{0,1\},$$

and

$$G_n^1 = \prod_{j=1}^{n-1} \{0,1\} \times \{1\} \times \prod_{j=n+1}^{\infty} \{0,1\}.$$

Suppose it was zero. Setting $E_l = E \cap F_l$, we get $\mu(g_n(E_l \cap G_n^0) \cap E_l \cap G_n^1) \neq 0$ for each l. It implies $\mu(E_l) \leq 1/2^{k+1}$. Thus $\mu(F_l - E_l) \geq 1/2^{k+1}$, and we get

$$\mu(E \triangle F) \geq \sum_{l=1}^{L} \mu(F_l - E_l) \geq \sum_{l=1}^{L} 1/2^{k+1} = \mu(F)/2,$$

which implies

$$(\sqrt{2}+1)\mu(E)^{1/2}/3 \geq \mu(E)^{1/2},$$

but it is a contradiction. Thus $\mu(g_n(E \cap G_n^0) \cap E \cap G_n^1) \neq 0$. Because $c(x, g_n x) = \mu_n$ for $x \in E \cap G_n^0 \cap g_n^{-1}(E \cap G_n^1)$, we get $\mu_n \in r^*(c_E)$. Thus we now have $\Gamma(\sigma) = \hat{H}$.

It is known that the dual action of the free action of $\mathbf{Z}$ on $\mathcal{R}$ is conjugate to the infinite tensor product type action $\bigotimes_{j=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$, $z \in \mathbf{T}$, of $\mathbf{T}$. Because each element of $\hat{G}$ appears for infinitely many times in $\{\mu_n\}$, we can write $\mathcal{R} = \bigotimes_{j=1}^{\infty} \mathcal{R}_j$, $\bigotimes_{j=1}^{\infty} \mathcal{P}_j \subseteq \mathcal{R}^{\sigma}$, where $\mathcal{R}_j \cong \mathcal{P}_j \cong \mathcal{R}$, $\mathcal{P}_j' \cap \mathcal{R}_j = \mathbf{C}I$ for all j. This implies $(\mathcal{R}^{\sigma})' \cap \mathcal{R} = \mathbf{C}I$. Q.E.D.

Now we can prove the following theorem.

Theorem 2.2. *For a separable locally compact abelian group G, a faithful almost periodic prime action α with $(\mathcal{R}^{\alpha})' \cap \mathcal{R} = \mathbf{C}I$ is unique up to cocycle conjugacy.*

Proof. By Proposition 7.3 in Thomsen [24], $\mathrm{Sp}_d(\alpha)$ is a countable dense subgroup of $\hat{G}$, and if we set $H = \mathrm{Sp}_d(\alpha)\hat{}$, there exists an action β of the separable compact abelian group H on $\mathcal{R}$ such that $\alpha_g = \beta_{\iota(g)}$, where ι is a natural dense embedding $G \subseteq H$. Because $\mathcal{R}^\alpha$ is a factor by assumption and ι is a dense embedding, $\mathcal{R}^\beta = \mathcal{R}^\alpha$ is also a factor, and we have $(\mathcal{R}^\beta)' \cap \mathcal{R} = \mathbf{C}I$. By Theorem 5.2 in Thomsen [24], this β is conjugate to the action σ in Lemma 2.1. By $\alpha_g = \beta_{\iota(g)}$, α is also of infinite tensor product type. Because $\Gamma(\alpha) = \hat{G}$ by the almost periodicity and primeness, we know that α is unique up to cocycle conjugacy by Corollary 1.9. Q.E.D.

Remark 2.3. A faithful almost periodic ergodic action is a particular case of almost periodic prime actions. A classification of this type of actions up to conjugacy was given by Theorem 7.4 in Olesen-Pedersen-Takesaki [16], and this result was extended to faithful almost periodic prime actions by Theorem 7.4 in Thomsen [24]. Their invariants are $\mathrm{Sp}_d(\alpha)$ and a symplectic bicharacter χ_α in Olesen-Pedersen-Takesaki [16], and a fixed point algebra $\mathcal{M}^\alpha$ and $N(\alpha) = \mathrm{Sp}_d(\alpha|_{(\mathcal{M}^\alpha)'\cap\mathcal{M}})$ in addition to these two in Thomsen [24]. Theorem 7.5 in Thomsen [24] claims that there is only one action of the type of Theorem 2.2 up to conjugacy for each $\mathrm{Sp}_d(\alpha)$. Our Theorem 2.2 shows cocycle conjugacy class of the action does not depend on the embedding into $H = \mathrm{Sp}_d(\alpha)\hat{}$, and these actions are unique in the case $(\mathcal{R}^\alpha)' \cap \mathcal{R} = \mathbf{C}I$ if we consider the cocycle conjugacy.

§3 One-parameter automorphism groups of $\mathcal{R}$.

In this section, we study the case where the group G in §1 and §2 is the real number group $\mathbf{R}$.

In Theorem 0.1 in Kawahigashi [13], we classified one-parameter automorphism group α of the hyperfinite type II_1 and II_∞ factors $\mathcal{R}$, $\mathcal{R}_{0,1}$ up to stable conjugacy under the assumption $\Gamma(\alpha) \neq \mathbf{R}$. Thus we have a complete classification for one-parameter automorphism groups of $\mathcal{R}$, $\mathcal{R}_{0,1}$ fixing a Cartan subalgebra by this result, Corollary 1.9, and Proposition 1.10. In Kawahigashi [13], we considered the ergodic flow on $\mathcal{Z}(\mathcal{M} \rtimes_\alpha \mathbf{R})$ given by $\hat{\alpha}$ and the type of $\mathcal{M} \rtimes_\alpha \mathbf{R}$ as complete invariants for $\mathcal{M} = \mathcal{R}, \mathcal{R}_{0,1}$. In this section, we show all the ergodic flows occur as this invariant by a similar construction to actions in §1. Note that we showed in Kawahigashi [13] the type of the crossed product algebra is of II_∞ or I_∞ unless α_t is inner for every $t \in \mathbf{R}$. Because we have an invariant trace on the crossed product algebra, if the crossed product is of type I_∞, only measure preserving ergodic flows can occur. In the following, we show this is the only restriction on these complete invariants.

Proposition 3.1. *In the above context, all the measure preserving ergodic flows occur as $\hat{\alpha}$ on $\mathcal{Z}(\mathcal{M} \rtimes_\alpha \mathbf{R})$ if $\mathcal{M} \rtimes_\alpha \mathbf{R}$ is of type I_∞, and all the ergodic flows occur as $\hat{\alpha}$ on $\mathcal{Z}(\mathcal{M} \rtimes_\alpha \mathbf{R})$ if $\mathcal{M} \rtimes_\alpha \mathbf{R}$ is of type II_∞.*

Proof. Suppose an ergodic flow T_t on a measure space (Y, ν) is given. First assume this is measure preserving. Then by the Theorem of Ambrose-Kakutani (see Katok [12]), there exist an ergodic measure preserving transformation T on a measure space (X, μ) and a positive measurable function h on X such that T_t on Y is conjugate to the flow under the ceiling function h over the base X. Construct a one-parameter automorphism group α for this X, T, and h as in §1. Then by Proposition 1.3, we know that $\hat{\alpha}$ on $\mathcal{Z}(\mathcal{R} \rtimes_\alpha \mathbf{R})$ is conjugate to T_t. It can be shown as in the argument after Lemma 1.1 in Kawahigashi [13] that the crossed product $\mathcal{R} \times_\alpha \mathbf{R}$ is of type I_∞. If we consider $\alpha \otimes i$ on $\mathcal{R} \bar{\otimes} \mathcal{R}$, where i is the trivial action of $\mathbf{R}$ on the second copy of $\mathcal{R}$, we get a type II_∞ crossed product algebra and the same flow as $\hat{\alpha}$ on $\mathcal{Z}(\mathcal{M} \rtimes_\alpha \mathbf{R})$.

Next assume there is no measure on Y that is equivalent to ν and preserved by T_t. By the Theorem of Ambrose-Kakutani-Krengel-Kubo (see Katok [12]), there exist an ergodic measurable transformation T on a measure space (X, μ) and a positive measurable function h on X such that T_t on Y is conjugate to the flow under the ceiling function h over the base X. Take and fix an action θ of $\mathbf{R}$ on $\mathcal{R}_{0,1}$ such that we have $\mathrm{tr} \circ \theta_t = e^t \mathrm{tr}$ where tr is the trace on $\mathcal{R}_{0,1}$ and $t \in \mathbf{R}$. (See Takesaki [22].) We define an automorphism σ of $L^\infty(X, \mu) \bar{\otimes} \mathcal{R}_{0,1}$ by $\sigma(y) = \theta_{-\log m(x)}$ for $x \in X$, where this define a map from $\mathcal{R}_{0,1}(x)$ to $\mathcal{R}_{0,1}(Tx)$ in

$$L^\infty(X, \mu) \bar{\otimes} \mathcal{R}_{0,1} = \int_X^\oplus \mathcal{R}_{0,1}(x)\, dx, \qquad \mathcal{R}_{0,1}(x) = \mathcal{R}_{0,1},$$

and $m(x)$ is the value of Radon-Nikodym derivative of T at $x \in X$. Then this σ is trace preserving on $L^\infty(X, \mu) \bar{\otimes} \mathcal{R}_{0,1}$, where the trace is given by μ and tr. By Lemma 7.11.10 in Pedersen [19] and the ergodicity of T, we know that $(L^\infty(X, \mu) \bar{\otimes} \mathcal{R}_{0,1}) \rtimes_\sigma \mathbf{Z}$ is a factor. Because it has a trace and it is infinite, it is isomorphic to $\mathcal{R}_{0,1}$. On this $(L^\infty(X, \mu) \bar{\otimes} \mathcal{R}_{0,1}) \rtimes_\sigma \mathbf{Z} \cong \mathcal{R}_{0,1}$, we can define a one-parameter automorphism α_t by

$$\begin{cases} \alpha_t(\varphi) = \varphi, & \text{for } \varphi \in L^\infty(X, \mu) \bar{\otimes} \mathcal{R}_{0,1} \\ \alpha_t(u) = \langle t, h(x) \rangle u, & \end{cases}$$

where u is the implementing unitary in the crossed product. Then by a similar argument to Proposition 1.3, we know that $\hat{\alpha}$ on $\mathcal{Z}(\mathcal{R}_{0,1} \rtimes_\alpha \mathbf{R})$ is conjugate to T_t. By choosing an invariant projection e with finite trace in $L^\infty(X, \mu) \bar{\otimes} \mathcal{R}_{0,1}$, we can also consider α^e which is a one-parameter automorphism group of the hyperfinite type II_1 factor $\mathcal{R}$ and has the desired property. Q.E.D.

In group actions, the trivial relative commutant property $\mathcal{R}' \cap \mathcal{R} \rtimes G = \mathbf{C}I$ has been important. We prove this property for one-parameter automorphism groups of the type in §1.

Proposition 3.2. *If a one-parameter automorphism group α of the hyperfinite type II_1 factor $\mathcal{R}$ fixes a Cartan subalgebra and $\Gamma(\alpha) = \mathbf{R}$, then this α has the trivial relative commutant property, $\mathcal{R}' \cap \mathcal{R} \rtimes_\alpha \mathbf{R} = \mathbf{C}I$.*

Proof. Because the trivial relative commutant property is invariant under cocycle conjugacy, we may assume α is of the infinite tensor product type by Theorem 1.6. Thus we have an increasing sequence $\mathcal{M}_n$ of matrix algebras in $\mathcal{R}$ such that $\alpha(\mathcal{M}_n) = \mathcal{M}_n$ and $\bigvee_n \mathcal{M}_n = \mathcal{R}$. Suppose $x \in \mathcal{R}' \cap \mathcal{R} \rtimes_\alpha \mathbf{R}$. Let $\mathcal{E}_n$ be the conditional expectation from $\mathcal{R} \rtimes_\alpha \mathbf{R}$ onto $\mathcal{M}_n \rtimes_\alpha \mathbf{R}$. Now $x \in \mathcal{R}' \cap \mathcal{R} \rtimes_\alpha \mathbf{R}$ implies $\mathcal{E}_n(x) \in \mathcal{M}_n' \cap \mathcal{M}_n \rtimes_\alpha \mathbf{R} = \lambda(\mathbf{R})$, where λ denotes the representation of $\mathbf{R}$ in the crossed product algebra. (Note that $\alpha|\mathcal{M}_n$ is inner.) Thus $x = \lim_{n\to\infty} \mathcal{E}_n(x) \in \lambda(\mathbf{R}) = \lambda(\mathbf{R})'$, hence we have $x \in \mathcal{R}' \cap \lambda(\mathbf{R})' = \mathbf{C}I$ because $\mathcal{R} \rtimes_\alpha \mathbf{R}$ is a factor by $\Gamma(\alpha) = \mathbf{R}$. Q.E.D.

In general, it is not very easy to compute the asymptotic range in Proposition 1.2. But we have the following example for one-parameter automorphism groups.

Example 3.3. Let θ be an irrational number, $0 < \theta < 1$, and consider the torus $\mathbf{T} = [0, 1)$ with the Lebesgue measure, and an ergodic transformation T on X defined by $Tx = x + \theta$. Take a number c, $0 < c < 1$, $c \notin \mathbf{Q} + \mathbf{Q}\theta$, and define a function $h(x) = \chi_{[0,c]}(x) - c$. Define an action α on $\mathcal{R} = L^\infty(\mathbf{T}, \mu) \rtimes \mathbf{Z}$ by h as above. Then by Theorem A in Oren [17], we have $r(h, T) = E$, where E is the closed subgroup of $\mathbf{R}$ generated by 1 and c, which is $\mathbf{R}$, and we get an example for $\Gamma(\alpha) = \mathbf{R}$. Corollary 1.9 shows that the cocycle conjugacy class of this action does not depend on choice of an irrational c.

We also have the following for almost periodic prime actions of $\mathbf{R}$.

Corollary 3.4. *For an almost periodic prime one-parameter automorphism group α on the hyperfinite type II_1 factor $\mathcal{R}$ with $(\mathcal{R}^\alpha)' \cap \mathcal{R} = \mathbf{C}I$, we have the trivial relative commutant property $\mathcal{R}' \cap \mathcal{R} \rtimes_\alpha \mathbf{R} = \mathbf{C}I$.*

Proof. Immediate by Theorem 2.2. Q.E.D.

We consider the next example of an almost periodic prime one-parameter automorphism group on the hyperfinite type II_1 factor $\mathcal{R}$.

Example 3.5. Take a free action β of $\mathbf{Z}^2$ on $\mathcal{R}$, and make the crossed product algebra $\mathcal{R} \rtimes_\beta \mathbf{Z}^2$, which is isomorphic to $\mathcal{R}$. A one-parameter automorphism group α can be defined by $\alpha_t(x) = x$ for $x \in \mathcal{R}$, $\alpha_t(u) = e^{2\pi i t\mu}u$ and $\alpha_t(v) = e^{2\pi i t\lambda}v$, where u and v are the implementing unitaries for $\mathbf{Z}^2$, and λ and μ are nonzero numbers with $\lambda/\mu \notin \mathbf{Q}$. Then it is easy to show this is faithful, almost periodic, and $(\mathcal{R}^\alpha)' \cap \mathcal{R} = \mathbf{C}I$. Thus by Theorem 2.2, this is cocycle conjugate to the one-parameter automorphism group in Example 3.3.

§4 The CAR C^*-algebra and quasi-free actions of $\mathbf{R}$

As an application of the theorem in §1, we will classify quasi-free actions arising from the CAR C^*-algebra.

We introduce a quasi-free action of $\mathbf{R}$ on the hyperfinite type II_1 factor $\mathcal{R}$ as follows.

Take a separable Hilbert space $\mathcal{H}$. There exists the Fock representation $f \mapsto a(f) \in \mathcal{L}(\mathcal{K})$ on another Hilbert space $\mathcal{K}$, which satisfies

(1) $a(\alpha f + \beta g) = \alpha a(f) + \beta a(g), \qquad \alpha, \beta \in \mathbf{C}$,

(2) $a(f)a(g) + a(g)a(f) = 0$,

(3) $a(f)^* a(g) + a(g)a(f)^* = (f|g)_{\mathcal{H}} I_{\mathcal{K}}$.

Then $a(f)$'s generate a C^*-algebra, and if $\{f_n\}_{n\geq 1}$ is a complete orthonormal basis for $\mathcal{H}$, we get the following correspondence between this C^*-algebra and the 2^∞ UHF algebra.

$$\begin{aligned}
a(f_1) &\longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
a(f_2) &\longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
a(f_3) &\longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&\vdots
\end{aligned}$$

We get the hyperfinite type II_1 factor $\mathcal{R}$ by taking a weak closure with respect to the trace. If we have a one-parameter unitary group $\{U_t\}_{t\in\mathbf{R}}$ on $\mathcal{H}$, then

$$\alpha_t(P(a(f_1), \ldots, a(f_n)^*)) = P(a(U_t f_1), \ldots, a(U_t f_n)^*),$$

where P is a (non-commutative) polynomial, defines a one-parameter automorphism group on the CAR C^*-algebra. This extends to a one-parameter automorphism group on $\mathcal{R}$, and we denote it by α, too. We call this a quasi-free action of $\mathbf{R}$ on $\mathcal{R}$, and classify this type of actions in this section.

In the above context, let H be a self-adjoint operator such that $e^{iHt} = U_t$. Then by von Neumann's theorem (see Theorem X.2.1 in Kato [11]) there exists a Hilbert-Schmidt class self-adjoint operator V such that a self-adjoint operator $K = H + V$ has pure point spectrum. Then one-parameter unitary group e^{iKt} defines another one-parameter automorphism group β on $\mathcal{R}$. We have the following for these two actions. (See also p. 315 in Saka [21].)

Theorem 4.1. *Let α and β be quasi-free actions of $\mathbf{R}$ on $\mathcal{R}$ corresponding to e^{iHt} and e^{iKt} respectively as above. Then α is cocycle conjugate to β, and β is of the infinite tensor product type.*

Proof. Let $V = \sum_{n=1}^{\infty} \lambda_n E_n$, where E_n is a mutually orthogonal rank-one projection onto a subspace spanned by f_n. We may assume $\{f_n\}_{n\geq 1}$ is a complete orthonormal basis of $\mathcal{H}$. We know that λ_n's are real and $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Let $\alpha_t^{(n)}$ be the one-parameter automorphism group on $\mathcal{R}$ corresponding to a one-parameter unitary group $\exp it(H + \sum_{j=1}^{n} \lambda_j E_j)$. (We use a convention $\alpha^{(0)} = \alpha$.) We use the above correspondence between the CAR C^*-algebra and the infinite tensor product of copies of the 2×2 matrix algebra for the orthonormal sequence $\{f_n\}_{n\geq 1}$. Define for $n \geq 0$,

$$u_t^{(n)} = \operatorname{Exp}_r \left(\int_0^t ; \alpha_s^{(n)} \left(1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} i\lambda_{n+1}/2 & 0 \\ 0 & -i\lambda_{n+1}/2 \end{pmatrix} \right) ds \right),$$

where Exp_r means an expansional. (See §2 in Araki [1].) By Theorem 2 in Araki

[1], it is an $\alpha^{(n)}$-unitary cocycle. If g is in $\mathrm{Dom}(H) = \mathrm{Dom}(K) \subseteq \mathcal{H}$, then we have

$$\begin{aligned}
&\delta_{\alpha^{(n+1)}}(a(g)) \\
=&a(i(H + \sum_{j=1}^{n} \lambda_j E_j + \lambda_{n+1} E_{n+1})g) \\
=&\delta_{\alpha^{(n)}}(a(g)) + \frac{d}{dt}\left(\mathrm{Ad}\left(\exp it\left(1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} \lambda_{n+1}/2 & 0 \\ 0 & -\lambda_{n+1}/2 \end{pmatrix}\right)\right) a(g)\right)\Big|_{t=0} \\
=&\delta_{\alpha^{(n)}}(a(g)) + \left[1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} i\lambda_{n+1}/2 & 0 \\ 0 & -i\lambda_{n+1}/2 \end{pmatrix}, a(g)\right],
\end{aligned}$$

where δ's stand for derivations. Thus $\alpha_t^{(n+1)}(a(g)) = \mathrm{Ad}(u_t^{(n)})\alpha_t^{(n)}(a(g))$ by Theorem 2 in Araki [1], and thus we get $\alpha_t^{(n+1)} = \mathrm{Ad}(u_t^{(n)})\alpha_t^{(n)}$. Hence if we set $v_t^{(n)} = u_t^{(n)} \cdots u_t^{(0)}$, it is an α-unitary cocycle. Note that we have

$$v_t^{(n)} = \mathrm{Exp}_r\left(\int_0^t ; \alpha_s\left(i\begin{pmatrix} \lambda_1/2 & 0 \\ 0 & -\lambda_1/2 \end{pmatrix} \odot \cdots \odot \begin{pmatrix} \lambda_{n+1}/2 & 0 \\ 0 & -\lambda_{n+1}/2 \end{pmatrix}\right) ds\right),$$

where the operation $\odot$ is define by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \odot \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a+c & 0 & 0 & 0 \\ 0 & a+d & 0 & 0 \\ 0 & 0 & b+c & 0 \\ 0 & 0 & 0 & b+d \end{pmatrix}.$$

(This operation is defined similarly for higher dimensional matrices.)

Now we claim that there exists an α-unitary cocycle v_t such that $v_t^{(n)} \longrightarrow v_t$ as $n \longrightarrow \infty$ uniformly for t on every compact set in $\mathbf{R}$.

For a given ε, choose N such that $\left(\sum_{n>N} \lambda_n^2\right)^{1/2} < 2\varepsilon$. We will show that $\|v_t^{(n)} - v_t^{(m)}\|_2 \leq \varepsilon t$ for $n > m > N$. Set

$$A_{m,n} = 0 \odot \cdots \odot 0 \odot \begin{pmatrix} \lambda_{m+2}/2 & 0 \\ 0 & -\lambda_{m+2}/2 \end{pmatrix} \odot \cdots \odot \begin{pmatrix} \lambda_{n+1}/2 & 0 \\ 0 & -\lambda_{n+1}/2 \end{pmatrix}.$$

Then by Theorem 1 in Araki [1]

$$\begin{aligned}
\|v_t^{(n)} - v_t^{(m)}\|_2 &= \|\mathrm{Exp}_r \left(\int_0^t ; \alpha_s^{(m+1)}(A_{m,n})\, ds \right) - 1\|_2 \\
&= \| \int_0^t \mathrm{Exp}_r \left(\int_0^s ; \alpha_r^{(m+1)}(A_{m,n})\, dr \right) \alpha_s^{(m+1)}(A_{m,n})\, ds\|_2 \\
&\leq \int_0^t \|A_{m,n}\|_2\, ds \\
&= \left(\sum_{j=m+2}^{n+1} \lambda_j^2 \right)^{1/2} t/2 < \varepsilon t.
\end{aligned}$$

Thus v_t^n converges to an α-unitary cocycle v_t. Letting $n \to \infty$ in

$$\mathrm{Ad}(v_t^{(n)})\alpha_t(a(g)) = \alpha_t^{(n)}(a(g)) = a(\exp it(H + \sum_{j=1}^{n} \lambda_j E_j)g),$$

we get $\mathrm{Ad}(v_t)\alpha_t = \beta_t$.

Because K has a pure point spectrum, we may assume $K = \sum_{n=1}^{\infty} \mu_n F_n$, where μ_n's are real numbers, F_n's are mutually orthogonal rank-one projections, and $\sum_{n=1}^{\infty} F_n = I_{\mathcal{H}}$. Thus β_t is of the form $\bigotimes_{n=1}^{\infty} \mathrm{Ad} \left(\exp it \begin{pmatrix} \mu_n/2 & 0 \\ 0 & -\mu_n/2 \end{pmatrix} \right)$.

Q.E.D.

Thus we can apply the result in §1 to quasi-free actions. In the following, we consider the above α, and β which arises from $K = \sum_{n=1}^{\infty} \mu_n F_n$. Define a groupoid whose unit is $X = \prod_{j=1}^{\infty} \{0, 1\}$ as in the proof of Lemma 2.1. We also define an $\mathbf{R}$-valued 1-cocycle c by $c(x, y) = \sum_{j=1}^{\infty} (x_j - y_j)\mu_j$ for $x \sim y \in X$. The obtained one-parameter automorphism group as in §1 is β, hence we get the following by Corollary 1.9 and Proposition 1.2.

Corollary 4.2. *A one-parameter automorphism group α on $\mathcal{R}$ arising from the CAR C^*-algebra in the above way is unique up to cocycle conjugacy if* $\Gamma(\alpha) = \mathbf{R}$.

Proposition 4.3. *In the above contest, the Connes spectra* $\Gamma(\alpha)$ *and* $\Gamma(\beta)$ *are equal to the asymptotic range* $r^*(c)$.

We can also show the trivial relative commutant property (Proposition 3.2) for this type of α if $\Gamma(\alpha) = \mathbf{R}$.

For the computation of examples, we prove the following.

Proposition 4.4. *For a one-parameter automorphism group α on $\mathcal{R}$ arising from one-parameter unitary group e^{iHt} in the above way, the essential spectrum $\sigma_e(H)$ is contained in $\Gamma(\alpha)$.*

Proof. By von Neumann's theorem again, we get $H + V = K$ as above. Because $\sigma_e(H) = \sigma_e(K)$, we consider $K = \sum_{n=1}^{\infty} \mu_n P_n$ and β instead of H and α.

Then we can prove the statement exactly as in the proof of Lemma 2.1.

Q.E.D.

In particular, if the spectrum of H contains a continuous part, it implies $\Gamma(\alpha) = \mathbf{R}$ for α coming from e^{iHt}, and this type of α is unique up to cocycle conjugacy.

Example 4.5. Let $\mathcal{H} = L^2(\mathbf{R})$, and define a one-parameter unitary group U_t by $U_t f(x) = f(x - t)$ for $f \in L^2(\mathbf{R})$. Then we can define a one-parameter automorphism group α on $\mathcal{R}$. (See pp. 4–5 in Powers-Robinson [20]. They construct a similar *endo*morphism *semi*group for the CAR C^*-algebra.) In this context, above H is $i\dfrac{d}{dx}$, which has spectrum $\mathbf{R}$. Thus by Proposition 4.4, we know $\Gamma(\alpha) = \mathbf{R}$.

Take another $\mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are infinite dimensional separable Hilbert spaces. We consider $H = \lambda_1 I_{\mathcal{H}_1} \oplus \lambda_2 I_{\mathcal{H}_2}$, where λ_1 and λ_2 are non-zero real numbers, and λ_1/λ_2 is irrational. Then we can consider one-parameter unitary group $e^{iH't}$, and it induces a one-parameter automorphism group α' of $\mathcal{R}$ as above. Because $\sigma_e(H') = \{\lambda_1, \lambda_2\}$, λ_1/λ_2 is irrational, and $\Gamma(\alpha)$ is a closed subgroup of $\mathbf{R}$, we know $\Gamma(\alpha') = \mathbf{R}$ again by Proposition 4.4. Note that this α' is of the form

$$\bigotimes_{j=1}^{\infty} \mathrm{Ad}\left(\exp it \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}\right) \otimes \bigotimes_{k=1}^{\infty} \mathrm{Ad}\left(\exp it \begin{pmatrix} \lambda_2 & 0 \\ 0 & 0 \end{pmatrix}\right).$$

Then by Corollary 4.2, we know that α and α' are cocycle conjugate. We also know that these are cocycle conjugate to the one-parameter automorphism groups in Examples 3.3 and 3.5.

Remark 4.6. The inclusion in Proposition 4.4 is not the best possible. Actually by the very similar arguments to Lemma 4.2, Lemma 4.3, Lemma 5.4 in Araki-Woods [2], and Théorème 1 in Connes [3] with Corollaries 4.2 and 4.3 here, we get the following characterization. (This corresponds to writing the asymptotic range in the original formulation of the asymptotic ratio set of Araki-Woods.)

A real number λ is in $\Gamma(\alpha)$ if and only if the following conditions are satisfied.

(1) $\{I_n\}_{n\in\mathbf{N}}$ *is a family of mutually disjoint finite subsets of* $\mathbf{N}$,

(2) K_n^1, K_n^2 *are mutually disjoint subsets of* $\{\sum_{j\in F}\mu_j \mid F \subseteq I_n\}$,

(3) ψ_n *is a bijective map from* K_n^1 *onto* K_n^2,

(4) $\sum_{n=1}^{\infty} |K_n^1|/2^{|I_n|} = \infty$,

(5) $\lim_{n\to\infty} \max_{\rho\in K_n^1} |\lambda - (\psi_n(\rho) - \rho)| = 0$.

In (2), μ_j's are eigenvalues of K as above.

By this criterion, we can show that the following H has $\sigma_e(H) = \{0\}$, but the one-parameter automorphism group it induces has the Connes spectrum $\mathbf{R}$. (Hence it is cocycle conjugate to the actions in Example 4.5.)

Let $\{a_n\}_{n\in\mathbf{N}}$ be the following sequence.

$$1, 1, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{8 \text{ times}}, \ldots, \underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n2^n \text{ times}}, \ldots$$

Define $a_{n,m} = \dfrac{1}{m} a_n$. We define $H = \sum_{n,m} a_{n,m} E_{n,m}$, where $E_{n,m}$'s are mutually orthogonal rank-one projections in $\mathcal{H}$. Then H is a compact operator, and $\sigma_e(H) = \{0\}$, but by the above criterion, we can show $1/m$ is in $\Gamma(\alpha)$ for every positive integer m. Thus we have $\Gamma(\alpha) = \mathbf{R}$.

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