HOW DOES THE RESTRICTION OF REPRESENTATIONS CHANGE UNDER TRANSLATIONS? A STORY FOR THE GENERAL LINEAR GROUPS AND THE UNITARY GROUPS

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Dedicated to Harish-Chandra whose pioneering work is a great inspiration for us

Abstract. We present a new approach to symmetry breaking for pairs of real forms of $(GL(n,\mathbb{C}), GL(n-1,\mathbb{C}))$. While translation functors are a useful tool for studying a family of representations of a single reductive group G, when applied to a pair of groups $G \supset G'$, translation functors can significantly alter the nature of symmetry breaking between the representations of G and G', even within the same Weyl chamber of the direct product group $G \times G'$. We introduce the concept of "fences for the interlacing pattern", which provides a refinement of the usual notion of "walls for Weyl chambers". We then present a theorem that states that multiplicity is constant unless these "fences" are crossed. This general theorem is illustrated with examples of both tempered and non-tempered representations. Additionally, we provide a new non-vanishing theorem of period integrals for pairs of reductive symmetric spaces, which is further strengthened through this approach.

Keywords and phrases: reductive group, symmetry breaking, representation, restriction, branching law, fence.

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1 Introduction

Any finite-dimensional representation of a compact Lie group G decomposes into a direct sum of irreducible representations when restricted to a subgroup of G. In contrast, the restriction of an irreducible admissible representation of a reductive Lie group to a *non-compact* subgroup G' is usually *not* a direct sum of irreducible representations. Therefore, it is useful to consider symmetry breaking operators (SBOs) which are continuous G'-homomorphisms from a topological G-module to a topological G'-module.

In this article, we are concerned mainly with the category $\mathcal{M}(G)$ of admissible smooth representations of G of finite length having moderate growth, which are defined on topological Fréchet vector spaces [33, Chap. 11]. Let $\mathrm{Irr}(G)$ denote the set of irreducible objects in $\mathcal{M}(G)$.

Let us denote by

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \tag{1.1}$$

the space of SBOs, that is, G'-homomorphisms from $\Pi \in \mathcal{M}(G)$ to $\pi \in \mathcal{M}(G')$, where the operators are continuous in the corresponding topology. The dimension of (1.1) is referred to as the *multiplicity*, which we denote by $[\Pi|_{G'} : \pi]$.

Explicit results about symmetry breaking for individual non-tempered representations are still sparse. For recent works, see [16, 18, 19, 24] for instance. If both G and G' are classical linear reductive Lie groups with complexified Lie algebras $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) = (\mathfrak{gl}_{n+1}, \mathfrak{gl}_n)$ or $(\mathfrak{so}_{n+1}, \mathfrak{so}_n)$, and they satisfy Harish-Chandra's rank conditions, the GGP conjectures/theorems are mostly concerned with non-zero symmetry breaking for an L-packet or a Vogan-packet of discrete series representations.

For a pair of groups (G, G'), where

$$G = GL(n, \mathbb{R}), \quad G' = GL(n-1, \mathbb{R}),$$

the dimension of the space of symmetry breaking operators is at most 1 [28]. In this article, we introduce a new approach to prove the non-vanishing of SBOs between irreducible representations that are not necessarily tempered. We begin by illustrating this new approach with known cases for the pair $(U(p,q), U(p-1,q))$ in Section 3, focusing on tempered representations. In this section, we also introduce "fences" for interlacing patterns, rather than the usual concept of "walls" for the Weyl chambers. In contrast to the fact that translation functors can significantly alter the nature of symmetry breaking even inside the Weyl chamber, the concept of "fences" plays a crucial role in understanding the behavior of "symmetry breaking" under translations. We formulate this in Theorem 3.3 for general irreducible representations of G and G', which are not necessarily tempered, where (G, G') is any real form of the pair $(GL(n,\mathbb{C}), GL(n-1,\mathbb{C}))$, stating that the multiplicity is constant unless we cross "fences".

In Section 4, we apply this approach to the branching of special unitary representations of $GL(2m,\mathbb{R})$ to the subgroups $GL(2m-1,\mathbb{R})$. In Section 7, we discuss symmetry breaking between irreducible representations in the discrete spectrum of

$$L^2(GL(n,\mathbb{R})/GL(p,\mathbb{R}) \times GL(n-p,\mathbb{R}))$$

and of

$$L^2(GL(n-1,\mathbb{R})/GL(p,\mathbb{R}) \times GL(n-p-1,\mathbb{R})).$$

These representations are not tempered if $2p < n-1$.

For this analysis, we provide a new non-vanishing theorem of period integrals related to discrete series representations of a pair of reductive symmetric spaces (Theorem 6.3), and examine the phenomenon of "jumping fences" in Section 7.4.

In Section 8, we discuss symmetry breaking between the irreducible representations in the discrete spectrum of

$$L^2(U(p,q)/U(r,s) \times U(p-r, q-s))$$

and of

$$L^2(U(p-1,q)/U(r,s) \times U(p-r-1, q-s)).$$

Details and proofs will be published in forthcoming articles [8, 20, 21].

Notation: $\mathbb{N} = \{0, 1, 2, \ldots, \}$, $\mathbb{N}_+ = \{1, 2, 3, \ldots, \}$, $\mathbb{R}^n_> = \{x \in \mathbb{R}^n : x_1 > \cdots > x_n\}$, $\mathbb{R}^n_\geq = \{x \in \mathbb{R}^n : x_1 \geq \cdots \geq x_n\}$

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2 Symmetry Breaking Under Translations

Let $G \supset G'$ be any real forms of $GL(n, \mathbb{C}) \supset GL(n-1, \mathbb{C})$.

In this section, we discuss "translation functors" for symmetry breaking operators (SBOs) between representations of G and G'.

2.1. Harish-Chandra Isomorphism and Translation Functor. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(N, \mathbb{C})$. We shall use N to refer to n or $n-1$ later. We set

$$\rho_N = (\tfrac{N-1}{2}, \tfrac{N-3}{2}, \dots, \tfrac{1-N}{2}). \tag{2.1}$$

Let $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ denote the center of the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$. We normalize the Harish-Chandra isomorphism

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}), \mathbb{C}) \simeq \mathbb{C}^N / \mathfrak{S}_N,$$

in such a way that the trivial one-dimensional $\mathfrak{g}_{\mathbb{C}}$-module has the infinitesimal character $\rho_N \mod \mathfrak{S}_N$.

For a $\mathfrak{g}$-module V and for $\tau \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}), \mathbb{C}) \simeq \mathbb{C}^N/\mathfrak{S}_N$, let $P_\tau(V)$ denote the τ-primary component of V, that is,

$$P_\tau(V) = \bigcup_{k=0}^{\infty} \bigcap_{z \in \mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})} \mathrm{Ker}(z - \tau(z))^k.$$

Let $\{f_i : i = 1, \dots, N\}$ be the standard basis of $\mathbb{Z}^N$. We focus on the following translation functors in the category $\mathcal{M}(G)$ or in the category of Harish-Chandra modules:

$$\phi_\tau^{\tau+\varepsilon f_i}(\cdot) := \begin{cases} P_{\tau+f_i}(P_\tau(\cdot) \otimes \mathbb{C}^N) & \text{if } \varepsilon = +, \\ P_{\tau-f_i}(P_\tau(\cdot) \otimes (\mathbb{C}^N)^\vee) & \text{if } \varepsilon = -1. \end{cases}$$

2.2. Non-vanishing condition for translating SBOs. Suppose that $\Pi \in \mathcal{M}(G)$ (resp., $\pi \in \mathcal{M}(G')$) has a $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character $\tau \in \mathbb{C}^n/\mathfrak{S}_n$ (resp. $\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})$-infinitesimal character $\tau' \in \mathbb{C}^{n-1}/\mathfrak{S}_{n-1}$).

In a forthcoming article [8], we prove the following theorems, which provide useful information on "symmetry breaking" under translations.

Theorem 2.1. *Let $\Pi \in \mathcal{M}(G)$ and $\pi \in \mathcal{M}(G')$. Suppose that any generalized eigenspaces of $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ in $\Pi \otimes \mathbb{C}^n$ are eigenspaces.*
(1) *If $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \neq \{0\}$, then $\mathrm{Hom}_{G'}(\phi_\tau^{\tau+f_i}(\Pi)|_{G'}, \pi) \neq \{0\}$ for any i such that $\tau_i \notin \{\tau_1' - \frac{1}{2}, \tau_2' - \frac{1}{2}, \ldots, \tau_{n-1}' - \frac{1}{2}\}$.*
(2) *If $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) = \{0\}$, then $\mathrm{Hom}_{G'}(\phi_\tau^{\tau+f_i}(\Pi)|_{G'}, \pi) = \{0\}$ for any i such that $\tau_i \notin \{\tau_1' - \frac{1}{2}, \tau_2' - \frac{1}{2}, \ldots, \tau_{n-1}' - \frac{1}{2}\}$.*

Theorem 2.2. *Let $\Pi \in \mathcal{M}(G)$ and $\pi \in \mathcal{M}(G')$. Suppose that any generalized eigenspaces of $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ in $\Pi \otimes (\mathbb{C}^n)^\vee$ are eigenspaces.*
(1) *If $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \neq \{0\}$, then $\mathrm{Hom}_{G'}(\phi_\tau^{\tau-f_i}(\Pi), \pi) \neq \{0\}$ for any i such that $\tau_i \notin \{\tau_1' + \frac{1}{2}, \tau_2' + \frac{1}{2}, \ldots, \tau_{n-1}' + \frac{1}{2}\}$.*
(2) *If $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) = \{0\}$, then $\mathrm{Hom}_{G'}(\phi_\tau^{\tau-f_i}(\Pi)|_{G'}, \pi) = \{0\}$ for any i such that $\tau_i \notin \{\tau_1' + \frac{1}{2}, \tau_2' + \frac{1}{2}, \ldots, \tau_{n-1}' + \frac{1}{2}\}$.*

Theorems 2.1 and 2.2 reveal an intrinsic reason why interlacing patterns appear in certain branching laws, such as Weyl's branching law and the Gan–Gross–Prasad conjecture, as we discuss in the following section. See also Theorem 3.3.

3 Known examples for $(G, G') = (U(p,q), U(p-1,q))$

We begin in this section by demonstrating how Theorems 2.1 and 2.2 clarify the interlacing patterns that appear in well-known examples of branching laws, such as Weyl's branching law for finite-dimensional representations regarding the restriction $U(n) \downarrow U(n-1)$ and the patterns [10] in the Gan–Gross–Prasad conjecture regarding the branching of discrete series representations for the restriction $U(p,q) \downarrow U(p-1,q)$.

The cases in the branching of non-tempered representations for the restriction $GL(n,\mathbb{R}) \downarrow GL(n-1,\mathbb{R})$ are more involved, which we will discuss in Section 4 through Section 7, along with the phenomenon of *jumping fences*. We revisit the branching for $U(p,q) \downarrow U(p-1,q)$ by considering non-tempered representations in Section 8.

3.1. Interlacing pattern.

We set

$$\begin{aligned} \mathbb{R}^n_> &:= \{x \in \mathbb{R}^n : x_1 > \cdots > x_n\}, \\ \mathbb{R}^n_\geq &:= \{x \in \mathbb{R}^n : x_1 \geq \cdots \geq x_n\}. \\ \mathbb{Z}^n_\geq &:= \mathbb{Z}^n \cap \mathbb{R}^n_\geq. \end{aligned}$$

We introduce the notion of “fences” as combinatorial objects. This will serve as a refinement of the “walls” of the Weyl chambers when we consider the branching for the restriction $G \downarrow G'$, where (G, G') are any real forms of $(GL(n,\mathbb{C}), GL(n-1,\mathbb{C}))$.

Definition 3.1 (Interlacing Pattern and Fence)**.** For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, an *interlacing pattern* D in $\mathbb{R}^n_> \times \mathbb{R}^m_>$ is a total order among $\{x_1, \dots, x_n,\ y_1, \dots, y_m\}$, which is compatible with the underlying inequalities $x_1 > x_2 > \cdots > x_n$ and $y_1 > y_2 > \cdots > y_m$. For an adjacent inequality between x_i and y_j such as $x_i > y_j$ or $y_j > x_i$, we refer to the hyperplane in $\mathbb{R}^{n+m}$ defined by $x_i = y_j$ as a *fence*.

By an abuse of notation, we also use the same letter D to denote the region in $\mathbb{R}^n_> \times \mathbb{R}^m_>$ given by its defining inequalities. We define $m(D) \in \{0, 1, \dots, n\}$ as follows: $m(D) := 0$ if D implies $y_1 > x_1$, and otherwise,

$$m(D) := \text{the largest } i \text{ such that } x_i > y_1 \text{ in } D. \tag{3.1}$$

Let $\mathfrak{P} \equiv \mathfrak{P}(\mathbb{R}^{n,m})$ denote the set of all interlacing patterns in $\mathbb{R}^n_> \times \mathbb{R}^m_>$.

Example 3.2. There are 35 interlacing patterns for $\mathbb{R}^4_> \times \mathbb{R}^3_>$, such as

$$\begin{aligned} D_1 &= \{(x,y) \in \mathbb{R}^{4+3} : x_1 > y_1 > x_2 > x_3 > y_2 > y_3 > x_4\}, \\ D_2 &= \{(x,y) \in \mathbb{R}^{4+3} : y_1 > y_2 > x_1 > x_2 > x_3 > y_3 > x_4\}. \end{aligned}$$

We also consider interlacing patterns in $\mathbb{R}^n_\geq \times \mathbb{R}^m_\geq$ such as $x_1 > y_1 \geq y_2 > x_2$, or those including equalities such as $x_1 = y_1 > x_2 = y_2$ or

$x_1 \geq y_1 \geq x_2 > y_2$. These interlacing patterns will be called *weakly interlacing patterns.*

3.2. Weyl's branching law for $U(n) \downarrow U(n-1)$.

We begin by illustrating the concept of *fences* with the classical branching for finite-dimensional representations.

Let $F^{U(n)}(x)$ denote the irreducible finite-dimensional representation of $G := U(n)$ with highest weight $x \in \mathbb{Z}^n_{\geq}$ in the standard coordinates. Similarly, $F^{U(n-1)}(y)$ denotes the irreducible representation of $G' = U(n-1)$ with highest weight $y \in \mathbb{Z}^{n-1}_{\geq}$.

Weyl's branching law tells us that $[F^G(x)|_{G'} : F^{G'}(y)] \neq 0$, or equivalently $[F^G(x)|_{G'} : F^{G'}(y)] = 1$, if and only if the following relation holds:

$$x_1 \geq y_1 \geq \cdots \geq x_{n-1} \geq y_{n-1} \geq x_n. \tag{3.2}$$

This section discusses the relationship of this classical branching law to Theorems 2.1 and 2.2 regarding "translation for symmetry breaking". To explore this, we reformulate the condition (3.2) in terms of the infinitesimal characters.

We recall from (2.1) $\rho_n := \frac{1}{2}(n-1, \ldots, 1-n)$ and $\rho_{n-1} := \frac{1}{2}(n-2, \ldots, 2-n)$. Then the $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character τ of $F^G(x)$ is given by $x + \rho_n \mod \mathfrak{S}_n$, while the $\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})$-infinitesimal character τ' of $F^{G'}(y)$ is $y + \rho_{n-1} \mod \mathfrak{S}_{n-1}$. Thus the inequality (3.2) for highest weights is equivalent to the following strict inequality:

$$\tau_1 > \tau'_1 > \cdots > \tau_{n-1} > \tau'_{n-1} > \tau_n. \tag{3.3}$$

We now explain how Theorems 2.1 and 2.2 reproduce the general interlacing property (3.2) of highest weights (or equivalently, (3.3) in terms of infinitesimal characters) from a simple and specific case. To see this, suppose that we are given any $y \in \mathbb{Z}^{n-1}_{\geq}$ and any x_n such that $y_{n-1} \geq x_n$. We set $\widetilde{x} := (y_1, \ldots, y_{n-1}, x_n) \in \mathbb{Z}^n_{\geq}$. Clearly, $[F^G(\widetilde{x})|_{G'} : F^{G'}(y)] \neq 0$ because the highest weight vector of $F^G(\widetilde{x})$ generates the G'-submodule $F^{G'}(y)$.

We now apply Theorem 2.2 to $\pi := F^{G'}(y) \in \mathcal{M}(G')$, and consider the translation functors for $\mathcal{M}(G)$. Due to the integral condition $\tau_i -$

$\tau_j \in \mathbb{Z}$ for all $1 \le i, j \le n$, the translation $\tau \rightsquigarrow \tau + \varepsilon f_i$ ($\varepsilon = +1$ or -1) does not cross the wall in $\mathcal{M}(G)$; hence the translation $\phi_\tau^{\tau+\varepsilon f_i}(F^G(x))$ is either 0 or irreducible. More precisely, $\phi_\tau^{\tau+\varepsilon f_i}(F^G(x)) \simeq F^G(x+\varepsilon f_i)$ if $x_i \neq x_{i-\varepsilon}$. Therefore, an iterated application of Theorem 2.2 implies that

$$[F^G(x)|_{G'} : F^{G'}(y)] \neq 0$$

as long as the pair $(x + \rho_n, y + \rho_{n-1})$ satisfies (3.3), or equivalently, if the classical interlacing property (3.2) is satisfied.

3.3. Coherent continuation and symmetry breaking.

Let $(G, G') = (GL(n, \mathbb{R}), GL(n-1, \mathbb{R}))$ or $(U(p,q), U(p-1,q))$.

Let $\mathcal{V}(G)$ be the Grothendieck group of $\mathcal{M}(G)$, that is, the abelian group generated by $X \in \mathcal{M}(G)$ modulo the equivalence relation $X \sim Y + Z$, whenever there is a short exact sequence $0 \to Y \to X \to Z \to 0$.

Let $\Pi \colon \xi + \mathbb{Z}^n \to \mathcal{V}(G)$ be a coherent family of G-modules, specifically, Π satisfies the following properties:

(1) Π_λ has a $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character λ if $\lambda \in \xi + \mathbb{Z}^n$;

(2) $\Pi_\lambda \otimes F \simeq \sum_{\nu \in \Delta(F)} \Pi_{\lambda+\nu}$ in $\mathcal{V}(G)$ for any finite-dimensional representation F of G.

Theorem 3.3. *Suppose that $\Pi \in \mathrm{Irr}(G)$ has an infinitesimal character ξ satisfying $\xi_i - \xi_{i+1} \ge 1$ $(1 \le i \le n-1)$. Let $\Pi \colon \xi + \mathbb{Z}^n \to \mathcal{V}(G)$ be the coherent family starting from $\Pi_\xi := \Pi$. Let ν be the infinitesimal character of $\pi \in \mathrm{Irr}(G')$. If (ξ, ν) satisfies an interlacing pattern D in $\mathbb{R}^n_> \times \mathbb{R}^m_\ge$, then we have*

$$[\Pi|_{G'} : \pi] = [\Pi_\lambda|_{G'} : \pi]$$

for all $\lambda \in \xi + \mathbb{Z}^n$ such that (λ, ν) satisfies the same interlacing pattern D.

Remark 3.4. (1) Such a coherent family exists uniquely because our assumption guarantees that ξ is non-singular.
(2) The concept of "fences" is a refinement of the Weyl chambers. Hence, if we do not cross the fence, that is, if $(\lambda, \nu) \in D$, then λ is non-singular and remains in the same Weyl chamber with ξ. Consequently, Π_λ is irreducible for any such λ.

We recall our notation that $\{f_i\}_{1\le i\le n}$ is the standard basis of $\mathbb{Z}^n$. To prove Theorem 3.3, we introduce the finite set defined by

$$\mathcal{E} := \{\pm f_i : 1 \le i \le n\} \subset \mathbb{Z}^n.$$

Lemma 3.5. *Let $D \in \mathfrak{P}_n$. For any (ξ,ν) and $(\lambda,\nu) \in D$ such that $\lambda - \xi \in \mathbb{Z}^n$, there exists a sequence $\lambda^{(j)} \in \xi + \mathbb{Z}^n$ $(j = 0,1,2,\dots,N)$ with the following properties:*

$$\lambda^{(0)} = \xi,\ \lambda^{(N)} = \lambda,\ \lambda^{(j)} - \lambda^{(j-1)} \in \mathcal{E},\ (\lambda^{(j)},\nu) \in D \quad \textit{for } 1 \le j \le N.$$

Proof. There exists a unique element $\mu \in \xi + \mathbb{Z}^n$ such that $(\mu,\nu) \in D$ and that μ satisfies the following property for any $\lambda \in \xi + \mathbb{Z}^n$ with $(\lambda,\nu) \in D$:

$$\begin{aligned} &\mu_i \le \lambda_i \quad \text{if } i \le m(D), \\ &\mu_i \ge \lambda_i \quad \text{if } m(D) < i \le n, \end{aligned}$$

where we recall (3.1) for the definition of $m(D) \in \{0,1,\dots,n\}$.

First, we assume that $\lambda = \mu$. Then it is readily verified by an inductive argument that Lemma 3.5 holds for $\lambda = \mu$.

Second, since the existence of the sequences $\{\lambda^{(j)}\}_{0\le j\le N}$ in Lemma 3.5 defines an equivalence relation $\sim$ among non-singular dominant elements in $\xi + \mathbb{Z}^n$, we have $\xi \sim \mu \sim \lambda$, whence the lemma. □

Proof of Theorem 3.3. By Lemma 3.5, it suffices to prove Theorem 3.3 when $\lambda - \xi \in \mathcal{E}$. For example, suppose that $\lambda - \xi = f_i$ for some $1 \le i \le n$. Then, we have

$$\xi_i \notin \{\nu_1 - \frac{1}{2}, \nu_2 - \frac{1}{2}, \dots, \nu_n - \frac{1}{2}\}$$

because (λ,ν) and (ξ,ν) satisfy the same interlacing property.

On the other hand, since ξ is non-singular and $\xi_a - \xi_b \in \mathbb{Z}$ for any $1 \le a \le b \le n$, $\xi + f_j$ $(1 \le j \le n)$ lies in the same Weyl chamber as ξ. Therefore, $\Pi_{\xi+f_j} \simeq \phi_\xi^{\xi+f_j}(\Pi)$ is either irreducible or zero. Thus, all the assumptions in Theorem 2.1 are satisfied, and we conclude

$$[\Pi_\lambda|_{G'} : \pi] = [\phi_\xi^{\xi+f_j}(\Pi) : \pi] \ne 0.$$

The multiplicity-freeness theorem concludes that $[\Pi_\lambda|_{G'} : \pi] = 1$. The case $\lambda - \xi = -f_i$ can be proven similarly by using Theorem 2.2. □

3.4. Gan–Gross–Prasad conjecture for $U(p,q) \downarrow U(p-1,q)$. In the non-compact setting $(G,G') = (U(p,q), U(p-1,q))$, an analogous interlacing property to (3.3) arises, which we now recall.

Let $G = U(p,q)$ and $K = U(p) \times U(q)$. The complexifications are given by $G_{\mathbb{C}} = GL(p+q,\mathbb{C})$ and $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$, respectively. Let $W_G = \mathfrak{S}_{p+q}$ and $W_K = \mathfrak{S}_p \times \mathfrak{S}_q$ be the Weyl groups for the root systems $\Delta(\mathfrak{g}_{\mathbb{C}})$ and $\Delta(\mathfrak{k}_{\mathbb{C}})$, respectively. We define

$$W^{\mathfrak{k}} := \{w \in W_G : w\nu \text{ is } \Delta^+(\mathfrak{k})\text{-dominant for any } \Delta(\mathfrak{g})\text{-dominant } \nu\}.$$

This means that $w \in W^{\mathfrak{k}}$ if $w \in W_G = \mathfrak{S}_{p+q}$ satisfies $w^{-1}(i) < w^{-1}(j)$ whenever $1 \le i < j \le p$ or $p+1 \le i < j \le p+q$.

Then $W^{\mathfrak{k}}$ is the set of complete representatives of $W_K \backslash W_G$, which parametrizes closed $K_{\mathbb{C}}$-orbits on the full flag variety of $G_{\mathbb{C}}$. We further define

$$C_+ := \{x \in \mathbb{R}^{p+q} : x_1 > \cdots > x_{p+q}\}.$$

For $w \in W$, the set wC_+ defines an interlacing pattern $\{x \in \mathbb{R}^{p+q} : x_{i_1} > x_{i_2} > \cdots > x_{i_{p+q}}\}$ in $\mathbb{R}^p_> \times \mathbb{R}^q_>$.

For $\varepsilon \in \frac{1}{2}\mathbb{Z}$, we define

$$\begin{aligned}
\mathbb{Z}_\varepsilon &:= \mathbb{Z} + \varepsilon. \\
(\mathbb{Z}_\varepsilon)^{p+q}_{\mathrm{reg}} &:= \{x \in (\mathbb{Z}_\varepsilon)^{p+q} : x_i \ne x_j \quad \text{if } i \ne j\}, \\
(\mathbb{Z}_\varepsilon)^{p+q}_{>} &:= \{x \in (\mathbb{Z}_\varepsilon)^{p+q} : x_1 > \cdots > x_{p+q}\},
\end{aligned}$$

$$(\mathbb{Z}_\varepsilon)^{p,q}_{>} := \{x \in (\mathbb{Z}_\varepsilon)^{p+q}_{\mathrm{reg}} : x_1 > \cdots > x_p \text{ and } x_{p+1} > \cdots > x_{p+q}\}. \tag{3.4}$$

Let $\mathrm{Disc}(G)$ denote the set of discrete series representations of G, which is parametrized for $G = U(p,q)$ as follows: let $\varepsilon := \frac{1}{2}(p+q-1)$.

$$\mathrm{Disc}(G) \simeq (\mathbb{Z}_\varepsilon)^{p,q}_{>} \simeq (\mathbb{Z}_\varepsilon)^{p+q}_{>} \times W^{\mathfrak{k}}, \quad \Pi_\lambda = \Pi^w(\lambda^+) \leftrightarrow \lambda \leftrightarrow (\lambda^+, w),$$

where $\lambda = w\lambda^+$. The geometric meaning of w is that the support of the localization of the $(\mathfrak{g},K)$-module $\Pi^w(\lambda^+)_K$ via the Beilinson–Bernstein correspondence using $\mathcal{D}$-modules is the closed $K_{\mathbb{C}}$-orbit that corresponds to w, while λ is the Harish-Chandra parameter, that is, $\lambda \equiv \lambda^+ \mod \mathfrak{S}_n$ is the $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character.

Let $G = U(p,q)$ and $G' = U(p-1,q)$. We set

$$\varepsilon = \frac{1}{2}(p+q-1) \text{ and } \varepsilon' = \frac{1}{2}(p+q-2).$$

The classification of a pair $(\Pi, \pi) \in \mathrm{Disc}(G) \times \mathrm{Disc}(G')$ such that $[\Pi^\infty|_{G'} : \pi^\infty] \neq 0$ is reduced to the classification of

$$(\lambda^+, \nu^+) \in (\mathbb{Z}_\varepsilon^{p+q})_> \times (\mathbb{Z}_{\varepsilon'}^{p+q-1})_>$$

such that $[\Pi^w(\lambda^+)|_{G'} : \pi^{w'}(\nu^+)] \neq 0$ for each $(w, w') \in W^{\mathfrak{k}}$.

He [10] determined all such pairs (λ^+, ν^+), relying on the combinatorics of the theta correspondence. In his theorem, certain interlacing patterns of (λ^+, ν^+) appears. The following theorem explains an intrinsic reason for these interlacing patterns, from a different perspective, using "translation functor for symmetry breaking", and reveals why interlacing patterns occur in the context of the Gan–Gross–Prasad conjecture.

For an interlacing pattern $D \in \mathfrak{P}(\mathbb{R}^{p+q,p+q-1}))$ (Definition 3.1), we set

$$D_{\mathrm{int}} := D \cap (\mathbb{Z}_\varepsilon^{p+q} \times \mathbb{Z}_{\varepsilon'}^{p+q-1}).$$

Theorem 3.6 $(U(p,q) \downarrow U(p-1,q))$**.** *Fix $w \in W^{\mathfrak{k}}$, $w' \in W^{\mathfrak{k}'}$ and an interlacing pattern $D \in \mathfrak{P}_{p+q}$. Then the following two conditions on the triple (w, w', D) are equivalent:*
(i) $[\Pi^w(\lambda^+)|_{G'} : \pi^{w'}(\nu^+)] \neq 0$ *for some* $(\lambda^+, \nu^+) \in D_{\mathrm{int}}$,
(ii) $[\Pi^w(\lambda^+)|_{G'} : \pi^{w'}(\nu^+)] \neq 0$ *for all* $(\lambda^+, \nu^+) \in D_{\mathrm{int}}$.

Theorem 3.6 is derived from the iterated application of Theorems 2.1 and 2.2, along with the use of a spectral sequence for cohomological parabolic induction. A key aspect in applying these theorems is a parity condition that $\lambda_i - \nu_j \in \mathbb{Z} + \frac{1}{2}$ for every $1 \leq i \leq p+q$ and $1 \leq j \leq p+q-1$.

We shall see in Section 7.4 that a phenomenon of "jumping the fences of interlacing patterns" naturally arises for the parity conditions on λ and ν such that $\lambda_i - \nu_j \notin \mathbb{Z} + \frac{1}{2}$.

Example 3.7. For $\nu \in \mathbb{Z}_{\varepsilon'}^{p+q-1}$ with $\nu_p > \cdots > \nu_{p+q-1} > \nu_1 > \cdots > \nu_{p-1}$, let $\pi(\nu)$ denote the corresponding holomorphic discrete series representation of G'. We take $\widetilde{\lambda} \in \mathbb{Z}_\varepsilon^{p+q}$ such that

$$\widetilde{\lambda}_{p+j} := \nu_{p+j-1} - \frac{1}{2} \quad (1 \leq j \leq q),$$
$$\widetilde{\lambda}_1 > \nu_1 > \cdots > \nu_{p-1} > \widetilde{\lambda}_p.$$

Then it is easy to see that $\pi(\nu)$ occurs in $\Pi(\widetilde{\lambda})|_{G'}$ as the "bottom layer". Therefore, an iterated application of Theorems 2.1 and 2.2 implies that $[\Pi(\lambda)|_{G'} : \pi(\nu)] \neq 0$ as long as $\lambda \in \mathbb{Z}_{\varepsilon}^{p+q}$ satisfies

$$\nu_p > \lambda_{p+1} > \cdots > \nu_{p+q-1} > \lambda_{p+q} > \lambda_1 > \nu_1 > \cdots > \nu_{p-1} > \lambda_p.$$

4 Branching of some special representations for $GL(2m, \mathbb{R}) \downarrow GL(2m-1, \mathbb{R})$

In this section, we explore an application of Theorem 3.3 to a family of non-tempered representations of $G = GL(2m, \mathbb{R})$, see [27], when restricted to the subgroup $G' = GL(2m-1, \mathbb{R})$.

4.1. Setup for a family of representations of $GL(2m, \mathbb{R})$. For $\varepsilon \in \{0, 1\}$, let

$$\Pi \colon (\mathbb{Z} + \varepsilon)^{2m} \to \mathcal{V}(G)$$

be the coherent family of smooth representations such that $\Pi(\lambda)$ is the smooth representation of a special unitary representation studied in [27], sometimes referred to as the *ℓ-th Speh representation*, if

$$\lambda = \frac{1}{2}(\ell, \ldots, \ell, -\ell, \ldots, -\ell) + (\rho_m, \rho_m) \quad \text{for } 1 \leq \ell.$$

Here, we recall from (2.1) $\rho_m = (\frac{m-1}{2}, \ldots, \frac{1-m}{2})$. The parity ε and ℓ is related by $\ell + 2\varepsilon + m + 1 \in 2\mathbb{Z}$.

There is a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(2m, \mathbb{C})$, unique up to an inner automorphism of $G = GL(2m, \mathbb{R})$, such that the real Levi subgroup $N_G(\mathfrak{q})$ is isomorphic to $L := GL(m, \mathbb{C})$. The underlying $(\mathfrak{g}, K)$-module of $\Pi(\lambda)$ is obtained by a cohomological parabolic induction from an irreducible finite-dimensional representation F_λ of $\mathfrak{q}$, on which the unipotent radical $\mathfrak{u}$ acts trivially and L acts by $F^{GL(m,\mathbb{C})}(\lambda' - \rho_m) \otimes \overline{F^{GL(m,\mathbb{C})}(\lambda'' - \rho_m)}$. Here $\lambda = (\lambda', \lambda'') \in (\mathbb{Z} + \varepsilon)^m \times (\mathbb{Z} + \varepsilon)^m$.

The representation $\Pi(\lambda)$ of G is irreducible if

$$\lambda_1 > \lambda_2 > \cdots > \lambda_{2m},$$

and is unitarizable if $\lambda_1 = \cdots = \lambda_m = -\lambda_{m+1} = \cdots = -\lambda_{2m}$.

4.2. Setup for a family of representations of $GL(2m-1,\mathbb{R})$. Let $L' := GL(1,\mathbb{R})\times GL(m-1,\mathbb{C})$ be a subgroup of $G' := GL(m-1,\mathbb{R})$. For $\nu \equiv (\nu', \nu_m, \nu'') \in (\mathbb{Z}+\varepsilon+\frac{1}{2})^{m-1}_{>}\times\mathbb{C}\times(\mathbb{Z}+\varepsilon+\frac{1}{2})^{m-1}_{>}$ and $\kappa \in \{0,1\}$, let $F'_\kappa(\nu)$ denote an irreducible finite-dimensional L'-module given by $\chi_{\nu_m,\kappa} \boxtimes W_{\nu',\nu''}$ where

$$\chi_{\nu_m,\kappa}(x) := |x|^{\nu_m}(\operatorname{sgn} x)^\kappa \qquad \text{for } x \in GL(1,\mathbb{R}) \simeq \mathbb{R}^\times,$$

and $W_{\nu'} := F^{GL(m-1,\mathbb{C})}(\nu' - \rho_{m-1}) \otimes \overline{F^{GL(m-1,\mathbb{C})}(\nu'' - \rho_{m-1})}$.

There is a θ-stable parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}'_{\mathbb{C}} + \mathfrak{u}'$ of $\mathfrak{g}'_{\mathbb{C}} = \mathfrak{gl}(2m-1,\mathbb{C})$, unique up to an inner automorphism of $G' = GL(2m-1,\mathbb{R})$, such that the real Levi subgroup $N_{G'}(\mathfrak{q}')$ is isomorphic to L'. Let $\pi_\kappa(\nu)$ be a smooth admissible representation of G' such that its underlying $(\mathfrak{g}', K')$-module is isomorphic to the cohomological parabolic induction from the irreducible finite-dimensional representation $F'_\kappa(\nu)$. In our normalization, ν is the $\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})$-infinitesimal character of $\pi_\kappa(\nu)$. The $(\mathfrak{g}', K')$-module $\pi_\kappa(\nu)$ is unitarizable if

$$\nu_m \in \sqrt{-1}\mathbb{R},\ \nu' = c\mathbf{1}_{m-1} + \rho_{m-1}, \ \text{ and } \nu'' = -c\mathbf{1}_{m-1} + \rho_{m-1}$$

for some $c \in \frac{1}{2}\mathbb{N}$.

We write simply $\pi(\nu)$ for $\pi_\kappa(\nu)$ when $\nu_m \in \mathbb{Z}$ and when

$$\kappa + \nu_m + 2\varepsilon + m - 1 \in 2\mathbb{Z}. \tag{4.1}$$

4.3. Branching for $GL(2m,\mathbb{R}) \downarrow GL(2m-1,\mathbb{R})$. In the same spirit as the reinterpretation of Weyl's classical branching laws from the perspective of "translation for symmetry breaking", as explained in Section 3.2, we derive the following theorem starting from a "simpler case", that is, when

(4.2)

$$\lambda_1 > \nu_1, \quad \nu_{2m-1} > \lambda_{2m}, \quad \lambda_1 + \lambda_{2m} = \nu_m,$$
$$\lambda_{i+1} = \nu_i - \frac{1}{2}\ (1 \le i \le m-1),\ \ \lambda_i = \nu_i + \frac{1}{2}\ (m+1 \le i \le 2m-1).$$

We note that such (λ, ν) lies in the interlacing pattern:

(4.3)

$$\lambda_1 > \nu_1 > \lambda_2 > \cdots > \nu_{m-1} > \lambda_m > \lambda_{m+1} > \nu_{m+1} > \cdots > \nu_{2m-1} > \lambda_{2m}.$$

Theorem 4.1 ([21]). *Let* $(G, G') = (GL(2m, \mathbb{R}), GL(2m-1, \mathbb{R}))$. *Let* $\varepsilon \in \{0, \frac{1}{2}\}$, $\lambda \in (\mathbb{Z}+\varepsilon)^{2m}_{>}$, *and* $\nu \in (\mathbb{Z}+\varepsilon+\frac{1}{2})^{m-1}_{>} \times \mathbb{Z} \times (\mathbb{Z}+\varepsilon+\frac{1}{2})^{m-1}_{>}$ *satisfying* $\nu_{m-1} > \nu_m > \nu_{m+1}$ *and* $\nu_{m-1} - \nu_{m+1} \neq 1$.

If (λ, ν) *satisfies* (4.3), *then*

$$[\Pi(\lambda)|_{G'} : \pi(\nu)] = 1.$$

5 A family of representations of $GL(n, \mathbb{R})$

In this section we consider branching of yet another family of irreducible unitary representations of $GL(N, \mathbb{R})$ for $N = n$ or $n-1$ that are not necessarily tempered.

5.1. Weyl's notation for $\widehat{O(N)}$.

The maximal compact subgroup $O(N)$ of $GL(N, \mathbb{R})$ is not connected. For the description of the set $\widehat{O(N)}$ of equivalence classes of irreducible representations of $O(N)$, we refer to Weyl [34, Chap. V, Sect. 7] as follows.

Let $\Lambda^+(O(N))$ be the set of $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ in one of the following forms.

$$\text{Type I}: (\lambda_1, \cdots, \lambda_k, \underbrace{0, \cdots, 0}_{N-k}),$$
$$\text{Type II}: (\lambda_1, \cdots, \lambda_k, \underbrace{1, \cdots, 1}_{N-2k}, \underbrace{0, \cdots, 0}_{k}),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $0 \leq 2k \leq N$.

For any $\lambda \in \Lambda^+(O(N))$, let v_λ be the highest weight vector of the irreducible $U(N)$-module $F^{U(N)}(\lambda)$. Then there exists a unique $O(N)$-irreducible submodule containing v_λ, which we denote by $F^{O(N)}(\lambda)$. Weyl established the following bijection:

$$\Lambda^+(O(N)) \xrightarrow{\sim} \widehat{O(N)}, \quad \lambda \mapsto F^{O(N)}(\lambda). \tag{5.1}$$

5.2. Relative discrete series of $GL(2,\mathbb{R})$.

Let σ_a $(a \in \mathbb{N}_+)$ denote the relative discrete series representation of $GL(2,\mathbb{R})$ with the following property:

infinitesimal character $\frac{1}{2}(a,-a)$ (Harish-Chandra parameter);

minimal K-type $F^{O(2)}(a+1,0)$ (Blattner parameter).

We note that the restriction $\sigma_a|_{SL(2,\mathbb{R})}$ splits into the direct sum of a holomorphic (resp. anti-holomorphic) discrete series with minimal K-type $\mathbb{C}_{a+1}$ (resp. $\mathbb{C}_{-(a+1)}$).

5.3. Some irreducible representations of $GL(n,\mathbb{R})$.

Let $G = GL(n,\mathbb{R})$. For $0 \le 2\ell \le n$, let P_ℓ be a real parabolic subgroup of G with Levi part $L_\ell := GL(2,\mathbb{R})^\ell \times GL(n-2\ell,\mathbb{R})$.

For $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{N}_+^\ell$, we define a unitary representation of G by means of normalized smooth parabolic induction:

$$\Pi_\ell(\lambda) := \operatorname{Ind}_{P_\ell}^G(\bigotimes_{j=1}^{\ell} \sigma_{\lambda_j} \otimes \mathbf{1}). \tag{5.2}$$

Then $\Pi_\ell(\lambda)$ is an irreducible unitary representation of $GL(n,\mathbb{R})$ (*cf.* [32]). Moreover, it is a tempered unitary representation if and only if $n = 2\ell - 1$ or 2ℓ.

For $2k \le n-1$, and for $\nu = (\nu_1, \dots, \nu_k)$, we shall use an analogous notation $\pi_k(\nu)$ for a family of irreducible unitary representations of $G' = GL(n-1,\mathbb{R})$.

5.4. Cohomological parabolic induction for $GL(n,\mathbb{R})$.

An alternative construction of the representations $\Pi_{\mathfrak{q}}(\lambda)$ is given by cohomological induction.

Let $2\ell \le n$ and $\mathfrak{q}_\ell$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{gl}(n,\mathbb{C})$ with the real Levi subgroup

$$L \equiv N_G(\mathfrak{q}_\ell) \simeq (\mathbb{C}^\times)^\ell \times GL(n-2\ell,\mathbb{R}).$$

We set

$$S_\ell := \frac{1}{2}\dim K/L = \ell(n-\ell-1).$$

Suppose that $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{Z}^\ell$ satisfies $\lambda_1 > \cdots > \lambda_\ell > 0$. We adopt a normalization such that the cohomological parabolic induction $\mathcal{R}_{\mathfrak{q}_\ell}^{S_\ell}(\mathbb{C}_\lambda)$ has a $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character given by

$$\frac{1}{2}(\lambda_1, \dots, \lambda_\ell, n-2\ell-1, \dots, 1+2\ell-n, -\lambda_\ell, \dots, -\lambda_1) \in \mathbb{C}^n/\mathfrak{S}_n, \tag{5.3}$$

via the Harish-Chandra isomorphism. Then its minimal K-type is given by

$$\mu_\lambda = (\lambda_1 + 1, \dots, \lambda_\ell + 1, 0, \dots, 0) \in \Lambda^+(O(n)) \tag{5.4}$$

in Weyl's notation.

The underlying $(\mathfrak{g}, K)$-module of the G-modules $\Pi_\ell(\lambda)$ can be described in terms of cohomological parabolic induction:

$$\Pi_\ell(\lambda)_K \simeq \mathcal{R}_{\mathfrak{q}_\ell}^{S_\ell}(\mathbb{C}_\lambda). \tag{5.5}$$

If $n > 2\ell$ then the $O(n)$-module $F^{O(n)}(\mu_\lambda)$ stays irreducible when restricted to $SO(n)$, and its highest weight is given by $(\lambda_1+1, \dots, \lambda_\ell + 1, \underbrace{0, \dots, 0}_{[\frac{n}{2}]-\ell})$ in the standard notation. If $n = 2\ell$, then $F^{O(n)}(\mu_\lambda)$ splits into the direct sum of two irreducible $SO(n)$-modules with highest weights $(\lambda_1+1, \dots, \lambda_{\ell-1}+1, \lambda_\ell+1)$ and $(\lambda_1+1, \dots, \lambda_{\ell-1}+1, -\lambda_\ell-1)$.

The parameter $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is in the "good range" with respect to $\mathfrak{q}_\ell$ in the sense of [31] if the following condition is satisfied:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > \max(n - 2\ell - 1, 0). \tag{5.6}$$

6 Non-vanishing theorem for period integrals

The general result in Theorems 2.1 and 2.2, see also Theorem 3.3, raises the problem of developing a method to detect the existence of a non-zero symmetry breaking operator for representations with *specific parameters* (λ, ν). We address this problem using the idea of period integrals for a pair of reductive symmetric pairs.

The main result of this section is Theorem 6.3, which provides a sufficient condition for the non-vanishing of period integrals in the general setting where $G \supset G'$ are *arbitrary pairs* of real reductive Lie groups.

6.1. Discrete series representations for $X = G/H$.
Let (X, μ) be a measure space and suppose that a group G acts on X in a measure-preserving fashion. Then, there is a natural unitary representation of G on the Hilbert space $L^2(X)$ of square-integrable functions.

An irreducible unitary representation Π is called a *discrete series representation* for X, if Π can be realized in a closed subspace of $L^2(X)$. Let $\mathrm{Disc}(X)$ denote the set of discrete series representations for X. Then $\mathrm{Disc}(X)$ is a (possibly, empty) subset of the unitary dual $\widehat{G}$ of G.

For $\Pi \in \widehat{G}$, let $\Pi^\vee$ (resp. $\overline{\Pi}$) denote the contragredient (resp. complex conjugate) representation of Π. Then $\Pi^\vee$ and $\overline{\Pi}$ are unitarily equivalent representations. Moreover, the set $\mathrm{Disc}(X)$ is closed under taking contragredient representations.

6.2. Reductive symmetric spaces.
Let G be a linear real reductive Lie group, σ be an involutive automorphism of G, and H an open subgroup of $G^\sigma := \{g \in G : \sigma g = g\}$. The homogeneous space $X = G/H$ is called a *reductive symmetric space*.

We take a Cartan involution θ of G that commutes with σ. Let K be the corresponding maximal compact subgroup of G. Flensted-Jensen [4] and Matsuki–Oshima [22] proved that $\mathrm{Disc}(G/H) \neq \emptyset$ if and only if

$$\operatorname{rank} G/H = \operatorname{rank} K/H \cap K, \tag{6.1}$$

generalizing the Harish-Chandra rank condition [7], $\operatorname{rank} G = \operatorname{rank} K$, for the existence of discrete series representations of the group manifold G.

In contrast to Harish-Chandra's discrete series representations for group manifolds, not every $\Pi \in \mathrm{Disc}(G/H)$ has a non-singular $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character. This means that if we realize the underlying $(\mathfrak{g}, K)$-module Π_K in terms of cohomological parabolic induction, the parameter is not necessarily in "good range" for this induction. In case the parameters are in the "good range", the minimal K-type of $\Pi \in \mathrm{Disc}(G/H)$, which we denote by $\mu(\Pi) \in \widehat{K}$, can easily be computed.

6.3. Period integrals: Generalities.
Let $X = G/H$ be a reductive symmetric space, as in Section 6.2. We now consider a pair $Y \subset X$ of symmetric spaces as below. Suppose that G' is a reductive subgroup of G, stable under the involutions σ and θ of G. Let $H' := H \cap G'$. Then $Y := G'/H'$ is also a reductive symmetric space, and there is a natural inclusion $\iota\colon Y \hookrightarrow X$, which is G'-equivariant.

Let Π be a discrete series representation for $X = G/H$. By convention, we identify Π with its corresponding representation space in $L^2(X)$. Then, the smooth representation $\Pi^\infty \in \mathcal{M}(G)$ is realized as a subspace of $(L^2 \cap C^\infty)(X)$.

The first step is to prove the convergence of period integrals in this general setting [20]:

Theorem 6.1. *For any* $\Pi \in \mathrm{Disc}(X)$ *and any* $\pi \in \mathrm{Disc}(Y)$*, the following period integral*

$$B\colon \Pi^\infty \times \pi^\infty \to \mathbb{C}, \quad (F, f) \mapsto \int_Y (\iota^* F)(y) f(y) dy \tag{6.2}$$

converges. Hence, it defines a continuous G'-invariant bilinear form. In particular, the bilinear form (6.2) *induces a symmetry breaking operator*

$$T_B\colon \Pi^\infty \to (\pi^\vee)^\infty, \quad F \mapsto B(F, \cdot), \tag{6.3}$$

where $\pi^\vee$ denotes the contragredient representation of π.

The second step is to detect when the period integral T_B does not vanish. It should be noted that the period integral can vanish, even when $\mathrm{Hom}_{G'}(\Pi^\infty|_{G'}, \pi^\infty) \neq 0$. This leads to the following question:

Question 6.2. Find a sufficient condition for the period integral (6.2) not to vanish.

Some sufficient conditions have been derived in the special cases when both X and Y are group manifolds [9, 29], and when X is a certain rank-one symmetric space [24, 25]. In a forthcoming paper [20], we will prove the following theorem for the general pair of reductive

Lie groups $G' \subset G$ and for their reductive symmetric spaces $Y \subset X$ of higher rank:

Theorem 6.3. *Let $Y \subset X$ be as in the beginning of this section. Additionally, we assume that G is contained in a connected complex reductive Lie group $G_{\mathbb{C}}$ and that K and K' are in the Harish-Chandra class. Let $\Pi \in \mathrm{Disc}(X)$ and $\pi \in \mathrm{Disc}(Y)$ both have non-singular infinitesimal characters. Suppose that the minimal K-types $\mu(\Pi) \in \widehat{K}$ and $\mu'(\pi) \in \widehat{K'}$ satisfy the following two conditions:*

$$[\mu(\Pi)|_{K'} : \mu'(\pi)] = 1; \tag{6.4}$$

(6.5)
a non-zero highest weight vector of $\mu(\Pi)$ is contained in $\mu'(\pi)$.

Then the period integral (6.2) *is non-zero, and consequently, the corresponding symmetry breaking operator (SBO) in* (6.3) *is non-zero.*

Remark 6.4. In the case where $(G, G') = (GL(n,\mathbb{R}), GL(n-1,\mathbb{R}))$, one of $K = O(n)$ or $K' = O(n-1)$ is not in Harish-Chandra class. However, Theorem 6.3 holds in this case as well, provided that we define minimal K-types in terms of their irreducible $\mathfrak{k}$-summands.

Remark 6.5. Theorem 6.3 applies to general pairs of real reductive Lie groups, (G, G'). In the specific cases where $(G, G') = (GL(n,\mathbb{R}), GL(n-1,\mathbb{R}))$ or $(U(p,q), U(p-1,q))$, the assumption (6.4) is utomatically derived from (6.5).

Remark 6.6. Yet another sufficient condition for the non-vanishing of the period integral (6.2) is

$$\dim \mathrm{Hom}_K(\mu(\Pi), C^\infty(K/M_H)) = \dim \mathrm{Hom}_{K'}(\mu'(\pi), C^\infty(K'/M'_H)) = 1,$$

where M_H is the centralizer of a generic element in $\mathfrak{g}^{-\theta,-\sigma}$ in $H \cap K$, and M'_H is that of $\mathfrak{g}'^{-\theta,-\sigma}$ in $H' \cap K'$. This condition is satisfied, in particular, when $K_{\mathbb{C}}/M_{H,\mathbb{C}}$ and $K'_{\mathbb{C}}/M'_{H,\mathbb{C}}$ are spherical. However, the settings that we will treat in Sections 7 and 8 are more general.

We give some examples of Theorem 6.1 in Sections 7 and 8 in settings where X is a symmetric space of $G = GL(n,\mathbb{R})$ and $G = U(p,q)$, respectively.

7 Restricting discrete series representations of the symmetric space $GL(p+q,\mathbb{R})/(GL(p,\mathbb{R})\times GL(q,\mathbb{R}))$

In this section, we prove the existence of a non-zero G'-homomorphism from Π^∞ to π^∞, where $\Pi \in \mathrm{Disc}(X)$ and $\pi \in \mathrm{Disc}(Y)$, by using a trick of "jumping fences" in the translation theorems for symmetry breaking, as explained in Section 2.

Throughout this section, we consider the following setup: $X = G/H$, $Y = G'/(H\cap G')$, where $p+q=n$ and

$$(G,H) = (GL(n,\mathbb{R}), GL(p,\mathbb{R})\times GL(q,\mathbb{R})), \tag{7.1}$$

$$(G',H') = (GL(n-1,\mathbb{R}), GL(p,\mathbb{R})\times GL(q-1,\mathbb{R})). \tag{7.2}$$

The first two subsections focus on describing $\mathrm{Disc}(K/H\cap K)$ and $\mathrm{Disc}(G/H)$. We then apply Theorem 6.3 to prove the non-vanishing of the period integral under the assumption on the minimal K-types, as described in (6.5). We shall see that the parity condition allows us to "jump the fences" for this interlacing pattern by iteratively applying Theorems 2.1 and 2.2. This leads to the whole range of parameters (λ,ν) for the non-vanishing of symmetry breaking in the restriction $G\downarrow G'$, as detailed in Theorem 7.8.

7.1. Description of $\mathrm{Disc}(K/H\cap K)$.

In the setting (7.1), the pair $(K, H\cap K)$ of maximal compact subgroups $(K,H\cap K)$ is given by $(O(p+q), O(p)\times O(q))$. The following result extends the Cartan–Helgason theorem, which was originally formulated for connected groups, to the case of disconnected groups.

Proposition 7.1. *Let $\ell := \min(p,q)$. In Weyl notation (see Section 5.1),* $\mathrm{Disc}(O(p+q)/O(p)\times O(q))$ *is given by*

$$\{F^{O(p+q)}(\mu) : \mu = (\mu_1,\dots,\mu_\ell,\overbrace{0,\dots,0}^{\max(p,q)}) \in (2\mathbb{Z})^{p+q}, \mu_1 \geq \cdots \geq \mu_\ell \geq 0\}.$$

If $p\neq q$, or if $\mu_\ell = 0$, then the $O(p+q)$-module $F^{O(p+q)}(\mu)$ remains irreducible when restricted to $SO(p+q)$. If $p=q$ and $\mu_\ell \neq 0$, then $F^{O(p+q)}(\mu)$ decomposes into the direct sum of two irreducible $SO(p+q)$-modules.

7.2. Discrete series for $GL(p+q,\mathbb{R})/GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$. In this subsection, we provide a complete description of discrete series representations for G/H in the setting (7.1).

Proposition 7.2. *Let $\ell := \min(p,q)$. Then* $\mathrm{Disc}(G/H)$ *is given by*

$$\{\Pi_\ell(\lambda) : \lambda = (\lambda_1, \dots, \lambda_\ell) \in (2\mathbb{Z}+1)^\ell, \lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0\}.$$

The $\mathfrak{Z}(\mathfrak{g}_\mathbb{C})$-infinitesimal character of the G-module $\Pi_\ell(\lambda)$ is non-singular if (5.6) holds, or equivalently, if $\lambda_\ell > n - 2\ell - 1$.

To verify Proposition 7.2, we utilize Matsuki–Oshima's description [22] of discrete series representations which may vanish, along with a detailed computation of cohomological parabolic induction beyond "good range", specifically, when $\lambda_\ell \le n - 2\ell - 1$ as discussed in [11]. We note that for such a singular parameter λ, neither the irreducibility nor the non-vanishing of cohomological parabolic induction is guaranteed by the general theory [31]. However, it turns out that both non-vanishing and irreducibility do hold in our specific setting.

We also derive an explicit formula for the minimal K-type $\mu(\Pi_\ell)$ of the G-module $\Pi_\ell(\lambda)$: it is given in Weyl's notation as follows.

$$\mu(\Pi_\ell(\lambda)) = F^{O(n)}(\lambda_1 + 1, \dots, \lambda_\ell + 1, 0, \dots, 0).$$

7.3. Comparison of minimal K-types for two groups $G' \subset G$. Let $n = p + q$. We realize $H = GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$ in standard block form as a subgroup of $G = GL(n,\mathbb{R})$, and realize $G' = GL(n-1,\mathbb{R})$ as a subgroup of G, corresponding to the partition $n = (n-1) + 1$. Accordingly, we obtain an embedding of the reductive symmetric space of G' into $X = G/H$, that is,

$$Y = GL(n-1,\mathbb{R})/(GL(p,\mathbb{R}) \times GL(q-1,\mathbb{R})).$$

We recall from Proposition 7.2 that any discrete series representation or X with a non-singular $\mathfrak{Z}(\mathfrak{g}_\mathbb{C})$-infinitesimal character is of the form $\Pi_\ell(\lambda)$, where $\ell = \min(p,q)$ and $\lambda = (\lambda_1, \dots, \lambda_\ell) \in (2\mathbb{Z}+1)^\ell$, satisfying (5.6).

We now assume $2p \leq n-1$. In this case, $\ell = p$, and any discrete series representation for the small symmetric space Y is of the form $\pi_\ell(\nu)$, with $\nu \in (2\mathbb{Z}+1)^\ell$, satisfying $\nu_1 > \cdots > \nu_\ell > 0$.

Next, we apply Theorems 6.1 and 6.3 to the pair $(\Pi_\ell(\lambda), \pi_\ell(\nu)) \in \widehat{G} \times \widehat{G'}$. The assumption (6.4) for minimal K-types is automatically satisfied for the pair $(K, K') = (O(n), O(n-1))$, while the condition (6.5) is computed explicitly as follows.

Lemma 7.3. *The condition* (6.5) *holds if and only if*

$$\lambda_1 = \nu_1 > \lambda_2 = \nu_2 > \cdots > \lambda_\ell = \nu_\ell > 0. \tag{7.3}$$

By Theorem 6.3, we obtain the following.

Proposition 7.4. *Suppose $2\ell \leq n-1$. Then we have*

$$\dim \mathrm{Hom}_{G'}(\Pi_\ell(\lambda)^\infty|_{G'}, \pi_\ell(\nu)^\infty) = 1$$

for any $\lambda \in (2\mathbb{Z}+1)^\ell$ and $\nu \in (2\mathbb{Z}+1)^\ell$ satisfying (7.3).

Remark 7.5. Alternatively, we can prove Proposition 7.4 by relying on the isomorphism (5.5) and Mackey theory. To do this, we use the fact that the G'-action on the generalized real flag manifold G/P_ℓ has an open dense orbit, and that the isotropy subgroup is contained in P'_ℓ, which is a parabolic subgroup of G of the same type.

7.4. Jumping the fences.

In this section, we analyze a phenomenon in which a certain parity condition allows us to "jump the fence", of the interlacing pattern in Theorems 2.1 and 2.2. We discover that this phenomenon indeed occurs for some geometric settings in the context of symmetry breaking for $GL(n,\mathbb{R}) \downarrow GL(n-1,\mathbb{R})$. As a result, we provide a refinement of the (non-)vanishing results of symmetry breaking.

We begin with the setting where $\Pi_\ell(\lambda)$ are irreducible unitary representations of G, and $\pi_k(\nu)$ are those of G', with $0 \leq 2\ell \leq n$ and $0 \leq 2k \leq n-1$, as introduced in Sections 5.3 and 5.4. In this generality, we impose a slightly stronger than the good range condition (5.6), that is, the following condition on the parameter $\lambda_1, \ldots, \lambda_\ell$:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > \max(n-2\ell-1, n-2k-3, 0). \tag{7.4}$$

Remark 7.6. For the application of Corollary 7.7 to Theorem 7.8, we use the case where $\ell = k$. In this case, or more generally, if $\ell \leq k+1$, the condition (7.4) reduces to the good range condition (5.6).

Corollary 7.7. *Let $\nu \in (2\mathbb{Z}+1)^k$ satisfying*

$$\nu_1 > \nu_2 > \cdots > \nu_k > \max(0, n-2k-2). \tag{7.5}$$

Then the following two conditions are equivalent:

(i) *there exists $\lambda \in (2\mathbb{Z}+1)^\ell$ satisfying* (7.4) *such that*

$$\mathrm{Hom}_{G'}(\Pi_\ell(\lambda)^\infty|_{G'}, \pi_k(\nu)^\infty) \neq \{0\};$$

(ii) *for every $\lambda \in (2\mathbb{Z}+1)^\ell$ satisfying* (7.4)*, one has*

$$\mathrm{Hom}_{G'}(\Pi_\ell(\lambda)^\infty|_{G'}, \pi_k(\nu)^\infty) \neq \{0\}.$$

Thus, Corollary 7.7 allows us to tear down all the "fences" of the weakly interlacing pattern given by Lemma 7.3, resulting in the following result:

Theorem 7.8. *Suppose $2\ell < n$. Then*

$$\dim \mathrm{Hom}_{G'}(\Pi_\ell(\lambda)^\infty|_{G'}, \pi_\ell(\nu)^\infty) = 1 \tag{7.6}$$

for any $\lambda, \nu \in (2\mathbb{Z}+1)^\ell$ satisfying the regularity conditions:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > n-2\ell-1,$$
$$\nu_1 > \nu_2 > \cdots > \nu_\ell > n-2\ell-1.$$

We already know that the left-hand side of (7.6) is either 0 or 1, according to the multiplicity-freeness theorem [28] for $GL(n,\mathbb{R}) \downarrow GL(n-1,\mathbb{R})$, since $\Pi_\ell(\lambda)$ and $\pi_\ell(\nu)$ are irreducible as G- and G'-modules, respectively. Our claim is that the multiplicity is non-zero, as a consequence of "jumping all the fences".

8 Restricting discrete series representations of symmetric spaces $U(p,q)/(U(r,s) \times U(p-r,q-s))$ to the subgroup $U(p-1,q)$

In this section, we revisit the case where

$$(G, G') = (U(p,q), U(p-1,q)),$$

and discuss the branching of the restriction $\Pi|_{G'}$, where Π is a *non-tempered* irreducible representation of G. Specifically, we consider a discrete series representation Π for the symmetric space

$$G/H = U(p,q)/(U(r,s) \times U(p-r, q-s)),$$

and prove that $\mathrm{Hom}_{G'}(\Pi^\infty|_{G'}, \pi^\infty) \neq 0$ for some family of irreducible representations $\pi \in \widehat{G'}$, which are not necessarily tempered.

The irreducible unitary representations π of the subgroup G' for which $\mathrm{Hom}_{G'}(\Pi^\infty|_{G'}, \pi^\infty) \neq 0$ were completely determined when $(r,s) = (0,1)$, as a particular case of [12, Thm. 3.4], which corresponds to the discretely decomposable case. In the case where π occurs as a discrete series representation for a symmetric space G'/H', a non-vanishing result was recently proven in [25] when $(r,s) = (1,0)$.

We provide a non-vanishing theorem in Theorem 8.6 for the general case of (p,q,r,s) under a certain interlacing condition on parameters. Our proof again utilizes the non-vanishing theorem of the period integral for specific parameters, as stated in Theorem 6.3, as well as the non-vanishing result of symmetry breaking under translations inside "fences", as stated in Theorem 3.3.

8.1. A family of (non-tempered) irreducible unitary representations of $U(p,q)$.

In this subsection, we define a family of irreducible unitary representations of $G = U(p,q)$. In the next subsection, we see in Proposition 8.5 that any discrete series representation for the symmetric space $X = U(p,q)/(U(r,s) \times U(p-r,q-s))$ is of this form when $2r \leq p$ and $2s \leq q$.

Let $\mathfrak{j}$ be a compact Cartan subalgebra, $\{H_1, \dots, H_{p+q}\}$ be the standard basis $\sqrt{-1}\mathfrak{j}$, and $\{f_1, \dots, f_{p+q}\}$ its dual basis. We fix a positive system of $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ by defining

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}) = \{f_i - f_j : 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq p+q\}.$$

Given $Z = (z_1, \dots, z_{p+q}) \in \sqrt{-1}\mathfrak{j} \simeq \mathbb{R}^{p+q}$, we define a θ-stable parabolic subalgebra $\mathfrak{q} \equiv \mathfrak{q}(Z) = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(p+q, \mathbb{C})$ such that the set of weights of the unipotent radical $\mathfrak{u}$ is given by

$$\Delta(\mathfrak{u}, \mathfrak{j}_{\mathbb{C}}) = \{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}) : \alpha(Z) > 0\}.$$

Any θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ is K-conjugate to $\mathfrak{q}(Z)$ for some $Z \in \mathbb{R}^p_{\geq} \times \mathbb{R}^q_{\geq}$. We are particularly interested in the following:

Setting 8.1. Let $0 \leq 2r \leq p$, $0 \leq 2s \leq q$, and

(8.1)
$$Z = (x_1, \dots, x_r, \underbrace{0, \dots, 0}_{p-2r}, -x_r, \dots, -x_1; y_1, \dots, y_s, \underbrace{0, \dots, 0}_{q-2s}, -y_s, \dots, -y_1)$$

with $x_1 > \cdots > x_r > 0$, $y_1 > \cdots, y_s > 0$, and $x_i \neq y_j$ for any i, j.

In this case, the (real) Levi subgroup L, the normalizer of $\mathfrak{q}(Z)$ in G, depends only on r and s, and is given by

$$L \equiv L^U_{p,q;r,s} \simeq \mathbb{T}^{2r+2s} \times U(p-2r, q-2s). \tag{8.2}$$

Lemma 8.2. *Let $G = U(p,q)$. We fix r and s such that $2r \leq p$ and $2s \leq q$. Then, there is a one-to-one correspondence among the following three objects:*
(i) *θ-stable parabolic subalgebras $\mathfrak{q} \equiv \mathfrak{q}(Z)$, where Z is of the form as given in Setting 8.1.*
(ii) *Interlacing patterns $D \in \mathfrak{P}(\mathbb{R}^{r,s})$ in $\mathbb{R}^r_{>} \times \mathbb{R}^s_{>}$.*
(iii) *Data $\kappa = \{(r_j), (s_j), M\}$ with $1 \leq M \leq \min(r,s)$ and*

$$0 \leq r_1 < \cdots < r_{M-1} < r_M = r, \ \ 0 < s_1 < \cdots < s_{M-1} \leq s_M = s. \tag{8.3}$$

Remark 8.3. We allow the cases $r_1 = 0$ or $s_{M-1} = s_M$, but assume that $s_1 > 0$ and $r_{M-1} < r_M$.

Proof. We describe the natural morphisms, which establish the one-to-one correspondence among (i), (ii) and (iii).
(i) $\Leftrightarrow$ (ii) By definition, an interlacing pattern D in $\mathfrak{P}(\mathbb{R}^{r,s})$ defines a θ-stable parabolic subalgebra $\mathfrak{q}(Z)$ via (8.1). Conversely, it is clear that the θ-stable parabolic subalgebra $\mathfrak{q}(Z)$ associated with Z in Setting 8.1 depends solely on the interlacing pattern of x, y in $\mathbb{R}^r_{>} \times \mathbb{R}^s_{>}$.
(ii) $\Leftrightarrow$ (iii) Given a condition in (8.3), we associate the following interlacing pattern D in $\mathbb{R}^r_{>} \times \mathbb{R}^s_{>}$ defined by

(8.4)
$$x_1 > \cdots > x_{r_1} > y_1 > \cdots > y_{s_1} > x_{r_1+1} > \cdots > x_{r_2} > y_{s_1+1} > \cdots$$
$$\cdots > y_{s_{M-1}} > x_{r_{M-1}+1} > \cdots > x_{r_M} > y_{s_{M-1}+1} > \cdots > y_{s_M},$$

and vice versa. $\square$

Let D be an interlacing pattern in $\mathbb{R}^r_> \times \mathbb{R}^s_>$ as in (8.4). For $A \in \mathbb{R}$, we set

$$D_{>A} := \{(x, y) \in D : x_i > A, y_j > A \text{ for any } i,\ j\}.$$

Suppose that $L = \mathbb{T}^{2r+2s} \times U(p-2r, q-2s)$ is the real Levi subgroup for the θ-stable parabolic subalgebra $\mathfrak{q}$, which is associated to an interlacing pattern $D \in \mathfrak{P}(\mathbb{R}^{r,s})$ in Lemma 8.2. For $\lambda = (x, y) \in (\mathbb{Z} + \frac{p+q-1}{2})^{r+s}$, we define a one-dimensional representation of the double covering group of the torus $\mathbb{T}^{2(r+s)}$, to be denoted by $\mathbb{C}_{\widetilde{\lambda}}$, such that its differential is given by the formula (8.1). We extend it to a one-dimensional representation of $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, by letting $\mathfrak{u}(p-2r, q-2s) + \mathfrak{u}$ act trivially. The character $\mathbb{C}_{\widetilde{\lambda}}$ is in the fair range (respectively, in the good range) with respect to $\mathfrak{q}$ in the sense of [31], if $\lambda \in D_{>0}$ (respectively, $\lambda \in D_{>Q}$), where we set

$$Q := \frac{1}{2}(p+q-1) - r - s.$$

When $\lambda \in D_{>0}$, cohomological parabolic induction gives a unitarizable $(\mathfrak{g}, K)$-module, which is possibly zero ([31]). It is irreducible if non-zero. Let Π_λ denote the unitarization. The unitary representation Π_λ is non-tempered if $p \neq 2r$ and $q \neq 2s$.

In our normalization, the $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character of the G-module Π_λ is given by

$$(8.1) \oplus (Q, Q-1, \ldots, 1-Q, -Q) \in \mathbb{C}^{p+q}/\mathfrak{S}_{p+q}.$$

When $D_{>Q}$, the general theory guarantees that Π_λ is non-zero and that the highest weight of its minimal K-type is given as follows:

$$\begin{aligned}
(\mu_\lambda)_i &= -(\mu_\lambda)_{p+1-i} = \lambda_i + \frac{-p+q+1}{2} + \ell_i && \text{for } 1 \le i \le r,\\
(\mu_\lambda)_{p+i} &= -(\mu_\lambda)_{p+q+1-i} = \lambda_{r+i} + \frac{p-q+1}{2} - \ell_{r+i} && \text{for } 1 \le i \le s,\\
(\mu_\lambda)_i &= 0 && \text{otherwise.}
\end{aligned}$$

Here, we define $\ell_i \equiv \ell_i(D) \in \mathbb{Z}$ for $1 \le i \le r+s$, depending on the interlacing pattern D, by

$$\begin{aligned} \ell_i(D) &:= \sharp\{x_k : x_k > x_i\} - \sharp\{y_k : y_k > x_i\} && \text{for } 1 \le i \le r, \\ \ell_{r+i}(D) &:= \sharp\{x_k : x_k > y_i\} - \sharp\{y_k : y_k > y_i\} && \text{for } 1 \le i \le s. \end{aligned}$$

Example 8.4. Let $(r,s) = (3,2)$ and $D = \{x_1 > y_1 > y_2 > x_2 > x_3\}$. Then

$$\ell_1(D) = 0,\ \ell_2(D) = -1,\ \ell_3(D) = 0; \ell_4(D) = 1,\ \ell_5(D) = 0.$$

8.2. Discrete series representations for the symmetric space $U(p,q)/(U(r,s) \times U(p-r,q-s))$.

Let $H = U(p_1,q_1) \times U(p_2,q_2)$ be a natural subgroup of $G = U(p,q)$, where $p_1 + p_2 = p$ and $q_1 + q_2 = q$. The symmetric space G/H has a discrete series representation if and only if the rank condition (6.1) holds, that is,

$$\min(p_1,p_2) + \min(q_1,q_2) = \min(p_1+q_1, p_2+q_2). \tag{8.5}$$

From now on, without loss of generality, we assume that

$$H = U(r,s) \times U(p-r,q-s) \qquad \text{with } 2r \le p \text{ and } 2s \le q.$$

Discrete series representations for a reductive symmetric space G/H are decomposed into families corresponding to H^d-closed orbits on the real flag variety of G^d. Here, (G^d, H^d) is the dual symmetric pair of (G,H), see [4, 22]. In the above setting, we have

$$(G^d, H^d) = (U(r+s, p+q-r-s), U(r,p-r) \times U(s,q-s)),$$

and there are $\frac{(r+s)!}{r!s!}$ closed orbits of the subgroup H^d on the real flag variety of G^d. These orbits are parametrized by interlacing patterns $\mathfrak{P}(\mathbb{R}^{r,s})$ in $\mathbb{R}^r_> \times \mathbb{R}^s_>$.

Proposition 8.5. *Suppose $0 \le 2r \le p$ and $0 \le 2s \le q$. Then the set of discrete series representations*

$$\mathrm{Disc}(U(p,q)/(U(r,s) \times U(p-r,q-s)))$$

is given by the disjoint union

$$\coprod_{D\in\mathfrak{P}(\mathbb{R}^{r,s})}\{\Pi_\lambda : \lambda \in D_{>0}\cap(\mathbb{Z}+\frac{p+q-1}{2})^{r+s}\}.$$

As mentioned in the previous subsection, Π_λ may vanish if $\mathbb{C}_{\tilde{\lambda}}$ is not in the good range, specifically, if $\lambda \in D_{>0}\setminus D_{>Q}$. The condition for the non-vanishing of Π_λ involves a number of inequalities of λ that depend heavily on $D\in\mathfrak{P}(\mathbb{R}^{r,s})$ (see [11, Chap. 5]).

8.3. Branching for $U(p,q)\downarrow U(p-1,q)$.

We are ready to state our main results of this section.

Theorem 8.6. *Suppose that* $0\le 2r\le p-1$, $0\le 2s\le q$ *and* $D, D'\in\mathfrak{P}(\mathbb{R}^{r,s})$. *Let* $\mathfrak{q}$ *be the* θ*-stable parabolic subalgebra of* $\mathfrak{g}_{\mathbb{C}}$ *and* $(\{r_i\},\{s_i\},M)$ *be the data, associated with* D*, as in Lemma 8.2. Similarly, let* $\mathfrak{q}'$ *be the* θ*-stable parabolic subalgebra of* $\mathfrak{g}'_{\mathbb{C}}$ *and* $(\{r'_j\},\{s'_j\},M')$ *be the data, associated with* D'*. We set* $Q=\frac{1}{2}(p+q-1)-r-s$ *and* $Q'=Q-\frac{1}{2}$.

Assume that $D=D'$*, or equivalently that* $M'=M$, $r'_i=r_i$ $(1\le i\le M)$ *and* $s'_i=s_i$ $(1\le i\le M)$*. Then we have the following identity:*

$$\dim \operatorname{Hom}_{G'}(\Pi_\lambda^\infty|_{G'},\pi_\nu^\infty)=1, \tag{8.6}$$

if $\lambda=(x,y)\in D_{>Q}\cap(\mathbb{Z}+Q)^{r+s}$ *and* $\nu=(\xi,\eta)\in D'_{>Q'}\cap(\mathbb{Z}+Q')^{r'+s}$ *satisfy the following interlacing pattern:*

$$\begin{aligned}&x_1>\xi_1>\cdots>x_{r_1}>\xi_{r_1}>\eta_1>y_1>\cdots>\eta_{s_1}>y_{s_1}>\\ &>x_{r_1+1}>\xi_{r_1+1}>\cdots>x_{r_2}>\xi_{r_2}>\eta_{s_1+1}>y_{s_1+1}>\cdots>\eta_{s_2}>y_{s_2}>\\ &\qquad\cdots>x_{r_M}>\xi_{r_M}>\eta_{s_{M-1}+1}>y_{s_{M-1}+1}>\cdots>\eta_{s_M}>y_{s_M}.\end{aligned}$$

Remark 8.7. The interlacing pattern on $\lambda=(x,y)$ and $\nu=(\xi,\eta)$ in Theorem 8.6 is equivalent to that $[DD'+]\in\mathfrak{P}(\mathbb{R}^{r+1,s})\times\mathfrak{P}(\mathbb{R}^{r,s})$ is a *coherent pair*, where $[DD'+]$ is an interlacing pattern of (λ, x_{r+1},ν), defined by the inequalities D for the entries of λ and ν, along with the condition that x_{r+1} is smaller than any of the entries of λ and ν. For various equivalent definitions of "coherent pairs", we refer to [8].

Owing to Theorem 3.3, which describes the nice behavior of symmetry breaking under translations inside the fences, the proof of Theorem

8.6 reduces to the following proposition, which is derived from Theorem 6.3 on the period integral for discrete series representations of the symmetric spaces G/H and G'/H'.

Proposition 8.8. *In the setting and assumptions of Theorem 8.6, the equality* (8.6) *holds if* $\lambda = (x, y) \in \mathbb{R}^{r+s}$ *and* $\nu = (\xi, \eta) \in \mathbb{R}^{r'+s}$ *satisfy following conditions:*

$$\begin{cases} x_i = \xi_i + \frac{1}{2} & (1 \le i \le r), \\ y_i = \eta_i - \frac{1}{2} & (1 \le i \le s). \end{cases} \tag{8.7}$$

Proof. We apply Theorem 6.3 to $X = G/H$ and $Y = G'/H'$, where we realize $G' = U(p-1, q)$ as a subgroup of G such that $H' := H \cap G \simeq U(r, s) \times U(p-r-1, q-s)$. Then, condition (8.7) ensures that assumption (6.5) for minimal K-types in Theorem 6.3 holds, while (6.4) is obvious. $\square$

In turn, Theorem 3.3 extends the specific parameters in Proposition 8.8 to all the parameters stated in Theorem 8.6, using the translations within the initial fences. Thus, the non-vanishing of symmetry breaking is guaranteed, and the proof of Theorem 8.6 is complete.

In contrast to the $GL(n, \mathbb{R})$ case in Section 7.4, we note that jumping the fences is not allowed in the $U(p, q)$ case due to a different parity condition.

9 Arthur packets, a "GGP theorem" for some non tempered representations and a conjecture.

9.1. Some geometric observations. We recall from [17] a generalized notion of "Borel subalgebras" for reductive symmetric spaces G/H associated with involutive automorphisms σ of G.

Let G_U be a maximal compact subgroup of $G_{\mathbb{C}}$ so that $G_U \cap G$ and $G_U \cap H$ are also maximal compact subgroups of G and H, respectively. We fix an $\mathrm{Ad}(G)$-invariant, non-degenerate symmetric bilinear form on the Lie algebra $\mathfrak{g}$, which is also non-degenerate on the subalgebra $\mathfrak{h}$. We write $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^{\perp}$ for direct sum decomposition, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \mathfrak{h}_{\mathbb{C}}^{\perp}$ for its complexification. Recall that to a given hyperbolic element Y

in $\mathfrak{g}$, one associates a parabolic subalgebra of $\mathfrak{g}$, defined as the sum of eigenspaces of $\mathrm{ad}(Y)$ with non-negative eigenvalues.

Definition 9.1 (Relative Borel subalgebra for G/H, *cf.* [17])**.** Let (G, H) be a reductive symmetric pair. A *Borel subalgebra* $\mathfrak{b}_{G/H}$ for G/H is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. It is defined by a generic element of $\mathfrak{h}_{\mathbb{C}}^{\perp} \cap \sqrt{-1}\mathfrak{g}_U$ or by its conjugate under an inner automorphism of $G_{\mathbb{C}}$.

The relative Borel subalgebra $\mathfrak{b}_{G/H}$ which is not necessarily solvable, and thus its Levi subalgebra $\mathfrak{l}_{G/H}$ is not always abelian. We note that $\mathfrak{b}_{G/H}$ and $\mathfrak{l}_{G/H}$ are determined solely from the complexified symmetric pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

The Levi subalgebra for the symmetric space

$$G/H = GL(n, \mathbb{R})/GL(\ell, \mathbb{R}) \times GL(n - \ell, \mathbb{R})$$

is given by

$$\mathfrak{l}_{G/H} = \mathbb{C}^{2\ell} \oplus \mathfrak{gl}(p + q - 2\ell, \mathbb{C}) \tag{9.1}$$

if $2\ell \leq n$.

On the other hand, for the group $G = U(p, q)$, the symmetric spaces

$$U(p, q)/(U(r, s) \times U(p - r, q - s)) \quad \text{for } 2r \leq p \text{ and } 2s \leq q$$

for different (r, s) are not isomorphic to each other, whereas they share the same complex Levi subalgebra as long as $r + s$ is constant, say $= \ell$; they are also isomorphic to the complex Levi subalgebra (9.1) of the symmetric space $GL(n, \mathbb{R})/(GL(\ell, \mathbb{R}) \times GL(n-\ell, \mathbb{R}))$. However, the real Levi subgroups, that are used in cohomological parabolic induction, are different: $L^U_{p,q;r,s}$ for the symmetric spaces $U(p, q)/(U(r, s) \times U(p-r, q-s)$ is $\mathbb{T}^{2r+2s} \times U(p - 2r, q - 2s)$, while $L^{\mathbb{R}}_{n;p}$ for $GL(n, \mathbb{R})/(GL(\ell, \mathbb{R}) \times GL(n - \ell, \mathbb{R}))$ is $(\mathbb{C}^{\times})^{\ell} \times GL(n - 2\ell, \mathbb{R})$.

Remark 9.2. For example, in the rank one case, we obtain non-compact symmetric spaces

$$L^U_{p,q;1,0}/L^U_{p,q;1,0} \cap (U(1, 0) \times U(p - 1, q))$$

and

$$L^{\mathbb{R}}_{n,1}/L^{\mathbb{R}}_{n,1} \cap (GL(1,\mathbb{R}) \times GL(n-1,\mathbb{R}))$$

which are not isomorphic.

9.2. Arthur packets of non-tempered representations. We recall some results about Arthur packets and representations in the discrete spectrum of the symmetric spaces.

The representations in the discrete spectrum of the symmetric spaces $U(p,q)/(U(1,0)\times U(p-1,q))$ and $U(p,q)/(U(0,1)\times U(p,q-1))$, which have the same non-singular infinitesimal character, are not isomorphic. However, both are in the same Arthur packet [25].

Let $2\ell \leq n$. Given a fixed non-singular integral infinitesimal character (5.3), C. Moeglin and D. Renard showed in [23] that the representations with this infinitesimal character, which are in the discrete spectrum of $GL(n,\mathbb{R})/(GL(\ell,\mathbb{R}) \times GL(n-\ell,\mathbb{R}))$, are in the same Arthur packet $\mathcal{A}(\lambda)$. The Arthur packets for $GL(n,\mathbb{R})$ each contain only one representation [1]. In particular, the irreducible unitary representation $\Pi_\ell(\lambda)$ in Proposition 7.2, with the regularity condition $\lambda_1 > \cdots > \lambda_{\ell-1} > \lambda_\ell > \max(n-2\ell-1, 0)$, is the only representation in the Arthur packet.

9.3. "Operations on the unitary dual".

In the article [30], A. Venkatesh also discusses the restriction of representations $\Pi_\ell(\lambda)$ of $GL(n,\mathbb{R})$ to a subgroup $GL(n-1,\mathbb{R})$ embedded in the upper left corner as the stabilizer of the last coordinate vector. More generally in this paper, he discusses for $GL(n)$, the effect on the unitary dual of the following operations: restriction to a Levi subgroup, induction from Levi subgroups and tensor product. Without explicitly computing symmetry breaking operators or referring to symmetric spaces, using only the Mackey machine, A. Venkatesh considers representations induced from the trivial representation of $GL(p,\mathbb{R}) \times GL(n-q,\mathbb{R})$ to $GL(n,\mathbb{R})$ and their restriction to the subgroup $GL(n-1,\mathbb{R})$ proving conjectures by L. Clozel. We cite from the abstract of the article by L. Clozel [3]: "The Burger–Sarnak principle states that the restriction to a reductive subgroup of an automorphic representation of a reductive group has automorphic support. Arthur's conjectures parametrize automorphic representations by means of the (Langlands) dual group. Taken together, these principles, combined with some new arguments, imply that unipotent orbits in a Langlands dual behave functorially with respect to arbitrary morphisms $H \to G$ of semisimple groups. The existence of this functoriality is proven for $SL(n)$, and combinatorial descriptions of it (due to Kazhdan, Venkatesh, and Waldspurger) are proposed".

In this article, we have discussed the restriction of a family of non-tempered unitary representations of $GL(n,\mathbb{R})$ to $GL(n-1,\mathbb{R})$ and have shown the existence of non-trivial SBOs. We also provided a proof for symmetry breaking for some tempered representations.

9.4. A GGP theorem for the symmetric space $GL(n,\mathbb{R})/(GL(p,\mathbb{R}) \times GL(n-p,\mathbb{R}))$?

Around 1992, B. Gross and D. Prasad published conjectures concerning the restriction of discrete series representations of orthogonal groups to smaller orthogonal groups [6]. These have been generalized to unitary groups and have been proven by H. He [10] for individual discrete series representations.

These ideas can be generalized in 2 directions:

- discrete series of symmetric spaces
- representations in Arthur packets

Discrete series representations of a symmetric space G/H are generally not tempered representations. See [2] for the classification of G/H such that the regular representation on $L^2(G/H)$ is non-tempered. In [23] D. Renard and C. Moeglin examine the relationship between Arthur packets and discrete series representations of symmetric spaces of classical groups. The discrete representations of symmetric spaces are members of an Arthur packet [23]. Not all members of such an Arthur packet are discrete series representations of a symmetric space, and an Arthur packet may contain discrete series representations of several symmetric spaces [23]. Generalizing the GGP conjectures to symmetric spaces involves generalizing them to the subset of representations in a given Arthur packet which are discrete series of a symmetric space. Moeglin and Renard showed that if a representation in an Arthur packet is in the discrete spectrum of a symmetric space, then another representation in the same packet is either in the discrete spectrum of no symmetric space or in the discrete spectrum of a unique symmetric space [23]. The results in [25] suggest that it may be possible to generalize the GGP conjectures to discrete series representations of symmetric spaces.

Using the observation that Arthur packets for $GL(n,\mathbb{R})$ contain exactly one representation, these ideas lead to the following reformulation of the conclusion of Theorem 7.8 as follows.

Let Π and π be discrete series representations in

$$L^2(GL(n,\mathbb{R})/GL(\ell,\mathbb{R}) \times GL(n-\ell,\mathbb{R})),$$

and

$$L^2(GL(n-1,\mathbb{R})/GL(\ell,\mathbb{R}) \times GL(n-\ell-1,\mathbb{R})),$$

respectively, where $2\ell \leq n-1$. We also assume that they have nonsingular infinitesimal characters. Let $\mathcal{A}_\Pi$ and $\mathcal{A}_\pi$ be Arthur packets, such that

$$\Pi \in \mathcal{A}_\Pi \quad \text{and} \quad \pi \in \mathcal{A}_\pi.$$

We can summarize our discussion as follows.

Corollary 9.3. *Under the above assumptions, we have:*

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) = \mathbb{C}$$

for all pairs of representations $\Pi \in \mathcal{A}_\Pi$ *and* $\pi \in \mathcal{A}_\pi$.

9.5. Restricting representations in the discrete spectrum of $U(p,q)/(U(r,s) \times U(p-r,q-s))$ to $U(p-1,q)$.

Next, we consider the restriction from $G = U(p,q)$ to $G' = U(p-1,q)$ of representations in the discrete spectrum of a symmetric space.

Let $2r \leq p$ and $2s \leq q$. By Proposition 8.5, representations in the discrete spectrum of the symmetric space

$$U(p,q)/(U(r,s) \times U(p-r,q-s))$$

can be parametrized by interlacing patterns $\mathfrak{P}(\mathbb{R}^{r,s})$ in $\mathbb{R}^r_> \times \mathbb{R}^s_>$. Recall that discrete series representations of $U(r,s)$ are also parametrized by interlacing patterns $\mathfrak{P}(\mathbb{R}^{r,s})$ in $\mathbb{R}^r_> \times \mathbb{R}^s_>$.

Observation 1
An interlacing pattern of the infinitesimal character of a representation in the discrete spectrum of $U(p,q)/(U(r,s) \times U(p-r,q-s))$ *defines an interlacing pattern of a discrete series representation of* $U(r,s)$. This is given in Proposition 8.5. Furthermore, every interlacing pattern that defines a discrete series representation of $U(r,s)$ can be extended to an interlacing pattern defining a representation in the discrete spectrum of $U(p,q)/(U(r,s) \times U(p-r,q-s))$.
Observation 2 *The interlacing rules by H. He* [10] *also suggest the non-vanishing of the multiplicity* $[\Pi|_{U(p-1,q)} : \pi]$ *for pairs* (Π, π) *of representations in the discrete spectrum of*

$$U(p,q)/(U(r,s) \times U(p-r,q-s))$$

and in the discrete spectrum of

$$U(p-1,q)/(U(r,s) \times U(p-r-1,q-s))$$

.

An Example: We consider discrete series representations in $U(2,1)$, $U(1,1)$ and their multiplicities, as well as representations in the discrete spectrum of $U(5,6)/(U(2,1) \times U(3,5))$, respectively $U(4,6)/(U(1,1) \times U(3,5))$. In accordance with the conventions from Proposition 8.5, we use discrete series representations of the small groups $U(2,1)$ and $U(1,1)$ to parametrize representations in the discrete spectrum of the symmetric spaces. Next, we investigate whether the branching of the discrete series representations is reflected in the branching of the representations in the discrete spectrum of the symmetric spaces.

In this example, we use the conventions from Sections 3 and 8.

The group $U(2,1)$ has 3 families of discrete series representations. They are parametrized by interlacing patterns of $(x_1, x_2, y_1) \in (\mathbb{Z})^2_> \times \mathbb{Z}$:

$$x_1, x_2, y_1 \quad x_1, y_1, x_2 \quad y_1, x_1, x_2.$$

The group $U(1,1)$ has 2 families of discrete series representations. They are are parametrized by an interlacing pattern of $w, v \in \mathbb{Z}+1/2$:

$$w, v \quad v, w.$$

The interlacing patterns identified by H. He which correspond to nonzero SBO for the discrete series representations are

- H: x_1, w, x_2, y_1, v, AH: v, y_1, x_1, w, x_2, A: x_1, w, v, y_1, x_2,
- B: x_1, y_1, x_2, w, v, C: x_1, y_1, v, w, x_2, D: v, w, x_1, y_1, x_2

We now fix a representation π of $U(1,1)$ corresponding to w, v. The interlacing patterns corresponding to discrete series reps Π of $U(2,1)$ satisfying $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \neq 0$ are H, A, and B. Of these, only the interlacing pattern A is coherent.

For $U(5,6)$ the infinitesimal characters of the representations in the discrete spectrum of $U(5,6)/U(2,1)\times U(3,5)$ correspond to interlacing pattern

- $x_1+4, x_2+4, y_1+4, 2, 1, 0, -1, -2, -y_1-4, -x_2-4, -x_1-4$ $\longleftrightarrow\ x_1, x_2, y_1$
- $x_1+4, y_1+4, x_2+4, 2, 1, 0, -1, -2, -x_2-4, -y_1-4, -x_1-4$ $\longleftrightarrow\ x_1, y_1, x_2$
- $y_1+4, x_1+4, x_2+4, 2, 1, 0, -1, -2, -x_2-4, -x_1-4, -y_1-4$ $\longleftrightarrow\ y_1, x_1, x_2$

For $U(4,6)$, the representations in the discrete spectrum of $U(4,6)/(U(1,1)\times U(3,5))$ correspond to the interlacing pattern:

- $w+4, v+4, 5/2, 3/2, 1/2, -1/2, -3/2, -5/2, -v-4, -w-4$ $\longleftrightarrow w, v$
- $v+4, w+4, 5/2, 3/2, 1/2, -1/2, -3/2, -5/2, -w-4, -v-4$ $\longleftrightarrow\ v, w$

We expect that there are non-trivial SBO for the restriction from $U(5,6)$ to $U(4,6)$ corresponding to the interlacing patterns,

- $x_1, w, x_2, y_1, v,$
  $5/2, 2, 3/2, 1, 1/2, 0, -1/2, -1, -3/2, -2, -5/2,$
  $-v, -y_1, -x_2, -w, -x_1$

- v, y_1, x_1, w, x_2
  $$5/2, 2, 3/2, 1, 1/2, 0, -1/2, -1, -3/2, -2, -5/2,$$
  $$-x_2, -w, -x_1, -y_1, -v$$
- x_1, y_1, v, w, x_2
  $$5/2, 2, 3/2, 1, 1/2, 0, -1/2, -1, -3/2, -2, -5/2,$$
  $$-x_2, -w, -v, -y_1, -x_1$$
- $x_1, w, v, y_1, x_2,$
  $$5/2, 2, 3/2, 1, 1/2, 0, -1/2, -1, -3/2, -2, -5/2,$$
  $$-x_2, -y_1, -v, -w, -x_1$$

Theorem 8.6 does not apply, and so we do not receive information about SBO.

On the other hand, consider the symmetric spaces

$$U(5,6)/U(1,1) \times U(4,5), \quad U(4,6)/U(1,1) \times U(3,5).$$

By Theorem 8.6, there exists a non-trivial SBO as we can see in this example. For $U(4,6)$, the representations in the discrete spectrum of $U(4,6)/U(1,1) \times U(3,5)$ correspond to

- $w' + 4, v' + 4, 7/2, 5/2, 3/2, 1/2, -1/2, -3/2, -5/2, -7/2, -v' - 4, -w' - 4$
- $v' + 4, w' + 4, 7/25/2, 3/2, 1/2, -1/2, -3/2, -5/2, -7/2, -w' - 4, -v' - 4$

For $U(5,6)/(U(1,1) \times U(4,5))$, the representations in the discrete spectrum correspond to

- $w + 9/2, v + 9/2, 4, 3, 2, 1, 0, -1, -2, -3, -4, -v - 9/2, -w - 9/2$
- $v + 9/2, w + 9/2, 4, 3, 2, 1, 0, -1, -2, -3, -4, -w - 9/2, -v - 9/2$

We have a non-trivial SBO if $w, v,\ w', v'$ satisfies the interlacing pattern w, w', v', v.

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