Exotic TQFT in the theory of operator algebras

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1 Introduction

The relations between operator algebras and low-dimensional topology originate from the celebrated Jones polynomial of links [J], and since A. Ocneanu has invented *paragroup theory* [O1], topological quantum field theories (TQFT) arising from subfactors —certain pairs of von Neumann algebras— have been interesting topics. We survey a relation between 3-dimensional topological quantum field theory and subfactor theory and present some interesting examples due to the first author and U. Haagerup in [AH] which do not seem to arise from quantum or classical groups.

2 The Turaev-Viro type TQFT's

Several mathematically rigorous formulations of (3-dimensional) topological quantum field theory have been worked out by many authors. From the operator algebraic viewpoint, the most important methods are the one by Reshetikhin-Turaev [RT] and the one by Turaev-Viro [TV]. In this section, we explain the Turaev-Viro type TQFT first. (See [EK, Chapter 12] for details.) In the case of compact oriented manifolds of dimension three without boundary, the Turaev-Viro TQFT gives a complex number as a topological invariant of the manifold, using quantum $6j$-symbols. The $6j$-symbols are defined as an assignment of a complex number to each tetrahedron labelled by elements of a certain finite set as in Fig. 1. Then we triangulate a given 3-manifold, label each tetrahedron in all the possible ways, multiply all the values of the $6j$-symbols over the tetrahedra, and sum the the products over all the possible ways of labelling. Such a sum is called a *state sum*. In order to get the topological invariance, we need to know how different triangulations of a same manifold are related. Such a relation has been classically known and two triangulations give a same manifold if and only if they are transformed to each other by a finite number of *Alexander moves*. In order for a state sum to be invariant under the Alexander moves, we have to require some axioms for $6j$-symbols. The axioms are called *unitarity*, *tetrahedral symmetry*, and the *pentagon identity*. If a finite set of data for $6j$-symbols satisfying these axioms, we can construct a TQFT with this method of Turaev-Viro. We now explain how to get such data from the theory of operator algebras. (Actually, one can come back to operator algebras from $6j$-symbols, but we do not explain it here.)

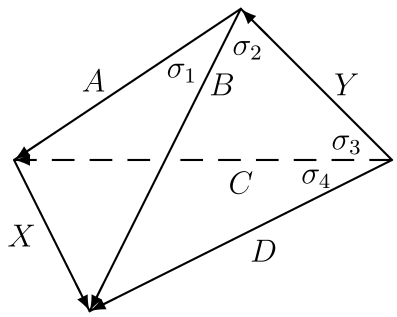

Figure 1: A labelled tetrahedron

We start with a hyperfinite II_1 subfactor $N \subset M$ of finite index and finite depth. Here, a "factor" means a "simple" von Neumann algebra which is a $*$-algebra of bounded linear operators on a Hilbert space closed in a certain topology. The adjective "hyperfinite" means that the operator algebra is represented as a closure of a union of finite dimensional algebras, and the term "II_1" specifies a certain category in the classification theory of Murray and von Neumann. Actually, it means that the von Neumann algebra is infinite dimensional and it has a unique trace with $\mathrm{tr}(1) = 1$. Our setting means that a von Neumann algebra M and its subalgebra N are both II_1 factors. We will explain the other two finiteness conditions below.

Using the trace on M, we define the inner product $\langle \cdot, \cdot \rangle$ on M by $\langle x, y \rangle = \mathrm{tr}(y^* x)$ for $x, y \in M$. By completing M with respect to the inner product, we get a Hilbert space. We denote this Hilbert space by $L^2(M)$. Since N is in M, we have a natural left action of N on M by usual multiplication of the algebra. We extend the action to $L^2(M)$, and then we have a left N-module ${}_N L^2(M)$. We also have the right action of M on $L^2(M)$ similarly, and then we have an N-M bimodule ${}_N L^2(M)_M$. For simplicity, we write this bimodule ${}_N M_M$. In a similar way, we have bimodules ${}_N N_N$, ${}_M M_M$, and ${}_M M_N$. We say a bimodule ${}_N X_M$ is *irreducible* when its bimodule endomorphism space $\mathrm{End}({}_N X_M)$ is trivial. The bimodule ${}_N M_M$ is irreducible if and only if $N' \cap M = \mathbf{C}$.

Next we consider *fusion rules*. In general, suppose that we have three von Neumann algebras A, B, C and two bimodules ${}_A X_B$, ${}_B Y_C$. Then there is a method to define a *relative tensor product* ${}_A X \otimes_B Y_C$. This is an A-C bimodule and we have some technical subtlety in the definition, but we omit details. Suppose X, Y are irreducible. We can make an irreducible decomposition of $X \otimes Y$, if we have a certain finiteness condition, called the finite index condition. This decomposition rule of the relative tensor product is called a fusion rule.

We start with a bimodule ${}_N M_M$. The finite index condition for $N \subset M$ is assumed and it is used to make irreducible decomposition as mentioned above. We make

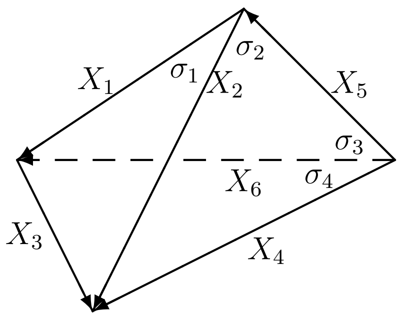

Figure 2: A labelled tetrahedron

finite tensor powers $\cdots {}_N M \otimes_M M \otimes_N M \otimes_M M \otimes_N M_M \cdots$ and make irreducible decompositions. We consider all the irreducible bimodules arising in this way. Note that we have four kinds of bimodules; N-N, N-M, M-N, M-M. In general, we have infinitely many bimodules in this way, but we say that the subfactor $N \subset M$ has a *finite depth* if we get only finitely many irreducible bimodules in this way.

Under the finite index and finite depth assumption, we concentrate on one kind of bimodules. Let

$$\mathcal{X} = \{X = {}_N X_N \,|N\text{-}N \text{ bimodules arising as above}\}$$

It is easily shown that a relative tensor product of two bimodules in $\mathcal{X}$ decomposes into a sum of bimodules in $\mathcal{X}$. We can also define a *dual* or a *contragredient* bimodule $\overline{X}$ for X, and for $X \in \mathcal{X}$, we have $\overline{X} \in \mathcal{X}$.

We use this set $\mathcal{X}$ as labels for edges of a tetrahedron. Next, we set

$$\mathcal{H}^Z_{X,Y} = \mathrm{Hom}(X \otimes Y, Z)$$

and fix an orthonormal system $\mathcal{S}^Z_{X,Y}$, where we have dropped the subscripts N for simplicity. We use this set for labels of the faces of tetrahedron. Now we define $6j$-symbol using $\mathcal{X}$ and $\{\mathcal{S}^Z_{X,Y}\}_{X,Y,Z\in\mathcal{X}}$ as follows. We label a tetrahedron as in Fig. 2, where X_i's are bimodules in $\mathcal{X}$ and $\sigma_1 \in \mathcal{S}^{X_3}_{X_1,X_2}$, $\sigma_2 \in \mathcal{S}^{X_4}_{X_5,X_2}$, $\sigma_3 \in \mathcal{S}^{X_5}_{X_6,X_1}$, and $\sigma_4 \in \mathcal{S}^{X_4}_{X_6,X_3}$. We then assign a complex number $\sigma_2 \cdot (\sigma_3 \otimes \mathrm{id}_{X_2}) \cdot (\mathrm{id}_{X_6} \otimes \sigma_1)^* \cdot \sigma_4^* \in \mathrm{End}(X_4) = \mathbf{C}$ to this labelled tetrahedron. By using this $6j$-symbol, we have a TQFT_3 constructed from a subfactor.

It is easy to construct two different subfactors giving the same TQFT of Turaev-Viro type. So there is a natural question to determine when two subfactors give the same TQFT. A sufficient condition called *equivalence of subfactors* is known and it is conjectured that this is also necessary.

3 Exotic TQFT's

Now, we have $6j$-symbols and the corresponding TQFT constructed from each subfactor. We are certainly interested in finding new kinds of TQFT's in this way. Many subfactors have been constructed from quantum groups or rational conformal field theory (see [EK, Chapters 12, 13]), but TQFT's for such subfactors are already known in topology and there is no advantage in using subfactors for topological studies on them. (The orbifold construction and conformal inclusions may be useful to some extent, but still, they do not give an entirely new TQFT.) So we have a problem whether we have a subfactor not arising from quantum groups, classical groups, or rational conformal field theory. (Here by a "subfactor", we mean a subfactor of the special class specified in the previous section.)

Until recently, no such examples of subfactors were known. U. Haagerup then gave a list of several candidates for subfactors which do not come from quantum groups, etc. in 1993. We have used the word "candidates" because they seemed to exist, but he could prove existence rigorously only for one particular example in the list. In 1998, the first author and Haagerup gave a proof of existence of another one in the list [AH]. These two are the only examples known today which do not seem to come from quantum groups, etc. So they are called "exotic" subfactors. (The exact meaning of "do not seem to come from quantum groups, etc." is as follows. There have been several known constructions of subfactors from quantum or classical groups, or rational conformal field theory. None of these do not give these two subfactors and these subfactors look "remote" from these constructions.) We have an analogue of the quantum dimension for bimodules and those for these two subfactors are algebraic numbers involving $\sqrt{13}$ and $\sqrt{17}$. We expect that the corresponding TQFT's are also "exotic", but no concrete results have been obtained about them. Theoretically, there is a possibility that these TQFT's coincide with TQFT's arising from some quantum group, though we feel it is rather unlikely. We can also construct a complex number valued invariant for links from these "exotic" subfactors. (See the next section.) So these examples might give some useful information about a famous problem whether all the Vassiliev invariants come from quantum groups or not.

4 The Reshetikhin-Turaev type TQFT's

Another famous construction of TQFT is due to Reshetikhin-Turaev [RT]. It can be regarded as a rigorous realization of a physical idea of Witten. It is possible to construct this type of TQFT from a subfactor and we explain this construction. See [O2] for more details.

The Reshetikhin-Turaev construction is based on surgery of links. They start with a link invariant of special type and prove invariance under Kirby moves. For this construction, we need a braiding, which is a special type of commutativity of the relative tensor product from a viewpoint of subfactor theory. In general, the relative tensor product of the N-N bimodules arising from $N \subset M$ as above is not commutative at all. So it seems that we cannot make a Reshetikhin-Turaev type construction from a given subfactor. But there is a method analogous to the quantum double construction of Drinfel′d. That is, we start with a subfactor $N \subset M$ (of a special type as above) and construct a new subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$ called the *asymptotic inclusion*. (We do not explain details here. See [EK, Chapter 12].) Then use this new subfactor instead of the original one $N \subset M$. One can show that all the axioms for the Reshetikhin-Turaev construction are satisfied and we get a TQFT in this way.

A natural problem is to find a relation between the Turaev-Viro type TQFT arising from a subfactor $N \subset M$ and the Reshetikhin-Turaev type TQFT arising from the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$. Ocneanu claims that they are equal, but no proof has been published on this equality.

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