Bounded multiplicity branching for symmetric pairs

Dedicated to Dr. Karl H. Hofmann with admiration and heartfelt gratitude for his contributions to mathematics and his devotion to the community

Toshiyuki KOBAYASHI *

Abstract

We prove that any simply connected non-compact semisimple Lie group G admits an infinite-dimensional irreducible representation Π with bounded multiplicity property of the restriction $\Pi|_{G'}$ for *all* symmetric pairs (G, G'). We also discuss which irreducible representations Π satisfy the bounded multiplicity property.

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1 Introduction

We initiated in [18, 19] the general study of *multiplicities* in the branching problem of reductive groups, and this article is a continuation of the work [27, 29, 31, 32, 33, 38]. The goal is to prove the following theorem.

Theorem 1.1. *Any simply connected non-compact semisimple Lie group G admits an infinite-dimensional irreducible representation Π with the bounded multiplicity property of the restriction $\Pi|_{G'}$ for all symmetric pairs (G, G'):*

$$\sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty. \tag{1.1}$$

*Graduate School of Mathematical Sciences, The University of Tokyo, Japan.

Let us explain some terminologies. We denote by $\mathrm{Irr}(G)$ the set of irreducible objects in the category $\mathcal{M}(G)$ of smooth admissible representations of a real reductive Lie group G of finite length with moderate growth, which are defined on Fréchet topological vector spaces [51, Chap. 11].

Suppose that G' is a reductive subgroup of G. For $\Pi \in \mathcal{M}(G)$, the *multiplicity* of $\pi \in \mathrm{Irr}(G')$ in the restriction $\Pi|_{G'}$ is defined by

$$[\Pi|_{G'} : \pi] := \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}, \tag{1.2}$$

where $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ denotes the space of *symmetry breaking operators*, *i.e.*, continuous G'-homomorphisms between the Fréchet representations.

By a *symmetric pair* (G, G'), we mean that G' is an open subgroup of the fixed point group G^σ of an involutive automorphism σ of G. The Riemannian symmetric pair (G, K) with σ being a Cartan involution θ and the *group manifold case* $({}^\backprime G \times {}^\backprime G, \mathrm{diag}\, {}^\backprime G)$ are typical examples. The pair $(SL(n, \mathbb{R}), SO(p, q))$ with $p + q = n$ is another example of symmetric pairs. The infinitesimal classification of irreducible symmetric pairs was accomplished by Berger [2].

Theorem 1.1 may look quite surprising, in view of the theorem [27] revealing that for "many" symmetric pairs (G, G') with G' non-compact

$$[\Pi|_{G'} : \pi] = \infty \text{ for some } \Pi \in \mathrm{Irr}(G) \text{ and } \pi \in \mathrm{Irr}(G'). \tag{1.3}$$

See [34] for the classification of such symmetric pairs (G, G').

We refer to [28] and [33, Sect. 2] for some motivation and perspectives of the general branching problems and the role of bounded multiplicity property.

The tensor product of two representations is a special case of the restriction with respect to a symmetric pair $(G \times G, \mathrm{diag}\, G)$. We also prove:

Theorem 1.2. *For any simply connected, non-compact semisimple Lie group G, there exist infinite-dimensional representations $\Pi_1, \Pi_2 \in \mathrm{Irr}(G)$ such that the tensor product representation has the bounded multiplicity property:*

$$\sup_{\Pi \in \mathrm{Irr}(G)} [\Pi_1 \otimes \Pi_2 : \Pi] < \infty. \tag{1.4}$$

When Π is a unitary representation of G, the restriction $\Pi|_{G'}$ decomposes into the direct integral of irreducible unitary representations of the subgroup G':

$$\Pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} m_\Pi(\pi) \pi \, d\mu(\pi), \tag{1.5}$$

where $\widehat{G'}$ denotes the set of equivalence classes of irreducible unitary representations (*unitary dual*) of G' equipped with Fell topology, μ is a Borel measure on $\widehat{G'}$, and $m_\Pi \colon \widehat{G'} \to \mathbb{N} \cup \{\infty\}$ is a measurable function. The irreducible decomposition (1.5) is called the *branching law* of the restriction $\Pi|_{G'}$. By the theory of nuclear spaces, the multiplicity in the category of admissible representations of moderate growth dominates the one in the category of unitary representations, namely, one has the inequality

$$m_\Pi(\pi) \leq [\Pi^\infty|_{G'} : \pi^\infty] \quad \text{a.e. } \pi \in \widehat{G'} \text{ with respect to } \mu, \tag{1.6}$$

where $\Pi^\infty \in \mathrm{Irr}(G)$ and $\pi^\infty \in \mathrm{Irr}(G')$ denote the Fréchet representations of smooth vectors of $\Pi \in \widehat{G}$ and $\pi \in \widehat{G'}$, respectively. Since we can take unitarizable representations in Theorems 1.1 and 1.2 as the proof in Sections 3–6 below shows, one has the following:

Corollary 1.3. (1) *Any simply connected non-compact semisimple Lie group G admits an infinite-dimensional irreducible unitary representation Π such that the branching law* (1.5) *of the restriction $\Pi|_{G'}$ satisfies the bounded multiplicity property for all symmetric pairs (G, G'): there exists $C > 0$ such that*

$$m_\Pi(\pi) \leq C \quad \textit{a.e. } \pi \in \widehat{G'} \textit{ with respect to } \mu.$$

(2) *For any simple connected, non-compact semisimple Lie group G, there exist infinite-dimensional irreducible unitary representations Π_1 and Π_2 such that the tensor product representation decomposes into the direct integral*

$$\Pi_1 \otimes \Pi_2 \simeq \int_{\widehat{G}}^{\oplus} m_{\Pi_1,\Pi_2}(\Pi) d\mu(\Pi),$$

with the bounded multiplicity property: there exists $C > 0$ such that

$$m_{\Pi_1,\Pi_2}(\Pi) \leq C \quad \textit{a.e. } \Pi \in \widehat{G} \textit{ with respect to } \mu.$$

These results concern with the *restriction*. On the other hand, the multiplicity occurring in the *induction* $\mathrm{Ind}_{G'}^{G}(\mathbf{1}) \simeq C^\infty(G/G')$ is finite for any reductive symmetric pair (G, G'), where $\mathbf{1}$ denotes the one-dimensional trivial representation of G', see van den Ban [1]:

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\Pi, C^\infty(G/G')) < \infty \quad \text{for every } \Pi \in \mathrm{Irr}(G). \tag{1.7}$$

More generally, it was proved in [38, Thm. A] that the finite multiplicity property (1.7) is characterized by the *real sphericity* (see Section 2.4 for the definition). We note that any reductive symmetric space is real spherical. When G/G' is a symmetric space, one has a stronger estimate than (1.7), namely, the following bounded multiplicity property holds:

$$\sup_{\Pi\in\mathrm{Irr}(G)} \dim_{\mathbb{C}} \mathrm{Hom}_G(\Pi, C^\infty(G/G')) < \infty. \tag{1.8}$$

More broadly, it was proved in [38, Thm. B] that the bounded multiplicity property (1.8) is characterized by the sphericity of the complexification $G_{\mathbb{C}}/G'_{\mathbb{C}}$. We note that $G_{\mathbb{C}}/G'_{\mathbb{C}}$ is spherical when G/G' is a symmetric space.

By Frobenius reciprocity $\mathrm{Hom}_G(\Pi, C^\infty(G/G')) \simeq \mathrm{Hom}_{G'}(\Pi|_{G'}, \mathbf{1})$, the estimate (1.8) is equivalent to

$$\sup_{\Pi\in\mathrm{Irr}(G)} [\Pi|_{G'} : \mathbf{1}] < \infty,$$

which may be compared with (1.1) and (1.3).

We prove Theorems 1.1 and 1.2 as well as Corollary 1.3 not merely as the existence theorem but also by exhibiting explicitly which $\Pi \in \mathrm{Irr}(G)$ satisfies the bounded multiplicity property (1.1) for (G, G') in scope of further detailed analysis (*e.g.,* "Stages B and C" in the branching program [28], see Section 2.1).

The proof of Theorems 1.1 and 1.2 is reduced to the case where $\mathfrak{g}$ is simple. We explore in more details in the setting that G satisfies one of the following:

- automorphism groups of Hermitian symmetric spaces (Section 3);
- automorphism groups of para-Hermitian symmetric spaces (Section 4);
- the complex minimal nilpotent orbit has real points (Section 5);
- the complex minimal nilpotent orbit has no real point (Section 6).

Correspondingly, we shall see the bounded multiplicity property holds for the restriction $\Pi|_{G'}$ when Π is a "geometric quantization" of certain elliptic, hyperbolic, or (real) minimal nilpotent coadjoint orbits, see Theorem 3.2, Corollary 4.5, and Theorems 5.5 and 6.1, respectively.

The paper is organized as follows. Section 2 explains some basic notions and known results as preliminaries, and Sections 3–6 provide a family of irreducible representations Π of G that satisfy the bounded multiplicity property (1.1) of the restriction of Π. Theorems 1.1 and 1.2 will be proved in Section 5 except for $\mathfrak{g} = \mathfrak{sp}(p, q)$ or $\mathfrak{f}_{4(-20)}$, which will be treated in Section 6.

2 Preliminaries

In this section, we explain some background, basic notions, and known theorems in proving our main results.

2.1 Branching problems

By branching problems in representation theory, we mean the broad problem of understanding how irreducible representations of a group behave when restricted to a subgroup. As viewed in [28], we may divide the branching problems into the following three stages:
Stage A. Abstract features of the restriction;
Stage B. Branching law;
Stage C. Construction of symmetry breaking operators.

The role of Stage A is to develop a theory on the restriction of representations as generally as possible. In turn, we may expect a detailed study of the restriction in Stages B (decomposition of representations) and C (decomposition of vectors) in the "promising" settings that are suggested by the general theory in Stage A.

Theorems 1.1 and 1.2 answer a question in Stage A of branching problems. In turn, we may expect a detailed analysis on the restriction $\Pi|_{G'}$ in Stages B and C. See [7, 29, 35, 36, 41, 42] *e.g.*, for some recent developments in Stage C in the setting where the bounded multiplicity (1.1) holds.

2.2 Harish-Chandra's admissibility theorem

Harish-Chandra's admissibility theorem plays a fundamental role in the algebraic study of representations of real reductive linear Lie groups G, which guarantees a finiteness property of multiplicities for the restriction $G \downarrow G'$ if G' is a maximal compact subgroup K of G. That is, one has the following:

Fact 2.1 ([51, Thm. 3.4.10])**.** Let $G' = K$. For any irreducible unitary representation Π of G, one has

$$[\Pi|_{G'} : \pi] < \infty \quad \text{for all } \pi \in \mathrm{Irr}(G'). \tag{2.1}$$

We explain two directions for generalizations of Fact 2.1.

One is to highlight *G'-admissible restriction* (Definition 2.2), namely, *discrete decomposability* as well as finite multiplicity property, see Fact 2.3 below.

The other direction is to focus on the finiteness property of the multiplicity, as we shall treat in Fact 2.6 (1).

2.3 Discretely decomposable restrictions

The notion and the results of this section will be used in Sections 3.3 and 6.4 for the proof of the bounded multiplicity results.

Definition 2.2 ([18, Sect. 1])**.** A unitary representation Π of G is G'-*admissible* if the restriction $\Pi|_{G'}$ splits into a direct sum of irreducible unitary representations of G':

$$\Pi|_{G'} \simeq {\sum_{\pi \in \widehat{G'}}}^{\oplus} m_{\Pi}(\pi)\pi, \tag{2.2}$$

with multiplicity $m_{\Pi}(\pi) < \infty$ for all $\pi \in \widehat{G'}$.

Fact 2.1 tells us that any $\Pi \in \widehat{G}$ is K-admissible. We begin with the case where G' is compact but is not necessarily a maximal compact subgroup K. In this case, discrete decomposability is obvious because G' is compact, and the finiteness of $m_{\Pi}(\pi)$ is non-trivial. We review a necessary and sufficient condition for (2.1) when G' is compact. In the following statement, we use the letter K' instead of G' to emphasize that K' is compact.

Fact 2.3 ([20, 30])**.** Suppose that K' is a subgroup of K. Let $\Pi \in \mathcal{M}(G)$. Then the following two conditions on the triple (G, K', Π) are equivalent:
(i) The finite multiplicity property (2.1) holds.
(ii) $\mathrm{AS}_K(\Pi) \cap C_K(K') = \{0\}$.

Here $\mathrm{AS}_K(\Pi)$ is the asymptotic K-support of Π. There are only finitely many possibilities of asymptotic K-supports $\mathrm{AS}_K(\Pi)$ for $\Pi \in \mathcal{M}(G)$. The closed cone $C_K(K')$ is the momentum set for the Hamiltonian action on the cotangent bundle $T^*(K/K')$. There are two proofs for the implication (ii) $\Rightarrow$ (i): by using the singularity spectrum (or the wave front set) of the character [20] and by using symplectic geometry [30]. The proof for the implication (i) $\Rightarrow$ (ii) is given in [30].

Fact 2.3 plays a crucial role in the study of discretely decomposable restriction with respect to *non-compact* reductive subgroups G' [18, 20, 21, 30].

Proposition 2.4 ([28, Thm. 4.5])**.** *Let $G \supset G'$ be a pair of real reductive Lie groups, and $K \supset K'$ maximal compact subgroups modulo centers. For an irreducible unitary representation Π of G, we denote by $\Pi^\infty \in \mathrm{Irr}(G)$ the Fréchet representation of smooth vectors, and by Π_K the underlying $(\mathfrak{g}, K)$-module. If one of the equivalent conditions in Fact 2.3 holds, then the restriction $\Pi|_{G'}$ is G'-admissible.*

Moreover, the multiplicity $m_\Pi(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi, \Pi|_{G'})$ of the discrete spectrum is finite, and satisfies the following equalities.

$$m_\Pi(\pi) = [\Pi^\infty|_{G'} : \pi^\infty] = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}',K'}(\Pi_K, \pi_{K'}). \tag{2.3}$$

Remark 2.5. (1) The bounded multiplicity property (1.1) does not hold in general even for the case $G' = K$. We shall see in Theorem 6.1 that (1.1) holds if $\Pi \in \mathrm{Irr}(G)$ is "small" in the sense that the Gelfand–Kirillov dimension of Π equals half the dimension of a *real* minimal coadjoint orbit.
(2) The first equality in (2.3) is not true in general when there is continuous spectrum in the restriction $\Pi|_{G'}$.
(3) ([25, Ex. 6.3]) The multiplicity $m_\Pi(\pi)$ of discrete spectrum may be infinite even for reductive symmetric pairs (G, G') if one of the equivalent conditions in Fact 2.3 fails.

See [10, 39, 40] for a classification theory of the triple (G, G', Π) satisfying the equivalent conditions in Fact 2.3.

2.4 Spherical spaces and real spherical spaces

In [27] and [38, Thms. C and D] we proved the following geometric criteria that concern *all* $\Pi \in \mathrm{Irr}(G)$ and *all* $\pi \in \mathrm{Irr}(G')$:

Fact 2.6. Let $G \supset G'$ be a pair of real reductive algebraic Lie groups.
(1) **Finite multiplicity** for a pair (G, G'):

$$[\Pi|_{G'} : \pi] < \infty, \quad {}^\forall \Pi \in \mathrm{Irr}(G), {}^\forall \pi \in \mathrm{Irr}(G')$$

if and only if $(G \times G')/\operatorname{diag} G'$ is real spherical.
(2) **Bounded multiplicity** for a pair (G, G'):

$$\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty \tag{2.4}$$

if and only if $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag} G'_{\mathbb{C}}$ is spherical.

Here a complex $G_{\mathbb{C}}$-manifold X is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in X, and that a G-manifold Y is called *real spherical* ([19]) if a minimal parabolic subgroup of G has an open orbit in Y.

A remarkable discovery in [38] includes that the bounded multiplicity property (2.4) is determined only by the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$. In particular, the classification of such pairs (G, G') is quite simple, because it is reduced to a classical result when G is compact [43]: the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is the direct sum of the following ones up to abelian ideals:

$$(\mathfrak{sl}_n, \mathfrak{gl}_{n-1}), (\mathfrak{so}_n, \mathfrak{so}_{n-1}), \text{ or } (\mathfrak{so}_8, \mathfrak{spin}_7). \tag{2.5}$$

On the other hand, the finite multiplicity property in Fact 2.6 (1) depends on real forms G and G'. For instance, it is fulfilled for any Riemannian symmetric pair (G, K) because the Iwasawa decomposition tells us that $(G \times G')/\operatorname{diag} G'$ is real spherical if $G' = K$, whereas the finiteness of the K-multiplicity traces back to Harish-Chandra's admissibility theorem (Fact 2.1). (Actually, [38] in this specific case gives a proof that a quasi-simple irreducible representation of G is K-admissible by using the boundary value problem of a system of partial differential equations.)

2.5 Visible actions on complex manifolds

Suppose a (real) Lie group G acts holomorphically on a connected complex manifold D.

Definition 2.7 ([23, Def. 3.3.1])**.** The action is called *strongly visible* if there exist a non-empty G-invariant open subset D' of D, a totally real submanifold S, and an anti-holomorphic diffeomorphism σ of D' such that

$$D' = G \cdot S,\ \sigma|_S = \mathrm{id}, \text{ and } \sigma \text{ preserves each } G\text{-orbit in } D'.$$

Loosely speaking, the significance of this definition is that, for any G-equivariant holomorphic vector bundle $\mathcal{V} \to D$ on which G acts strongly visibly on D, the multiplicity-free property propagates from fibers to sections, see [23, Thm. 4] for a rigorous formulation.

We shall utilize the following results in Sections 3 and 4.

Fact 2.8 ([24, Thm. 1.5])**.** Let G/K be a Hermitian symmetric space, either of compact type or of non-compact type. Then the G'-action on G/K is strongly visible for any symmetric pair (G, G').

2.6 Coisotropic action on coadjoint orbits

Let V be a vector space equipped with a symplectic form ω. A subspace W is called *coisotropic* if $\{v \in V : \omega(v, \cdot)$ vanishes on $W\}$ is contained in W.

The concept of coisotropic actions is defined infinitesimally as follows.

Definition 2.9 (Huckleberry–Wurzbacher [13]). Let H be a connected Lie group, and X a Hamiltonian H-manifold. The H-action is called *coisotropic* if there is an H-stable open dense subset U of X such that $T_x(H \cdot x)$ is a coisotropic subspace in the tangent space T_xX for all $x \in U$.

Suppose that $\mathbb{O}$ is a coadjoint orbit of a connected Lie group G through $\lambda \in \mathfrak{g}^*$. Denote by G_λ the stabilizer subgroup of λ in G, and by $\mathfrak{Z}_{\mathfrak{g}}(\lambda)$ its Lie algebra. The Kirillov–Kostant–Souriau symplectic form ω on $\mathbb{O} \simeq G/G_\lambda$ is given at the tangent space $T_\lambda\mathbb{O} \simeq \mathfrak{g}/\mathfrak{Z}_{\mathfrak{g}}(\lambda)$ by

$$\omega \colon \mathfrak{g}/\mathfrak{Z}_{\mathfrak{g}}(\lambda) \times \mathfrak{g}/\mathfrak{Z}_{\mathfrak{g}}(\lambda) \to \mathbb{R}, \quad (X, Y) \mapsto \lambda([X, Y]).$$

Suppose G is semisimple. Then the Killing form induces an isomorphism $\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$, $\lambda \mapsto X_\lambda$. The following result is useful in later argument.

Lemma 2.10 ([33, Lem. 2]). *Let H be a connected subgroup with Lie algebra $\mathfrak{h}$. The H-action on a coadjoint orbit $\mathbb{O}$ is coisotropic if there exists a subset S (slice) in $\mathbb{O}$ with the following two properties:*

$$\begin{aligned} &\mathrm{Ad}^*(H)S \text{ is open dense in } \mathbb{O}, \\ &(\mathfrak{h} + \mathfrak{Z}_{\mathfrak{g}}(\lambda))^\perp \subset [X_\lambda, \mathfrak{h}] \qquad \text{for any } \lambda \in S. \end{aligned} \tag{2.6}$$

Here $\perp$ stands for the orthogonal subspace with respect to the Killing form.

The original proof of Fact 2.6 in [38] utilized hyperfunction boundary maps for the "if" part (*i.e.,* the sufficiency of the finite multiplicity property) and a generalized Poisson transform [27] for the "only if" part. An alternative approach in [32, 50] for the proof of the 'if' part of Fact 2.6 (2) used a theory of holonomic $\mathcal{D}$-modules, which is also the method of Theorems 4.1 and 4.2 below. Our proof in this article still uses a theory of $\mathcal{D}$-modules, and more precisely, the following:

Theorem 2.11 ([17]). *Let* $\mathrm{Ann}\,\Pi$ *be the annihilator of* $\Pi \in \mathcal{M}(G)$ *in the universal enveloping algebra* $U(\mathfrak{g}_{\mathbb{C}})$. *Assume that the* $G'_{\mathbb{C}}$*-action on the associated variety* $\mathcal{V}(\mathrm{Ann}\,\Pi)$ *is coisotropic. Then the restriction* $\Pi|_{G'}$ *has the bounded multiplicity property* (1.1).

The associated variety $\mathcal{V}(\mathrm{Ann}\,\Pi)$ is the closure of a single nilpotent coadjoint orbit if $\Pi \in \mathrm{Irr}(G)$ [4, 15]. We note that the assumption in Theorem 2.11 depends only on the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ of the complexified Lie algebras as in Fact 2.6 (2).

3 Restriction of highest weight modules

In this section we discuss the bounded multiplicity property (1.1) for a symmetric pair (G, G') when Π is a highest weight module of G. We shall see Theorem 3.2 implies Theorems 1.1 and 1.2 when G is the automorphism group of a Hermitian symmetric space, see (3.1) below for the list of such simple Lie algebras $\mathfrak{g}$.

3.1 Preliminaries for highest weight modules

Let G be a non-compact simple Lie group, θ a Cartan involution of G, and $K := \{g \in G : \theta g = g\}$. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G.

We assume that G is of *Hermitian type*, that is, the Riemannian symmetric space G/K carries the structure of a Hermitian symmetric space, or equivalently, the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is non-trivial. The classification of simple Lie algebras $\mathfrak{g}$ of Hermitian type is given as follows:

$$\mathfrak{su}(p,q)\,,\ \mathfrak{sp}(n,\mathbb{R})\,,\ \mathfrak{so}^*(2m)\,,\ \mathfrak{so}(m,2)\ (m \neq 2)\,,\ \mathfrak{e}_{6(-14)}\,,\ \mathfrak{e}_{7(-25)}\,. \tag{3.1}$$

In this case, there exists a characteristic element $Z \in \mathfrak{c}(\mathfrak{k})$ such that

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_- \tag{3.2}$$

is the eigenspace decomposition of $\mathrm{ad}(Z)$ with eigenvalues 0, $\sqrt{-1}$ and $-\sqrt{-1}$, respectively, and that $\mathfrak{c}(\mathfrak{k}) = \mathbb{R}Z$.

Suppose V is an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module. We set

$$V^{\mathfrak{p}_+} := \{v \in V : Yv = 0 \ \text{ for any } Y \in \mathfrak{p}_+\}\,. \tag{3.3}$$

Since K normalizes $\mathfrak{p}_+$, $V^{\mathfrak{p}_+}$ is a K-submodule. Further, $V^{\mathfrak{p}_+}$ is either zero or an irreducible finite-dimensional representation of K. We say V is a *highest weight module* if $V^{\mathfrak{p}_+} \neq \{0\}$, and of *scalar type* if $\dim_{\mathbb{C}} V^{\mathfrak{p}_+} = 1$. For any non-compact simple Lie group G of Hermitian type, there exist infinitely many irreducible unitary highest weight representations of scalar type.

For any symmetric pair (G, G'), the G'-action on the Hermitian symmetric space G/K is strongly visible (Fact 2.8). Correspondingly, we proved in [25, Thms. A and C] the following multiplicity-free theorems.

Fact 3.1 (multiplicity-free theorem)**.** Let G be a non-compact simple Lie group of Hermitian type, and Π, Π_1, Π_2 irreducible unitary highest weight representations of scalar type.
(1) The restriction $\Pi|_{G'}$ is multiplicity-free for any symmetric pair (G, G').
(2) The tensor product $\Pi_1 \otimes \Pi_2$ is multiplicity-free.

The following theorem asserts that the multiplicities are still uniformly bounded even if we drop the assumption that π is of scalar type.

Theorem 3.2 (uniformly bounded multiplicities)**.** *Let* Π*,* Π_1*,* Π_2 *be the smooth representations of irreducible unitary highest weight representations of* G*.*
(1) *The restriction* $\Pi|_{G'}$ *satisfies bounded multiplicity property* (1.1) *for any symmetric pair* (G, G')*.*
(2) *The tensor product* $\Pi_1 \otimes \Pi_2$ *satisfies the bounded multiplicity property* (1.4)*.*

3.2 Involutions on Hermitian symmetric spaces

The branching law in Fact 3.1 formulated in the category of unitary representations may and may not contain discrete spectra. To clarify this, we observe that there are two types of involutions σ of a non-compact simple Lie group G of Hermitian type. Without loss of generality, we may assume that σ commutes with the Cartan involution θ. We use the same letter σ to denote its differential. Then σ stabilizes $\mathfrak{k}$ and also $\mathfrak{c}(\mathfrak{k})$. Because $\sigma^2 = \mathrm{id}$ and $\mathfrak{c}(\mathfrak{k}) = \mathbb{R}Z$, there are two possibilities:

$$\sigma Z = Z\,, \tag{3.4}$$

$$\sigma Z = -Z\,. \tag{3.5}$$

Geometrically, the condition (3.4) implies:

1-a) σ acts **holomorphically** on the Hermitian symmetric space G/K,
1-b) $G^\sigma/K^\sigma \hookrightarrow G/K$ defines a complex submanifold,

whereas the condition (3.5) implies:

2-a) σ acts **anti-holomorphically** on G/K,

2-b) $G^\sigma/K^\sigma \hookrightarrow G/K$ defines a totally real submanifold.

Definition 3.3. We say the involutive automorphism σ is *of holomorphic type* if (3.4) is satisfied, and is of *anti-holomorphic type* if (3.5) is satisfied. The same terminology will be applied also to the symmetric pair (G, G') (or its Lie algebras $(\mathfrak{g}, \mathfrak{g}')$) corresponding to the involution σ.

The restriction $\Pi|_{G'}$ is discretely decomposable if (G, G') is of holomorphic type for any unitary highest weight representation Π of G ([20] or [22, Thm. 7.4]).

3.3 Proof of Theorem 3.2

The bounded multiplicity property for symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$ of holomorphic type was established in [25, Thm. B] in the category of unitary representations. Since Theorem 3.2 is formulated in the category of smooth admissible representations, we need some additional argument.

Proof of Theorem 3.2. First, suppose that the symmetric pair (G, G') is of holomorphic type. In this case, any irreducible highest weight module of G is K'-admissible, hence G'-admissible (Definition 2.2), see [20] or [22, Thm. 7.4]. In turn, the bounded multiplicity theorem ([25, Thm. B]) in the category of unitary representations implies the one in the category of smooth admissible representations by Proposition 2.4.

Next suppose that (G, G') is of anti-holomorphic type. Via the identification $\mathfrak{g}^* \simeq \mathfrak{g}$, the associated variety is the closure of an adjoint orbit $\mathrm{Ad}(G_{\mathbb{C}})X$ for some $X \in \mathfrak{p}_+$. Then Theorem 3.2 reduces to the following geometric results owing to Theorem 2.11. □

Theorem 3.4. *Let G be a non-compact simple Lie group of Hermitian type. Retain the notation as in* (3.2).
(1) *If σ is of anti-holomorphic type, then the $G_{\mathbb{C}}^\sigma$-action on* $\mathrm{Ad}(G_{\mathbb{C}})X$ *is coisotropic for any $X \in \mathfrak{p}_+$.*
(2) *The diagonal $G_{\mathbb{C}}$-action on* $\mathrm{Ad}(G_{\mathbb{C}})X \times \mathrm{Ad}(G_{\mathbb{C}})Y$ *is coisotropic for any $X \in \mathfrak{p}_+$ and $Y \in \mathfrak{p}_-$.*

Proof. (1) For any non-zero $X \in \mathfrak{p}_+$, one can take $Y \in \mathfrak{p}_-$ and $H \in \mathfrak{k}_{\mathbb{C}}$ such that $\{X, H, Y\}$ forms an $\mathfrak{sl}_2$-triple. We write $\mathfrak{sl}_2^X$ for the corresponding complex subalgebra in $\mathfrak{g}_{\mathbb{C}}$.

Since $\sigma Z = -Z$, one has $\sigma \mathfrak{p}_+ = \mathfrak{p}_-$. Moreover, one has $\mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X) \supset \mathfrak{p}_+$ because $\mathfrak{p}_+$ is abelian. Hence the decomposition (3.2) yields

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} =& \sigma \mathfrak{p}_+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+ \\ =& \sigma(\mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X)) + \mathfrak{k}_{\mathbb{C}} + \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X) \\ =& \mathfrak{g}_{\mathbb{C}}^{\sigma} + \mathfrak{k}_{\mathbb{C}} + \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X). \end{aligned} \tag{3.6}$$

We set $S := \mathrm{Ad}(K_{\mathbb{C}})X$. The equality (3.6) implies that $\mathrm{Ad}(G_{\mathbb{C}}^{\sigma})S$ is open in $\mathrm{Ad}(G_{\mathbb{C}})X$. By Lemma 2.10, it suffices to show

$$(\mathfrak{g}_{\mathbb{C}}^{\sigma} + \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(W))^{\perp} \subset [W, \mathfrak{g}_{\mathbb{C}}^{\sigma}] \quad \text{for all } W \in S.$$

Without loss of generality, we may replace $W = \mathrm{Ad}(k)X$ $(\in \mathfrak{p}_+)$ with X.

We claim the following equality

$$[X, \mathfrak{p}_-] = \mathfrak{Z}_{\mathfrak{k}_{\mathbb{C}}}(X)^{\perp} \quad \text{in } \mathfrak{k}_{\mathbb{C}}, \tag{3.7}$$

where the right-hand side stands for the orthogonal complement of $\mathfrak{Z}_{\mathfrak{k}_{\mathbb{C}}}(X)$ in $\mathfrak{k}_{\mathbb{C}}$ with respect to the Killing form B of $\mathfrak{g}_{\mathbb{C}}$. The inclusion $[X, \mathfrak{p}_-] \subset \mathfrak{Z}_{\mathfrak{k}_{\mathbb{C}}}(X)^{\perp}$ is direct because $B([X, \mathfrak{p}_-], W) = B([X, W], \mathfrak{p}_-) = \{0\}$ for any $W \in \mathfrak{Z}_{\mathfrak{k}_{\mathbb{C}}}(X)$. On the other hand, since $\dim \mathrm{Ad}(G_{\mathbb{C}})X = 2 \dim \mathrm{Ad}(K_{\mathbb{C}})X$, one has $\dim [X, \mathfrak{p}_{\mathbb{C}}] = \dim \mathfrak{k}_{\mathbb{C}} - \dim \mathfrak{Z}_{\mathfrak{k}_{\mathbb{C}}}(X)$. As X is an element of the abelian subalgebra $\mathfrak{p}_+$, one has $[X, \mathfrak{p}_-] = [X, \mathfrak{p}_{\mathbb{C}}]$, and thus the equality (3.7) is proved.

Since $\mathfrak{p}_- = \sigma(\mathfrak{p}_+)$ and $\mathfrak{p}_+ \subset \mathfrak{Z}_{\mathfrak{p}_{\mathbb{C}}}(X)$, one has

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_- \oplus \mathfrak{p}_+ = \mathfrak{p}_{\mathbb{C}}^{\sigma} + \mathfrak{Z}_{\mathfrak{p}_{\mathbb{C}}}(X),$$

hence $\mathfrak{g}_{\mathbb{C}}^{\sigma} + \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X) = \mathfrak{p}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}^{\sigma} + \mathfrak{Z}_{\mathfrak{k}_{\mathbb{C}}}(X)$. Therefore

$$\begin{aligned} (\mathfrak{g}_{\mathbb{C}}^{\sigma} + \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X))^{\perp} =& (\mathfrak{k}_{\mathbb{C}}^{\sigma} + \mathfrak{Z}_{\mathfrak{k}_{\mathbb{C}}}(X))^{\perp} \quad \text{in } \mathfrak{k}_{\mathbb{C}} \\ =& [X, \mathfrak{p}_-]^{-\sigma}. \end{aligned} \tag{3.8}$$

Since $[X, \mathfrak{p}_-] = \{[X, W + \sigma W] : W \in \mathfrak{p}_-\} = [X, \mathfrak{p}_{\mathbb{C}}^{\sigma}]$, we have shown the desired inclusive relation $(\mathfrak{g}_{\mathbb{C}}^{\sigma} + \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X))^{\perp} \subset [X, \mathfrak{g}_{\mathbb{C}}^{\sigma}]$.

(2) We apply the same argument as in (1) and obtain

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}} &= \operatorname{diag} \mathfrak{g}_{\mathbb{C}} + (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}) + (\mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X) \oplus \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(Y)) \\ &= \operatorname{diag} \mathfrak{g}_{\mathbb{C}} + [\operatorname{diag} \mathfrak{g}_{\mathbb{C}}, (X, Y)] + (\mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X) \oplus \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(Y)). \end{aligned}$$

Thus, by setting a submanifold $S := \mathrm{Ad}(K_{\mathbb{C}} \times K_{\mathbb{C}})(X, Y)$, one sees that $\mathrm{Ad}(\operatorname{diag} G_{\mathbb{C}})S$ is open in $\mathrm{Ad}(G_{\mathbb{C}} \times G_{\mathbb{C}})(X, Y)$ and that

$$(\operatorname{diag} \mathfrak{g}_{\mathbb{C}} + (\mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(X') \oplus \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}(Y'))^{\perp} \subset [(X', Y'), \operatorname{diag} \mathfrak{g}_{\mathbb{C}}]$$

for any $(X', Y') \in S$. Now the second assertion follows from Lemma 2.10. $\square$

4 Degenerate principal series representations

In this section we discuss which degenerate principal series representation Π of G satisfies the bounded multiplicity property (1.1) for a symmetric pair (G, G'). In particular, we shall see that Theorems 1.1 and 1.2 hold if G is the automorphism group of a para-Hermitian symmetric space, see Table 4.1 below for the list of such simple Lie algebras $\mathfrak{g}$.

4.1 Bounded multiplicity theorems for the restriction of degenerate principal series representations

For a reductive Lie group G, we write G_U for the compact real form of the complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$.

For a Lie group P, we write $\mathrm{Char}(P)$ for the set of the equivalence classes of one-dimensional representations of P, and $\mathrm{Irr}(P)_f$ for that of finite-dimensional irreducible representations of P.

The following theorems are special cases of the general results [32, Thm. 1.4].

Theorem 4.1 ([32, Ex. 4.5]). *Let $G \supset G'$ be a pair of real reductive algebraic Lie groups, and P a parabolic subgroup of G. Then one has the equivalence on the triple $(G, G'; P)$:*
(i) *One has*

$$\sup_{\chi \in \mathrm{Char}(P)} \sup_{\pi \in \mathrm{Irr}(G')} [\mathrm{Ind}_P^G(\chi)|_{G'} : \pi] < \infty.$$

(ii) *There exists $C > 0$ such that*

$$\sup_{\pi \in \mathrm{Irr}(G')} [\mathrm{Ind}_P^G(\xi)|_{G'} : \pi] < C \dim \xi$$

for any $\xi \in \mathrm{Irr}(P)_f$.
(iii) *$G_{\mathbb{C}}/P_{\mathbb{C}}$ is strongly G'_U-visible (Definition 2.7).*
(iv) *$G_{\mathbb{C}}/P_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.*

Theorem 4.2 ([32, Cor. 4.10]). *Let G be a real reductive algebraic Lie group, and P_j $(j = 1, 2)$ parabolic subgroups. Then the following four conditions on the triple (G, P_1, P_2) are equivalent:*
(i) *One has*

$$\sup_{\chi_1 \in \mathrm{Char}(P_1)} \sup_{\chi_2 \in \mathrm{Char}(P_2)} \sup_{\Pi \in \mathrm{Irr}(G)} [\mathrm{Ind}_{P_1}^G(\chi_1) \otimes \mathrm{Ind}_{P_2}^G(\chi_2) : \Pi] < \infty. \tag{4.1}$$

(ii) *There exists* $C > 0$ *such that*

$$\sup_{\Pi \in \mathrm{Irr}(G)} [\mathrm{Ind}_{P_1}^G(\xi_1) \otimes \mathrm{Ind}_{P_2}^G(\xi_2) : \Pi] \le C \dim \xi_1 \dim \xi_2$$

for any $\xi_1, \xi_2 \in \mathrm{Irr}(P)_f$.
(iii) $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(P_{1\mathbb{C}} \times P_{2\mathbb{C}})$ *is* $\mathrm{diag}(G_U)$*-strongly visible.*
(iv) $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(P_{1\mathbb{C}} \times P_{2\mathbb{C}})$ *is* $\mathrm{diag}(G_{\mathbb{C}})$*-spherical.*

Remark 4.3. (1) A distinguished feature in Theorem 4.1 is that the necessary and sufficient condition of the bounded multiplicity property is given only by the triple $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}})$ of complexified Lie algebras, which traces back to [19, 38].
(2) For each complex symmetric pair $(G_{\mathbb{C}}, G'_{\mathbb{C}})$, parabolic subgroups $P_{\mathbb{C}}$ satisfying the sphericity condition (iv) were classified in [12]. See also [23, 49] for some classification of strongly visible actions.
(3) Littelmann [44] classified the pairs of parabolic subgroups $(P_{1\mathbb{C}}, P_{2\mathbb{C}})$ satisfying (iv) in Theorem 4.2 under the assumption that $P_{1\mathbb{C}}$ and $P_{2\mathbb{C}}$ are maximal, whereas all the pairs $(P_{1\mathbb{C}}, P_{2\mathbb{C}})$ satisfying the strong visibility condition (iii) in Theorem 4.2 were classified in [23] for type A and in Tanaka [49] for the other cases.

By Theorem 4.1, we are interested in the following question in connection with Theorem 1.1.

Question 4.4. For which simple Lie group G, does there exist a parabolic subgroup P such that $G_{\mathbb{C}}/P_{\mathbb{C}}$ is G'_U-strongly visible (or equivalently, $G'_{\mathbb{C}}$-spherical) for *all* symmetric pairs (G, G')?

We give an affirmative answer to this question if G is the automorphism group of a para-Hermitian symmetric space.

Let $P = LN$ be a Levi decomposition of a parabolic subgroup P. Without loss of generality, we may and do assume that both G' and L are stable under the Cartan involution θ of G. We write G_U, G'_U, and L_U for the connected subgroups of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_U = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$, $\mathfrak{g}'_U := \mathfrak{g}'_{\mathbb{C}} \cap \mathfrak{g}_U$, and $\mathfrak{l}_U := \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}_U$. We note that $L_U = P_{\mathbb{C}} \cap G_U$. If the unipotent radical N is abelian, or equivalently, if (G, L) is a para-Hermitian symmetric pair, then G_U/L_U is a compact Hermitian symmetric space and the strong visibility condition (ii) in Theorem 4.1 for $G_U/L_U \simeq G_{\mathbb{C}}/P_{\mathbb{C}}$ is satisfied for all symmetric pairs (G_U, G'_U) by Fact 2.8. Similarly the strong visibility condition (ii) in Theorem

4.2 for the tensor product case is satisfied if the unipotent radicals of P_1 and P_2 are abelian [24, Thm. 1.7].

Thus we have proved the following in answer to Question 4.4:

Corollary 4.5. *Let G be a non-compact simple Lie group, and G/L a para-Hermitian symmetric space. Then Theorem 1.1 holds for any symmetric pair (G, G') by taking Π to be $\mathrm{Ind}_P^G(\xi)$ for $\xi \in \mathrm{Irr}(P)_f$. Likewise, Theorem 1.2 holds by taking Π_1 and Π_2 to be $\mathrm{Ind}_P^G(\xi_1)$ and $\mathrm{Ind}_P^G(\xi_2)$ for $\xi_1, \xi_2 \in \mathrm{Irr}(P)_f$.*

4.2 Para-Hermitian symmetric spaces

Kaneyuki–Kozai [16] gave a classification of para-Hermitian symmetric pairs $(\mathfrak{g}, \mathfrak{l})$ for simple Lie algebras $\mathfrak{g}$ as in Table 4.1 below.

$\mathfrak{g}$	$\mathfrak{l}$
$\mathfrak{sl}(p+q, \mathbb{R})$	$\mathfrak{sl}(p, \mathbb{R}) + \mathfrak{sl}(q, \mathbb{R}) + \mathbb{R}$
$\mathfrak{su}^*(2p+2q)$	$\mathfrak{su}^*(2p) + \mathfrak{su}^*(2q) + \mathbb{R}$
$\mathfrak{sl}(p+q, \mathbb{C})$	$\mathfrak{sl}(p, \mathbb{C}) + \mathfrak{sl}(q, \mathbb{C}) + \mathbb{C}$
$\mathfrak{su}(n, n)$	$\mathfrak{sl}(n, \mathbb{C}) + \mathbb{R}$
$\mathfrak{so}(n, n)$	$\mathfrak{sl}(n, \mathbb{R}) + \mathbb{R}$
$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n) + \mathbb{R}$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) + \mathbb{C}$
$\mathfrak{so}(p+1, q+1)$	$\mathfrak{so}(p, q) + \mathbb{R}$
$\mathfrak{so}(n+2, \mathbb{C})$	$\mathfrak{so}(n, \mathbb{C}) + \mathbb{C}$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) + \mathbb{R}$
$\mathfrak{sp}(n, n)$	$\mathfrak{su}^*(2n) + \mathbb{R}$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) + \mathbb{C}$
$\mathfrak{e}_{6(6)}$	$\mathfrak{so}(5, 5) + \mathbb{R}$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{so}(1, 9) + \mathbb{R}$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{so}(10, \mathbb{C}) + \mathbb{C}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(6)} + \mathbb{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} + \mathbb{R}$
$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{6,\mathbb{C}} + \mathbb{C}$

Table 4.1: List of para-Hermitian symmetric pairs with $\mathfrak{g}$ simple

In particular, Theorem 1.1 holds if $\mathfrak{g}$ is in Table 4.1. Similarly, Theorem 1.2 for the tensor product representations hold if $\mathfrak{g}$ is in Table 4.1 (see [32, Cor. 4.11]).

5 Restriction of "smallest" representations

This section provides a bounded multiplicity theorem for the restriction $\Pi|_{G'}$ when the associated variety $\mathcal{V}(\operatorname{Ann}\Pi)$ of $\Pi \in \operatorname{Irr}(G)$ is the closure of the complex minimal nilpotent orbits. The main result of this section is Theorem 5.5 which was proved in [33] under the assumption that $\mathfrak{g}$ is absolutely simple. We shall see that the same line of argument works when $\mathfrak{g}$ is a complex simple Lie algebra. At the end of this section, we give a proof of Theorems 1.1 and 1.2 for simple Lie algebras $\mathfrak{g}$ except for $\mathfrak{sp}(p,q)$ and $\mathfrak{f}_{4(-20)}$.

5.1 Complex minimal nilpotent orbits $\mathbb{O}_{\min,\mathbb{C}}$

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra. There exists a unique non-zero minimal nilpotent $(\operatorname{Int}\mathfrak{g}_{\mathbb{C}})$-orbit in $\mathfrak{g}_{\mathbb{C}}^*$, which we denote by $\mathbb{O}_{\min,\mathbb{C}}$. We write $n(\mathfrak{g}_{\mathbb{C}})$ for half the (complex) dimension of $\mathbb{O}_{\min,\mathbb{C}}$. Here is the formula of $n(\mathfrak{g}_{\mathbb{C}})$, see [9] for example.

$\mathfrak{g}_{\mathbb{C}}$	A_n	B_n $(n \geq 2)$	C_n	D_n	$\mathfrak{g}_2^{\mathbb{C}}$	$\mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{e}_6^{\mathbb{C}}$	$\mathfrak{e}_7^{\mathbb{C}}$	$\mathfrak{e}_8^{\mathbb{C}}$
$n(\mathfrak{g}_{\mathbb{C}})$	n	$2n-2$	n	$2n-3$	3	8	11	17	29

Let G be a non-compact connected simple Lie group with Lie algebra $\mathfrak{g}$. We set $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We note that $\mathfrak{g}_{\mathbb{C}}$ is simple if $\mathfrak{g}$ does not have a complex structure. For a complex simple Lie algebra $\mathfrak{g}$, we set $n(\mathfrak{g}_{\mathbb{C}}) := 2n(\mathfrak{g})$. To see its meaning, we write J for the complex structure on $\mathfrak{g}$, and decompose $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ into the direct sum of the eigenspaces $\mathfrak{g}^{\mathrm{hol}}$ and $\mathfrak{g}^{\mathrm{anti}}$ of J with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Then one has a direct sum decomposition:

$$\mathfrak{g} \oplus \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^{\mathrm{hol}} \oplus \mathfrak{g}^{\mathrm{anti}} = \mathfrak{g}_{\mathbb{C}}, \quad (X, Y) \mapsto \frac{1}{2}(X - \sqrt{-1}JX, Y + \sqrt{-1}JY).$$

Accordingly, the complexification $G_{\mathbb{C}}$ of the complex Lie group G is given by the totally real embedding

$$\operatorname{diag}\colon G \hookrightarrow G \times G =: G_{\mathbb{C}}, \tag{5.1}$$

where the second factor is equipped with the reverse complex structure. In this case, we set $\mathbb{O}_{\min,\mathbb{C}} := \mathbb{O}_{\min} \times \mathbb{O}_{\min}$ where $\mathbb{O}_{\min}$ is the minimal nilpotent orbit for $\mathfrak{g}$.

5.2 Real minimal nilpotent orbits

Let G be a connected non-compact simple Lie group. Denote by $\mathcal{N}$ the nilpotent cone in $\mathfrak{g}$, and $\mathcal{N}/G$ the set of nilpotent orbits, which may be identified with nilpotent coadjoint orbits in $\mathfrak{g}^*$ via the Killing form. The finite set $\mathcal{N}/G$ is a poset with respect to the closure ordering, and there are at most two minimal elements in $(\mathcal{N} \setminus \{0\})/G$, which we refer to as *real minimal nilpotent (coadjoint) orbits*. See [5, 9, 40, 45] and references therein. The relationship with the complex minimal nilpotent orbits $\mathbb{O}_{\min,\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ is summarized as below.

Fact 5.1 (see *e.g.*, [45])**.** Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of a simple Lie algebra $\mathfrak{g}$. Then exactly one of the following cases occurs.
(1) $(\mathfrak{g}, \mathfrak{k})$ is not of Hermitian type, and $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g} = \emptyset$.
(2) $(\mathfrak{g}, \mathfrak{k})$ is not of Hermitian type, and $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g}$ is a single orbit of G.
(3) $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian type, and $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g}$ consists of two orbits of G.

Correspondingly, we write $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g} = \{\mathbb{O}_{\min,\mathbb{R}}\}$ in Case (2) of Fact 5.1, $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g} = \{\mathbb{O}^+_{\min,\mathbb{R}}, \mathbb{O}^-_{\min,\mathbb{R}}\}$ in Case (3). Then they exhaust all real minimal nilpotent orbits in Cases (2) and (3). Real minimal nilpotent orbits are unique in Case (1), to be denoted by $\mathbb{O}_{\min,\mathbb{R}}$. We set

$$m(\mathfrak{g}) := \begin{cases} \frac{1}{2}\dim \mathbb{O}_{\min,\mathbb{R}} & \text{in Cases (1) and (2),} \\ \frac{1}{2}\dim \mathbb{O}^+_{\min,\mathbb{R}} = \frac{1}{2}\dim \mathbb{O}^-_{\min,\mathbb{R}} & \text{in Case (3).} \end{cases} \tag{5.2}$$

$$\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}} := \begin{cases} \mathrm{Ad}(G_{\mathbb{C}})\mathbb{O}_{\min,\mathbb{R}} & \text{in Cases (1) and (2),} \\ \mathrm{Ad}(G_{\mathbb{C}})\mathbb{O}^+_{\min,\mathbb{R}} = \mathrm{Ad}(G_{\mathbb{C}})\mathbb{O}^-_{\min,\mathbb{R}} & \text{in Case (3).} \end{cases}$$

Then $m(\mathfrak{g}) = n(\mathfrak{g}_{\mathbb{C}})$ in Cases (2) and (3), and $m(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$ in Case (1). The formula of $m(\mathfrak{g})$ in Case (1) is given in [45] as follows.

$\mathfrak{g}$	$\mathfrak{su}^*(2n)$	$\mathfrak{so}(n-1,1)$	$\mathfrak{sp}(m,n)$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-26)}$
$m(\mathfrak{g})$	$4n-4$	$n-2$	$2(m+n)-1$	11	16

Here is a summary about when $m(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$.

Fact 5.2 ([5], [40, Cor. 5.9], [45, Prop. 4.1])**.** Suppose that $\mathfrak{g}$ is absolutely simple. Then the following six conditions on $\mathfrak{g}$ are equivalent:

(i) $\mathbb{O}_{\min} \cap \mathfrak{g} = \emptyset$.

(ii) $\mathbb{O}_{\min,\mathbb{C}} \neq \mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$.

(iii) $\theta\beta \neq -\beta$.

(iv) $m(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$.

(v) $\mathfrak{g}$ is compact or is isomorphic to $\mathfrak{su}^*(2n)$, $\mathfrak{so}(n-1,1)$ $(n \geq 5)$, $\mathfrak{sp}(m,n)$, $\mathfrak{f}_{4(-20)}$, or $\mathfrak{e}_{6(-26)}$.

(vi) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}$ or the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ is isomorphic to $(\mathfrak{sl}(2n,\mathbb{C}), \mathfrak{sp}(n,\mathbb{C}))$, $(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(n-1,\mathbb{C}))$ $(n \geq 5)$, $(\mathfrak{sp}(m+n,\mathbb{C}), \mathfrak{sp}(m,\mathbb{C}) \oplus \mathfrak{sp}(n,\mathbb{C}))$, $(\mathfrak{f}_4^{\mathbb{C}}, \mathfrak{so}(9,\mathbb{C}))$, or $(\mathfrak{e}_6^{\mathbb{C}}, \mathfrak{f}_4^{\mathbb{C}})$.

Remark 5.3. The equivalence (i) $\iff$ (v) was stated in [5, Prop. 4.1] without proof. One may find a proof in [45].

5.3 Gelfand–Kirillov dimension

The Gelfand–Kirillov dimension serves as a coarse measure of the "size" of representations. Let G be a real reductive Lie group. We recall from Section 2 that for $\Pi \in \mathcal{M}(G)$, we denote by $\operatorname{Ann}\Pi$ the annihilator of Π in the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The associated variety $\mathcal{V}(\operatorname{Ann}\Pi)$ is the closure of a single nilpotent coadjoint orbit in $\mathfrak{g}^*_{\mathbb{C}}$ if $\Pi \in \operatorname{Irr}(G)$. The Gelfand–Kirillov dimension $\operatorname{DIM}(\Pi)$ of Π is defined to be half the dimension of $\mathcal{V}(\operatorname{Ann}\Pi)$. The same notation will be applied for Harish-Chandra modules of finite length.

By definition, the Gelfand–Kirillov dimension has the following property:

$$\operatorname{DIM}(\Pi) = 0 \iff \Pi \text{ is finite-dimensional.}$$

For any infinite-dimensional $\Pi \in \operatorname{Irr}(G)$, one has

$$(n(\mathfrak{g}_{\mathbb{C}}) \leq)\ m(\mathfrak{g}) \leq \operatorname{DIM}(\Pi). \tag{5.3}$$

5.4 Coisotropic action on $\mathbb{O}_{\min,\mathbb{C}}$

As we saw in Section 2.6, any coadjoint orbit of a Lie group G is a Hamiltonian G-manifold with the Kirillov–Kostant–Souriau symplectic form. We consider the holomorphic setting, and have proved in [33, Thm. 23] the following:

Fact 5.4. Let $\mathbb{O}_{\min,\mathbb{C}}$ be the minimal nilpotent coadjoint orbit of a connected complex simple Lie group $G_{\mathbb{C}}$.

(1) For any symmetric pair $(G_{\mathbb{C}}, K_{\mathbb{C}})$, the $K_{\mathbb{C}}$-action on $\mathbb{O}_{\min,\mathbb{C}}$ is coisotropic.

(2) The diagonal action of $G_{\mathbb{C}}$ on $\mathbb{O}_{\min,\mathbb{C}} \times \mathbb{O}_{\min,\mathbb{C}}$ is coisotropic.

In Section 6, we give a generalization of this statement, see Theorems 6.7 and 6.8.

5.5 Bounded multiplicity theorems

In view of the inequality (5.3), one may think of $\Pi \in \mathrm{Irr}(G)$ satisfying $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$ as the "smallest" amongst infinite-dimensional irreducible representations of G. Minimal representations [11, 14, 48] are unitarizable and have this property. For $G = SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, or $SU(p, q)$ $(p, q > 0)$, the Joseph ideal is not defined, but there exist infinitely many irreducible unitary representations Π with $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$. In general, the coherent continuation of such representations obtained by the tensor product with finite-dimensional representations also satisfy $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$.

The restriction of such Π to arbitrary symmetric pairs (G, G') has a bounded multiplicity property as follows.

Theorem 5.5. *Let G be a connected simple Lie group, and Π, Π_1, $\Pi_2 \in \mathrm{Irr}(G)$.*

(1) *If* $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$, *then for any symmetric pair* (G, G'), *one has*

$$\sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$$

(2) *If* $\mathrm{DIM}(\Pi_1) = \mathrm{DIM}(\Pi_2) = n(\mathfrak{g}_{\mathbb{C}})$, *then one has*

$$\sup_{\Pi \in \mathrm{Irr}(G)} [\Pi_1 \otimes \Pi_2 : \Pi] < \infty.$$

Remark 5.6. (1) When (G, G') is a Riemannian symmetric pair, namely, $G' = K$, Theorem 5.5 (1) for minimal representations Π is known by Kostant in a stronger from that the supremum is one, see [11, Prop. 4.10].
(2) Theorem 5.5 was proved in [33, Thms. 7 and 8] by using Fact 5.4 when $\mathfrak{g}$ is absolutely simple.

Remark 5.7. We shall see in Theorem 6.1 and Remark 6.3 that Theorem 5.5 still holds by replacing $n(\mathfrak{g}_{\mathbb{C}})$ with $m(\mathfrak{g})$.

Proof of Theorem 5.5. As we saw in Remark 5.6, it suffices to consider when G is a complex Lie group. In this case there are two types of involutions σ of G:

(1) (σ is holomorphic) G^{σ} is a complex subgroup of G,

(2) (σ is anti-holomorphic) G^{σ} is a real form of G.

For simplicity, suppose that G' is the identity component of G^{σ}. Then via the identification $G_{\mathbb{C}} \simeq G \times G$ in (5.1), one has

$$\begin{aligned} G'_{\mathbb{C}} &\simeq G' \times G' && \text{for (1)},\\ G'_{\mathbb{C}} &\simeq \mathrm{diag}_{\sigma}(G) := \{(g, \sigma g) : g \in G\} && \text{for (2)}. \end{aligned}$$

Then $G'_{\mathbb{C}}$ acts on $\mathbb{O}_{\min,\mathbb{C}} \times \mathbb{O}_{\min,\mathbb{C}}$ coisotropically in both cases (1) and (2) by Fact 5.4 (1) and (2), respectively. This implies the first statement of Theorem 5.5 by Theorem 2.11. On the other hand, $G_{\mathbb{C}} \times G_{\mathbb{C}}$ acts on $(\mathbb{O}_{\min,\mathbb{C}} \times \mathbb{O}_{\min,\mathbb{C}}) \times (\mathbb{O}_{\min,\mathbb{C}} \times \mathbb{O}_{\min,\mathbb{C}})$ coisotropically by Fact 5.4 (2), whence the second statement of Theorem 5.5 follows. □

5.6 Proof of Theorems 1.1–1.2 except $\mathfrak{sp}(p, q)$ and $\mathfrak{f}_{4(-20)}$

In order to apply Theorem 5.5, we need the existence of $\Pi \in \mathrm{Irr}(G)$ satisfying $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$. However, we know from the inequality (5.3) that there is no such Π if $m(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$, namely, if $\mathfrak{g}$ is in the list of Fact 5.2 (v). The converse is not true, but "almost" holds as follows.

Lemma 5.8. *Let G be a simply-connected non-compact simple Lie group. Then there exist an infinite-dimensional irreducible and unitarizable representation Π of G such that $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$, if $\mathfrak{g}$ is not isomorphic to the following:*

$$\begin{aligned} &\mathfrak{so}(n, 1)\ (n \geq 6), \quad \mathfrak{so}(p, q)\ (p, q \geq 4, p + q \text{ odd}),\\ &\mathfrak{su}^*(2n), \quad \mathfrak{sp}(p, q)\ (p, q \geq 1), \quad \mathfrak{e}_{6(-26)}, \quad \mathfrak{f}_{4(-20)}. \end{aligned}$$

Proof. When $\mathfrak{g}$ is of type A, one may take Π to be a degenerate principal series representation induced from a mirabolic subgroup for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$), and a highest weight module of the smallest Gelfand–Kirillov dimension for $\mathfrak{g} = \mathfrak{su}(p, q)$. When $\mathfrak{g}$ is not of type A, one may take Π to be a minimal representation [48]. $\square$

Proof of Theorems 1.1 and 1.2 except for $\mathfrak{g} = \mathfrak{sp}(p, q)$ *and* $\mathfrak{f}_{4(-20)}$. If $\mathfrak{g}$ is not in the list of Lemma 5.8, there exists $\Pi \in \mathrm{Irr}(G)$ such that $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$. Hence Theorem 5.5 applies. For $\mathfrak{g} = \mathfrak{su}^*(2n)$, $\mathfrak{so}(p, q)$ or $\mathfrak{e}_{6(-26)}$, one sees from Table 4.1 that G is the transformation group of a para-Hermitian symmetric space, hence Corollary 4.5 applies. $\square$

6 Restriction of "small" representations

This section completes the proof of Theorems 1.1 and 1.2. As we have seen, the remaining cases are when $\mathfrak{g} = \mathfrak{sp}(p, q)$ and $\mathfrak{f}_{4(-20)}$, for which there is no $\Pi \in \mathrm{Irr}(G)$ with $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$ and for which G does not admit a Hermitian or para-Hermitian symmetric space, hence none of Theorem 3.2, Corollary 4.5, or Theorem 5.5 applies. By the classification of irreducible symmetric pairs (Berger [2]), we need to treat the following symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$:

$\mathfrak{g}$	$\mathfrak{g}'$
$\mathfrak{sp}(p, q)$	$\mathfrak{u}(p, q)$, $\mathfrak{sp}(p_1, q_1) + \mathfrak{sp}(p - p_1, q - q_1)$
$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9)$, $\mathfrak{so}(8, 1)$, $\mathfrak{sp}(2, 1) + \mathfrak{sp}(1)$

Table 6.1: Remaining symmetric pairs

The main results of this section is Theorem 6.1, which guarantees the bounded multiplicity property for the restriction $\Pi|_{G'}$ for any $\Pi \in \mathrm{Irr}(G)$ satisfies $\mathrm{DIM}(\Pi) = m(\mathfrak{g})$ $(> n(\mathfrak{g}_{\mathbb{C}}))$, and we complete the proof of Theorems 1.1 and 1.2 in the end.

6.1 Bounded multiplicity theorems

Suppose that (G, G') is a symmetric pair defined by an involution σ of G. We use the same letter σ to denote its holomorphic extension to a simply connected complexification $G_{\mathbb{C}}$, and also its differential. We set $\mathfrak{g}^{-\sigma} := \{Y \in \mathfrak{g} : \sigma Y = -Y\}$. We take a Cartan involution θ commuting with σ, and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition. We take a maximal split abelian subspace $\mathfrak{a}$ to be σ-split, namely, $\mathfrak{a}^{-\sigma} := \mathfrak{a} \cap \mathfrak{g}^{-\sigma}$ is a maximal abelian subspace in $\mathfrak{p} \cap \mathfrak{g}^{-\sigma}$, and $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ to be compatible with a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a}^{-\sigma})$. Let μ be the highest element in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. We prove:

Theorem 6.1. *Suppose $\Pi \in \mathrm{Irr}(G)$ satisfies $\mathrm{DIM}(\Pi) = m(\mathfrak{g})$. If $\sigma\mu = -\mu$, then the restriction $\Pi|_{G'}$ has the bounded multiplicity property* (1.1).

This theorem extends [33, Thm. 34], which treated $\sigma = \theta$ (Cartan involution) or its conjugation by $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$.

Example 6.2. (1) The assumption $\sigma\mu = -\mu$ in Theorem 6.1 is automatically satisfied if $\mathfrak{a}^{-\sigma} = \mathfrak{a}$, namely, if $\mathrm{rank}_{\mathbb{R}}\, G/G' = \mathrm{rank}_{\mathbb{R}}\, G$. This is the case $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{sp}(p,q), \mathfrak{u}(p,q))$ or $\mathfrak{g} = \mathfrak{f}_{4(-20)}$.
(2) A direct computation shows $\sigma\mu = -\mu$ for $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{sp}(p,q), \mathfrak{sp}(p_1,q_1) + \mathfrak{sp}(p-p_1, q-q_1))$.

Remark 6.3. (1) If $m(\mathfrak{g}) = n(\mathfrak{g}_{\mathbb{C}})$, the conclusion of Theorem 6.1 holds without the assumption $\sigma\mu = -\mu$, see Theorem 5.5.
(2) Okuda [46] verified that the assumption $\sigma\mu = -\mu$ is satisfied for all symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$ if $\mathfrak{g}$ is one of the five simple Lie algebras in Fact 5.2 (v), namely, if $m(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$.

Remark 6.4. When $m(\mathfrak{g}) = n(\mathfrak{g}_{\mathbb{C}})$, it may happen that $\sigma\mu \neq -\mu$. Here are examples of such symmetric pairs.
(1) $(\mathfrak{sl}(2n,\mathbb{R}), \mathfrak{sp}(n,\mathbb{R}))$
(2) $(\mathfrak{su}(2p,2q), \mathfrak{sp}(p,q))$, $(\mathfrak{su}(n,n), \mathfrak{sp}(n,\mathbb{R}))$
(3) $(\mathfrak{sp}(p+q,\mathbb{R}), \mathfrak{sp}(p,\mathbb{R}) \oplus \mathfrak{sp}(q,\mathbb{R}))$, $(\mathfrak{sp}(2n,\mathbb{R}), \mathfrak{sp}(n,\mathbb{C}))$,
(4) $(\mathfrak{so}(p,q), \mathfrak{so}(p-1,q))$ or $(\mathfrak{so}(p,q), \mathfrak{so}(p,q-1))$ for "$p \geq q \geq 4$ and $p \equiv q \mod 2$", "$p \geq 5$ and $q = 2$", or "$p \geq 4$ and $q = 3$".
(5) $(\mathfrak{f}_{4(4)}, \mathfrak{so}(5,4))$,
(6) $(\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$, $(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$, or $(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$,
(7) complex symmetric pairs in Fact 5.2 (vi).

The condition $\sigma\mu \neq -\mu$ yields an interesting phenomenon that the restriction $\Pi|_{G'}$ stays almost irreducible ([33, Thm. 10]) for any $\Pi \in \mathrm{Irr}(G)$ such that $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$. (Such Π exists in the above cases (1)–(7).) This gives a uniform explanation of the phenomena that have been observed in various literatures, for instance, as a well-known property of the Segal–Shale–Weil representation of the metaplectic group for the pair (3), the branching law of the minimal representation of $O(p,q)$ in [35, Thm. A] for (4), that of a degenerate principal series representation from a mirabolic in [37, Thm. 7.3] for (1) (see also [6] for the complex case), that of a cohomological parabolic induction $A_{\mathfrak{q}}(\lambda)$ in [26, Thm. 3.5] for (2), and that of a minimal highest weight module in Binegar–Zierau [3] for $(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$ of (6), etc.

6.2 Structural results on real minimal nilpotent orbits

Retain the notation as in Section 5.2. The assumption $\mathrm{DIM}(\Pi) = m(\mathfrak{g})$ means that the associated variety $\mathcal{V}(\mathrm{Ann}\,\Pi)$ of Π is the closure of $\mathbb{O}^{\mathbb{C}}_{\mathrm{min},\mathbb{R}}$. In this section, we recall some basic facts on real minimal nilpotent orbits.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of a simple Lie algebra $\mathfrak{g}$. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and fix a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Denote by μ the highest element in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, and $A_\mu \in \mathfrak{a}$ the coroot of μ. Any (real) minimal nilpotent coadjoint orbit $\mathbb{O}$ is of the form $\mathbb{O} = \mathrm{Ad}(G)X$ for some non-zero element

$$X \in \mathfrak{g}(\mathfrak{a}; \mu) := \{X \in \mathfrak{g} : [H, X] = \mu(H)X \text{ for all } H \in \mathfrak{a}\}$$

via the identification $\mathfrak{g}^* \simeq \mathfrak{g}$, and vice versa (*e.g.*, [45]). Let G_X be the stabilizer subgroup of X in G, and $\mathfrak{z}_{\mathfrak{g}}(X)$ its Lie algebra. We take $Y \in \mathfrak{g}(\mathfrak{a}; -\mu)$ such that $\{X, A_\mu, Y\}$ forms an $\mathfrak{sl}_2$-triple. We write $\mathfrak{sl}_2^X$ for the corresponding subalgebra in $\mathfrak{g}$. Since μ is the highest root in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, the representation theory of $\mathfrak{sl}_2(\mathbb{R})$ tells us that possible eigenvalues of $\mathrm{ad}(A_\mu)$ are 0, ±1, or ±2. Hence one has the eigenspace decomposition of $\mathrm{ad}(A_\mu)$ as

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{6.1}$$

where $\mathfrak{g}_j := \mathrm{Ker}(\mathrm{ad}(A_\mu) - j)$. We note that $\mathfrak{g}_{\pm2} = \mathfrak{g}(\mathfrak{a}; \pm\mu)$. Let $a := \dim_{\mathbb{R}} \mathfrak{g}_1$ and $b := \dim_{\mathbb{R}} \mathfrak{g}_2$. We denote by $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{sl}_2^X)$ the centralizers of $\mathfrak{sl}_2^X$ in $\mathfrak{g}$.

Example 6.5. (1) For $\mathfrak{g} = \mathfrak{sp}(p, q)$, $a = 4(p + q - 2)$, $b = 3$, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{sl}_2^X) \simeq \mathfrak{sp}(p-1, q-1) \oplus \mathbb{R}$, and $\mathfrak{g}_0 \simeq \mathfrak{sp}(p-1, q-1) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}$.

(2) For $\mathfrak{g} = \mathfrak{f}_{4(-20)}$, one has $a = 8$, $b = 7$, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{sl}_2^X) \simeq \mathfrak{spin}(6)$, and $\mathfrak{g}_0 \simeq \mathfrak{spin}(7) \oplus \mathbb{R}$.

Lemma 6.6. (1) *The Lie algebra $\mathfrak{g}$ decomposes as an $\mathfrak{sl}_2^X$-module:*

$$\mathfrak{g} \simeq \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{sl}_2^X) \oplus a\mathbb{R}^2 \oplus b\mathbb{R}^3, \tag{6.2}$$

where $\mathbb{R}^2$ and $\mathbb{R}^3$ stand for the natural representation and the adjoint representation of $\mathfrak{sl}_2(\mathbb{R})$, respectively.

(2) *One has a direct sum decomposition as a vector space:*

$$\mathfrak{Z}_{\mathfrak{g}}(X) = \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{sl}_2^X) \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2. \tag{6.3}$$

(3) *The dimension of the adjoint orbit* $\mathrm{Ad}(G)X$ *is equal to* $a + 2b$.

(4) $\mathfrak{g}_0 = \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{sl}_2^X) + (\mathfrak{m} \oplus \mathbb{R}A_\mu)$.

Proof. The first two assertions are immediate consequences of the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, whence the dimension formula of $\mathfrak{g}/\mathfrak{Z}_{\mathfrak{g}}(X)$. For the last assertion, the inclusion $\mathfrak{g}_0 \supset \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{sl}_2^X) + (\mathfrak{m} \oplus \mathbb{R}A_\mu)$ is obvious. By the irreducible decomposition (6.2) of $\mathfrak{g}$ as an $\mathfrak{sl}_2^X$-module, one sees that $\mathfrak{g}_0 = \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{sl}_2^X) \oplus [X, \mathfrak{g}(\mathfrak{a}; -\mu)]$. Since $[\mathfrak{g}(\mathfrak{a}; \mu), \mathfrak{g}(\mathfrak{a}; -\mu)] \subset \mathfrak{m} + \mathfrak{a}$ and since $\mathfrak{a} \subset \mathbb{R}A_\mu + \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{sl}_2^X)$, the opposite inclusion follows. $\square$

6.3 Coisotropic actions of $G_{\mathbb{C}}^\sigma$ on $\mathbb{O}_{\min,\mathbb{R}}^{\mathbb{C}}$.

The proof of Theorem 6.1 is reduced to the following geometric properties by Theorem 2.11.

Theorem 6.7. *Assume σ satisfies $\sigma\mu = -\mu$ as in Theorem 6.1. Then the $G_{\mathbb{C}}^\sigma$-action on $\mathbb{O}_{\min,\mathbb{R}}^{\mathbb{C}}$ is coisotropic.*

Theorem 6.8. *The diagonal action of $G_{\mathbb{C}}$ on $\mathbb{O}_{\min,\mathbb{R}}^{\mathbb{C}} \times \mathbb{O}_{\min,\mathbb{R}}^{\mathbb{C}}$ is coisotropic.*

Remark 6.9. (1) Theorem 6.7 generalizes [33, Thm. 29] which treats the case $\sigma = \theta$ (Cartan involution).
(2) Theorem 6.8 generalizes Fact 5.4 (2) which treats the case $m(\mathfrak{g}) = n(\mathfrak{g}_{\mathbb{C}})$.

We take $X \in \mathfrak{g}(\mathfrak{a}; \mu)$ such that $\mathbb{O}_{\min,\mathbb{R}} = \mathrm{Ad}(G)X$, hence $\mathbb{O}_{\min,\mathbb{R}}^{\mathbb{C}} = \mathrm{Ad}(G_{\mathbb{C}})X$ via the isomorphism $\mathfrak{g}_{\mathbb{C}}^* \simeq \mathfrak{g}_{\mathbb{C}}$.

For the proof of Theorem 6.7, we begin with the following:

Lemma 6.10. *If $\sigma\mu = -\mu$ then*

$$\mathfrak{g} = \mathfrak{g}^\sigma + (\mathfrak{m} + \mathbb{R}A_\mu) + \mathfrak{Z}_{\mathfrak{g}}(X) = \mathfrak{g}^\sigma + \mathfrak{Z}_{\mathfrak{g}}(X) + [X, \mathfrak{g}_{-2}].$$

Proof. Since $\sigma(A_\mu) = -A_\mu$, one has $\sigma(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ $(j = 0, 1, 2)$. We set $Y := \sigma X \in \mathfrak{g}_{-2} = \mathfrak{g}(\mathfrak{a}; -\mu)$. Then $[X, Y] \in \mathfrak{g}(\mathfrak{a}; 0) \cap \mathfrak{g}^{-\sigma} = \mathfrak{a}^{-\sigma}$. Hence, after an appropriate normalization, $\{X, \sigma X, [X, \sigma X]\}$ forms an $\mathfrak{sl}_2$-triple. In particular, $[X, \mathfrak{g}_{-2}] = [Y, \mathfrak{g}_2]$ is σ-stable. Since $\mathfrak{Z}_{\mathfrak{g}}(X) \supset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ by Lemma 6.6 (2), the decomposition (6.1) yields

$$\begin{aligned}\mathfrak{g} =&\sigma(\mathfrak{Z}_{\mathfrak{g}}(X)) + \mathfrak{g}_0 + \mathfrak{Z}_{\mathfrak{g}}(X)\\ =&\mathfrak{g}^\sigma + \mathfrak{g}_0 + \mathfrak{Z}_{\mathfrak{g}}(X).\end{aligned}$$

By Lemma 6.6 (4), the first equality of Lemma 6.10 follows because $\mathfrak{Z}_{\mathfrak{g}}(X) \supset \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{sl}_2^X)$. Since $\mathfrak{g}_0 + \mathfrak{Z}_{\mathfrak{g}}(X) = [X, \mathfrak{g}_{-2}] + \mathfrak{Z}_{\mathfrak{g}}(X)$ by the irreducible decomposition (6.2), the second equality holds. □

We now give a proof of Theorems 6.7 and 6.8. For a complex simple Lie algebra $\mathfrak{g}$, the statement is reduced to Fact 5.4 as we have seen in the proof of Theorem 5.5. So it suffices to treat the case where $\mathfrak{g}$ is absolutely simple.

Proof of Theorem 6.7. We set $L := M \exp(\mathbb{R}A_\mu)$, and

$$S := \mathrm{Ad}(L)X \subset \mathbb{O}_{\mathrm{min},\mathbb{R}} = \mathrm{Ad}(G)X.$$

Then $\mathrm{Ad}(G^\sigma)S$ is open in $\mathbb{O}_{\mathrm{min},\mathbb{R}}$ by Lemma 6.10. We now verify the condition of Lemma 2.10:

$$(\mathfrak{g}^\sigma + \mathfrak{Z}_{\mathfrak{g}}(W))^\perp \subset [W, \mathfrak{g}^\sigma] \qquad \text{for all } W \in S. \tag{6.4}$$

Since $S \subset \mathfrak{g}(\mathfrak{a}; \mu)$, it suffices to show (6.4) for $W = X$. By the second equality in Lemma 6.10, one has $(\mathfrak{g}^\sigma + \mathfrak{Z}_{\mathfrak{g}}(X))^\perp \subset [X, \mathfrak{g}_{-2}]$. Since $\sigma(\mathfrak{g}_{-2}) = \mathfrak{g}_2$ is abelian, one has

$$[X, \mathfrak{g}_{-2}] = \{[X, V + \sigma(V)] : V \in \mathfrak{g}_{-2}\} \subset [X, \mathfrak{g}^\sigma].$$

Thus (6.4) is shown. □

Proof of Theorem 6.8. The coadjoint orbit $\mathbb{O}^{\mathbb{C}}_{\mathrm{min},\mathbb{R}}$ is of the form $\mathbb{O}^{\mathbb{C}}_{\mathrm{min},\mathbb{R}} = \mathrm{Ad}(G_{\mathbb{C}})X \simeq G_{\mathbb{C}}/(G_{\mathbb{C}})_X$ for any non-zero $X \in \mathfrak{g}(\mathfrak{a}; \mu)$ via the identification $\mathfrak{g}_{\mathbb{C}}^* \simeq \mathfrak{g}_{\mathbb{C}}$. We take $Y \in \mathfrak{g}(\mathfrak{a}; -\mu)$ such that $\{X, A_\mu, Y\}$ forms an $\mathfrak{sl}_2$-triple as before. Since $\mathbb{O}^{\mathbb{C}}_{\mathrm{min},\mathbb{R}}$ contains Y, one can also write as $\mathbb{O}^{\mathbb{C}}_{\mathrm{min},\mathbb{R}} = \mathrm{Ad}(G_{\mathbb{C}})Y \simeq G_{\mathbb{C}}/(G_{\mathbb{C}})_Y$. Then Lemma 6.6 implies that

$$\mathfrak{g} = \mathfrak{Z}_{\mathfrak{g}}(Y) + (\mathfrak{m} + \mathbb{R}A_\mu) + \mathfrak{Z}_{\mathfrak{g}}(X). \tag{6.5}$$

We take any nonzero $Y' \in \mathfrak{g}(\mathfrak{a}; -\mu)$. We claim

$$(\operatorname{diag}\mathfrak{g} + \mathfrak{Z}_{\mathfrak{g}\oplus\mathfrak{g}}(X, Y'))^{\perp} \subset [(X, Y'), \operatorname{diag}\mathfrak{g}]. \tag{6.6}$$

In fact, by using the decomposition (6.1) via the $\mathfrak{sl}_2$-triple $\{X, A_\mu, Y\}$, one has $\mathfrak{Z}_{\mathfrak{g}}(Y') \supset \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$, hence

$$(\mathfrak{Z}_{\mathfrak{g}}(X) + \mathfrak{Z}_{\mathfrak{g}}(Y'))^{\perp} \subset (\mathfrak{Z}_{\mathfrak{g}}(X) \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2})^{\perp} = [X, \mathfrak{g}_{-2}] = [X, \mathfrak{g}(\mathfrak{a}; -\mu)]$$

by the representation theory of $\mathfrak{sl}_2^X$. Switching the role of X and Y', one sees

$$(\mathfrak{Z}_{\mathfrak{g}}(X) + \mathfrak{Z}_{\mathfrak{g}}(Y'))^{\perp} \subset [Y', \mathfrak{g}(\mathfrak{a}; \mu)].$$

Hence the left-hand side of (6.6) is contained in

$$\{(Z, -Z) : Z \in [X, \mathfrak{g}(\mathfrak{a}; -\mu)] \cap [Y', \mathfrak{g}(\mathfrak{a}; \mu)]\},$$

which is a subspace of $[(X, Y'), \operatorname{diag}\mathfrak{g}]$ because both $\mathfrak{g}(\mathfrak{a}; \mu)$ and $\mathfrak{g}(\mathfrak{a}; -\mu)$ are abelian. Hence (6.6) is shown.

We set $L_{\mathbb{C}} := M_{\mathbb{C}} \exp(\mathbb{C}A_\mu)$ and $S := \{(\operatorname{Ad}(\ell)X, \operatorname{Ad}(\ell^{-1})Y) : \ell \in L_{\mathbb{C}}\}$. By (6.5), $\operatorname{diag}(G_{\mathbb{C}})S$ is open dense in $\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}} \times \mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$ in light of the identification $\operatorname{diag}(G_{\mathbb{C}})\backslash(G_{\mathbb{C}} \times G_{\mathbb{C}}) \simeq G_{\mathbb{C}}$, $(x, y) \mapsto x^{-1}y$.

Similarly to (6.6), one obtains the following inclusion:

$$(\operatorname{diag}(\mathfrak{g}_{\mathbb{C}}) + \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}\oplus\mathfrak{g}_{\mathbb{C}}}(\operatorname{Ad}(\ell)X, \operatorname{Ad}(\ell^{-1})Y))^{\perp} \subset [(\operatorname{Ad}(\ell)X, \operatorname{Ad}(\ell^{-1})Y), \operatorname{diag}(\mathfrak{g}_{\mathbb{C}})]$$

for any $\ell \in L_{\mathbb{C}}$. Thus Theorem 6.8 follows from Lemma 2.10. $\square$

6.4 Singular representations of $Sp(p, q)$ and $F_{4(-20)}$

In this section, we verify the existence of $\Pi \in \operatorname{Irr}(G)$ satisfying $\operatorname{DIM}(\Pi) = m(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sp}(p, q)$ or $\mathfrak{f}_{4(-20)}$. Actually, one can take Π to be the globalization of Zuckerman's module $A_{\mathfrak{q}}(\lambda)$, a cohomological parabolic induction for some θ-stable parabolic subalgebra $\mathfrak{q}$ in $\mathfrak{g}_{\mathbb{C}}$.

In what follows, we write $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ for the Levi decomposition of a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$, where $\mathfrak{l}_{\mathbb{C}}$ is the complexified Lie algebra of $L = N_G(\mathfrak{q})$, the normalizer of $\mathfrak{q}$ in G. Then the Gelfand–Kirillov dimension of $A_{\mathfrak{q}}(\lambda)$ is the complex dimension of $\operatorname{Ad}(K_{\mathbb{C}})(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$, see *e.g.*, [21].

Lemma 6.11. *Let $G = Sp(p,q)$, and $\mathfrak{q}$ be a θ-stable parabolic subalgebra with $L \simeq \mathbb{T} \times Sp(p-1,q)$ or $Sp(p,q-1) \times \mathbb{T}$. Then* $\mathrm{DIM}(A_{\mathfrak{q}}(\lambda)) = 2(p+q)-1$.

Proof. We identify $\mathfrak{p}_{\mathbb{C}} \simeq M(2p, 2q; \mathbb{C})$. Then $\mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$ is contained in the variety of rank one matrices for the above parabolic subalgebra $\mathfrak{q}$, which is of complex dimension $2(p+q)-1$. Hence $\mathrm{DIM}(A_{\mathfrak{q}}(\lambda)) \leq 2(p+q)-1$. Since $\mathrm{DIM}(A_{\mathfrak{q}}(\lambda)) \geq m(\mathfrak{g}) = 2(p+q)-1$, we obtain the desired equality. □

Lemma 6.12. *Let $G = F_{4(-20)}$, and $\mathfrak{q}$ be one of θ-stable parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$ in* [39, Table C.4]. *Then* $\mathrm{DIM}(A_{\mathfrak{q}}(\lambda)) = 11$.

Proof. For $\mathfrak{g} = \mathfrak{f}_{4(-20)}$, there are three real nilpotent coadjoint orbits, and their dimensions are 0, 22, 30. This implies that $\mathrm{DIM}(\Pi) \in \{0, 11, 15\}$ for any $\Pi \in \mathrm{Irr}(G)$.

The asymptotic K-support (Section 2.3) of $A_{\mathfrak{q}}(\lambda)$ has the following upper estimate $\mathrm{AS}_K(A_{\mathfrak{q}}(\lambda)) \subset \mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} \rangle$, see [20, Ex. 3.2] and the notation therein.

As we saw in [39], the asymptotic K-support $\mathrm{AS}_K(A_{\mathfrak{q}}(\lambda))$ for the parabolic subalgebra $\mathfrak{q}$ under consideration is strictly smaller than that of a principal series representation of G. In turn, by [30, Prop. 2.6], we conclude that $\mathrm{DIM}(A_{\mathfrak{q}}(\lambda)) < m(\mathfrak{g}) = 15$. Hence $\mathrm{DIM}(A_{\mathfrak{q}}(\lambda)) = 11$. □

Remark 6.13. The Gelfand–Kirillov dimensions of irreducible representations are known for the group of real rank one. In particular, one may observe from [8, Fig. 8.16] that $\mathrm{DIM}\colon \mathrm{Irr}(F_{4(-20)}) \to \{0, 11, 15\}$ is surjective.

Proof of Theorems 1.1 and 1.2. We have shown at the end of Section 5 that the remaining cases are $\mathfrak{g} = \mathfrak{sp}(p,q)$ or $\mathfrak{f}_{4(-20)}$. For these Lie algebras, the assumption $\sigma\mu = -\mu$ in Theorem 6.1 is satisfied (see Example 6.2). On the other hand, we also have verified that there exists $\Pi \in \mathrm{Irr}(G)$ with $\mathrm{DIM}(\Pi) = m(\mathfrak{g})$ for these Lie algebras $\mathfrak{g}$. Hence Theorem 6.1 covers the remaining cases, and completes the proof of Theorems 1.1 and 1.2. □

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