Proper Actions and Representation Theory. IV
— Tempered homogeneous spaces

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Mini-courses of Mini-lectures
AIM Research Community
Representation Theory & Noncommutative Geometry
Organizers: P. Clare, N. Higson, and B. Speh
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Proper actions and representation theory

Plan

Tempered subgroup $(G \downarrow H)$ *vs* tempered homogeneous space

Recall from 3rd talk: Restriction of reps $G \downarrow H$

Definition (Margulis)* $G \supset H$ is a G-tempered subgroup

$\iff$ Matrix coefficients of $\widehat{G}_K \setminus \{\mathbf{1}\}$ are bounded uniformly by $^{\exists} q \in L^1(H)$ when restricted to the subgroup H.

Today: $G \curvearrowright (X, \mu)$ measure preserving. *E.g.,* $H \uparrow G$ for $X = G/H$.

Definition** X is a G-tempered space

$\iff$ The rep $G \curvearrowright L^2(X)$ is a tempered representation of G.

Note the terminology does NOT match exactly because $q \in L^1(H)$ in Definition 1.

* Margulis, Bull. Soc. Math. France **125** (1997), 447–456.

** Benoist–Kobayashi, Tempered homogeneous spaces I (2015), II (2022), III (2021), IV, arXiv:2009.10391.

Reminder of tempered representation — Definition

Let G be a locally compact group.

Def A unitary rep π of G is called tempered if $\pi \ll L^2(G)$.

$\ll$ $\cdots$ weakly contained

i.e., every matrix coefficient of π is a uniform limit on every compacta of G of a sequence of sum of coefficients of $L^2(G)$.

- Any unitary rep π can be disintegrated (Mautner) (*e.g.*, branching law, Plancherel-type theorem).

$\pi \simeq \int^{\oplus} \sigma$ with σ irreducible

π is tempered $\iff$ σ is tempered, almost everywhere

Just one irred non-tempered discrete spectrum σ would change the temperedness of π.

Irreducible tempered reps — semisimple Lie groups

Recall

<u>Def</u> A unitary representation π is called tempered if $\pi \ll L^2(G)$.

- For a solvable Lie group, all unitary reps π are tempered.
- For a semisimple Lie group G and irreducible $\pi \in \widehat{G}$, tempered representations π have been studied extensively.

<u>Known results</u> on tempered reps and <u>beyond</u> ...

- Many equivalent definitions, *e.g.*, $L^{2+\varepsilon}(G)$,
- Harish-Chandra's theory towards Plancherel formula,
- Knapp–Zuckerman's classification (∼1982),
- Building blocks of Langlands classification,
- Selberg $\frac{1}{4}$ eigenvalue conjecture (1965-),
- Gan–Gross–Prasad conjecture, $\cdots$

Tempered representations — Examples (irreducible cases)

V. Bargmann (1947): Irreducible unitary reps of $SL(2,\mathbb{R})$

$= \{ \mathbf{1} \} \amalg \{$ principal series $\} \amalg \{$ complementary series $\}$

$\amalg\{$ discrete series $\} \amalg \{$ limit of discrete series $\}$

$-\frac{1}{2}$ Casimir operator acts on them as scalars

$\{0\}$, $[\frac{1}{4}, \infty)$, $(0, \frac{1}{4})$, $\{\frac{1}{4}(n^2 - 1) : n \in \mathbb{N}_+\}$, $\{0\}$

Γ: congruence subgroup of $G = SL(2,\mathbb{R})$

Selberg's $\frac{1}{4}$ eigenvalue conjecture *:

All eigenvalues of Δ on Maas wave forms for $\Gamma \geq \frac{1}{4}$?

$\Leftrightarrow$ The unitary rep of $G \curvearrowright L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ is tempered ?

Just one irred non-tempered rep would deny the conjecture.

* A. Selberg, On the estimate of Fourier coefficients of modular forms, Proc. Symp. Pure Math. 1965.

Temperedness criterion for $G \curvearrowright L^2(G/H)$

$G \curvearrowright (X, \mu)$ measure preserving

$\rightsquigarrow G \curvearrowright L^2(X)$ unitary representation.

<u>Question 1</u> When is the unitary rep on $L^2(X)$ tempered?

If semisimple $G \curvearrowright X$ algebraically, then Question 1 is reduced to the case of principal orbits G/H :

<u>Question 1'</u> When is the unitary rep on $L^2(G/H)$ tempered?

<u>Examples</u>

1. H compact $\Longrightarrow L^2(G/H)$ is tempered.
2. H amenable $\Longrightarrow L^2(G/H)$ is tempered.

Tempered homogeneous space G/H, *i.e.*, $L^2(G/H) \ll L^2(G)$

Question 1' When is the unitary rep on $L^2(G/H)$ tempered?

Remark Do not confuse with a classical result that $L^2(X)$ can be disintegrated by irred X-tempered reps (this is almost 'tautology'). (Harish-Chandra, Oshima, Bernstein $\sim$ 80s).

Example (X-tempered reps $\neq$ tempered reps of G) Let $X = G/H$ be a reductive symmetric space. $L^2(X)$ can be disintegrated by irreducible X-tempered reps π, *i.e.*, those π satisfying if

$$\mathrm{Hom}_{\mathfrak{g},K}(\pi_K, C^\infty(X) \cap \bigcap_{\varepsilon>0} L^{2+\varepsilon}(X)) \neq \{0\}.$$

But we are interested in finding the criterion for $L^2(X) \ll L^2(G)$.

- Selberg $\frac{1}{4}$ eigenvalue conjecture

  $\Longleftarrow$ tempered rep of G, and not $\Gamma\backslash G$-tempered.

Proper actions and representation theory

Plan

Approach from dynamical system

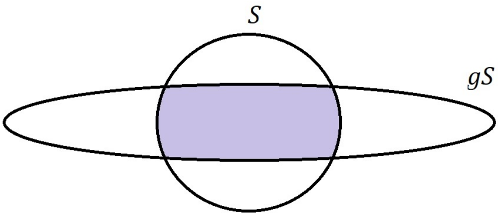
Locally compact group $G \curvearrowright X$ locally compact space

$G \curvearrowright X$ proper $\overset{\text{def}}{\Leftrightarrow} \{g \in G : S \cap gS \neq \emptyset\}$ is compact ${}^{\forall}S \subset X$ compact,

$\Leftrightarrow \mathrm{vol}(S \cap gS) \in C_c(G)$ ${}^{\forall}S \subset X$ compact.

Idea: Quantify non-properness of the actions.

Look at asymptotic behavior of $\mathrm{vol}(S \cap gS)$ as g goes to infinity.



Volume estimate $\mathrm{vol}(t \cdot S \cap S)$: Prototype $\mathbb{R} \curvearrowright \mathbb{R}^2$

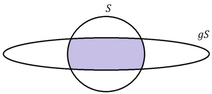
<u>Example</u> Let $G := \mathbb{R} \ni t$ act on $X = \mathbb{R}^2$ by

$$(x, y) \mapsto (e^t x, e^{-t} y)$$

- This action is not proper (1st talk).
- This action is measurably proper (3rd talk).
- <u>Asymptotic behavior of volume</u> <u>(today)</u>.

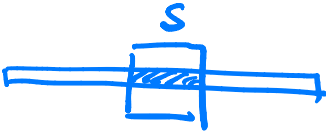
For any compact neighbourhood S of the origin in $\mathbb{R}^2$,

$$C_1 e^{-|t|} \le \mathrm{vol}(t \cdot S \cap S) \le C_2 e^{-|t|}.$$



E.g., if $S = \{(x, y) \in \mathbb{R}^2 : |x| \le 1, |y| \le 1\}$, then one has

$$\mathrm{vol}(t \cdot S \cap S) = 4e^{-|t|}.$$



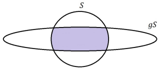
Volume estimate: Example $\mathbb{R} \curvearrowright \mathbb{R}^n$ and $SL(2,\mathbb{R}) \curvearrowright \mathbb{R}^n$

We let $\mathbb{R} \ni t$ act on $\mathbb{R}^n$ by

$$(x_1, x_2, \cdots, x_n) \mapsto (e^{(n-1)t} x_1, e^{(n-3)t} x_2, \cdots, e^{(1-n)t} x_n).$$

For any compact neighbourhood S of the origin in $\mathbb{R}^n$, one has

$$C_1 e^{-A(n)|t|} \leq \mathrm{vol}(t \cdot S \cap S) \leq C_2 e^{-A(n)|t|}.$$



n	2	3	4	5	6	$\cdots$
$\exp(-A(n)\|t\|)$	$e^{-\|t\|}$	$e^{-2\|t\|}$	$e^{-4\|t\|}$	$e^{-6\|t\|}$	$e^{-9\|t\|}$	$\cdots$

Since the Haar measure on $G = SL(2,\mathbb{R})$ is of the form

$$dg = \sinh 2t \, dk_1 dt dk_2 \quad \text{for } g = k_1 \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} k_2,$$

$\mathrm{vol}(g \cdot S \cap S) \in L^1(G)$ for any compact $S \subset \mathbb{R}^n \iff n \geq 4$,

$\mathrm{vol}(g \cdot S \cap S) \in L^{2+\varepsilon}(G)$ for any compact $S \subset \mathbb{R}^n \iff n \geq 2$

via an irreducible rep $\phi \colon G \to SL(n,\mathbb{R})$.

Piecewise linear function ρ_V

$\mathfrak{h}$: Lie algebra/$\mathbb{R}$

Definition For a finite-dimensional $\tau: \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$,

$$\rightsquigarrow \quad \rho_V : \mathfrak{h} \to \mathbb{R}_{\geq 0}, \quad Y \mapsto \frac{1}{2} \sum |\mathrm{Re}\, \lambda(Y)|.$$

gen. eigenvalues of $\tau(Y) \in \mathrm{End}(V_{\mathbb{C}})$

Let $\mathfrak{a} \subset \mathfrak{h}$ a maximal split abelian subalgebra.
Then ρ_V is determined by its restriction to $\mathfrak{a}$, and $\rho_V|_{\mathfrak{a}}$ is a piecewise linear function.

Remark For $\mathfrak{h}$ semisimple and for $(\tau, V) = (\mathrm{ad}, \mathfrak{h})$,
$\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ = twice the usual ρ on the dominant Weyl chamber,
however, our $\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ is not linear whereas the usual ρ is linear.

Baby case: $H \curvearrowright V$ (linear action)

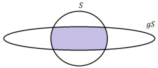
Let $H \subset SL_{\mathbb{R}}(V)$, and $\mathfrak{a}$ a maximally split abelian subalgebra of $\mathfrak{h}$.

<u>Definition</u> $p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$.

<u>Proposition 1</u> Suppose H is reductive in $SL_{\mathbb{R}}(V)$, and $p > 0$.
One has the equivalence (i) $\iff$ (ii).
(i) $p > p_V$.
(ii) $\mathrm{vol}(hS \cap S) \in L^{p+\varepsilon}(H)$ for any compact set $S \subset V$, $^\forall \varepsilon > 0$.

Proof Similarly to the aforementioned $SL(2,\mathbb{R})$ example, one has

$$\mathrm{vol}(hS \cap S) \sim e^{-\rho_V(Y)} \quad \text{for } h = k_1 e^Y k_2.$$



The Haar measure on $H \ni h = k_1 e^Y k_2$ is of the form (not precise)

$$dh \sim e^{+\rho_{\mathfrak{h}}(Y)} dk_1 dY dk_2.$$

Example: $H = SL(n,\mathbb{R}) \curvearrowright \mathbb{R}^n = V$

$$
\begin{aligned}
p_V :=& \max_{Y \in \mathfrak{a}\setminus\{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} \\
=& 2 \max_{x_1+\cdots+x_n=0} \frac{\sum_{1\le i<j\le n} |x_i - x_j|}{\sum_{i=1}^{n} |x_i|} \\
=& 2(n-1).
\end{aligned}
$$

Proposition 1 implies that for any compact $S \subset \mathbb{R}^n$,

$$\mathrm{vol}(gS \cap S) \in L^{2(n-1)+\varepsilon}(SL(n,\mathbb{R})) \quad {}^\forall \varepsilon > 0.$$

Temperedness criterion : Baby case $H \curvearrowright V$ linear

Let $\phi\colon H \to SL_{\pm}(V)$ be a finite-dimensional rep $/\mathbb{R}$.

Theorem 1* Suppose that H is a real reductive linear group.
One has the equivalence (i) $\iff$ (ii)
(i) (rep theory) The unitary rep $\pi\colon H \curvearrowright L^2(V)$ is tempered.
(ii) (criterion) $p_V \le 2$, *i.e.*, $\rho_{\mathfrak{h}}(Y) \le 2\rho_V(Y) \quad {}^{\forall} Y \in \mathfrak{h}$.

Proof. Observe that for $\varphi_1, \varphi_2 \in C_c(X)$ one has

$$(\pi(g)\varphi_1, \varphi_2)_{L^2(X)} \le \|\varphi_1\|_\infty \|\varphi_2\|_\infty \operatorname{vol}(gS \cap S)$$

where $S := \operatorname{Supp}\varphi_1 \cup \operatorname{Supp}\varphi_2$ $(\subset X)$. Hence if $\operatorname{vol}(gS \cap S) \in L^p(G)$ then the matrix coefficient $(\pi(g)\varphi_1, \varphi_2) \in L^p(G)$.
Now combine Proposition 1 for $p = 2 + \varepsilon$ $(\varepsilon > 0)$ and a theorem of Cowling–Haagerup–Howe below. □

Definition $p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$.

*

Tempered rep *vs* $L^{2+\varepsilon}(G)$ (matrix coefficients)

Recall A unitary rep π of G is tempered. $\underset{\text{def}}{\iff}$ $\pi \prec L^2(G)$.

Theorem 2 (Cowling–Haagerup–Howe)* For a unitary rep $(\pi, \mathcal{H})$ of a semisimple Lie group G, one has (i) $\Leftrightarrow$ (ii).

(i) π is tempered.

(ii) π is almost L^2, *i.e.,* there exists a dense subspace D in $\mathcal{H}$ such that

$$g \mapsto (\pi(g)u_1, u_2) \text{ belongs to } L^{2+\varepsilon}(G) \qquad ({}^\forall\varepsilon > 0, {}^\forall u_1, {}^\forall u_2 \in D).$$

Remark (i) $\Longrightarrow$ (ii) fails if G is non-compact amenable group for which $\mathbf{1}$ is a tempered representation, but is not almost L^2.

* M. Cowling–M. Haagerup– R. Howe, Almost L^2 matrix coefficients, J. Reine Angew. Math. **387**, (1988), 97–110.

The constant p_V appears also in restriction $SL(V) \downarrow H$

For $H \subset SL_{\mathbb{R}}(V)$, recall $p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} \quad (< \infty)$.

This number p_V appeared in Theorem 1 for $H \curvearrowright L^2(V)$:

$p_V \leq 2 \iff H \curvearrowright L^2(V)$ is a tempered rep of H.

The same p_V appears also for the restriction $SL(V) \downarrow H$.

<u>Theorem 3</u> ($SL(V) \downarrow H$) Let H be a reductive subgroup of $SL_{\mathbb{R}}(V)$. Then one has the equivalence:

(1) $p_V < 1 \iff$ H is an <u>$SL_{\mathbb{R}}(V)$-tempered subgroup</u> in the sense of Margulis (3rd talk).

(2) $p_V \leq 2 \iff$ $\pi|_H$ is a <u>tempered unitary rep of H</u> for all $\pi \in \widehat{SL_{\mathbb{R}}(V)} \setminus \{\mathbf{1}\}$.

* Margulis, Bull. Soc. Math. France **125** (1997), 447–456.

Example: $H := SL(p,\mathbb{R}) \times SL(q,\mathbb{R}) \times SL(r,\mathbb{R})$

Consider two homomorphisms:

$$H \hookrightarrow SL(p+q+r,\mathbb{R}) \qquad =: G, \tag{1}$$

$$H \to SL(pq+qr+rp,\mathbb{R}) =: \widetilde{G}. \tag{2}$$

(2) is defined via $\mathbb{R}^{pq+qr+rp} \simeq (\mathbb{R}^q)^* \otimes \mathbb{R}^p \oplus (\mathbb{R}^r)^* \otimes \mathbb{R}^p \oplus (\mathbb{R}^q)^* \otimes \mathbb{R}^p$.

We discuss three unitary reps $H \curvearrowright L^2(V)$, $\widetilde{G} \downarrow H$ and $L^2(G/H)$:

<u>Example</u> One has an equivalence (i) ⇔ (ii) ⇔ (iii) ⇔ (iv):

(i) $H \curvearrowright L^2(\mathbb{R}^{pq+qr+rp})$ is a tempered rep of H.

(ii) For any irred unitary rep π $(\neq \mathbf{1})$ of $\widetilde{G} = SL(pq+qr+rp,\mathbb{R})$, the restriction $\pi|_H$ via (2) is a tempered representation of H.

(iii) $L^2(G/H)$ is a tempered rep of $G = SL(p+q+r,\mathbb{R})$.

(iv)* $2\max(p,q,r) \le p+q+r+1$.

(i) $\cdots$ Theorem1, (ii) $\cdots$ Theorem 3, (iii) $\cdots$ Theorem 4 below.

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces III, J. Lie Theory (2021) for (iii) ⇔ (iv).

General case: Asymptotic estimate of $\mathrm{vol}(gS \cap S)$ for $G \curvearrowright G/H$

$G \supset H$ real reductive groups, $G \curvearrowright X = G/H$.

For any compact $S \subset G/H$, we want to find $m(g)$ and $M(g)$:

$m(g) \leq \mathrm{vol}(gS \cap S) \leq M(g)$ for all $g \in G$.

One could find a lower bound $m(h)$ for $h \in H$ by the linear case:

$$H \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}/\mathfrak{h} \underset{\text{infinitesimally}}{\fallingdotseq} G/H.$$

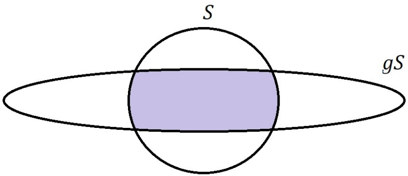
Some difficulties

- Need a lower bound $m(g)$ for $g \in G$, not only for $h \in H$.
- Find an upper bound $M(g)$.

General case — global estimate of volume

$G \curvearrowright X$, $S \subset X$ compact subset.

Asymptotic behavior of $\mathrm{vol}(S \cap gS)$ as $g \in G$ tends to the infinity.



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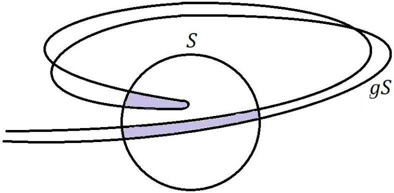


G/H

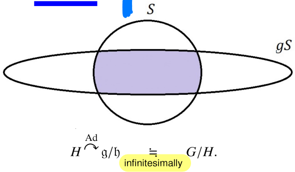
General case: Asymptotic estimate of $\mathrm{vol}(gS \cap S)$ for $G \curvearrowright G/H$

$G \supset H$ real reductive groups, $G \curvearrowright X = G/H$.

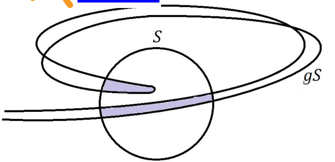
For any compact $S \subset G/H$, we want to find $m(g)$ and $M(g)$:

$$m(g) \le \mathrm{vol}(gS \cap S) \le M(g) \quad \text{for all } g \in G.$$

for $h \in H$



? $h \in H$



Some difficulties

- Need a lower bound $m(g)$ for $g \in G$, not only for $h \in H$.
- Find an upper bound $M(g)$.

Use an idea of 1st lecture (about $\pitchfork$ and $\sim$).

Temperedness criterion for $L^2(V)$

$G \supset H$: real reductive algebraic groups.

Recall $p_{\mathfrak{g}/\mathfrak{h}} = \max\limits_{\mathfrak{h} \ni Y \neq 0} \dfrac{\rho_{\mathfrak{h}}(Y)}{\rho_{\mathfrak{g}/\mathfrak{h}}(Y)}$ is defined for $H \curvearrowright \mathfrak{g}/\mathfrak{h}$.

<u>Proposition</u> For any $p \geq 1$, one has the equivalence (i) $\iff$ (ii):

(i) $p > p_{\mathfrak{g}/\mathfrak{h}} + 1$

(ii) $\mathrm{vol}(gS \cap S) \in L^p(G)$ $\quad {}^\forall S \subset G/H$ compact.

<u>Theorem 4</u>* One has the equivalence (i) $\iff$ (ii):

(i) $p_{\mathfrak{g}/\mathfrak{h}} \leq 1$, namely, $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ on $\mathfrak{h}$.

(ii) $G \curvearrowright L^2(G/H)$ is a tempered rep of G.

<u>Remark</u> $\rho_{\mathfrak{g}} = \rho_{\mathfrak{h}} + \rho_{\mathfrak{g}/\mathfrak{h}}$ on $\mathfrak{h}$.

* Y. Benoist, T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

Temperedness criterion — general case

G, H reductive $\underset{\text{generalization}}{\rightsquigarrow}$ G, H algebraic

G : real algebraic Lie group $\supset$ G_{ss}: max semisimple subgroup,
H : algebraic subgroup.

$$G_{ss} \subset G \curvearrowright L^2(G/H)$$

Theorem 5* $L^2(G/H)$ is G_{ss}-tempered $\iff$ $2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}}$ on $\mathfrak{h}$.

Method of Proof

Theorem 4 (dynamical approach + geometry)

+ Herz majoration principle

+ $\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h}$ (“limit algebra”).

* Benoist–Kobayashi, Tempered homogeneous spaces IV, arXiv:2009.10391.

A tour around temperedness criterion 1: symmetric space G/H

The Plancherel theorem* for G/H gives a unitary equivalence:

$$L^2(G/H) \simeq \bigoplus_{j=1}^{N} \int_{\nu}^{\oplus} \sum_{\lambda}^{\oplus} \mathrm{Ind}_{L_j N_j}^{G} (\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)}) d\nu.$$

$\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)} \cdots$ relative discrete series for $L_j/(L_j \cap H)$.

$L^2(G/H)$ is tempered	$\overset{\text{Plancherel}}{\Longleftrightarrow}$	$\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)}$ is a tempered rep of L_j for all λ, a.e. ν,
$\not\Uparrow$ $\Downarrow$ Theorem 4 + combinatorics**		$\Downarrow$ obvious
the set of points in $\mathfrak{g}/\mathfrak{h}$ with amenable stabilizer in H in dense.	$\overset{\text{Quantization}}{\Longleftrightarrow}$	$\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)}$ is a tempered rep of L_j, $^\forall \lambda \gg 0$, a.e. ν.

- Difficulties arising from "singular" λ.

* T. Oshima; Delorme, Ann. Math. 1998; van den Ban–Schlichtkrull, Invent. Math. 2005.

** Y. Benoist–T. Kobayashi, Tempered homogeneous spaces III, J. Lie Theory (2021).

Delicate example: reductive symmetric case

$$G/H := Sp(p_1 + p_2, q_1 + q_2)/(Sp(p_1, q_1) \times Sp(p_2, q_2))$$
$$(p_1 \geq 1, q_1 \geq 1, p_1 + q_1 = p_2 + q_2 + 1).$$

- A "large part" of $\pi \in \widehat{G}$ in the support of the Plancherel formula for G/H is a tempered representation of G.

$$L^2(G/H) \simeq \bigoplus_{j=1}^{N} \int_{\nu}^{\oplus} \sum_{\lambda}^{\oplus} \mathrm{Ind}_{L_j N_j}^{G}(\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)}) d\nu.$$

$\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)} \cdots$ relative discrete series for $L_j/(L_j \cap H)$.

- But $G \curvearrowright L^2(G/H)$ is NOT a tempered unitary representation by Theorem 4. This means that at least one of singular representations $\tau_\lambda^{(j)}$ of $[L_j, L_j]$ (Zuckerman's $A_{\mathfrak{q}}(\lambda)$ modules outside the weakly good range of parameter λ) survive after wall crossings and non-tempered rep $\mathrm{Ind}_{L_j N_j}^{G}(\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)})$ contribute to $L^2(G/H)$.

A tour around temperedness criterion 2: Vogan's theory

The temperedness criterion $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ may remind us a formula

$$\underbrace{\lambda}_{\text{Harish-Chandra parameter}} \rightsquigarrow \underbrace{\lambda + \rho_{\mathfrak{g}} - 2\rho_{\mathfrak{k}}}_{\text{Blattner parameter}}$$

for discrete series representations of G.

One may recall Vogan's thesis (1976)* of the classification of irreducible admissible representations of real reductive groups by using "lowest (minimal) K-type".

* Vogan, Green book, Chapter 6, 1981; D. Vogan, The algebraic structure of the representation of semisimple Lie groups I, Ann. Math. (1979) 1–60.

Application of Vogan's theory of minimal K-type

minimal K-type μ $\overset{\text{Vogan}^*}{\rightsquigarrow}$ $\lambda = \mathrm{proj}(\mu + 2\rho_{\mathfrak{k}} - \rho_{\mathfrak{g}})$

$\rightsquigarrow$ θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ in $\mathfrak{g}_{\mathbb{C}}$.

Then $\mathfrak{l}$ is quasi-split, in particular, if $\lambda = 0$, $\mathfrak{g}$ is quasi-split.

Corollary Let G/H be a reductive symmetric space with H split.
$G \curvearrowright L^2(G/H)$ is tempered $\iff$ $G_{\mathbb{R}}$ is quasi-split.

$G_{\mathbb{R}}$ is defined by the pair (G, H) via

$G/H \underset{\text{complexification}}{\subset} G_{\mathbb{C}}/H_{\mathbb{C}} \underset{\text{real form}}{\supset} G_{\mathbb{R}}/K_{\mathbb{R}}$ ($K_{\mathbb{R}}$: max compact subgp).

Proof. $L^2(G/H)$ is a tempered rep of G $\underset{\text{Theorem 4}}{\iff}$ $2\rho_{\mathfrak{h}_{\mathbb{C}}} \leq \rho_{\mathfrak{g}_{\mathbb{C}}}$

$\underset{\text{Vogan's theory}}{\iff}$ $G_{\mathbb{R}}$ is quasi-split.

Ex. $GL(p+q,\mathbb{R})/GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$ is tempered $\Leftrightarrow$ $U(p,q)$ is quasi-split $\Leftrightarrow$ $|p-q| \leq 1$.

$*$ Vogan, Green book, Chapter 6, 1981; D. Vogan, The algebraic structure of $\cdots$, Ann. Math. (1979).

A tour around temperedness criterion 3: Orbit philosophy

Orbit philosophy by Kirillov–Kostant

$$
\begin{array}{ccc}
\mathrm{Supp}(L^2(G/H)) & \fallingdotseq & \mathrm{Ad}^*(G)\mathfrak{h}^\perp / \mathrm{Ad}^*(G) \\
\cap & & \cap \\
\widehat{G} & \fallingdotseq & \mathfrak{g}^* / \mathrm{Ad}^*(G) \\
\cup & & \cup \\
\widehat{G}_{\mathrm{temp}} & \fallingdotseq & \mathfrak{g}^*_{\mathrm{reg}} / \mathrm{Ad}^*(G)
\end{array}
$$

$\mathfrak{g}^* \supset \mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G)\lambda$ is of maximal dimension$\}$,

$\mathfrak{g}^* \supset \mathfrak{h}^\perp := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}$.

Theorem 6* Suppose G is a complex reductive group, and H an algebraic subgroup. Then temperedness criterion (Theorem 4) gives the equivalence:

$$G \curvearrowright L^2(G/H) \text{ is tempered} \iff \mathfrak{h}^\perp \cap \mathfrak{g}^*_{\mathrm{reg}} \underset{\text{dense}}{\subset} \mathfrak{h}^\perp.$$

* Benoist–Kobayashi, Tempered homogeneous spaces IV, arXiv:2009.10391.

Yet another relation: proper actions and representation theory

Example $Spin(1,8) \subset SO(8,8) \supset SO(7,8) \rightsquigarrow X = SO(8,8)/SO(7,8)$.

(1)* There exists a compact 15-dim'l pseudo-Riemannian manifold of signature (8,7) with constant sectional curvature -1.

(2)**,† The restriction $\pi|_{Spin(1,8)}$ decomposes discretely and multiplicity freely for any $\widehat{SO(8,8)} \ni \pi \hookrightarrow L^2(X)$.

$G' \subset G \supset H$ real reductive groups $\rightsquigarrow$ $G' \subset G \curvearrowright X := G/H$

Theorem 7 Assume G' acts properly and spherically on $X = G/H$.

(1)(geometry)* X admits a cocompact discontinuous group.

(2)(rep)** Any $\pi \in \widehat{G}$ realized in $\mathcal{D}'(X)$ is discretely decomposable with finite multiplicity when restricted to the subgroup G'.

* Theorem 4 in 2nd lecture;

** T. Kobayashi, Global analysis by hidden symmetry, Progr. Math. (2017), (Howe 70th birthday volume);

† Schlichtkrull–Trapa–Vogan, São Paulo J. Math. Sci. (2018).

Proper actions and representation theory

Plan

1. Discontinuous dual and properness criterion (4/25)
2. The Mackey analogy and proper actions (5/2)
3. Tempered subgroups à la Margulis (5/9)
4. Tempered homogeneous spaces (5/16)

Thank you very much!

Another direction of generalization — Almost L^p representation

G: real reductive $\supset H$ real reductive, $p \in 2\mathbb{N}$.

Theorem 3′ $L^2(G/H)$ is almost $L^p \iff \frac{p}{p-1}\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ on $\mathfrak{h}$.

Example (arXiv:2108.12125) $G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})$
The smallest even integer p for which $L^2(G/H)$ is almost L^p amounts to $p = 2[\frac{n-1}{2(n-m)}]$ with $m = \max(n_1, \cdots, n_r)$.