Proper Actions and Representation Theory. II
—The Mackey analogy and proper actions

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Mini-courses of Mini-lectures
AIM Research Community
Representation Theory & Noncommutative Geometry
Organizers: P. Clare, N. Higson, and B. Speh
May 2, 2022

100 years ago · · ·

150 years ago Klein's Erlangen program (1872)

60 years ago Calabi–Markus (Ann Math 1962)

Any de Sitter manifold is non-compact.

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100 years ago Radon–Hurwitz number (1922)

Radon–Hurwitz number (1922)

One has the following formulæ:

$$(a^2+b^2)(x^2+y^2) = (\underline{ax-by})^2 + (\underline{ay+bx})^2$$

$$(a^2+b^2)(x^2+y^2+z^2+w^2) = (\underline{ay+bz})^2 + (\underline{ax+bw})^2 + (\underline{-aw+bx})^2 + (\underline{az-by})^2$$

$$(a^2+b^2+c^2)(x^2+y^2+z^2+w^2) = (\underline{ay+bz+cw})^2 + (\underline{ax+bw-cz})^2 + (\underline{-aw+bx+cy})^2 + (\underline{az-by+cx})^2$$

However, no such formula for

$$(a^2+b^2)(x^2+y^2+z^2) = (\underline{\quad})^2 + (\underline{\quad})^2 + (\underline{\quad})^2$$

bilinear forms on $\{(a,b,c,\cdots)\} \times \{(x,y,z,w,\cdots)\}$

Radon–Hurwitz number (1922)

$$(a^2+b^2)(x^2+y^2) = (ax-by)^2 + (ay+bx)^2$$

$$(a^2+b^2)(x^2+y^2+z^2+w^2) = (ay+bz)^2 + (ax+bw)^2 + (-aw+bx)^2 + (az-by)^2$$

$$(a^2+b^2+c^2)(x^2+y^2+z^2+w^2) = (ay+bz+cw)^2 + (ax+bw-cz)^2 + (-aw+bx+cy)^2 + (az-by+cx)^2$$

bilinear forms on $\{(a,b,c,\cdots)\} \times \{(x,y,z,w,\cdots)\}$

Question For which pairs (p,q) does there exist a bilinear map $f\colon \mathbb{R}^{p+1} \times \mathbb{R}^q \to \mathbb{R}^q$ such that

$$\|v\|\|w\| = \|f(v,w)\| \quad {}^\forall v \in \mathbb{R}^{p+1}, {}^\forall w \in \mathbb{R}^q.$$

Example $(p,q) = (1,2), (1,4), (2,4)$

Example $p+1 = q \in \{1,2,4,8\}$ corresponding to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

Observation (p,q) OK $\Rightarrow (p',q)$ OK $\quad {}^\forall p' \leq p$

Radon–Hurwitz number (1922)

Definition We define the Radon–Hurwitz number $\rho(q)$ defined by

$$\rho(q) := 8\alpha + 2^{\beta} \quad \text{if } q = 2^{4\alpha+\beta} \times \text{(odd integer)}.$$

Theorem 1 (Radon* (1922), Hurwitz* (1923))
The following two conditions on (p, q) are equivalent:

(i) There exists a bilinear map $f : \mathbb{R}^{p+1} \times \mathbb{R}^{q} \to \mathbb{R}^{q}$ such that

$$\|v\|\|w\| = \|f(v, w)\| \quad {}^{\forall}v \in \mathbb{R}^{p+1}, {}^{\forall}w \in \mathbb{R}^{q}.$$

(ii) $p < \rho(q)$.

p	$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
q	0	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	$4\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$16\mathbb{N}$	$32\mathbb{N}$	$64\mathbb{N}$	$\cdots$

* J. Radon, Abh. math. Sem. Hamburg **1** (1922); ** A. Hurwitz, Math. Ann. (1923).

Proper actions and representation theory

Plan

1. Discontinuous dual and properness criterion (4/25)
2. The Mackey analogy and proper actions (5/2)
3. Tempered subgroups (5/9)
4. Tempered homogeneous spaces (5/16)

Space forms in pseudo-Riemannian geometry (definition)

(M, g) : pseudo-Riemannian manifold of signature (p, q),
geodesically complete

Def. (M, g) is a space form
$\Longleftrightarrow$ sectional curvature κ is constant

J. A. Wolf, Spaces of Constant Curvature, 6th ed. AMS, 2011

Space forms (examples)

Space form $\cdots$ $\begin{cases} \text{Signature } (p,q) \text{ of pseudo-Riemannian metric} \\ \text{Curvature } \kappa \in \{+,0,-\} \end{cases}$

<u>Example</u> $q = 0$ (Riemannian manifold)

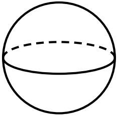
sphere S^n	$\mathbb{R}^n$	hyperbolic sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

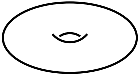
<u>Example</u> $q = 1$ (Lorentzian manifold)

de Sitter sp	Minkowski sp	anti-de Sitter sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$O(n+1,1)/O(n,1)$		$O(n,2)/O(n,1)$

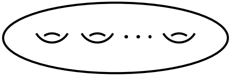
2-dim'l compact space forms

Riemannian case ($\Longleftrightarrow$ signature $(2,0)$)





Σ_g $(g \geq 2)$



curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

Lorenzian case ($\Longleftrightarrow$ signature $(1,1)$)

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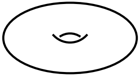

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curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

There do NOT exist compact forms if $\kappa > 0$ and $\kappa < 0$

Local to global problem in pseudo-Riemannian geometry

Space form problem for pseudo-Riemannian manifolds

Local Assumption

signature (p, q), curvature $\kappa \in \{+, 0, -\}$

$\Downarrow$

Global Results

- Do compact forms exist?
- What groups can arise as their fundamental groups?

Formulation in group language

Suppose $p \neq 1$ for simplicity.

Uniformization theorem: Any pseudo-Riemannian manifold M of signature (q, p) with $\kappa \equiv -1$ is of the form

$$\Gamma \backslash O(p+1, q)/O(p, q)$$

where Γ is a discontinuous group for $O(p+1, q)/O(p, q)$.

Γ is responsible for global properties:

e.g., $\#\Gamma = \infty$, $\Gamma \simeq \pi_1(\Sigma_g)$, Γ cocompact, etc.

Definition (discontinuous group for X) For a G-space X, we say Γ is a discontinuous group for X if Γ is a discrete subgroup of G and the Γ-action on X is proper.

Global properties of Space forms $\kappa \equiv -1$, signature (q, p)

Theorem 2* Let $G/H = O(p+1, q)/O(p, q)$.

(1) (Calabi–Markus phenomenon) G/H admits an infinite discontinuous group if and only if $p < q$.

(2) If G/H admits a cocompact discontinuous group, then either $q = 0$ or "$p < q$ and q is even".

(3) If (p, q) is in the table below, then G/H admits a cocompact discontinuous group.

p	$\mathbb{N}$	0	1	3	7
q	0	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	8

Conjecture** The converse of (3) holds.

* Calabi–Markus (1962), Wolf (1962), Kulkarni (1981), Kobayashi (1996), Tholozan (2015), Morita (2017).

** T. Kobayashi, Conjectures on reductive homogeneous spaces, arXiv:2204.08854.

Cohomological dimension $\mathrm{cd}(\Gamma)$

Consider $\mathbb{Z}^k \curvearrowright \mathbb{R}^n$ affine action

Observation

(1) If $\mathbb{Z}^k$ acts properly discontinuously on $\mathbb{R}^n$, then $k \leq n$.

(2) Furthermore, $\mathbb{Z}^k$ acts cocompactly on $\mathbb{R}^n$ if and only if $k = n$.

- What is k for $\mathbb{Z}^k$? Use cohomology of abstract groups.
- What is n for $\mathbb{R}^n$? (next slide)

Definition (cohomological dimension)

For an abstract group Γ, we define

$\mathrm{cd}_{\mathbb{R}}(\Gamma)$:= the projective dimension of $\mathbb{R}$ as the trivial $\mathbb{R}[\Gamma]$-module

$= \sup\{n \in \mathbb{N} : H^n(\Gamma; A) \neq 0$ for some left $\mathbb{R}[\Gamma]$-module $A\}$.

Example $\mathrm{cd}_{\mathbb{R}}(\mathbb{Z}^k) = k$

$\mathrm{cd}_{\mathbb{R}}(\pi_1(\Sigma_g)) = 2$

Handy criterion for cocompactness

K : maximal compact subgroup of a connected reductive Lie group G.

$G \approx K$ homotopy equivariant

$d(G) := \dim G - \dim K$ ("non-compactness dimension").

> Proposition*,** Suppose Γ is a torsion-free discontinuous gp for G/H.
> (1) $\mathrm{cd}_{\mathbb{R}}(\Gamma) \le d(G) - d(H)$.
> (2) The equality holds iff Γ is cocompact (i.e. $\Gamma \backslash G/H$ is compact).

Use Serre's spectral sequence for

$$G/H \simeq K \underset{H \cap K}{\times} \mathbb{R}^{d(G)-d(H)} \underset{\text{homotopic}}{\approx} K/H \cap K$$

$$\downarrow$$

$$\Gamma \backslash G/H$$

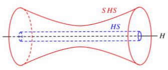
if H is reductive such that $H \cap K$ is a maximal compact subgp in H.

* J. P. Serre, Cohomologie des groupes discrètes, Annals of Math. Studies, 1971;

** Kobayashi, Proper action on homogeneous spaces of reductive type, Math. Ann. (1989).

Cohomological obstruction to cocompact discont gp

Recall $H \sim H' \overset{\text{def}}{\iff} H \subset SH'S$ and $H' \subset SHS$
($^{\exists}$ compact set $S \subset G$)



Theorem 3* (non-existence) If there exists $H' \subset G$ such that
$H \sim H'$ and $d(H) < d(H')$,
then G/H does not admit a cocompact discontinuous group Γ.

Proof If such Γ existed, then

$$\Gamma \pitchfork H \quad \text{and} \quad \Gamma \pitchfork H' \text{ because } H \sim H',$$

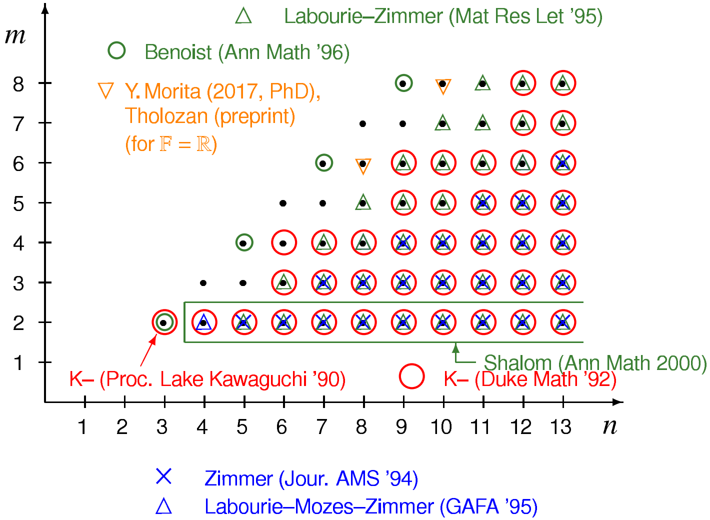
$$\mathrm{cd}_{\mathbb{R}}(\Gamma) = d(G) - d(H) > d(G) - d(H') \geq \mathrm{cd}_{\mathbb{R}}(\Gamma),$$

which would be a contradiction.

Example** $(G, H) = (SL(3,\mathbb{C}), SL(2,\mathbb{C}))$
Take $H' = SU(2,1)$. Then $H \sim H'$ and $d(H') = 4 > d(H) = 3$.
$\Rightarrow$ $SL(3,\mathbb{C})/SL(2,\mathbb{C})$ does not admit a cocompact discontinuous gp.

* T. Kobayashi, Duke Math. J., 1992; ** T. Kobayashi, Proc. ICM-1990 satellite (Kawaguchi Lake).

Nonexistence of Compact quotients for $SL(n, \mathbb{F})/SL(m, \mathbb{F})$



The figure is for $\mathbb{F} = \mathbb{C}$. Some results depend slightly on $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Construction of cocompact discontinuous groups

Suppose that $G \supset H$ are a pair of real reductive linear Lie groups.

Theorem 4* (existence)
If there exists a reductive subgroup L of G such that

$$d(L) + d(H) = d(G) \quad \& \quad \mathfrak{a}_L \cap W\mathfrak{a}_H = \{0\}$$

then both G/H and G/L admits cocompact discontinuous groups.

Example $\mathbb{Z}^k \subset \mathbb{R}^k \curvearrowright \mathbb{R}^n/\mathbb{R}^\ell$ if $k + \ell = n$.
$\Gamma \subset L \curvearrowright G/H$

Proof. Properness criterion ("Theorem 4" in 1st talk)

$$L \pitchfork H \iff H \pitchfork L \iff \mathfrak{a}_L \cap W\mathfrak{a}_H = \{0\}.$$

Take a cocompact lattice Γ in L, which exists by Borel's theorem **.
Then Γ is a cocompact discontinuous group for G/H.

* Kobayashi, Proper action on homogeneous spaces of reductive type, Math. Ann. (1989).

** A. Borel, Compact Clifford–Klein forms of symmetric spaces, Topology **2** (1963).

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Cartan motion group G_θ

G : real reductive linear Lie group

θ : Cartan involution, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$

$K = \{g \in G : \theta g = g\}$ maximal compact subgroup of G

$G = K \exp \mathfrak{p}$

Definition (Cartan motion group G_θ)

$$G_\theta := K \ltimes \mathfrak{p}$$

$$(k_1, X_1) \cdot (k_2, X_2) := (k_1 k_2, X_1 + \mathrm{Ad}(k_1) X_2)$$

$\mathfrak{g}_\theta = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{p}$ abelian subalgebra.

cf. Mackey analogy for the unitary dual

$$\widehat{G} \longleftrightarrow \widehat{G_\theta}$$

Mackey analogy — from geometric viewpoints

$$G = K \exp \mathfrak{p} \ \rightsquigarrow\ G_\theta = K \ltimes \mathfrak{p}$$ (Cartan motion group)

Example Riemannian manifold with constant sectional curvature κ

$\kappa < 0$	$\kappa = 0$	$\kappa > 0$
hyperbolic space H^n	$\mathbb{R}^n$	sphere S^n
$\Vert$	$\Vert$	$\Vert$
G/K	G_θ/K	G_U/K

κ

$$G = \mathrm{Isom}(H^n)_0 = K \exp \mathfrak{p} \simeq SO_0(n,1)$$

change

$$G_\theta = \mathrm{Isom}(\mathbb{R}^n)_0 = K \ltimes \mathfrak{p} \simeq SO(n) \ltimes \mathbb{R}^n$$

change

$$G_U = \mathrm{Isom}(S^n)_0 = K \exp(\sqrt{-1}\mathfrak{p}) \simeq SO(n+1)$$

"Tangential homogeneous space" G_θ / H_θ

Let H be a reductive subgroup of G.
Take a Cartan involution θ such that $\theta H = H$.

G		G_θ
reductive Lie gp	$\Longrightarrow$	Cartan motion gp

$$G = K \exp \mathfrak{p} \quad \Longrightarrow \quad G_\theta = K \ltimes \mathfrak{p}$$

$$H = (H \cap K) \exp(\mathfrak{h} \cap \mathfrak{p}) \quad \Longrightarrow \quad H_\theta = (H \cap K) \ltimes (\mathfrak{h} \cap \mathfrak{p})$$

$$G/H \quad \Longrightarrow \quad G_\theta / H_\theta$$

Mackey analogy for $G_\theta/H_\theta \longleftrightarrow G/H$

G	real reductive	$\rightsquigarrow$	$G_\theta = K \ltimes \mathfrak{p}$
$\cup$			$\cup$
L, H	reductive subgps	$\rightsquigarrow$	L_θ, H_θ
(θ-stable)			

Proposition 5 (Mackey analogy)*

(1) $^\exists K$-equivariant diffeomorphism $G/H \simeq G_\theta/H_\theta$.

(2) $L \pitchfork H$ in G $\Longleftrightarrow$ $L_\theta \pitchfork H_\theta$ in G_θ.

(3) $L \sim H$ in G $\Longleftrightarrow$ $L_\theta \sim H_\theta$ in G_θ.

(4) (Calabi–Markus phenomenon)

No infinite discont gp for G/H $\Longleftrightarrow$ No infinite discont gp for G_θ/H_θ.

(5) $L \curvearrowright G/H$ properly and cocompactly $\Longleftrightarrow$ $L_\theta \curvearrowright G_\theta/H_\theta$ properly and cocompactly.

* T. Kobayashi, T. Yoshino, Compact Clifford Klein forms of symmetric spaces — revisited, Pure and Appl. Math. Quarterly **1**, (2005), 603–684.

Compare G_θ/H_θ and G/H (space form conjecture)

Let $G/H = O(p+1,q)/O(p,q)$.

G/H is a $(p+q)$-dimensional space form with signature (q,p),

with sectional curvature $\kappa \equiv -1$

<u>Conjecture</u>* G/H admits a cocompact discontinuous group if and only if (p,q) is in the following list.

p	$\mathbb{N}$	0	1	3	7
q	0	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	8

"if" part is true (Theorem 2).

* T. Kobayashi, Conjectures on reductive homogeneous spaces, arXiv:2204.08854.

Existence of cocompact discontinuous group for G_θ/H_θ

$$G/H = O(p,q+1)/O(p,q) \rightsquigarrow G_\theta/H_\theta$$

space form

<u>Theorem 6</u>*

There exists a cocompact discontinuous group for G_θ/H_θ

$\Longleftrightarrow p < \rho(q)$ (Radon–Hurwitz number)

p	$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
q	0	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	$4\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$16\mathbb{N}$	$32\mathbb{N}$	$64\mathbb{N}$	$\cdots$

* T. Kobayashi, T. Yoshino, Pure and Appl. Math. Quarterly **1**, (2005), 603–684. Special Issue: In Memory of A. Borel.

Key lemma for Theorem 6

Let G_θ/H_θ be the "tangential homogeneous sp" of $G/H = O(p+1,q)/O(p,q)$

Proposition 7* The following conditions on (p,q) are equivalent.
(i) G_θ/H_θ admits a cocompact discontinuous group.
(ii) $^\exists$ bilinear map $f\colon \mathbb{R}^{p+1}\times\mathbb{R}^q \to \mathbb{R}^q$ such that

$$f(v,w) = 0 \text{ only if } v = 0 \text{ or } w = 0.$$

Trivial case $p = 0$. $H_\theta = O(q)$ compact!
(ii) holds by putting

$$f\colon \mathbb{R}\times\mathbb{R}^q \to \mathbb{R}^q, \quad (a,\vec{x}) \mapsto a\vec{x}.$$

Discontinuous group $\Gamma \rightsquigarrow$ Continuous analog W.

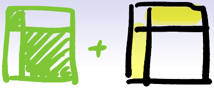
(i) $\iff$ $^\exists W \subset \mathfrak{p}$ subspace such that

$$W \pitchfork H_\theta \text{ in } G_\theta \quad \text{and} \quad d(W) + d(H_\theta) = d(G_\theta)$$

$\implies \dim_{\mathbb{R}} W = q$.

* T. Kobayashi, T. Yoshino, Pure and Appl. Math. Quarterly **1**, (2005), 603–684.

Proof of Proposition 7



Idea Use $\mathfrak{p} \simeq \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{p+1}, \mathbb{R}^q)$ for $\mathfrak{g} = \mathfrak{o}(p+1, q) = \mathfrak{k} + \mathfrak{p}$.

Given a subspace $W \subset \mathfrak{p}$, one obtains a bilinear map
$f_W \colon \mathbb{R}^{p+1} \times W \to \mathbb{R}^q$.

$\mathfrak{a} \subset \mathfrak{p}$ maximally abelian subspace w.r.t. [,] in $\mathfrak{g}$.

$$H = O(p,q) \subset G = O(p+1,q) \qquad (p \leq q)$$
$$\dim \mathfrak{a} = \mathrm{rank}_{\mathbb{R}} G = p+1 > \mathrm{rank}_{\mathbb{R}} H = p$$
$$\mu(H_\theta) = W_G \cdot \mathbb{R}^p \simeq \{X \in \mathfrak{a} : Xv = 0 \quad {}^\exists v \in \mathbb{R}^{p+1}\} \subset \mathfrak{p}.$$

Then $W \pitchfork H_\theta$ in $G_\theta \iff \mu(W) \cap \mu(H_\theta) = \{0\}$ in $\mathfrak{a}$.

$\iff f_W(v,w) = 0$ only if $v = 0$ or $w = 0$.

After Hurwitz–Radon–Eckmann–Adams

Theorem 8 The following conditions on (p, q) are equivalent.

(ii) $^{\exists}$ bilinear map $f\colon \mathbb{R}^{p+1} \times \mathbb{R}^{q} \to \mathbb{R}^{q}$ such that

$$f(v, w) = 0 \text{ only if } v = 0 \text{ or } w = 0.$$

(iii) $^{\exists}$ bilinear map $f\colon \mathbb{R}^{p+1} \times \mathbb{R}^{q} \to \mathbb{R}^{q}$ such that

$$\|f(v, w)\| = \|v\|\,\|w\| \ {}^{\forall}v \in \mathbb{R}^{p+1},\ {}^{\forall}w \in \mathbb{R}^{q}.$$

(iii) $\Longrightarrow$ (ii) Clear.

(ii) $\overset{\text{Lemma}}{\Longrightarrow}$ There exist p vector fields on the sphere S^{q-1} which are linearly independent at every point.

$\underset{\text{Adams}\,*}{\Longrightarrow}$ $p < \rho(q)$

$\underset{\text{Radon–Hurwitz}\,**\ (1922)}{\Longleftrightarrow}$ (iii)

* J. F. Adams, Vector fields on spheres, Ann. Math., **75** (1962), 603–632.

** J. Radon, Abn. math. Sem. Hamburg, **1** (1922); A. Hurwitz, Math. Ann., (1923).