Tempered representations and limit algebras

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Tempered representations and limit algebras

1. Solvable " limit algebras ".
2. Tempered representations .
3. (1) vs (2) for algebraic group $/\mathbb{C}$.
4. Approach from dynamical system for (2).
5. Coadjoint geometry vs (2).
6. Further theorems

Joint with Y. Benoist

Tempered representations and limit algebras — Introduction

G : connected Lie group with Lie algebra $\mathfrak{g}$,

H : connected closed subgroup with Lie algebra $\mathfrak{h}$.

Plan to discuss conditions on (G, H) from 4 different disciplines:

(i) (unitary rep) $G \curvearrowright L^2(G/H)$ is tempered.

(ii) (combinatorics) $2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}}$.

(iii) (orbit method) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^{*}_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.

(iv) (limit algebra) $\overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_{\infty}$ solvable.

Analysis (i)

Algebra (ii)

Geometry (iii)

Topology (iv)

Plan

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Limit algebras (1) — Example

Consider two equi-dimensional subalgebras of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$:

$$\mathfrak{k} = \mathfrak{so}(n), \qquad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$$

<u>Observation</u> $\exists$ sequence $g_j \in SL(n, \mathbb{R})$ such that $\lim_{j \to \infty} \mathrm{Ad}(g_j)\, \mathfrak{k} = \mathfrak{n}$

Proof $(n = 2)$

Take $g_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{-j} \end{pmatrix}$. Then

$$\mathrm{Ad}(g_j) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2^{2j} \\ 2^{-2j} & 0 \end{pmatrix}.$$

<u>Remark</u> $\nexists$ sequence $g_j \in SL(n, \mathbb{R})$ such that $\lim_{j \to \infty} \mathrm{Ad}(g_j)\, \mathfrak{n} = \mathfrak{k}$

Limit algebras (2) — Definition

$\mathfrak{g}$: Lie algebra.
$\mathfrak{h}$: k-dimensional subalgebra of $\mathfrak{g}$.

We regard $\mathfrak{h}$ as a point of the Grassmann variety $Gr_k(\mathfrak{g})$.

$$Gr(\mathfrak{g}) := \coprod_{k=0}^{\dim \mathfrak{g}} \mathrm{Gr}_k(\mathfrak{g}).$$

$Gr(\mathfrak{g}) \underset{\text{submanifold}}{\supset} \mathrm{Ad}(G)\mathfrak{h}$, which may or may not be closed.

$Gr(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty$ (limit algebra)

<u>Definition</u> (limit algebra) $\mathfrak{h}_\infty$ ($\subset \mathfrak{g}$) is a limit algebra of $\mathfrak{h}$ if $^\exists$ sequence $g_j \in G$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h} = \mathfrak{h}_\infty$ in $Gr(\mathfrak{g})$.

Limit algebras (3) — Properties

$\mathfrak{g} \supset \mathfrak{h}$ subalgebra $\rightsquigarrow$ $Gr(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty$ (limit algebra)

Remark $\mathfrak{h}$ itself is a limit algebra of $\mathfrak{h}$.

Basic properties

1) Any limit algebra $\mathfrak{h}_\infty$ is an equi-dimensional Lie algebra.

2) If $\mathfrak{h}$ is $\begin{cases} \text{abelian} \\ \text{nilpotent} \\ \text{solvable} \end{cases}$ then any limit algebra $\mathfrak{h}_\infty$ is also $\begin{cases} \text{abelian} \\ \text{nilpotent} \\ \text{solvable} \end{cases}$.

"Semisimple" may collapse to "solvable", but not vice versa.

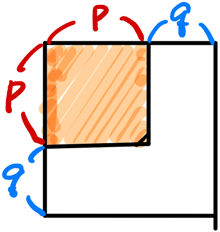
Question What can we say about $\mathfrak{h}$ if $\overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}$ solvable $\mathfrak{h}_\infty$?

Limit algebras (4) — Example

$\mathfrak{g} \supset \mathfrak{h}$ subalgebra $\rightsquigarrow$ $Gr(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty$ (limit algebra)

Remark $\mathfrak{h}_\infty$ is determined by how $\mathfrak{h}$ is embedded in $\mathfrak{g}$.

Exercise Fix p, and consider $\mathfrak{h} = \mathfrak{sl}_p \hookrightarrow \mathfrak{g} = \mathfrak{sl}_{p+q}$
Find a necessary and sufficient condition on q such that $\overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}$ solvable $\mathfrak{h}_\infty$.



$\Updownarrow$ (see later)

$q \geq p + 1$

Variety of all Lie algebras $\mathcal{L}$ and its subset $\mathcal{S}$

$\mathfrak{g}$: Lie algebra.

Formulation: Consider the variety of all subalgebras in $\mathfrak{g}$.

$\mathcal{L}$:= {subalgebras of $\mathfrak{g}$}

$\mathcal{S}$:= $\{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty$ solvable $\}$.

$$\{\text{solvable subalgs}\} \subset \mathcal{S} \subset \mathcal{L} \underset{\text{closed}}{\subset} Gr(\mathfrak{g}) \equiv \coprod_{N=0}^{\dim \mathfrak{g}} \mathrm{Gr}_N(\mathfrak{g}).$$

> Question (Topology of $\mathcal{S}$) Let $\mathfrak{g}$ be a Lie algebra.
>
> (1) Is $\mathcal{S}$ closed in $\mathcal{L}$?
>
> (2) Is $\mathcal{S}$ open in $\mathcal{L}$?

Topology of $\mathcal{S} = \{\mathfrak{h} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty$ solvable$\}$

<u>Theorem 1</u> Suppose $\mathfrak{g}$ is an algebraic Lie algebra $/\mathbb{C}$.

(1) $\mathcal{S}$ is closed in $\mathcal{L}$.

(2) $\mathcal{S}$ is open and closed in $\mathcal{L}$ if $\mathfrak{g}$ is semisimple.

Recall

$\mathcal{L}$:= {subalgebras of $\mathfrak{g}$}

$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty$ solvable $\}$.

$$\{\text{solvable subalgs}\} \subset \mathcal{S} \subset \mathcal{L} \underset{\text{closed}}{\subset} Gr(\mathfrak{g}) \equiv \coprod_{N=0}^{\dim \mathfrak{g}} \mathrm{Gr}_N(\mathfrak{g}).$$

Our proof for Theorem 1 uses unitary representation theory.

Plan

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Tempered representation — Definition

Let G be a locally compact group.

Def A unitary rep π of G is called tempered if $\pi \ll L^2(G)$.

$\ll$ $\cdots$ weakly contained

i.e., every matrix coefficient of π is a uniform limit on every compacta of G of a sequence of sum of coefficients of $L^2(G)$.

- Any unitary rep π can be disintegrated (Mautner) (*e.g.*, branching law, Plancherel thm).

$$\pi \simeq \int^{\oplus} \sigma \quad \text{with } \sigma \text{ irreducible}$$

π is tempered $\iff$ σ is tempered, almost everywhere

When is $L^2(X)$ tempered?

$G \curvearrowright L^2(X)$: L^2-sections for the half-density bundle on a G-space X

$X = G/H$ with H connected closed subgroup

Question When is the unitary rep on $L^2(X)$ tempered?

Examples

1. H compact $\Rightarrow L^2(G/H)$ is tempered.
2. H amenable $\Rightarrow L^2(G/H)$ is tempered.

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What can we say on $L^2(G/H)$ when $\overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^\exists\mathfrak{h}_\infty$ solvable ?

G : complex algebraic group $\supset$ G_{ss} : max semisimple subgroup

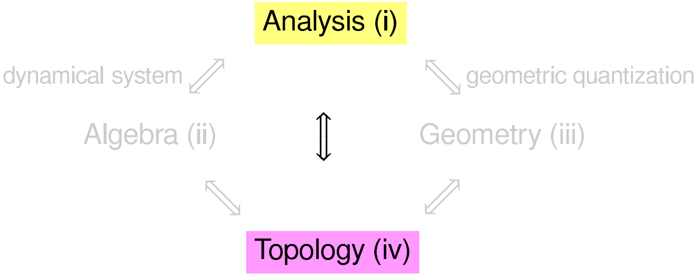
H : algebraic subgroup of G

<u>Theorem 2</u> The following conditions on (G, H) are equivalent:

(i) $\overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^\exists\mathfrak{h}_\infty$ solvable.

(ii) $L^2(G/H)$ is tempered as a G_{ss}-module.

For the proof, we begin with tempered reps of <u>real</u> Lie group G.



Irreducible tempered reps — semisimple Lie groups

Recall

Def A unitary representation π is called **tempered** if $\pi \ll L^2(G)$.

- For a solvable Lie group, all π are **tempered**.
- For a semisimple Lie group G and irreducible π, tempered representations π have been studied extensively.

Known results on **tempered reps** and beyond ...

- Many equivalent definitions, *e.g.*, $L^{2+\varepsilon}(G)$,
- Harish-Chandra's theory towards Plancherel formula,
- Knapp–Zuckerman's classification (1982),
- Building blocks of Langlands classification,
- Selberg $\frac{1}{4}$ conjecture,
- Margulis work for discontinuous groups Γ for G/H,
- Gan–Gross–Prasad conjecture, $\cdots$

Tempered representations — Examples (irreducible cases)

V. Bargmann (1947): Irreducible unitary reps of $SL_2(\mathbb{R})$

$= \{ \mathbf{1} \} \amalg \{$ principal series $\} \amalg \{$ complementary series $\}$

$\amalg\{$ discrete series $\} \amalg \{$ limit of discrete series $\}$

$-\frac{1}{2}$ Casimir operator acts on them as scalars

$\{0\}$, $[\frac{1}{4}, \infty)$, $(0, \frac{1}{4})$

$\{\frac{1}{4}(n^2 - 1) : n \in \mathbb{N}_+\}$, $\{0\}$

Γ: congruence subgroup of $G = SL(2, \mathbb{R})$

Selberg's $\frac{1}{4}$ eigenvalue conjecture:

Are all eigenvalues of Δ on Maas wave forms for $\Gamma \geq \frac{1}{4}$?

$\Leftrightarrow$ Is the unitary rep of $G \curvearrowright L^2_{\text{cusp}}(\Gamma \backslash G)$ is tempered ?

Just one irred non-tempered rep would deny the conjecture.

When is $L^2(X)$ tempered?

Question When is the unitary rep on $L^2(G/H)$ tempered?

Examples

1. H compact $\Rightarrow L^2(G/H)$ is tempered.
2. H amenable $\Rightarrow L^2(G/H)$ is tempered.
3. $L^2(SL_{p+q}(\mathbb{R})/SL_p(\mathbb{R}))$ is tempered $\iff q \geq p+1$ (later).
4. G/H a semisimple symmetric space (well-studied)?
5. Tensor product of two non-tempered reps?

We discuss when matrix coefficients for $L^2(X)$ belong to $L^{2+\varepsilon}(G)$.

cf. A classical result about $L^2(X)$ vs $L^{2+\varepsilon}(X)$
(Harish-Chandra, Oshima, Bernstein ~ 80s).

Example: G/H semisimple symmetric spaces

The known “Plancherel theorem” for symmetric spaces G/H may suggest:

$L^2(G/H)$ is tempered

$\overset{?}{\Leftrightarrow}$ the set of points in $\mathfrak{g}/\mathfrak{h}$ with amenable stabilizer in H is dense.

$\Rightarrow$ is true.

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$\Rightarrow$ is true, however, $\Leftarrow$ is false even when G/H is a symmetric space!

Just one irred non-tempered rep would deny the implication $\Leftarrow$.

Our criterion (to be explained soon) detects

Counterexample If $p_1 \geq 1$, $q_1 \geq 1$, $p_1 + q_1 = p_2 + q_2 + 1$,

then $Sp(p_1 + p_2, q_1 + q_2)/Sp(p_1, q_1) \times Sp(p_2, q_2)$ is not tempered.

Example: G/H semisimple symmetric spaces

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Just one irred non-tempered rep would deny the implication $\Leftarrow$.

A subtle point ("What is missing?"):

— It may happen that discrete series reps π_λ for G/H are tempered for generic λ, but are non-tempered for very singular λ if $\pi_\lambda \neq 0$.

— Langlands parameter of π_λ becomes unstable when λ is singular, after crossing many walls.

— Condition for which $\pi_\lambda \simeq A_{\mathfrak{q}}(\lambda - 2\rho) \neq 0$ is complicated if λ is very singular, (cf. K–, Memoirs of AMS 1992, Trapa 2001).

Change of approach: from PDE to dynamical system

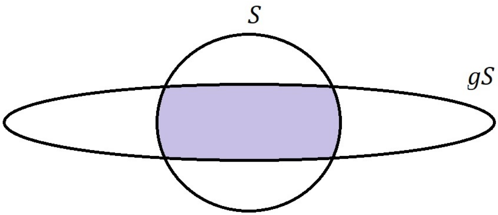
Locally compact group $G \curvearrowright X$ locally compact space

$G \curvearrowright X$ **proper** $\overset{\text{def}}{\Leftrightarrow} \{g \in G : S \cap gS \neq \emptyset\}$ is compact $^{\forall}S \subset X$ compact,

$\Leftrightarrow \mathrm{vol}(S \cap gS) \in C_c(G)$ $^{\forall}S \subset X$ compact.

Idea: Qualitative estimate of non-proper actions.

Look at asymptotic behavior of $\mathrm{vol}(S \cap gS)$ as g goes to infinity.



Piecewise linear function ρ_V

$\mathfrak{h}$: Lie algebra/$\mathbb{R}$

<u>Definition</u> For a finite-dim'l $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$,

$\rightsquigarrow$ $\rho_V : \mathfrak{h} \to \mathbb{R}_{\geq 0}, \quad Y \mapsto \frac{1}{2} \sum |\mathrm{Re}\, \lambda(Y)|.$

eigenvalues of $\tau(Y) \in \mathrm{End}(V)$

ρ_V is a piecewise linear function.

<u>Remark</u> For $\mathfrak{h}$ semisimple and for $(\tau, V) = (\mathrm{ad}, \mathfrak{h})$,
$\rho_{\mathfrak{h}}$ is twice the usual ρ on a dominant Weyl chamber.

Temperedness criterion — dynamical approach $+\alpha$

G : real algebraic Lie group $\supset G_{ss}$: max semisimple subgroup,
H : algebraic subgroup.

<u>Theorem 3</u> $L^2(G/H)$ is G_{ss}-tempered $\iff 2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ on $\mathfrak{h}$.

Condition on $2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}}$ — Example

Recall that for $\pi\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, ρ_V is a piecewise linear function:

Definition $\rho_V\colon \mathfrak{h} \to \mathbb{R}_{\ge 0}, \quad Y \mapsto \frac{1}{2}\sum |\mathrm{Re}\,\lambda(Y)|.$

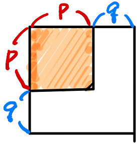
eigenvalues of $\pi(Y) \in \mathrm{End}(V)$

Example $\mathfrak{g} = \mathfrak{sl}(p+q,\mathbb{R}) \supset \mathfrak{h} = \mathfrak{sl}(p,\mathbb{R})$

$$2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}} \iff \sum_{1\le i<j\le p} |x_i - x_j| \le q\sum_{i=1}^{p} |x_i| \quad {}^{\forall}(x_i, \cdots, x_p) \text{ with } \sum_{j=i}^{p} x_j = 0$$

$$\iff p \le q+1$$

Theorem 3 tells $L^2(SL(p+q,\mathbb{R})/SL(p,\mathbb{R}))$ is tempered $\Leftrightarrow$ $p \le q+1$.



Sketch of proof (easier part) of Theorem 3

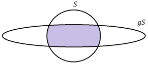
G : real algebraic $\supset H$: algebraic subgroup.

Theorem 3 $L^2(G/H)$ is G_{ss}-tempered $\Longleftrightarrow 2\rho_\mathfrak{h} \le \rho_\mathfrak{g}$ on $\mathfrak{h}$.

Sketch of the proof for the easier part $\Rightarrow$ in the reductive case:

For a small ball S near $o = eH \in G/H$ and $Y \in \mathfrak{h}$,

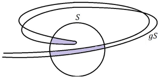
$$C_S e^{(\rho_\mathfrak{h}-\rho_\mathfrak{g})(Y)} \underset{\text{geometry}}{\le} \mathrm{vol}(e^Y S \cap S) \underset{\text{temperedness}}{\le} C'_S e^{-\rho_\mathfrak{g}(Y)}$$



$$\therefore \quad 2\rho_\mathfrak{h} \le \rho_\mathfrak{g} \quad \text{on } \mathfrak{h}.$$

The general case and the converse $\Leftarrow$ are much more involved.

- Global picture



Temperedness criterion — dynamical approach $+\alpha$

G : real algebraic Lie group $\supset$ G_{ss}: max semisimple subgroup,
H : algebraic subgroup.

Theorem 3 $L^2(G/H)$ is G_{ss}-tempered $\Longleftrightarrow 2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ on $\mathfrak{h}$.

References for Theorem 3
[BK] Tempered homogeneous spaces I,II,IV

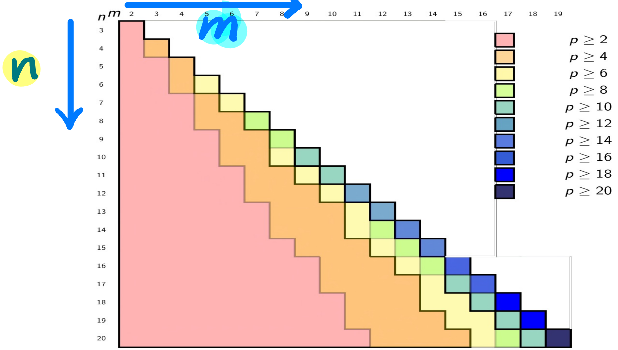
G reductive $\supset H$ reductive (I, J. Eur. Math., 2015)
G reductive $\supset H$ any (II, Margulis Festschrift, 2021)
G any $\supset H$ any (IV, preprint)

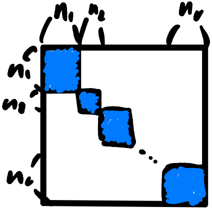
Another direction of generalization — Almost L^p representation

G: real reductive $\supset H$ real reductive, $p \in 2\mathbb{N}$.

Theorem 3′ $L^2(G/H)$ is almost $L^p \iff \frac{p}{p-1}\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}}$ on $\mathfrak{h}$.

Example (arXiv:2108.12125) $G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})$
The smallest even integer p for which $L^2(G/H)$ is almost L^p amounts to $p = 2[\frac{n-1}{2(n-m)}]$ with $m = \max(n_1, \cdots, n_r)$.





4 different disciplines — G complex reductive

Thm 4 Let $\mathfrak{g}$ be a complex reductive Lie algebra.
The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

(i) (unitary rep) $L^2(G/H)$ is tempered.

(ii) (combinatorics) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.

(iii) (orbit method) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.

(iv) (limit algebra) $\mathfrak{h} \in \mathcal{S}$, *i.e.*, $\overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_{\infty}$ solvable.

Analysis (i)

Algebra (ii)

Geometry (iii)

Topology (iv)

From orbit philosophy by Kirillov–Kostant

Explain (iii) (Geometric condition) $\mathfrak{h}^\perp \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$

$$\mathfrak{g}^* \supset \mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G)\lambda \text{ is of maximal dimension}\},$$
$$\mathfrak{g}^* \supset \mathfrak{h}^\perp := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}.$$

Orbit philosophy by Kirillov–Kostant

$$\begin{array}{ccc}
\mathrm{Supp}(L^2(G/H)) & \fallingdotseq & \mathrm{Ad}^*(G)\mathfrak{h}^\perp / \mathrm{Ad}^*(G) \\
\cap & & \cap \\
\widehat{G} & \fallingdotseq & \mathfrak{g}^* / \mathrm{Ad}^*(G) \\
\cup & & \cup \\
\widehat{G}_{\mathrm{temp}} & \fallingdotseq & \mathfrak{g}^*_{\mathrm{reg}} / \mathrm{Ad}^*(G)
\end{array}$$

<u>Remark</u> $\mathfrak{h}^\perp \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset \iff \mathfrak{h}^\perp \cap \mathfrak{g}^*_{\mathrm{reg}} \underset{\text{dense}}{\subset} \mathfrak{h}^\perp$

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The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

(i) (unitary rep) $L^2(G/H)$ is tempered.

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(iv) (limit algebra) $\overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_{\infty}$ solvable.

Analysis (i)

dynamical system $\Updownarrow$ geometric quantization

Algebra (ii) $\Longleftrightarrow$ Geometry (iii)

Topology (iv)

Proof of Theorem 1

Recall: Variety of all Lie algebras $\mathcal{L}$ and its subset $\mathcal{S}$

$\mathcal{L}$:= {subalgebras of $\mathfrak{g}$}

$\mathcal{S}$:= $\{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty$ solvable $\}$.

$$\{\text{solvable subalgs}\} \subset \mathcal{S} \subset \mathcal{L} \underset{\text{closed}}{\subset} Gr(\mathfrak{g}) \equiv \coprod_{N=0}^{\dim \mathfrak{g}} \mathrm{Gr}_N(\mathfrak{g}).$$

Recall

<u>Theorem 1</u> Suppose $\mathfrak{g}$ is an algebraic Lie algebra $/\mathbb{C}$.

(1) $\mathcal{S}$ is closed in $\mathcal{L}$.

(2) $\mathcal{S}$ is open and closed in $\mathcal{L}$ if $\mathfrak{g}$ is semisimple.

Theorem 4 says

(iv) $\mathcal{S} \ni \mathfrak{h} \Leftrightarrow$ (ii) $2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}} \Leftrightarrow$ (iii) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$.

Sketch of proof of Theorem 4 — complex case

Thm 4 Let $\mathfrak{g}$ be a complex reductive Lie algebra.
The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

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(iii) (orbit method) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.

(iv) (limit algebra) $\overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_{\infty}$ solvable.

Analysis (i)

dynamical system $\Updownarrow$ (Algebra–Analysis) $\Updownarrow$ geometric quantization (Analysis–Geometry)

Classification $\Leftarrow$ Algebra (ii) $\Longleftrightarrow$ Geometry (iii)

$\Updownarrow$ (Algebra–Topology) $\Updownarrow$ (Geometry–Topology)

Topology (iv)

Full classification of pairs of real reductive groups $G \supset H$ for which $L^2(G/H)$ is non-tempered, ([BKIII, 2021]).

More general setting — Real case

G : real algebraic Lie group $\supset H$: algebraic subgroup

Recall

(i) (unitary rep) $L^2(G/H)$ is G_{ss}-tempered.

(ii) (combinatorics) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.

(iii) (orbit method) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.

Theorem 3 tells (i) $\Leftrightarrow$ (ii) in this general setting.

Corollary If G is real reductive, then (iii) $\Rightarrow$ (i) & (ii).

Analysis (i)

dynamical system $\Updownarrow$ (Algebra (ii) $\Leftrightarrow$ Analysis (i))

$\Nwarrow$ G reductive $/\mathbb{R}$ (Geometry (iii) $\Rightarrow$ Analysis (i))

Algebra (ii) $\Longleftarrow$ Geometry (iii)

G reductive $/\mathbb{R}$

Remark $G/H = SL(3,\mathbb{H})/SL(2,\mathbb{H})$ satisfies (i) and (ii) but not (iii).

Thank you very much!

References

This work is joint with Yves Benoist.

For more details of the talk today, please see

Tempered Homogeneous Spaces IV (arXiv:2009.10391)

Related references:

——— I. (J. Euro Math., 2015)

Method(Dynamical System)

——— II. (Margulis Festschrift, 2021)

Representation Theory

——— III. (J. Lie Theory, 2021)

Classification Theory (Combinatorics)

cf. $\otimes$ product of GL_n (arXiv:2108.12125) (B-Inoue-K)