Asymptotic behaviour of the null variety for a convex domain in a non-positively curved space form

By Toshiyuki KOBAYASHI

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Asymptotic behaviour of the null variety for a convex domain in a non-positively curved space form

Dedicated to Professor Hiroshi Fujita on his sixtieth birthday

By Toshiyuki KOBAYASHI*)

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Chapter 1. Introduction

For a measurable bounded set Ω in $\boldsymbol{R}^n$, the Fourier transform of the characteristic function of Ω is given by,

$$\tilde{\chi}_\Omega(\zeta)=\int_\Omega \exp\left(\sqrt{-1}\sum_{j=1}^{n} x_j\zeta_j\right)dx_1\cdots dx_n,$$

which is an entire function of $\zeta=(\zeta_1, \cdots, \zeta_n)\in \boldsymbol{C}^n$. Then we associate an analytic set (*null variety*) in $\boldsymbol{C}^n$ (or $\boldsymbol{R}^n$) defined by,

$$\mathcal{N}(\Omega):=\{\zeta\in \boldsymbol{C}^n;\ \tilde{\chi}_\Omega(\zeta)=0\},$$

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or

$$\mathcal{N}(\Omega)_R := \mathcal{N}(\Omega) \cap \boldsymbol{R}^n = \{\zeta \in \boldsymbol{R}^n;\ \tilde{\chi}_\Omega(\zeta) = 0\}.$$

In this paper, we concern ourselves to the assignment:

$$\mathcal{N}: \{\text{Bounded sets}\} \ni \Omega \longmapsto \mathcal{N}(\Omega) \in \{\text{Analytic sets}\}.$$

Since the strength of the differentiability assumption will not be the issue, we suppose the boundary $\partial\Omega$ to be of class C^∞.

Our goal is to initiate a new line of investigation by posing the following problems:

Problem A. **How is $\mathcal{N}(\Omega)$?**
Describe $\mathcal{N}(\Omega)$ in terms of the geometrical invariants of Ω.

Problem B. **Does $\mathcal{N}(\Omega)$ determine Ω?**
Is the assignment $\mathcal{N}: \Omega \longmapsto \mathcal{N}(\Omega)$ one to one?

Or more strongly, we ask, "Does a *suitable* subset of $\mathcal{N}(\Omega)$ determine Ω?" Here, a 'suitable subset of $\mathcal{N}(\Omega)$' would mean 'some connected components of $\mathcal{N}(\Omega)$', 'a set of first zero points of $\tilde{\chi}_\Omega(\zeta)$' or 'the intersection of $\mathcal{N}(\Omega)$ and some fixed submanifold in $\boldsymbol{C}^n$', etc. and such definitions might require certain restrictions on Ω.

Since $\mathcal{N}(\Omega)$ is invariant under parallel displacements of Ω, the injectivity of $\mathcal{N}$ should be considered in the sense of 'up to parallel displacements'. A positive answer to the Problem B would enable us to translate the properties of Ω into those of $\mathcal{N}(\Omega)$ in principle. Thus,

Problem C. **Relate Ω with $\mathcal{N}(\Omega)$.**
What can you tell about Ω when $\mathcal{N}(\Omega)$ has some special properties?

Let us begin with typical examples.

Example (1.1) (Example (2.3.9)). Let Ω be a unit ball in $\boldsymbol{R}^n$. Then, we have

$$\tilde{\chi}_\Omega(\zeta) = (2\pi)^{n/2} \frac{J_{n/2}(t)}{t^{n/2}},$$

where $t := (\zeta_1^2 + \cdots + \zeta_n^2)^{1/2}$ for $\zeta = (\zeta_1, \cdots, \zeta_n) \in \boldsymbol{C}^n$, and $J_\nu(t)$ denotes the ν-th Bessel function. Let j_m $(m = 1, 2, \cdots)$ be the enumeration of the positive zeros of $J_{n/2}(t)$. Then,

$$\mathcal{N}(\Omega)=\coprod_{m=1}^{\infty}\left\{\zeta\in \boldsymbol{C}^n;\ \sum_{k=1}^{n}\zeta_k^2=j_m^2\right\}\qquad \text{(disjoint union)}.$$

So $\mathcal{N}(\Omega)_R\equiv\mathcal{N}(\Omega)\cap \boldsymbol{R}^n$ consists of countably many concentric hyperspheres.

Example (1.2). Let Ω be a cubic domain $\{x\in \boldsymbol{R}^n;\ |x_j|<1,\ (1\leq j\leq n)\}$. Then, $\tilde{\chi}_\Omega(\zeta)=\prod_{k=1}^{n}\frac{2\sin\zeta_k}{\zeta_k}$, for $\zeta=(\zeta_1,\cdots,\zeta_n)\in \boldsymbol{C}^n$. Hence $\mathcal{N}(\Omega)=\bigcup_{k=1}^{n}\bigcup_{m\in \boldsymbol{Z}\setminus\{0\}}\{\zeta\in \boldsymbol{C}^n;\ \zeta_k=\pi m\}$.

A clear distinction in these two example: the null variety $\mathcal{N}(\Omega)_R$ for a ball consists of infinitely many compact components, whereas the one for a cubic domain is connected and noncompact. The former case will be generalized to *strictly* convex domains in Euclidean space and hyperbolic space.

As for the Problem B on the injectivity of $\mathcal{N}$, we must pose a certain condition (connectedness, etc...) on Ω. In fact, we have the following example:

Example (1.3). Fix two positive numbers A and B such that $A\geq 5B$. Set $K:=\left[\frac{A-B}{2B}\right]$ (Gaussian integer). For each integer $j\in[1,K]$, put $a_j:=\frac{2j+2}{2j+1}A+B$, $b_j:=\frac{2j+2}{2j+1}A-B$, $c_j:=\frac{2j}{2j+1}A+B$, and $d_j:=\frac{2j}{2j+1}A-B$. Then $0<d_j<c_j<b_j<a_j$, $(1\leq j\leq K)$. Set $\Omega_j:=(-a_j,-b_j)\cup(-c_j,-d_j)\cup(d_j,c_j)\cup(b_j,a_j)\subset \boldsymbol{R}$. Then $\tilde{\chi}_{\Omega_j}(\zeta)=\frac{8}{\zeta}\sin(B\zeta)\cos(A\zeta)\cos\left(\frac{A\zeta}{2j+1}\right)$, and therefore $\mathcal{N}(\Omega_1)=\cdots=\mathcal{N}(\Omega_K)$. A Cartesian product of such Ω's yields the domains in $\boldsymbol{R}^n$ having the same null variety.

Thus we pose

Problem (B.1). Is the assignment

$\mathcal{N}$: {connected bounded sets in $\boldsymbol{R}^n$} $\ni\Omega\longmapsto\mathcal{N}(\Omega)\in$ {analytic sets in $\boldsymbol{C}^n$}

one to one up to parallel displacements?

When we restrict our interest on strictly convex domains, this problem contains the following two problems:

Problem (B.2). Is the assignment

$\mathcal{N}$: {strictly convex domains in $\boldsymbol{R}^n$} $\ni \Omega \longmapsto \mathcal{N}(\Omega) \in$ {analytic sets in $\boldsymbol{C}^n$} one to one up to parallel displacements?

Problem (B.3). Suppose the null variety $\mathcal{N}(\Omega)$ for a bounded domain be the (asymptotically) same with that of a strictly convex domain. Is this domain Ω convex?

As for the Problem C, we know a typical example, so called Pompeiu problem (or Schiffer conjecture) (see [20]).

FACT (1.4) (cf. [5]). *The following three conditions on a bounded domain Ω with a connected smooth boundary in $\boldsymbol{R}^n$ are equivalent:*
a) *There exists a nonzero function $u \in C^2(\bar{\Omega})$ and $\lambda \in \boldsymbol{C}$ such that*

$$\Delta u = \lambda u \text{ in } \Omega; \ \frac{\partial u}{\partial n} = 0 \text{ and } u \equiv \text{constant on } \partial\Omega. \tag{1}$$

b) *There exists a nonzero continuous function $f \in C(\boldsymbol{R}^n)$ such that*

$$\int_\Omega f(\sigma \cdot x)dx = 0,$$

for any element σ of the Euclidean motion group.
c) *$\mathcal{N}(\Omega)$ contains an $O(n, \boldsymbol{C})$ invariant set.*

Clearly, these conditions are satisfied when Ω is a ball. But whether the converse is true or not is still open even when Ω is assumed to be strictly convex in $\boldsymbol{R}^2$.

If Ω is a centrally symmetric domain, $\tilde{\chi}_\Omega(\zeta)$ is a real valued function on $\boldsymbol{R}^n$. In this case, $\partial\Omega \subset \boldsymbol{R}^n$ and $\mathcal{N}(\Omega)_{\boldsymbol{R}} \subset \boldsymbol{R}^n$ are both codimension one. This fact confirms us that $\mathcal{N}(\Omega)_{\boldsymbol{R}}$ has enough data to determine Ω (ex. Remark (2.3.25)). But unless Ω is centrally symmetric, $\mathcal{N}(\Omega)_{\boldsymbol{R}}$ would have more codimension and less information about Ω. For instance:

Example (1.5). Let $\Omega := \{(x, y) \in \boldsymbol{R}^2;\ 0 < y < \min(1-x, 1+x)\}$ and let $\Omega' := \{(x, y) \in \boldsymbol{R}^2;\ 0 < x < \min(1-y, 1+y)\}$. Then $\tilde{\chi}_\Omega(\xi, \eta) = \dfrac{-2\{\xi(\cos\xi - \cos\eta) + \sqrt{-1}(\eta\sin\xi - \xi\sin\eta)\}}{\xi(\xi^2-\eta^2)}$, and $\mathcal{N}(\Omega)_{\boldsymbol{R}} = \mathcal{N}(\Omega')_{\boldsymbol{R}} = \mathcal{N}(-\Omega)_{\boldsymbol{R}} = \mathcal{N}(-\Omega')_{\boldsymbol{R}} = \{(m\pi, n\pi) \in \boldsymbol{R}^2;\ m, n \in \boldsymbol{Z},\ m-n \in 2\boldsymbol{Z}, \text{ and } m \neq \pm n\}$.

In fact, some characterization of central symmetry will be given in terms of the codimension of $\mathcal{N}(\Omega)_{\boldsymbol{R}}$ in Corollary (2.3.11).

For a general domain, we will treat $\mathcal{N}(\Omega) \cap S$ instead of $\mathcal{N}(\Omega)_{\boldsymbol{R}} =$

$\mathcal{N}(\Omega)\cap\boldsymbol{R}^n$, where $S:=\{(\zeta\omega_1,\cdots,\zeta\omega_n)\in\boldsymbol{C}^n;\zeta\in\boldsymbol{C},\ (\omega_1,\cdots,\omega_n)\in S^{n-1}\}$. Note that $\boldsymbol{R}^n\subset S\subset\boldsymbol{C}^n$, and $S\setminus\{0\}$ is an $n+1$-dimensional smooth manifold. Then generalizing Example (1.1), we describe the asymptotic behaviour of $\mathcal{N}(\Omega)\cap S$ in Theorem (2.3.6), as a partial answer to Problem A. Let us state it briefly:

THEOREM (1.6) (see Theorem (2.3.6) for details). *Suppose Ω be a strictly convex domain in $\boldsymbol{R}^n$. Then there is a nonnegative integer $m_0\equiv m_0(\Omega)$ such that*

$$\mathcal{N}(\Omega)\cap S=\Big(\coprod_{m\geq m_0}\mathcal{N}_m\Big)\amalg\ (\text{compact set}),\qquad (\textit{disjoint union}),\tag{2}$$

where each $\mathcal{N}_m$ is analytically diffeomorphic to S^{n-1}. More precisely, there are analytic functions $F_m: S^{n-1}\longrightarrow\boldsymbol{C}$ $(m\geq m_0)$ such that

$$\mathcal{N}_m=\{F_m(\omega)\cdot\omega\in\boldsymbol{C}^n;\ \omega\in S^{n-1}\}$$

and

$$F_m(\omega)=\frac{4m+n-1}{2H_\Omega(\omega)}+\sqrt{-1}d_\Omega(\omega)+O(m^{-1}),\qquad \textit{as } m\longrightarrow\infty.\tag{3}$$

Here H_Ω and d_Ω are smooth functions defined on S^{n-1} which is represented by the supporting functions and the curvature of $\partial\Omega$ explicitly (Definition (2.1.14), (2.1.16)).

On the other hand, if Ω is sufficiently near to a ball in a Sobolev norm of the boundary $\partial\Omega$ (Ω needs not convex), the connected components of $\mathcal{N}(\Omega)\cap S$ sufficiently near to the origin are analytically diffeomorphic to S^{n-1}. Now we propose the following conjecture:

CONJECTURE (1.7). *Suppose Ω be a strictly convex domain in Euclidean space. Then* (2) *in Theorem* (1.6) *can be replaced by*

$$\mathcal{N}(\Omega)\cap S=\coprod_{m\geq 1}\mathcal{N}_m,\qquad (\textit{disjoint union}),\tag{2$'$}$$

where each $\mathcal{N}_m$ is analytically diffeomorphic to S^{n-1}.

As an application of Theorem (1.6), we obtain the following results.

COROLLARY (1.8) (see Corollary (2.3.11)). *We can characterize the geometric properties* (*centrally symmetric, with constant breadth, or globular*) *of a strictly convex domain Ω in $\boldsymbol{R}^n$ in terms of the asymptotics of $\mathcal{N}(\Omega)$.*

In particular, if there are infinitely many eigenvalues for the overdetermined Neumann problem (1), *or more weakly, if* $\mathcal{N}(\Omega)_R$ *contains infinitely many approximating hyperspheres,* Ω *must be a ball.*

A result similar to the last statement was first obtained by Berenstein [1], when Ω is a simply connected domain in $\boldsymbol{R}^2$. Compare also Proposition (1.11).

The preceding Corollary (1.8) is in the line of Problem C. Moreover, Theorem (1.6) itself almost characterize strict convexity. That is, the converse of Theorem (1.6) is almost true:

PROPOSITION (1.9) (cf. Problem (B.3)). *Let* Ω *be a multiply-connected bounded domain with analytic boundaries. If* $\mathcal{N}(\Omega)$ *has the expression* (2) *and* (3) *with some continuous functions* H, d *on* S^{n-1}, *then* Ω *must be convex.*

The proof of this proposition and Proposition (1.11) below will appear in another paper. Let us remark that a null variety for a convex polyhedron does not have even the property (2) in general (for instance, see Example (1.2)). On the other hand, there is a bounded domain Ω which is not convex but whose null variety $\mathcal{N}(\Omega)$ satisfies (2).

Example (1.10). Let $\Omega:=\{x\in\boldsymbol{R}^n; r<|x|<1\}$. Then, $\tilde{\chi}_\Omega(\zeta)=(2\pi)^{n/2}t^{-n/2}\times\{J_{n/2}(t)-r^{n/2}J_{n/2}(rt)\}$, where $t:=(\zeta_1^2+\cdots+\zeta_n^2)^{1/2}$ for $\zeta=(\zeta_1,\cdots,\zeta_n)\in\boldsymbol{C}^n$. Hence, $\mathcal{N}(\Omega)$ satisfies (2) in Theorem (1.6).

Applying Corollary (1.8) and Theorem (1.9) to a special case, we get,

PROPOSITION (1.11). *Let* Ω *be a multiply-connected bounded domain with Lipshitz boundary in* $\boldsymbol{R}^n$. *If the spectrum of* (1) *for* Ω *are asymptotically the same with that of* $B(R)$ *(ball with radius* R*) then* $\Omega=B(R)$.

As another application of Theorem (1.6), we also give a positive answer to Problem (B.2), when the dimension $n=2$. That is:

COROLLARY (1.12). *If two strictly convex domains in* $\boldsymbol{R}^2$ *have the same null variety, then these domains differ from each other by a parallel translation.*

Such injectivity of $\mathcal{N}$ holds in some other cases: First, when restricting to centrally symmetric convex domains in $\boldsymbol{R}^n$, one sees easily that $\mathcal{N}$ is injective (Remark (2.3.25)). Secondly, the injectivity also holds locally when one perturbs a ball. This idea is used in the last Chapter of [17].

When the asymptotic data of $\mathcal{N}(\Omega)\cap S$ is given, Theorem (1.6) enables us to deduce the injectivity problem (Problem (B.2)) from the positive solution of the following problem in differential geometry:

Problem (1.13). Let two strictly convex domain have the same breadth functions and the same ratio of the Gauss-Kronecker curvatures at each point and its antipodal point as a function of normal vectors. Then do these domains differ from each other by a parallel translation?

This problem is equivalent to a uniqueness problem of a certain single differential equation of the second order of the supporting function over S^{n-1} modulo first eigenfunctions of $\Delta_{S^{n-1}}$. This is treated in § 5 of Chapter 2 when $n=2$, but when $n\geq 3$ it is yet unsolved.

Since Corollary (1.12) assures the injectivity of $\mathcal{N}$ in $\boldsymbol{R}^2$ case, we also take an interest in its image. This seems hard, but more weakly we can do this *in an asymptotic sense,* after giving reformulations of Theorem (1.6) and Corollary (1.12) in terms of the coefficients $P_{\mathcal{N}(\Omega)}$, $Q_{\mathcal{N}(\Omega)}$, $R_{\mathcal{N}(\Omega)}, \cdots \in C^\infty(S^{n-1})$ of the asymptotic expansion of the Dirichlet series made from $\mathcal{N}(\Omega)$. Then we obtain,

COROLLARY (1.14) (Proposition (2.3.19), Proposition (2.3.20)). *The assignment:*

$$\left\{\begin{matrix}\textit{Strictly convex domain in } \boldsymbol{R}^2 \\ \textit{up to parallel displacements}\end{matrix}\right\} \ni \Omega \longmapsto (P_{\mathcal{N}(\Omega)}, R_{\mathcal{N}(\Omega)}) \in C^\infty(S^1, \boldsymbol{R}^2)$$

is one to one, and the image can be explicitly characterized.

On a Riemannian symmetric space X, the Fourier transform is also defined with similar properties to those on a Euclidean space ([11]). So the null variety $\mathcal{N}(\Omega)$ is also defined for a bounded domain $\Omega\subset X$. In Chapter 3 we will generalize Theorem (1.6) (Theorem (2.3.6)) to a hyperbolic space $SO_0(n,1)/SO(n)$ case. To do this, we introduce *H-convex domain,* a notion different from geodesic convexity. Then the analogue of *Gauss-maps* and *supporting functions* are introduced in a noncompact rank one Riemannian symmetric space, both of which are defined in pairs according to the order of the (little) Weyl group. After preparing basic properties of H-convex domains in § 2 and § 3, restricting ourselves to a hyperbolic space, we get the following theorem by using the horospherical method as in Euclidean case.

THEOREM (1.15) (Theorem (3.6.4)). *Suppose Ω be a strictly H-convex domain in a hyperbolic space $X=SO_0(n,1)/SO(n)$. Then*

$$\mathcal{N}(\Omega)=\Big(\coprod_{m\in N}\mathcal{N}_m\Big)\amalg\text{(compact set)}\qquad\textit{(disjoint union).}$$

More precisely, the first approximation of $\mathcal{N}_m$ $(m\to\infty)$ is explicitly expressed in terms of the curvatures and the supporting functions of $\partial\Omega$.

In a hyperbolic space, the null variety $\mathcal{N}(\Omega)$ is not invariant in general under an isometric transformation on X. So a hyperbolic space version of Problem (B.2) is formulated by,

Problem (B.4). Let Ω_1, and Ω_2 be strictly H-convex domains with $\mathcal{N}(\Omega_1)=\mathcal{N}(\Omega_2)$. Then
1) If Ω_1 is not a ball, is $\Omega_1=\Omega_2$?
2) If Ω_1 is a ball, is $\Omega_1=g\cdot\Omega_2$ for some $g\in SO_0(n,1)$?

Theorem (1.15) gives us a method to deal with the injectivity problem of $\mathcal{N}$. In the final section of Chapter 3, we shall prepare more detailed analysis of convex domains when the dimension $n=2$, and illustrate the idea for the injectivity problem which was used in a Euclidean space. In this special case, a uniqueness problem for a periodic solution of a special type of the Duffing equation appears.

In this paper we will use the standard notation N, N_+, $\boldsymbol{Z}$, $\boldsymbol{R}$, $\boldsymbol{R}_+$ and $\boldsymbol{C}$. Here N is the set of non-negative integers and $\boldsymbol{R}_+$ is the set of the positive real numbers and $N_+=N\cap\boldsymbol{R}_+$. For a smooth manifold M, we denote by $C(M)$, $C_0(M)$, $C^{k,\alpha}(M)$, $C^\infty(M)$ and $\mathcal{E}'(M)$ the space of continuous functions, continuous functions with compact support, functions with k-th derivatives satisfying Hölder's condition of order α locally, infinitely differentiable functions and distributions with compact support, defined on M respectively. If M is a complex manifold (resp. a real analytic manifold), we denote by $\mathcal{O}(M)$ (resp. $\mathcal{A}(M)$) the space of holomorphic (resp. real analytic) functions on M.

Parts of the results here were announced in [17]. The author expresses his sincere gratitude to Professor Toshio Oshima for awaking the author's interest in this subject and for his constant encouragement. Thanks are also due to Kaoru Ono and Takashi Kurose for helpful conversations.

Chapter 2. Null variety for a convex domain (Euclidean case)

§ 1. Convex domain in R^n

In this section, we shall review some standard facts about convex domain in Euclidean space.

(2.1.1) Let Ω be a bounded domain whose boundary $\partial\Omega$ is a connected $n-1$ dimensional smooth submanifold of R^n.

We fix an inner product (,) in R^n, and denote the unit sphere by S^{n-1}. Let the Gauss map be

$$\nu \equiv \nu_\Omega : \partial\Omega \longrightarrow S^{n-1}, \tag{2.1.2}$$

defined by its outer normal vector field, and the Gauss-Kronecker curvature be,

$$K \equiv K_\Omega : \partial\Omega \longrightarrow R, \tag{2.1.3}$$

where we adopt the signature of K so that K is everywhere positive if Ω is a ball. Then the following characterization of (strict) convexity is well known:

FACT (2.1.4) (Hadamard, Chern-Lashof). *Let Ω satisfy* (2.1.1). *Then the following three conditions on Ω are equivalent:*

1) *The Gauss-Kronecker curvature K is positive valued.*

2) *The second fundamental form of the imbedding $\partial\Omega \hookrightarrow R^n$ is nondegenerate.*

3) *The Gauss map ν gives a diffeomorphism from $\partial\Omega$ onto S^{n-1}.*

Moreover one of (therefore all of) the conditions 1)$\sim$3) *implies the following three equivalent conditions:*

4) *Ω is geodesically convex.*
i.e. for any $x, y \in \Omega$ and any $t \in [0, 1]$, $tx+(1-t)y \in \Omega$.

5) *Ω lies in one side with respect to any hyperplane tangent to $\partial\Omega$.*

6) *K is nonnegative valued, and the mapping degree of ν is* 1.

DEFINITION (2.1.5). For a domain $\Omega \subset R^n$ satisfying (2.1.1), Ω is called *strictly convex* if and only if the equivalent conditions 1)$\sim$3) are satisfied. Ω is called *convex* if and only if the equivalent conditions 4)$\sim$6) are satisfied.

REMARK (2.1.6). In Chapter 3, the definition of convexity will be generalized reasonably in a rank one noncompact Riemannian symmetric space, to two different ones, that is, horospherically convex (Definition (3.2.13), Proposition (3.4.2)) and geodesically convex (Remark (3.2.14)).

DEFINITION (2.1.7). *For a convex domain Ω, the supporting function* $h \equiv h_{\Omega} : S^{n-1} \longrightarrow R$ is given by,

$$h(\omega) := (\omega, \nu^{-1}(\omega)) \tag{2.1.8}$$

$$= \sup_{x \in \Omega} (x, \omega), \qquad \text{for } \omega \in S^{n-1}. \tag{2.1.9}$$

Let $\tilde{h} : \boldsymbol{R}^n \longrightarrow \boldsymbol{R}$ be a linear extension of h, that is,

$$\tilde{h}(x) := |x| h\left(\frac{x}{|x|}\right) \qquad \text{for } x \in \boldsymbol{R}^n \setminus \{0\}, \tag{2.1.10}$$

$$\tilde{h}(0) := 0.$$

Then the following lemma is well known, which states how a strictly convex domain is recovered by its supporting function.

LEMMA (2.1.11) (cf. Corollary (3.4.12), Proposition (3.7.23)). *Let Ω be a strictly convex domain and ν, h and $\tilde{h}$ be as defined in* (2.1.2), (2.1.8) *and* (2.1.10) *respectively. Then for any element ω of S^{n-1},*

$$\nu^{-1}(\omega) = \left(\frac{\partial \tilde{h}}{\partial x^i}(\omega)\right)_{i=1,\cdots,n}. \tag{2.1.12}$$

Or more directly, $\Omega = \bigcap_{\omega \in S^{n-1}} \{x \in \boldsymbol{R}^n; (x, \omega) < h(\omega)\}$.

REMARK (2.1.13). The definition of $\tilde{h}$ depends on the choice of the origin 0. More precisely, the difference of $\tilde{h}$ by parallel displacements of Ω is just linear functions on $\boldsymbol{R}^n$.

The breadth function of a convex domain Ω is given by,

$$H \equiv H_{\Omega} : S^{n-1} \longrightarrow \boldsymbol{R}_+, \quad H(\omega) := h(\omega) + h(-\omega), \quad \text{for } \omega \in S^{n-1}. \tag{2.1.14}$$

From the definition, H is a positive valued C^{∞} function which is invariant under parallel displacements of Ω, and clearly satisfies the following equality:

$$H(\omega) = H(-\omega), \qquad \text{for any } \omega \in S^{n-1}. \tag{2.1.15}$$

For a strictly convex domain Ω, we introduce a new function

$d \equiv d_\Omega : S^{n-1} \longrightarrow \boldsymbol{R}$, as follows:

$$d(\omega) := \frac{\log K \circ \nu^{-1}(-\omega) - \log K \circ \nu^{-1}(\omega)}{2H(\omega)}, \quad (\omega \in S^{n-1}). \tag{2.1.16}$$

Since Ω is strictly convex, the Gauss-Kronecker curvature K and the breadth function H are positive valued, so d is a well-defined C^∞ function on S^{n-1}, which is also invariant under parallel displacements of Ω. The following formula is derived from the definition of $d \equiv d_\Omega$ and from (2.1.15):

$$d(\omega) + d(-\omega) = 0, \qquad \text{for any } \omega \in S^{n-1}. \tag{2.1.17}$$

LEMMA (2.1.18). *Let Ω be a strictly convex domain and d defined in* (2.1.16). *Then the following two conditions on Ω are equivalent*:
1) $d(\omega) \equiv 0$, *for any* $\omega \in S^{n-1}$.
2) *Ω is centrally symmetric with respect to an inner point.*

PROOF. 1)$\rightarrow$2) Put $\Omega^\vee := \{x \in \boldsymbol{R}^n;\ -x \in \Omega\}$. The pullbacks of the Gauss-Kronecker curvatures of Ω and $\Omega^\vee$ by the Gauss maps, namely, $K_\Omega \circ \nu_\Omega^{-1}$ and $K_{\Omega^\vee} \circ \nu_{\Omega^\vee}^{-1}$ coincide because of the assumption 1). Now, by Alexandroff-Fenchel-Jessen's theorem (uniqueness of Minkowski's problem) (cf. [6]), Ω and $\Omega^\vee$ differ each other only by a parallel displacement. Therefore Ω is centrally symmetric with respect to the center of gravity.

The converse statement is clear, so the lemma is proved. Q.E.D.

§ 2. Asymptotic behavior of $\tilde{\chi}_\Omega(\zeta)$

We shall devote this section to the study of the asymptotic behaviour of $\tilde{\chi}_\Omega$ along some direction in $\boldsymbol{C}^n$. Using the classical Radon transform, we reduce it to the problem of one variable. Next, we show in Proposition (2.2.16) that any zero of a certain Fourier transform has bounded imaginary part, generalizing the well-known fact: *any zero of the ν-th Bessel function $J_\nu(t)$ $(\nu > -1)$ is real.* Finally we obtain in Proposition (2.2.32) the asymptotic behaviour of $\tilde{\chi}_\Omega(\omega, \zeta)$ with the imaginary part of ζ bounded. The results in this section will play a basic role in the proof of Theorem (2.3.6).

We denote the characteristic function of Ω by $\chi_\Omega(x)$, where Ω is a bounded measurable set in $\boldsymbol{R}^n$. Namely,

$$\chi_\Omega(x):=\begin{cases}1 & \text{if } x\in\Omega,\\ 0 & \text{otherwise.}\end{cases}$$

The Fourier-Laplace transform of χ_Ω is given by,

$$\tilde{\chi}_\Omega(z)\equiv\mathcal{F}\chi_\Omega(z):=\int_{R^n}\chi_\Omega(x)e^{i\langle x,z\rangle}dx \tag{2.2.1}$$
$$=\int_\Omega e^{i(x_1z_1+x_2z_2+\cdots+x_nz_n)}dx_1dx_2\cdots dx_n,$$

for $z=(z_1,\cdots,z_n)\in C^n$.

Since Ω is bounded, $\tilde{\chi}_\Omega(z)\in\mathcal{O}(C^n)$, where $\mathcal{O}(C^n)$ denotes the totality of entire functions on C^n. Set

$$\mathcal{N}(\Omega):=\{z\in C^n;\ \tilde{\chi}_\Omega(z)=0\}, \tag{2.2.2}$$

and,

$$\mathcal{N}(\Omega)_R:=\mathcal{N}(\Omega)\cap R^n=\{z\in R^n;\ \tilde{\chi}_\Omega(z)=0\}. \tag{2.2.3}$$

Then $\mathcal{N}(\Omega)$ (resp. $\mathcal{N}(\Omega)_R$) is an analytic set in C^n (resp. R^n). We call $\mathcal{N}(\Omega)$ *the null variety* for a given Ω.

REMARK (2.2.4). $\mathcal{N}(\Omega)$ and $\mathcal{N}(\Omega)_R$ are invariant under parallel displacements of Ω, because $\tilde{\chi}_{\Omega+x_0}(z)=e^{i\langle x_0,z\rangle}\tilde{\chi}_\Omega(z)$, where $\Omega+x_0:=\{x+x_0\in R^n;\ x\in\Omega\}$, for any fixed element x_0 of R^n.

Next, we introduce a collection of special complex lines in C^n with *'real direction'*. That is,

$$l_\omega:=C\omega=\{\zeta\omega=(\zeta\omega_1,\zeta\omega_2,\cdots,\zeta\omega_n)\in C^n;\zeta\in C\},$$

for each fixed element $\omega\equiv(\omega_1,\omega_2,\cdots,\omega_n)\in S^{n-1}\,(\subset R^n)$.

We shall study the asymptotic behavior of $\tilde{\chi}_\Omega$ along l_ω. Geometric invariants of Ω will be read from the asymptotics of the intersection of $\mathcal{N}(\Omega)$ and l_ω.

Now, by a little abuse of language, we will use the same letter $\tilde{\chi}_\Omega$ for its restriction to $S^{n-1}\times C$, that is:

$$\tilde{\chi}_\Omega:\ S^{n-1}\times C\ni(\omega,\zeta)\longmapsto\tilde{\chi}_\Omega(\omega,\zeta)\equiv\tilde{\chi}_\Omega(\zeta\omega_1,\zeta\omega_2,\cdots,\zeta\omega_n)\in C^n.$$

From the definition, $\tilde{\chi}_\Omega$ satisfies

$$\tilde{\chi}_\Omega(\omega,z)=\tilde{\chi}_\Omega(-\omega,-z),\qquad\text{for any }\omega\in S^{n-1}\text{ and }z\in C. \tag{2.2.5}$$

Since $\chi_\Omega(x)$ is real valued, $\tilde{\chi}_\Omega$ also satisfies

(2.2.6) $\overline{\tilde{\chi}_\Omega(\omega, \xi+i\eta)}=\tilde{\chi}_\Omega(\omega, -\xi+i\eta)$, for any $\omega\in S^{n-1}$ and $\xi, \eta\in \boldsymbol{R}$.

Here, $\bar{z}$ denotes the complex conjugation of $z\in \boldsymbol{C}$.

The classical method of decomposing a Fourier-Laplace transform into a Radon transform and a Fourier transform of one variable gives the following representation of $\tilde{\chi}_\Omega(\omega, \zeta)$:

(2.2.7)
$$\tilde{\chi}_\Omega(\omega, \zeta)=\int_{-\infty}^{+\infty} S(\omega, p)e^{ip\zeta}dp=\int_{-h(-\omega)}^{h(\omega)} S(\omega, p)e^{ip\zeta}dp,$$

where $S(\omega, p)\equiv S_\Omega(\omega, p):=\int_{\boldsymbol{R}^n}\chi_\Omega(x)\delta(p-\langle x, \omega\rangle)dx$ is nothing but the Euclidean area of sectional face of Ω by the hyperplane $\{x\in \boldsymbol{R}^n; \langle x, \omega\rangle=p\}$. Here δ denotes Dirac's delta function of a single valuable.

The next lemma essentially goes back to F. John (see [15]).

LEMMA (2.2.8). *Suppose Ω be strictly convex and retain notation as above. Then, $S\equiv S_\Omega: S^{n-1}\times \boldsymbol{R}\longrightarrow \boldsymbol{R}$ is a continuous function with compact support $\bar{V}$, and $S_\Omega|_V\in C^\infty(V)$. Here, $V:=\{(\omega, p)\in S^{n-1}\times \boldsymbol{R}; \omega\in S^{n-1}, -h(-\omega)<p<h(\omega)\}$, and $\bar{V}$ denotes its closure.*
More precisely, for a sufficiently small constant $\delta>0$, there are two C^∞ functions: $S_j: S^{n-1}\times(-\delta, \delta)\ni(\omega, p)\longmapsto S_j(\omega, p)\in \boldsymbol{R}$, $(j=1, 2)$, such that $S(\omega, p)$ is represented in the neighbourhood of ∂V as follows (see Notation (2.2.12)):

(2.2.9)
$$\begin{aligned} S(\omega, p) &=\frac{(2\pi)^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}(K\circ\nu^{-1}(-\omega))^{-1/2}(p+h(-\omega))_+^{(n-1)/2} \\ &\quad\times(1+S_1(\omega, p+h(-\omega))(p+h(-\omega))), \qquad \textit{for } |p+h(-\omega)|<\delta. \\ &=\frac{(2\pi)^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}(K\circ\nu^{-1}(\omega))^{-1/2}(p-h(\omega))_-^{(n-1)/2} \\ &\quad\times(1+S_2(\omega, h(\omega)-p)(h(\omega)-p)), \qquad \textit{for } |p-h(\omega)|<\delta. \end{aligned}$$

PROOF. The first statement is clear. We will first prove the smoothness of $S(\omega, p)$ as a function of ω and $(p-h(\omega))_-^{1/2}$. Since this is a local statement, we fix a trivializing neighbourhood U ($\subset\partial\Omega$) of the tangent bundle $T(\partial\Omega)\longrightarrow\partial\Omega$. Fix an orthonormal frame on $\partial\Omega$ ($\hookrightarrow \boldsymbol{R}^n$), which gives the bundle isomorphism

$$U\times \boldsymbol{R}^{n-1} \underset{\varphi}{\xrightarrow{\sim}} T(\partial\Omega)_{|U} \ (\underset{\mathrm{j}}{\hookrightarrow} U\times \boldsymbol{R}^n \underset{\mathrm{p}}{\longrightarrow} \boldsymbol{R}^n).$$

In the above parenthesis, j is induced from $T(\partial\Omega)\hookrightarrow\partial\Omega\times\boldsymbol{R}^n$ and p is the projection to the second factor.

Define a C^∞ map $\phi: U\times\boldsymbol{R}^{n-1}\times\boldsymbol{R}\longrightarrow\boldsymbol{R}^n$ by,

$$\phi(x; y, s):=x+\mathrm{p}\circ\mathrm{j}\circ\varphi(x, y)+s\ \nu(x), \quad \text{for } (x, y, s)\in U\times\boldsymbol{R}^{n-1}\times\boldsymbol{R}.$$

Here, we look upon an element x of U as in $\boldsymbol{R}^n$ and $\nu:\partial\Omega\longrightarrow S^{n-1}\hookrightarrow\boldsymbol{R}^n$ is the Gauss map (Definition (2.1.2)). From the definition, for each fixed element x of U, $\phi(x;\ ,\ ): \boldsymbol{R}^{n-1}\times\boldsymbol{R}\xrightarrow{\sim}\boldsymbol{R}^n$ gives a moving frame with x its origin. Since $\partial\Omega$ is locally represented by a graph, there is a C^∞ function $f: U\times W\longrightarrow\boldsymbol{R}$ with some open neighbourhood W $(\subset\boldsymbol{R}^{n-1})$ containing the origin 0 such that the following conditions (2.2.10) are satisfied for any $(x, y)\in U\times W$,

$$(2.2.10)\quad \begin{cases} \phi(x; y, f(x, y))\in\partial\Omega, \\ f(x, 0)=\dfrac{\partial f}{\partial y_j}(x, 0)=0 \qquad (1\leq j\leq n-1), \\ \text{The eigenvalues of } \left(\dfrac{\partial^2 f}{\partial y_j\partial y_k}(x, 0)\right)_{j,k} \text{ are } -\kappa_j(x)\ (1\leq j\leq n-1). \end{cases}$$

Here $\kappa_j(x)$ $(1\leq j\leq n-1)$ are the principal curvatures of $\partial\Omega$ at x. Set $\tilde{f}$ be the composition map of

$$U\times(-\varepsilon, \varepsilon)\times S^{n-2}\ni(x; r, \theta)\longmapsto(x, r\theta)\in U\times W,$$

and

$$U\times W\ni(x, y)\longmapsto f(x, y)\in\boldsymbol{R},$$

with a small fixed $\varepsilon>0$. Then from (2.2.10), there is a C^∞ function f_1 defined on $U\times(-\varepsilon, \varepsilon)\times S^{n-2}$ such that

$$\tilde{f}(x; r, \theta)=-r^2 f_1(x; r, \theta).$$

Since $f_1(x; 0, \theta)>0$, retaking U and ε if necessary, we may assume $f_1(x; r, \theta)>0$ for any $(x, r, \theta)\in U\times(-\varepsilon, \varepsilon)\times S^{n-2}$.

Then applying the implicit function theorem to the equation

$$r\,(f_1(x; r, \theta))^{1/2}=a,$$

we find a C^∞ function $R: U'\times(-\delta, \delta)\times S^{n-2}\longrightarrow(-\varepsilon, \varepsilon)$ such that for any $(x, a, \theta)\in U'\times(-\delta, \delta)\times S^{n-2}$,

$$R(x; a, \theta)\ (f_1(x; R(x; a, \theta), \theta))^{1/2}=a,$$

and

$$R(x; 0, \theta) = 0,$$

with some open neighbourhood $U' \subset U$ $(\subset \partial\Omega)$ and $\delta > 0$. The above equalities imply

$$\tilde{f}(x; R(x; a, \theta), \theta) = -a^2,$$

and

$$R(x; a, \theta) > 0, \qquad \text{if } a > 0.$$

On the other hand, the intersection of Ω and the hyperplane $\{z \in \boldsymbol{R}^n; \langle z, \omega \rangle = p\}$ is represented by the polar coordinate as follows:

$$\{\psi(\nu^{-1}(\omega); r\theta, p - h(\omega));\ 0 \leq r < R(\nu^{-1}(\omega); (h(\omega) - p)^{1/2}, \theta),\ \theta \in S^{n-2}\}.$$

The function $S(\omega, p)$ is the Euclidean area of this set, which is obtained by the integration over S^{n-2} of the radial part $R(\nu^{-1}(\omega); (h(\omega)-p)^{1/2}, \theta)$ and its derivatives with respect to θ. Therefore, $S(\omega, p)$ is a C^∞ function of $(\omega, (h(\omega)-p)^{1/2}) \in \nu(U') \times [0, \sqrt{\delta})$. This proves the smoothness of $S(\omega, p)$ as a function of ω and $(p - h(\omega))_-^{1/2}$.

The first term of the asymptotic expansion of $S(\omega, p)$ is obtained in [15]. Let us review it. It is the volume of the ellipsoid with the radii of curvature $((h(\omega)-p)/\kappa_i(\nu^{-1}(\omega)))^{1/2}$ $(1 \leq i \leq n-1)$, namely, $(2\pi)^{(n-1)/2} \times \Gamma((n+1)/2)^{-1}(K \circ \nu^{-1}(\omega))^{-1/2}(h(\omega)-p)^{(n-1)/2}$.

To complete the proof of the lemma, we only have to show that any coefficient of the power $(p - h(\omega))_-^{n/2+k}$ $(k = 0, 1, 2, \cdots)$ vanishes in the expansion of $S(\omega, p)$ at $p = h(\omega)$. Since this claim is $SO(n)$-invariant, we may assume that $\omega = \omega_0 \equiv (0, \cdots, 0, 1) \in S^{n-1}$. Set $f_0 : W \longrightarrow \boldsymbol{R}$ by $f_0(y) := f(\nu^{-1}(\omega_0), y)$ $(y \in W)$, $\tilde{f}_0 : (-\varepsilon, \varepsilon) \times S^{n-2} \longrightarrow \boldsymbol{R}$ by $\tilde{f}_0(r, \theta) := \tilde{f}(\nu^{-1}(\omega_0), r, \theta) = f_0(r \cdot \theta)$, and $R_0 : (-\delta, \delta) \times S^{n-2} \longrightarrow (-\varepsilon, \varepsilon)$ by $R_0(t, \theta) := R(\nu^{-1}(\omega_0), t, \theta)$. Let $f_0(y) \sim \sum_{|\alpha| \geq 2} a_\alpha y^\alpha$, $R_0(t, \theta) \sim \sum_{m=1}^{\infty} b_m(\theta) t^m$ be the Taylor expansions with coefficients a_α, $b_m(\theta) \in \boldsymbol{R}$ respectively. Here $\alpha = (\alpha_1, \cdots, \alpha_{n-1})$ $(\alpha_j \in N)$ is a multi-index and the length of α is defined by $|\alpha| := \sum_{j=1}^{n-1} \alpha_j$. Note that $\sum_{|\alpha|=2} a_\alpha y^\alpha$ are non-degenerate quadratic form because of (2.2.10).

Set $P_k(\theta) := \sum_{|\alpha|=k} a_\alpha \theta^\alpha$. Formal substitution of the above Taylor expansions into the identity $y = \tilde{f}_0(R_0(y^{1/2}, \theta), \theta)$ gives,

$$y\sim \sum_{|\alpha|\geq 2} a_\alpha \theta^\alpha\Big(\sum_{m=1}^{\infty} b_m(\theta) y^{m/2}\Big)^{|\alpha|}$$
$$=\sum_{k=2}^{\infty} P_k(\theta)\Big(\sum_{m=1}^{\infty} b_m(\theta) y^{m/2}\Big)^{k}.$$

Comparing the coefficients of the power $y^{N/2}$ $(N\geq 2)$ of the both side, we get,

$$\sum_{k=2}^{N} P_k(\theta)(\sum b_{n_1}(\theta)b_{n_2}(\theta)\cdots b_{n_k}(\theta))=\begin{cases}1 & (N=2)\\ 0 & (N\geq 3).\end{cases} \tag{2.2.11}$$

Here the sum is taken over $\Big\{(n_1, n_2, \cdots, n_k)\in N_+^k;\ \sum_{j=1}^{k} n_j=N\Big\}$.

Now let us show

$$b_k(-\theta)=(-1)^{k-1}b_k(\theta) \qquad (k\geq 1),$$

by the induction on k.

First note that $P_k(-\theta)=(-1)^k P_k(\theta)$ $(k\geq 1)$, from definition.

From (2.2.11) with $N=2$, we have,

$$P_2(\theta)b_1(\theta)^2=1.$$

Therefore, $b_1(\theta)=b_1(-\theta)$.

Suppose $b_k(-\theta)=(-1)^{k-1}b_k(\theta)$ for $1\leq k\leq N-1$. $(N\geq 2)$. From (2.2.11) with N replaced by $N+1$,

$$2P_2(\theta)b_1(\theta)b_N(\theta)$$
$$=-P_2(\theta)\sum_{j=2}^{N-1} b_j(\theta)b_{N+1-j}(\theta)-\sum_{k=3}^{N+1} P_k(\theta)(\sum b_{n_1}(\theta)b_{n_2}(\theta)\cdots b_{n_k}(\theta)).$$

Here the sum is taken over $\Big\{(n_1, n_2, \cdots, n_k)\in N_+^k;\ \sum_{j=1}^{k} n_j=N+1\Big\}$. In particular, b_j appears in the right hand side only when $j\leq N-1$.

Replacing θ by $-\theta$ in the above identity and using the assumption of the induction, we get,

$$2P_2(\theta)b_1(\theta)b_N(-\theta)$$
$$=-P_2(\theta)(-1)^{N+1}\sum_{j=2}^{N-1} b_j(\theta)b_{N+1-j}(\theta)-\sum_{k=3}^{N+1}(-1)^k P_k(\theta)$$
$$\times(\sum(-1)^{\Sigma(n_j-1)}b_{n_1}(\theta)\cdots b_{n_k}(\theta))$$
$$=(-1)^{N+1}\Big(-P_2(\theta)\sum_{j=2}^{N-1} b_j(\theta)b_{N+1-j}(\theta)-\sum_{k=3}^{N+1} P_k(\theta)(\sum b_{n_1}(\theta)\cdots b_{n_k}(\theta))\Big)$$
$$=(-1)^{N+1}2P_2(\theta)b_1(\theta)b_N(\theta).$$

Hence $b_N(-\theta)=(-1)^{N-1}b_N(\theta)$ and the induction is now completed.

Since the volume $\{r\theta\in \boldsymbol{R}^{n-1}; 0\leq r<R_0(t, \theta), \theta\in S^{n-2}\}$ is given by the pairing of $R_0(t, \cdot)$ and an $SO(n-2)$-invariant distribution F on S^{n-2}, we have

$$S(\omega_0, -t^2)=\int_{S^{n-1}}F(\theta)R_0(t, \theta)d\theta$$
$$=\frac{1}{2}\int_{S^{n-1}}F(\theta)(R_0(t, \theta)+R_0(t, -\theta))d\theta.$$

Here, we write the distribution $F(\theta)$ as if it were a usual function for convenience. Since the Taylor expansion of $\frac{1}{2}(R_0(t, \theta)+R_0(t, -\theta))$ is given by $\sim\sum_{n=0}^{\infty}b_{2n+1}(\theta)t^{2n+1}$, any coefficient of the power $(p-h(\omega_0))_-^{n/2+k}$ ($k=0, 1, 2, \cdots$) vanishes in the expansion of $S(\omega_0, p)$ at $p=h(\omega_0)$.

Hence the proof of the lemma is completed. Q.E.D.

Paley-Wiener's theorem asserts that the singularity of a function is reflected by the growth of its Fourier transform. We shall go into details of the asymptotic behaviour of the functions having special singularities. To do this, it is convenient to list up some notations here which will be used repeatedly throughout this section.

Notation (2.2.12). For $\mu>0$, we put

$$x_+^\mu:=\begin{cases}x^\mu & \text{if } x>0,\\ 0 & \text{if } x\leq 0.\end{cases}\qquad x_-^\mu:=\begin{cases}0 & \text{if } x\geq 0,\\ |x|^\mu & \text{if } x<0.\end{cases}$$
$$(\xi+i0)^{-\mu-1}:=\begin{cases}\xi^{-\mu-1} & \text{if } \xi>0,\\ e^{\pi i\mu}|\xi|^{-\mu-1} & \text{if } \xi<0.\end{cases}$$

(We will need this not in the case when ξ is near 0 but only when $|\xi|$ is very large.)

Let φ be a C^∞ function on $\boldsymbol{R}$ having the following three properties:

$$(2.2.13)\qquad \begin{cases}\varphi(x)=\varphi(-x) & \text{for any } x\in \boldsymbol{R},\\ \varphi(x)=1 & \text{if } x\in[-\delta, \delta],\\ \operatorname{supp}\varphi\subset[-2\delta, 2\delta].\end{cases}$$

Here δ is a positive constant.

Let λ be a real positive number. Let $f(x)$ be a continuous function on $\boldsymbol{R}$ with compact support contained in $[0, A]$ and have the following expression for some $N\in N$ and for some φ satisfying (2.2.13) with $4\delta<A$:

$$f(x)=\sum_{j=0}^{N} a_j x_+^{\lambda+j}\varphi(x)+\sum_{j=0}^{N} b_j(x-A)_-^{\lambda+j}\varphi(x-A)+g(x), \tag{2.2.14)-(N}$$

where a_j, $b_j \in \boldsymbol{C}$ ($a_0 \neq 0$), $g(x) \in C^{[\lambda]+N,1}(\boldsymbol{R})$, and $\operatorname{supp} g \subset [0, A]$. Note that A, a_j, and b_j are unique for a given function $f(x)$ and not dependent on the choice of φ. Clearly, $f(x)|_{\boldsymbol{R}\setminus[0,A]} \in C^\infty(\boldsymbol{R}\setminus[0, A])$.

Finally, set

$$p(\lambda):=\Gamma(\lambda+1)\exp\left(\frac{\pi i(\lambda+1)}{2}\right). \tag{2.2.15}$$

It is well known that *all the zeros of the ν-th Bessel function $J_\nu(t)$ ($\nu>-1$) are real*. The following proposition is a generalization of this fact (see Example (2.2.28), Example (2.2.29)). Recall that $\mathcal{F}f$ stands for the Fourier-Laplace transform of f (see (2.2.1)).

PROPOSITION (2.2.16). *Let f be a continuous function on $\boldsymbol{R}$ having an expression as in* (2.2.14) *with $N=1$. Then,*

$$\sup\{|\operatorname{Im}(\zeta)|;\ \mathcal{F}f(\zeta)=0\}<\infty. \tag{2.2.17}$$

PROOF. From the assumption, $f(x)$ has the expression as in (2.2.14). That is,

$$f(x)=\sum_{j=0}^{1} a_j x_+^{\lambda+j}\varphi(x)+\sum_{j=0}^{1} b_j(x-A)_-^{\lambda+j}\varphi(x-A)+g(x).$$

Here $g \in C^{[\lambda]+1,1}(\boldsymbol{R})$. Set $h(x):=f(A-x)$. Then h has also the expression (2.2.14), and $\mathcal{F}h(\zeta)=\exp(i\zeta A)\mathcal{F}f(-\zeta)$. Therefore it is enough to prove

$$\sup\{\operatorname{Im}(\zeta);\ \mathcal{F}f(\zeta)=0\}<\infty, \tag{2.2.18}$$

instead of (2.2.17). Before proving (2.2.18), we shall prepare the following three Lemmas (2.2.19), (2.2.22) and (2.2.24).

LEMMA (2.2.19). *Let φ be a function satisfying* (2.2.13). *For any $\lambda>0$, and any integer $k\geq\lambda$, there is a constant $C>0$ such that*

$$|\mathcal{F}(x_+^\lambda\varphi(x))(\zeta)-p(\lambda)\zeta^{-\lambda-1}|\leq C|\zeta|^{-k}\eta^{-1}\exp(-\delta\eta), \tag{2.2.20}$$

for any $\zeta=\xi+i\eta$ with $\eta>0$.

Note that if $\eta>1$, the right hand side of (2.2.20) *can be replaced by $C|\zeta|^{-k}$.*

PROOF. Set $\phi(x):=x_+^\lambda\varphi(x)-x_+^\lambda$. Then $\phi(x)$ is a smooth function on $\boldsymbol{R}$ with its support contained in $[\delta,\infty)$. Since $k>\lambda$,

$$C:=\sup_{x\in\boldsymbol{R}}\left|\left(\frac{d}{dx}\right)^k\phi(x)\right|<\infty.$$

If $\eta=\mathrm{Im}(\zeta)>0$, using integral by parts, we get,

$$\begin{aligned}|\zeta^k\mathcal{F}\phi(\zeta)|&\le\int_\delta^\infty\left|\left(\frac{d}{dx}\right)^k\phi(x)\right||\exp(i\zeta x)|dx\\&\le C\int_\delta^\infty\exp(-\eta x)dx\\&=C\eta^{-1}\exp(-\delta\eta).\end{aligned}$$

On the other hand, the next formula holds ([9] vol. 1 p. 171) when $\lambda>0$ and $\eta>0$.

(2.2.21) $$\int_{-\infty}^{+\infty}x_+^\lambda e^{ix(\xi+i\eta)}dx=p(\lambda)(\xi+i\eta)^{-\lambda-1}.\qquad\text{(Notation (2.2.15))}$$

Therefore

$$|\mathcal{F}(x_+^\lambda\varphi(x))(\zeta)-p(\lambda)\zeta^{-\lambda-1}|\le|\mathcal{F}\phi(\zeta)|\le C|\zeta|^{-k}\eta^{-1}\exp(-\delta\eta).$$

Hence the lemma. Q.E.D.

LEMMA (2.2.22). *Let $g\in C^{k-1,1}(\boldsymbol{R})$ $(k\in N_+)$ whose support is contained in $[0,A]$ $(0<A<\infty)$. Then,*

(2.2.23) $$|\mathcal{F}g(\zeta)|\le\left\|\left(\frac{d}{dx}\right)^k g\right\|_\infty|\zeta|^{-k}\eta^{-1},$$

for any $\zeta=\xi+i\eta$ with $\eta>0$.

Note that if $\eta>1$, the right hand side of (2.2.23) can be replaced by $\left\|\left(\frac{d}{dx}\right)^k g\right\|_\infty|\zeta|^{-k}$.

This lemma is proved in the same way as in the last lemma, so we omit the proof.

LEMMA (2.2.24). *Let φ be a function as in (2.2.13). Fix any $\lambda>0$, and any nonnegative integer N. Put $L:=[\lambda]+N+1\in N$. Then there is a constant $C>0$ such that*

(2.2.25)
$$\left|\mathcal{F}(x_-^\lambda\varphi(x))(\zeta)-\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}p(\lambda+n)(-\bar{\zeta})^{-\lambda-n-1}\right|$$
$$\leq C|\zeta|^{-L}\eta^{-1}\{(1+\eta^{[\lambda]})\eta^{N+1}\exp(4\delta\eta)+(1+\eta^N)\exp(-\delta\eta)\},$$

for any $\zeta=\xi+i\eta$ *with* $\eta>0$.

Note that if $\eta>1$, *the right hand side of* (2.2.20) *can be written (possibly after changing the constant* $C>0$) *as* $C|\zeta|^{-L}\exp(A\eta)$, *where* $A>0$ *is a constant larger than* 4δ.

PROOF. Set $h(x):=x_+^\lambda\varphi(x)\exp(2\eta)-\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}x_+^{\lambda+n}\varphi(x)$. Then $h(x)=\sum_{n=N+1}^{\infty}\frac{(2\eta)^n}{n!}x_+^{\lambda+n}\varphi(x)\in C^{L-1,1}(\boldsymbol{R})$, and $\operatorname{supp} h\subset[0,2\delta]$.

Let $x\in[0,2\delta]$,

$$\left|\left(\frac{d}{dx}\right)^L h(x)\right|\leq\sum_{n=0}^{\infty}\frac{(2\eta)^{n+N+1}}{(n+N+1)!}\left|\left(\frac{d}{dx}\right)^L\{x_+^{\lambda+n+N+1}\varphi(x)\}\right|$$
$$\leq 2^L\sup\left\{\left|\left(\frac{d}{dx}\right)^k\varphi(x)\right|;\ x\in\boldsymbol{R},\ 0\leq k\leq L\right\}$$
$$\times\sum_{n=0}^{\infty}\frac{(2\eta)^{n+N+1}}{(n+N+1)!}\max_{0\leq k\leq L}\left(\prod_{j=0}^{k}(\lambda+n+N+1-j)\right)x^{\lambda+n+N+1-k}$$
$$\leq C_1\eta^{N+1}\left(1+\sum_{n=1}^{\infty}\frac{n^{[\lambda]}}{n!}(2\eta x)^n\right)$$
$$\leq C_1\eta^{N+1}\left(1+\left(x\frac{d}{dx}\right)^{[\lambda]}\exp(2\eta x)\right)$$
(2.2.26)
$$\leq C_2\eta^{N+1}(1+\eta^{[\lambda]})\exp(2\eta x).$$

Therefore, if $\eta=\operatorname{Im}(\zeta)>0$,

$$\left|\mathcal{F}(x_-^\lambda\varphi(x))(\xi+i\eta)-\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}p(\lambda+n)(-\xi+i\eta)^{-\lambda-n-1}\right|$$
$$=\left|\mathcal{F}(x_+^\lambda\varphi(x))(-\xi-i\eta)-\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}p(\lambda+n)(-\xi+i\eta)^{-\lambda-n-1}\right|$$
$$=\left|\mathcal{F}(x_+^\lambda\varphi(x)\exp(2x\eta))(-\xi+i\eta)-\mathcal{F}\left(\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}x_+^{\lambda+n}\right)(-\xi+i\eta)\right|$$
$$\leq|\mathcal{F}h(-\xi+i\eta)|+\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}|\mathcal{F}(x_+^{\lambda+n}-x_+^{\lambda+n}\varphi(x))(-\xi+i\eta)|.$$

From Lemma (2.2.19) and Lemma (2.2.22),

$$\leq \left\| \left(\frac{d}{dx} \right)^L h(x) \right\|_\infty |\zeta|^{-L}\eta^{-1} + C' \left(\sum_{n=0}^{N} \frac{(2\eta)^n}{n!} \right) |\zeta|^{-L}\eta^{-1} \exp(-\delta\eta)$$
$$\leq C_2|\zeta|^{-L}\eta^{-1}(\eta^{N+1}(1+\eta^{[\lambda]})\exp(4\delta\eta) + (1+\eta^N)\exp(-\delta\eta)),$$

whence the lemma. Q.E.D.

Now, let us complete the proof of Proposition (2.2.16). It is sufficient for (2.2.18) to show

$$\text{(2.2.18)}' \qquad \sup\{\mathrm{Im}(\zeta);\ \mathcal{F}f(\zeta)=0,\ \mathrm{Im}(\zeta)>0, \text{ and } |\zeta| \geq K\} < \infty,$$

with some large constant $K>0$. (If the set in (2.2.18)′ is empty, there is nothing to prove!.) Now we will prove (2.2.18)′. From Lemma (2.2.19), Lemma (2.2.24), and Lemma (2.2.22), there is a constant $C>0$ such that the following three inequalities hold for any $\zeta=\xi+i\eta$ with $\eta>0$, and $0 \leq j \leq 1$,

$$|a_j| \left| \mathcal{F}(x_+^{\lambda+j}\varphi(x))(\zeta) - p\left(\lambda+\frac{j}{2}\right)\zeta^{-\lambda-j-1} \right| \leq C|\zeta|^{-[\lambda]-2},$$

$$|b_j| \left| \mathcal{F}((x-A)_-^{\lambda+j}\varphi(x-A))(\zeta) - \exp(iA\zeta)\left(p\left(\lambda+\frac{j}{2}\right)(-\bar{\zeta})^{-\lambda-j-1} \right.\right.$$
$$\left.\left. + 2\eta p\left(\lambda+1+\frac{j}{2}\right)(-\bar{\zeta})^{-\lambda-j-2}\right) \right| \leq C|\zeta|^{-[\lambda]-2},$$

and

$$|\mathcal{F}g(\zeta)| \leq C|\zeta|^{-[\lambda]-2}.$$

Therefore,

$$|\mathcal{F}f(\zeta) - a_0p(\lambda)\zeta^{-\lambda-1} - \exp(iA\zeta)b_0p(\lambda)(-\bar{\zeta})^{-\lambda-1}|$$
$$\leq |b_0||\mathcal{F}(2\eta(x-A)_-^{\lambda+1})(\zeta)| + |b_1||\mathcal{F}((x-A)_-^{\lambda+1}(1+2\eta(x-A)))(\zeta)|$$
$$+ |a_1||\mathcal{F}(x_+^{\lambda+1})(\zeta)| + \sum_{j=0}^{1} |a_j||\mathcal{F}(x_+^{\lambda+j}(\varphi(x)-1))(\zeta)|$$
$$+ \sum_{j=0}^{1} |b_j||\mathcal{F}((x-A)_-^{\lambda+j}(\varphi(x-A)-1-2\eta(x-A)))(\zeta)| + |\mathcal{F}g(\zeta)|$$
$$\leq |b_0 2\eta\exp(-A\eta)p(\lambda+1)(-\bar{\zeta})^{-\lambda-2}| + |a_1||p(\lambda+1)\zeta^{-\lambda-2}|$$
$$+ |b_1||\exp(-A\eta)(p(\lambda+1)(-\bar{\zeta})^{-\lambda-2} + 2\eta p(\lambda+2)(-\bar{\zeta})^{-\lambda-3})| + 5C|\zeta|^{-[\lambda]-2}.$$

Since the coefficients such as $\eta\exp(-A\eta)$ are bounded if $\eta>0$, there is a constant $M>0$ independent of η such that

$$\text{(2.2.27)} \qquad |\mathcal{F}f(\zeta) - a_0p(\lambda)\zeta^{-\lambda-1} - \exp(iA\zeta)b_0p(\lambda)(-\bar{\zeta})^{-\lambda-1}| \leq M|\zeta|^{-[\lambda]-2}.$$

Set

$$K:=\left(\frac{2M}{|a_0 p(\lambda)|}\right),$$

and

$$\varepsilon:=[\lambda]-\lambda+1 \quad (>0).$$

Then multiplying the last inequality (2.2.27) by $|\zeta|^{\lambda+1}$, we obtain

$$\left|\mathcal{F}f(\zeta)\zeta^{\lambda+1}-a_0 p(\lambda)-\exp(iA\zeta)b_0 p(\lambda)\left(-\frac{\bar{\zeta}}{\zeta}\right)^{-\lambda-1}\right|\leq M|\zeta|^{-\varepsilon},$$

for any $\zeta=\xi+i\eta$ with $\eta>1$.

Suppose $\mathcal{F}f(\zeta)=0$, $\eta=\mathrm{Im}(\zeta)>1$, and $|\zeta|\geq K^{1/2}$. Then we have,

$$|p(\lambda)|\left|a_0+b_0\exp(iA\zeta)\left(-\frac{\bar{\zeta}}{\zeta}\right)^{-\lambda-1}\right|\leq M|\zeta|^{-\varepsilon}.$$

Transposing the first term to the other side, and then using the assumption $|\zeta|\geq K^{1/2}$, we get,

$$|b_0|\exp(-A\eta)\geq|a_0|-\frac{M|\zeta|^{-\varepsilon}}{p(\lambda)}\geq\frac{1}{2}|a_0|,$$

which implies (2.2.18)′. Proposition (2.2.16) is thus proved. Q.E.D.

Now we give the following typical examples of Proposition (2.2.6).

Example (2.2.28). Let $\lambda>-1$ and

$$f(x):=\begin{cases}(1-x^2)^{\lambda} & \text{if } |x|<1,\\ 0 & \text{if } |x|\geq 1.\end{cases}$$

Then $\mathcal{F}f(\zeta)=\sqrt{\pi}\,\Gamma(\lambda+1)\left(\frac{\zeta}{2}\right)^{-\lambda-1}J_{\lambda+1/2}(\zeta)$, and all the zeros of $\mathcal{F}f(\zeta)$ are real.

This is derived from the following well-known fact: *All the zeros of the ν-th Bessel function $J_\nu(\zeta)$ $(\nu>-1)$ are real.*

Example (2.2.29). Let $\lambda>-1$ and

$$f(x):=\begin{cases}\cos^{\lambda}x & \text{if } |x|<\pi/2.\\ 0 & \text{if } |x|\geq\pi/2.\end{cases}$$