

RECENT ADVANCES IN BRANCHING PROBLEMS OF REPRESENTATIONS

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ABSTRACT. How does an irreducible representation of a group behave when restricted to a subgroup? This is part of *branching problems*, which are one of the fundamental problems in representation theory, and also interact naturally with other fields of mathematics.

This expository paper is an up-to-date account on some new directions in representation theory highlighting the branching problems for real reductive groups and their related topics ranging from global analysis of manifolds via group actions to the theory of discontinuous groups beyond the classical Riemannian setting.

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How does an irreducible representation of a group behave when restricted to a subgroup? How is its restriction decomposed (in a broad sense) into irreducible representations of the subgroup? These questions are part of *branching problems*, which naturally emerge from various fields of mathematics. The decomposition of the tensor product of two (or more) representations is such an example. The Plancherel-type theorem that gives the expansion of functions on a homogeneous space X is often equivalent to a special case of a branching problem via “hidden symmetry”. The theta correspondence, which plays a prominent role in number theory, is also a branching law in the broad sense. Further, when one tries to understand the geometry of a submanifold through the function space on it, the branching problems for a pair of transformation groups emerge in a natural way.

For finite-dimensional representations, branching laws are in principle computable. Combinatorial methods for computing branching laws exist, and various algorithms have been further developed. On the other hand, irreducible representations of non-compact reductive Lie groups such as $GL(n, \mathbb{R})$ are mostly infinite-dimensional and a “general algorithm” that computes branching laws is still far from being known. Indeed, it often happens that representations cannot be controlled well by a subgroup even when it is maximal (Section 6.2). Moreover, it could also be the case that the irreducible representations, which play as “receptacles”, of a subgroup is not well-understood. Indeed, historically, “new” irreducible representations were sometimes discovered through the branching laws of the restriction of the “known”

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representations of a larger group. Via the study of branching problems of a representation of a larger group, we may expect to explore deeper properties of irreducible representations of subgroups.

In the mid-1980s when the author started to challenge branching problems of infinite-dimensional representations, there seemed to be a widespread pessimism that it was “hopeless” to build a general theory of the restriction of infinite-dimensional representations of reductive Lie groups except for some specific cases. When one tries to obtain branching laws for groups larger than $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$, some “bad phenomena” such as infinite multiplicities appear and it was not easy to find a promising direction to develop a theory of the branching problems. In a thorough analysis of such “bad phenomena”, the author encountered not a few “mysterious and nice phenomena”. From there he was fortunate to have been able to reveal some general principles for branching problems. Through new themes such as “the theory of discretely decomposable restrictions”, “the theory of visible actions”, “the theory of finite/bounded/multiplicity-free multiplicities”, “global analysis of the minimal representations”, and “the construction of symmetry breaking operators”, which were born in this way, more mathematicians have become increasingly interested in these new directions and are advancing the study of branching problems. Now the **theory of branching problems for infinite-dimensional representations of reductive Lie groups**, which used to be regarded as hopeless, proceeds to a completely new developing stage.

It may be helpful to provide at this stage a brief overview of these developments in the last 20–30 years. In this expository article we give the ideas on the general theories of “restrictions of representations” and give some new perspectives with programs to advance further the study of branching problems. We collect some references in this area, most of which are closely related to the viewpoint in this article. The author apologizes to the many people whose works are not mentioned here because of the author’s ignorance. He would like to notice to the reader in advance that there are many other important topics that are not treated here.

1. BRANCHING LAWS OF REPRESENTATIONS—INTRODUCTION

As part of commemorative events for the 70th anniversary of the re-establishment of the Mathematical Society of Japan¹, the author was requested to give a plenary talk on an area of mathematics that are remarkably developed recently. The lecture should be addressed to the wide audience about “why”, “what”, and “how” have been achieved in the past together with future prospects. As it is not realistic to cover all the interesting topics, the lecture was dedicated to highlight a new the theory of branching problems of representations of reductive Lie groups, to which the author himself involved and also in which there has been significant development in the last 20–30 years. Besides, recent progress of two relevant topics, global analysis via representation theory and the theory of discontinuous groups, was also discussed. This article is written based on the lecture notes [75]. The author has attempted to clarify the ideas without technicalities and to write this article in a manner that would simultaneously be generally comprehensive to graduate students

¹The organization was established in 1877 as the Tokyo Sugaku Kaisha (Tokyo Mathematics Society). This was the first academic society in Japan, and was reorganized in its present form in 1946.

and scholars of diverse backgrounds and yet be of some value to experts desiring to advance this field.

1.1. What is a branching law of a representation? Hereafter, as a general rule, irreducible representations of groups and those of their subgroups are denoted by uppercase and lowercase Greek letters, respectively, such as Π and π . When a representation Π of a group G is defined on a vector space V , we write $\Pi|_{G'}$ for the representation of its subgroup G' , which is obtained by restricting the action to the subgroup G' on the same representation space V . Namely, $\Pi|_{G'}$ is the composition of two group homomorphisms

$$(1.1) \quad \Pi|_{G'} : G' \hookrightarrow G \xrightarrow{\Pi} GL(V).$$

Even if Π is an irreducible representation of G , the restriction $\Pi|_{G'}$ is in general not irreducible as a representation of the subgroup G' .²

We begin with a classical case, where Π is a finite-dimensional representation. If the restriction $\Pi|_{G'}$ is completely reducible, then it can be described as the finite direct sum of irreducible representations of G' :

$$(1.2) \quad \Pi|_{G'} \simeq \bigoplus_{\pi} \underbrace{(\pi \oplus \cdots \oplus \pi)}_{m(\Pi, \pi)} = \bigoplus_{\pi} m(\Pi, \pi) \pi.$$

The irreducible decomposition (1.2) is a prototype of the **branching law** of the restriction $\Pi|_{G'}$. The number of times that the irreducible representation π of G' appears in Π is called the **multiplicity** of the branching law. Under the assumption that the finite-dimensional representation $\Pi|_{G'}$ is completely reducible, the following identities hold:

$$(1.3) \quad m(\Pi, \pi) = \dim_{\mathbb{C}} \text{Hom}_{G'}(\pi, \Pi|_{G'}) = \dim_{\mathbb{C}} \text{Hom}_{G'}(\Pi|_{G'}, \pi).$$

In this article we deal with the case that Π is an infinite-dimensional representation. In contrast to the case that the representation space V is finite-dimensional, it may happen that there is no irreducible subrepresentation of the subgroup G' in V . The “multiplicity” of the branching law may also be infinite even when G' is a maximal subgroup of G . When talking about a **branching problem** for an infinite-dimensional representation, we consider not only finding the branching law (the irreducible decomposition of the restriction) but also the following broader problem, aiming to understand the “restriction of the representation” itself.

Problem 1.1 (Branching Problem). *How does the restriction $\Pi|_{G'}$ behave as a representation of a subgroup G' ?*

1.2. Restriction of a representation and examples of branching problems.

Let us give a few examples on a “restriction of a representation” that arise from different contexts. We would like the reader to feel the richness and diversity of the themes related to branching problems by browsing these examples and possibly by skipping unfamiliar topics if any.

Example 1.2 (Tensor product representation). The tensor product representations of two representations (π_j, V_j) ($j = 1, 2$) of a group H

$$\pi_1 \otimes \pi_2 : H \rightarrow GL(V_1 \otimes V_2), \quad h \mapsto \pi_1(h) \otimes \pi_2(h)$$

²It may happen that the restriction $\Pi|_{G'}$ of an irreducible representation Π of G remains irreducible as a representation of a subgroup G' , although such cases are rare. See [66] for a list of such triplets (G, G', Π) and some geometric explanations.

can be interpreted as an example of the “restriction of a representation”. Namely, the tensor product representation $\pi_1 \otimes \pi_2$ is identified with the restriction of the outer tensor product representation

$$\pi_1 \boxtimes \pi_2: H \times H \rightarrow GL(V_1 \otimes V_2), \quad (h_1, h_2) \mapsto \pi_1(h_1) \otimes \pi_2(h_2)$$

of the direct product $G := H \times H$ to a subgroup $G' = \text{diag}(H) := \{(h, h) : h \in H\}$, which is isomorphic to H . Fusion rules in theoretical physics are the irreducible decompositions of tensor product representations $\pi_1 \otimes \pi_2$. The Clebsch–Gordan rule and the Pieri rule are special cases of fusion rules.

Example 1.3 (Cartan–Weyl’s highest weight theory). Cartan–Weyl’s theory gives a classification of irreducible finite-dimensional representations of a connected compact Lie group G by their **highest weights**. One may interpret it as part of branching problems, where the highest weights arise as the “edges” of the irreducible decompositions (branching laws) of the representations when restricted to a maximal torus T of G , or equivalently, as the unique subrepresentation of the restriction to a Borel subalgebra of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (Example 2.7).

Example 1.4 (Vogan’s minimal K -type theory). Vogan’s classification theory of irreducible admissible representations Π of a reductive group G (Definition 2.12) utilizes **minimal K -type** and **\mathfrak{u} -cohomology** [125]. This algebraic approach is different from the analytic approach that was taken in Langlands’ classification (Sections 2.6 and 2.7). One may interpret Vogan’s method as a branching problem, where minimal K -types are the “edges” of the branching laws of the representations restricted to a maximal compact subgroup K of G and \mathfrak{u} -cohomologies are defined by the restriction to a nilpotent Lie subalgebra \mathfrak{u} of $\mathfrak{g}_{\mathbb{C}}$ as variants of highest weight vectors, see also Example 1.8.

Example 1.5 (Character theory). Let G be a reductive Lie group and let H_1, \dots, H_k be the complete system of representatives of its Cartan subgroups. The distribution character $\text{Trace}(\Pi)$ of irreducible admissible representation Π (Definition 2.12) is a locally integrable function on G , and thus, it is determined by the restrictions to Cartan subgroups H_j ($j = 1, \dots, k$) (Harish-Chandra). The study of the characters $\text{Trace}(\Pi)|_{H_j}$ is related to the understanding of the restriction of the representation Π to the subgroups H_j . When G is compact, one has $k = 1$ and the explicit formula $\text{Trace}(\Pi)|_{H_1} = \text{Trace}(\Pi|_{H_1})$ is known as the Weyl character formula.

Example 1.6 (Theta correspondence). Let Π be the Weil representation of metaplectic group $G = Sp^{\sim}$ and let $G' := G'_1 \cdot G'_2$ be a subgroup of G consisting of a dual pair, that is, G'_1 and G'_2 are the centralizers of each other in G . The restriction $\Pi|_{G'}$ yields the theta correspondence between irreducible representations of G'_1 and G'_2 (Howe [34]), which may be also thought of as a branching law (in a broad sense) from G to the subgroup $G' = G'_1 \cdot G'_2$.

Example 1.7 (Rankin–Cohen differential operator). The Rankin–Cohen differential operator [16, 110], which explicitly constructs modular forms of higher weights from ones of lower weights, is a “symmetry breaking operator” for the decomposition of the tensor product representation of holomorphic discrete series representations of $SL(2, \mathbb{R})$ (Section 8.2).

Example 1.8 (Cohomology of a representation). When $\text{Hom}_{G'}(\Pi|_{G'}, \pi)$ equals to zero, it is natural to consider higher order cohomologies $\text{Ext}_{G'}^*(\Pi|_{G'}, \pi)$. Especially, when G is a reductive Lie group and $\pi = \mathbf{1}$ (trivial representation), the cohomology for a maximal nilpotent Lie subalgebra \mathfrak{n} instead of a subgroup G' contains some information related to the asymptotic behavior of matrix coefficients on analytic representation theory.

Example 1.9 (Gross–Prasad conjecture). For a pair of groups $(G, G') = (O_n, O_{n-1})$ defined over real or p -adic fields, the restriction of an irreducible admissible representation Π of G to a subgroup G' is multiplicity-free ([3, 118]). The Gross–Prasad conjecture [27] describes the branching law (in a broad sense) for tempered representations Π (Definition 2.3), and is extended to the Gan–Gross–Prasad conjecture [26] including a pair of groups $(G, G') = (GL_n, GL_{n-1})$.

Example 1.10 (Modular variety). On a locally Riemannian symmetric space $X = \Gamma \backslash G/K$ obtained by taking a quotient of arithmetic subgroup Γ , a subgroup G' of G defines a cycle called a **modular variety**. Understanding modular variety is closely related to the branching law of the restriction of automorphic representations of G to the subgroup G' , as a dual notion between geometry and functions on it.

As observed in these examples, the problem of the branching law is deeply related to the structure of representations and it also arises widely in various fields of mathematics.

1.3. Branching laws of infinite-dimensional representations. In this article, with emphasis on the connection to global analysis, we consider infinite-dimensional representations of Lie groups. Loosely speaking, there are two approaches in representation theory; an algebraic approach which handles a representation without a topology on a representation space V and an analytic approach which handles a (continuous) representation by equipping V with a topology. A representation Π on a topological vector space V of a topological group G is called a **continuous representation** if the map

$$G \times V \rightarrow V, \quad (g, v) \mapsto \Pi(g)v$$

is continuous. Hereafter, we consider continuous representations on a topological vector space over \mathbb{C} , unless otherwise specified.

Definition 1.11 (Unitary representation). Let V be a Hilbert space. A continuous representation $\Pi: G \rightarrow GL(V)$ of a group G defined on V is called a **unitary representation** if $\Pi(g)$ is a unitary operator for all $g \in G$ on V .

The advantage of unitary representations is that the concept of “irreducible decompositions” makes sense. That is, if Π is a unitary representation of a locally compact topological group G , then the restriction $\Pi|_{G'}$ can be decomposed into a direct integral of irreducible representations of G , which may be considered as a branching law in the unitary case (see Theorem 2.2 below). In contrast to the case of $\dim_{\mathbb{C}} V < \infty$, the “multiplicity” of the branching law is not necessarily finite and further a “continuous spectrum” may also appear. On the other hand, in a more general case in which Π is not necessarily a unitary representation, e.g. V is a Fréchet space, irreducible decompositions have a less clear meaning. In such a case we may study continuous G' -homomorphisms between the irreducible representation Π of G and an irreducible representation π of its subgroup G' , instead

of the “irreducible decomposition” of the restriction $\Pi|_{G'}$. Then the following two concepts

$$\begin{array}{ll} \text{the space of symmetry breaking operators} & \text{Hom}_{G'}(\Pi|_{G'}, \pi), \\ \text{the space of holographic operators} & \text{Hom}_{G'}(\pi, \Pi|_{G'}) \end{array}$$

become important research objects. The dimensions of these spaces could largely vary depending on a choice of the topologies, e.g. the space Π^∞ of smooth vectors or the space $\Pi^{-\infty}$ of distribution vectors, on the representation spaces [73].

1.4. Branching laws of representations of reductive Lie groups. In completing this introduction, we shortly highlight the main theme of this article, “branching problems of reductive Lie groups”, by comparing the cases of finite-dimensional representations with those of infinite-dimensional representations.

- Branching laws of finite-dimensional representations
 - Irreducible representations, which appear as building blocks of branching laws, were classified in the early 20th century (Cartan–Weyl’s highest weight theory, Example 2.7).
 - There exist algorithms that compute branching laws, and combinatorial techniques have been further developed, e.g. the Littlewood–Richardson rule and Littelman’s path method.
- Branching laws of infinite-dimensional representations (unitary case)
 - The classification of irreducible unitary representations, which appear as building blocks of branching laws of unitary representations, has a rich history of study but is not completely understood (Section 2.6).
 - Algorithms that compute branching laws are not known except for some special cases (e.g. theta-correspondence or Howe correspondence [34] and the restriction of highest weight modules [61]).
 - A “continuous spectrum” may appear in branching laws (Section 2.3).
 - The “grip” of a subgroup G' may not be strong enough, involving a phenomenon of infinite multiplicities in branching laws (Section 6.2).
- Branching laws of infinite-dimensional representations (when no unitarity is imposed)
 - The classification of irreducible admissible representations, which appear as building blocks of (non-unitary) branching laws, was established in the 1970s to early 1980s (Section 2.6).
 - Nevertheless, it is generally a difficult problem to determine when symmetry breaking operators exist (e.g. the Gan–Gross–Prasad conjecture for specific pairs (G, G') , see Example 1.9). Recently, new theories not only for the existence but also for the construction and classification of symmetry breaking operators are emerging (Section 8.3).

In the following we shall introduce new programs in the branching problem and some recent developments, while explaining the basic notion and facts mentioned here.

2. SEARCH FOR FUNDAMENTAL OBJECTS AND “MORE” FUNDAMENTAL OBJECTS—THE CLASSIFICATION OF “IRREDUCIBLES” AND DECOMPOSITION TO “IRREDUCIBLES”

Let us consider what the possible roles of the branching problems are in the representation theory of Lie groups. The notion of Lie groups (continuous groups) was introduced in 1870s by Sophus Lie (1842–1899) in his attempt to develop an analogous theory for differential equations to the Galois theory for algebraic equations. Lie groups and their representation theory have been developed through numerous intersections with analysis, geometry, algebra, and theoretical physics. In this section, applying the philosophical notion “analysis and synthesis”:

- (1) understanding the “smallest objects” (e.g. classification) and
- (2) how things are built up from the “smallest objects”,

we give a brief account of the current status of what problems have been solved and what problems remain open in the representation theory of Lie groups. We then briefly describe the connection of these with other branches of mathematics. Part of Section 2 overlaps with the earlier exposition [50] of the author.

2.1. Lie groups and Lie algebras.

Lie algebras and their smallest objects: The “smallest objects” of Lie algebras are a one-dimensional (abelian) Lie algebra and simple Lie algebras which are of dimension ≥ 2 and do not have nontrivial ideals. Finite-dimensional simple Lie algebras over \mathbb{R} are classified as the 10 series of classical Lie algebras, namely, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{su}(p, q)$, $\mathfrak{su}^*(2n)$, $\mathfrak{so}^*(2n)$, $\mathfrak{so}(p, q)$, $\mathfrak{sp}(p, q)$, $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, and $\mathfrak{sp}(n, \mathbb{C})$, and 22 exceptional ones (É. Cartan, 1914).

Building up from the smallest objects: Any finite-dimensional Lie algebra is obtained by iterating extensions of simple Lie algebras or abelian Lie algebras. When the extension is trivial, that is, when the Lie algebras are expressed as the direct sum of simple Lie algebras and abelian Lie algebras, they are called **reductive Lie algebras**. When the Lie algebras are expressed as the direct sum of only simple Lie algebras, they are called **semisimple Lie algebras**.

Lie groups: A Lie group is a group that carries a manifold structure with continuous (equivalently, smooth) multiplication. Typical examples of Lie groups include **algebraic groups**, which are groups obtained as the zero sets of polynomials on $M(n, \mathbb{R})$. Lie algebras are the infinitesimal algebraic structures of Lie groups and all the local properties and some of global ones of Lie groups can be described in terms of Lie algebras (**Lie theory**). Lie groups whose corresponding Lie algebras are simple, semisimple, and reductive are called simple Lie groups, semisimple Lie groups, and **reductive Lie groups**, respectively.

Reductive Lie groups: Reductive Lie groups are locally isomorphic to the direct product of abelian Lie groups and simple Lie groups. **Classical groups** such as $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $O(p, q)$, $Sp(n, \mathbb{R})$, \dots are reductive algebraic Lie groups, that is, algebraic groups as well as reductive Lie groups. Any two maximal compact subgroups of a Lie group G are conjugate to each other by an inner automorphism. Let G be a semisimple Lie group of finite center, and K a maximal compact subgroup of G . Then the homogenous space G/K is simply connected, and has the structure

of a Riemannian symmetric space [31, Chap. VI]. The classification of simple Lie algebras is equivalent to that of simply connected irreducible Riemannian symmetric spaces.

2.2. Fundamental problems in representation theory. Since representations are defined on vector spaces that has linearity, the superposition principle may be considered in an equivariant fashion for group representations. Then the viewpoint of the “building up from the smallest units” given in the beginning of Section 2 raises the following two fundamental problems in representation theory:

- (1) classify the irreducible representations;
- (2) decompose given representations irreducibly.

Branching problems are one of the main themes related to the latter, “irreducible decomposition”. However, not only that, it provides a useful method on the former, “classification of irreducible representations”. We shall illustrate this idea with some examples in Section 2.7.

2.3. Irreducible decomposition. In general representations cannot always be decomposed into the direct sum of irreducible representations. Indeed, it sometimes happens that a representation Π does not contain any irreducible submodule when Π is infinite-dimensional. In a program for the branching problems described in Section 5, we shall propose a new direction of research, that is, the study of “symmetry breaking operators” between representations of two groups $G \supset G'$, which are not necessarily unitary and may not be decomposed into irreducibles in the usual sense. However, in this section, we first focus on a simpler situation in which irreducible decompositions make sense. For this we recall some classical results.

The best setting for irreducible decompositions is the case when the representations are unitary defined on a Hilbert space. The “smallest units” in this case are irreducible unitary representations. The “decomposition” of a Hilbert space is defined by using the notion of the direct integral of a family of Hilbert spaces introduced by von Neumann. Then a result of Mautner and Teleman (Theorem 2.2 given below) states that any unitary representation Π can be decomposed into irreducibles ([102], also [121, Chapter 6]).

Definition 2.1 (Unitary dual). The set \widehat{G} of equivalence classes of irreducible unitary representations of a topological group G is called the **unitary dual** of G . For a locally compact group G , the unitary dual \widehat{G} carries a natural topological structure called the Fell topology.

Theorem 2.2 (Irreducible decomposition of a unitary representation by the direct integral). *For any unitary representation Π of a locally compact group G , there exist a Borel measure μ on the unitary dual \widehat{G} and a measurable function $n_\Pi: \widehat{G} \rightarrow \mathbb{N} \cup \{\infty\}$ such that Π is unitarily equivalent to the direct integral of irreducible unitary representations:*

$$(2.1) \quad \Pi \simeq \int_{\widehat{G}}^{\oplus} n_\Pi(\pi) \pi d\mu(\pi).$$

Here $n_\Pi(\pi)\pi$ stands for a multiple of (π, V_π) as in (1.2), namely, a unitary representation of G on the Hilbert space $\mathcal{H}_\pi \otimes V_\pi$ where \mathcal{H}_π is a Hilbert space of dimension $n_\Pi(\pi)$ with trivial G -action. We may think of (2.1) as the irreducible

decomposition of the unitary representation Π . Theorem 2.2 includes the case that a “continuous spectrum” appears in the irreducible decomposition.

It should be noted that Theorem 2.2 assures the “existence” of an irreducible decomposition, but the uniqueness may fail in general. However, the irreducible decomposition (2.1) of a unitary representation is unique up to unitary equivalence if the group G is an algebraic Lie group, for which the proof of the uniqueness of the irreducible decomposition (2.1) is reduced to the case where G is a real reductive Lie group, and this case was proved by Harish-Chandra by using the K -admissibility theorem (see Remark 6.6 (1) below), see [127, Thm. 14.6.10] for instance. We shall work mainly with reductive algebraic Lie groups, hence the irreducible decomposition (2.1) is unique in our setting throughout the paper. In this case the function $n_\Pi: \widehat{G} \rightarrow \mathbb{N} \cup \{\infty\}$ is well-defined up to a measure zero set, and is referred to as the multiplicity for the unitary representation Π .

Definition 2.3.

- (1) (Support of an irreducible decomposition) Let $\text{Supp}_{\widehat{G}}(\Pi)$ be the subset of the unitary dual \widehat{G} defined as the support of the irreducible decomposition (2.1) of a unitary representation Π . If $\text{Supp}_{\widehat{G}}(\Pi)$ is a countable set, then Π is decomposed into the direct sum of irreducible unitary representations (Π is said to be **discretely decomposable**).
- (2) (Tempered representation) When Π is the regular representation $L^2(G)$ of G , the set $\text{Supp}_{\widehat{G}}(L^2(G))$ is denoted by $\widehat{G}_{\text{temp}}$. A unitary representation Π of G is called a **tempered representation** if Π satisfies $\text{Supp}_{\widehat{G}}(\Pi) \subset \widehat{G}_{\text{temp}}$, equivalently, if Π is weakly contained in the regular representation $L^2(G)$ in the sense that any matrix coefficient is approximated by a linear combination of matrix coefficients of $L^2(G)$ on compact sets.

Example 2.4. If a Lie group G is amenable, in particular, if G is solvable or compact, then $\widehat{G}_{\text{temp}} = \widehat{G}$.

Example 2.5. When G is a reductive Lie group, irreducible tempered representations can be characterized by the asymptotic behavior of their matrix coefficients. The classification of the set $\widehat{G}_{\text{temp}}$ was accomplished by Knapp–Zuckerman [45] in terms of the direct products of $\mathbb{Z}/2\mathbb{Z}$ called R groups.

2.4. Classification problem of the unitary dual—the orbit method and geometric quantization of symplectic manifolds. Do we know all irreducible unitary representations of Lie groups? Actually, for a general Lie group G , the classification of the unitary dual \widehat{G} has not been completely understood. Postponing a brief summary of the current status to Sections 2.5 and 2.6, we begin with some typical examples in which the unitary dual is completely classified.

Example 2.6 (Abelian group). All irreducible unitary representations of an abelian group G are one-dimensional. For instance, when $G = \mathbb{R}$, set $\chi_\xi: \mathbb{R} \rightarrow GL(1, \mathbb{C})$, $x \mapsto e^{ix\xi}$. Then we have $\widehat{\mathbb{R}} \simeq \{\chi_\xi : \xi \in \mathbb{R}\}$.

Example 2.7 (Compact Lie group, Cartan–Weyl 1925). For a compact group G , all irreducible unitary representations Π of G are finite-dimensional. Suppose G is a connected compact Lie group with Lie algebra \mathfrak{g} and \mathfrak{b} is a Borel subalgebra of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Then irreducible representations Π can

be classified by the unique one-dimensional subrepresentations χ (**highest weight**) of \mathfrak{b} .

Example 2.8 ($SL(n, \mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$). The unitary dual of $G = PSL(2, \mathbb{R})$ consists of the trivial representation, uncountable family of infinite-dimensional representations that are not equivalent to each other (spherical principal series representations, complementary series representations), and countable family of infinite-dimensional representations (discrete series representations and the limit of discrete series representations) (Bargmann 1947). This result is generalized to $G = SL(n, \mathbb{R})$ ($n = 3$: Vakhutinski 1968, $n = 4$: Speh 1981, and n general: Vogan 1986). The unitary dual of $G = SL(n, \mathbb{C})$ started with the case of $n = 2$ by Gelfand–Naimark (1947), Tsuchikawa 1968 ($n = 3$), Duflo 1980 ($n \leq 5$), and Barbasch 1985 (n general). The unitary dual of $G = SL(n, \mathbb{H})$ is classified by Hirai 1962 ($n = 2$) and Vogan 1986 (n general).

Example 2.9 (Nilpotent Lie group, Kirillov [41], 1962). For a simply connected nilpotent Lie group G , Kirillov discovered that there exists a natural bijection

$$(2.2) \quad \widehat{G} \simeq \mathfrak{g}^* / \text{Ad}^*(G),$$

where $\text{Ad}^*: G \rightarrow GL_{\mathbb{R}}(\mathfrak{g}^*)$ is the coadjoint representation of the Lie group G , namely, the contragredient representation of $\text{Ad}: G \rightarrow GL_{\mathbb{R}}(\mathfrak{g})$. This result is extended to exponential solvable Lie groups [24], see also [25].

The unitary dual \widehat{G} for non-compact non-commutative group is “huge” in a sense, as it may contain uncountably many equivalence classes of irreducible infinite-dimensional representations. Example 2.9 above asserts that when G is a nilpotent Lie group, the unitary dual \widehat{G} , though it looks “huge”, can be parametrized by just one finite-dimensional representation, namely, the coadjoint representation $(\text{Ad}^*, \mathfrak{g}^*)$. The right-hand side of (2.2) is nothing but the set of coadjoint orbits, and the idea to understand the unitary dual \widehat{G} via coadjoint orbits is referred to as the **orbit method** or the **orbit philosophy** by Kirillov–Kostant–Duflo [42]. For instance, when $G = GL(n, \mathbb{R})$, the set $\mathfrak{g}^* / \text{Ad}^*(G)$ of coadjoint orbits may be identified with the set of the Jordan normal forms of real square matrices of order n , which is “supposed to approximate” the parameter set of irreducible unitary representations of $GL(n, \mathbb{R})$ according to the orbit philosophy. Let us see that this is not just a coincidence and that the “correspondence” from coadjoint orbits to irreducible unitary representations may be interpreted as a “geometric quantization” at least to some extent.

The “quantization” in physics is the procedure from a classical understanding of physical phenomena to a newer understanding known as quantization theory such as

“classical mechanics \rightsquigarrow quantum mechanics”.

As its mathematical analogue, one may wish to define “**geometric quantization**” under suitable assumptions:

symplectic manifold $M \rightsquigarrow$ Hilbert space \mathcal{H} ;
symplectic transformations on $M \rightsquigarrow$ unitary operators on \mathcal{H} .

One may further expect the naturality and the functoriality of this correspondence, in particular, the following properties are supposed to hold for a family of transformations:

$$(2.3) \quad \begin{aligned} &\text{Hamiltonian actions on } M \rightsquigarrow \text{unitary representations on } \mathcal{H}; \\ &\text{transitivity } \rightsquigarrow \text{irreducibility.} \end{aligned}$$

Now, the right-hand side of (2.2) on the orbit method is identified with the space of coadjoint orbits $\mathcal{O}_\lambda = \text{Ad}^*(G)\lambda$ ($\lambda \in \mathfrak{g}^*$) of the Lie group G . Each coadjoint orbit \mathcal{O}_λ has a symplectic structure induced by the skew-symmetric bilinear form

$$\mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}, \quad (X, Y) \mapsto \lambda([X, Y])$$

and the group action of G is clearly Hamiltonian and transitive (Kirillov–Kostant–Souriau). Therefore, if the idea of the geometric quantization (2.3) works, then one expects to obtain irreducible unitary representations of G corresponding to the coadjoint orbits \mathcal{O}_λ , namely, the correspondence from the right-hand side to the left-hand side in the orbit method (2.2).

For a real reductive Lie group G , it has been observed by experts that there is no natural bijection between the unitary dual \widehat{G} and (a subset of) $\mathfrak{g}^*/\text{Ad}^*(G)$, and the orbit method does not work perfectly. For instance, it is notoriously difficult to give a natural interpretation of complementary series representations from the orbit philosophy.³ Nevertheless, the set $\mathfrak{g}^*/\text{Ad}^*(G)$ of coadjoint orbits gives a rough approximation of the unitary dual \widehat{G} . In particular, geometric quantization of semisimple coadjoint orbits is quite satisfactory, and this provides a considerable part of the unitary dual \widehat{G} . In fact, when the coadjoint orbit \mathcal{O}_λ is semisimple satisfying certain mild conditions including integrality of λ , one can define an irreducible unitary representation of G as a “geometric quantization” of \mathcal{O}_λ , as one can see by combining analytic/geometric results with algebraic representation theory [44, 125, 126, 127, 131] in the 1950s–1990s in the representation theory of Lie groups. Roughly speaking, the irreducible unitary representations obtained as the “geometric quantization” of semisimple orbits $\mathcal{O}_\lambda \in \mathfrak{g}^*/\text{Ad}^*(G)$ are as follows:

Spherical unitary principal series representation	⋯	\mathcal{O}_λ is a hyperbolic orbit;
Zuckerman’s derived functor module $A_q(\lambda)$	⋯	\mathcal{O}_λ is an elliptic orbit;
Tempered representation	⋯	the dimension of \mathcal{O}_λ is maximal ($= \frac{1}{2}(\dim \mathfrak{g} - \text{rank } \mathfrak{g})$).

Here, one conducts the geometric quantization by using the real polarization for hyperbolic orbits and the complex polarization for elliptic orbits. In particular, when G is a connected compact Lie group, orbits \mathcal{O}_λ are always elliptic orbits and further they are compact Kähler manifolds. In this case the geometric quantization of \mathcal{O}_λ that puts “integral conditions” to λ can be constructed by the Borel–Weil–Bott theorem. This is equivalent to the opposite manipulation of taking highest weights (Example 2.7). More details such as terminology, an explicit construction, and some delicate issues for singular parameters may be found in the expository paper [50, Sect. 2], see also [117, Commentary].

³There is some recent trial, e.g., a geometric construction of the full complementary series representations of $SO(n, 1)$ is proposed in [33, Thm. 3.4] along the orbit philosophy.

On the other hand, “geometric quantizations” of nilpotent orbits have yet to be fully elucidated. For minimal nilpotent orbits, there have been some new progress in constructing geometric models of the “corresponding” unitary representations (**minimal representations**) including the L^2 -model (the “Schrödinger model”) on the Lagrangian submanifold in the nilpotent orbits and the global analysis with motif in minimal representations in the last 20 years, in which the author himself has been also involved [9, 72, 82], see Section 4.4. A recent progress of the geometric quantization of nilpotent orbits shows some interesting interactions with other areas of mathematics, which would deserve a separate survey.

2.5. Classification of irreducible unitary representations: a role of reductive Lie groups. The Mackey theory analyzes (irreducible) unitary representations of a group via those of normal subgroups and of their quotients. The following theorem is proven by using the Mackey theory for group extension and by the structure of algebraic Lie groups.

Theorem 2.10 (Duflo [19]). *The classification of irreducible unitary representations of real algebraic groups reduces to the classification problem of the unitary duals of real reductive Lie groups.*

The reductive Lie groups required in Theorem 2.10 are abelian when G is a nilpotent Lie group as it is obtained as an iteration of group extensions by abelian Lie groups. Thus, any irreducible unitary representation of a nilpotent Lie group G is built up from irreducible unitary representations of abelian groups, namely, from one-dimensional unitary representations, and this is how Kirillov classified the unitary dual of simply-connected nilpotent Lie groups (Example 2.9). For the classification of the unitary dual for a general Lie group G , it is sufficient to determine the unitary dual when G is a simple Lie group by Theorem 2.10. We will review the current status of this long-standing problem in the next subsection.

2.6. Classification theory of irreducible representations of reductive Lie groups. In this subsection we overview what has been known about the classification problem for irreducible representations of simple Lie groups (or slightly more generally reductive Lie groups). Although we highlight the unitary dual \widehat{G} of a reductive Lie group G , we also consider a smaller set $\widehat{G}_{\text{temp}}$ (irreducible tempered representations, Definition 2.3) and a larger set $\text{Irr}(G)$ (irreducible admissible representations, Definition 2.12).

$$\widehat{G}_{\text{temp}} \subset \widehat{G} \subset \text{Irr}(G).$$

Example 2.11. For $G = \mathbb{R}$, we set $\chi_\xi: G \rightarrow \mathbb{C}^\times$, $x \mapsto e^{\sqrt{-1}x\xi}$ for $\xi \in \mathbb{C}$. Then we have the following bijections: $\widehat{G}_{\text{temp}} = \widehat{G} \simeq \{\chi_\xi : \xi \in \mathbb{R}\} \simeq \mathbb{R}$, $\text{Irr}(G) \simeq \{\chi_\xi : \xi \in \mathbb{C}\} \simeq \mathbb{C}$.

In order to define $\text{Irr}(G)$, the set of equivalence classes of “irreducible admissible representations”, we need to clarify what “admissibility” means. There are several ways for the characterization of admissibility ([127, Chap, 3], see also Remark 2.14 (1) below). We choose one of the equivalent definitions of admissibility, which fits with the branching law, a main theme of this article. For this, we recall that there are at most countably many equivalence classes of irreducible representations of compact Lie groups. We also recall that any continuous representation of a compact Lie group defined over a suitable topological vector space (for instance,

a Hilbert space, or more generally, a complete locally convex topological vector space) contains the algebraic direct sum of irreducible representations as a dense subspace (**discrete decomposition**), but the number of appearances of the same irreducible representations (**multiplicity**) may vary from 0 to infinity.

Definition 2.12 (Harish-Chandra’s admissible representation). A continuous representation Π of a reductive Lie group G is said to be *admissible* when the restriction $\Pi|_K$ to a maximal compact subgroup K of G contains each irreducible representation of K with at most finite multiplicity.

In Section 6.1, we shall extend the notion “admissibility” to the restriction with respect to a more general (non-compact) subgroup, and refer to Harish-Chandra’s admissibility as K -admissibility.

In analysis one may consider various function spaces and equip them with natural topologies such as the Banach space topology for L^p -functions or the Fréchet space topology for C^∞ -functions on a manifold. Analogously, in analytic representation theory, when a continuous representation Π of a Lie group G is defined over a Banach space V (or more generally, a complete locally convex topological vector space), one may consider the space V^∞ of C^∞ -vectors:

$$V^\infty := \{v \in V : G \rightarrow V, g \mapsto \Pi(g)v \text{ is a } C^\infty\text{-map}\}.$$

The space V^∞ is a G -invariant dense subspace of V and also the differential representation $d\Pi$ of the Lie algebra \mathfrak{g} can be defined on V^∞ . Then V^∞ is complete with respect to the family of seminorms given by $\|d\Pi(X_1) \cdots d\Pi(X_k)v\|$ for $X_1, \dots, X_k \in \mathfrak{g}$. With respect to this topology the map $G \times V^\infty \rightarrow V^\infty$, $(g, v) \mapsto \Pi(g)v$ is continuous, hence one obtains a continuous representation of G on the Fréchet space V^∞ , to be denoted by Π^∞ .

A vector $v \in V$ is called K -finite if $\dim_{\mathbb{C}} \mathbb{C}\text{-span}\{\Pi(k)v\} < \infty$. We write V_K for the space of K -finite vectors. If (Π, V) is an admissible representation on a Banach space V , then $V_K \subset V^\infty$, hence one can define the action of the Lie algebra \mathfrak{g} as well as that of the maximal compact subgroup K on V_K , to which we refer to as the *underlying* (\mathfrak{g}, K) -module of (Π, V) .

In this article, we adopt the Fréchet representation of *moderate growth* in defining the set $\text{Irr}(G)$, which fits with the study of symmetry breaking operators in later sections. Some justification is given in Remark 2.14 via (essentially) equivalent definitions.

Definition 2.13 (Irreducible admissible representation). Suppose G is a reductive Lie group. If (Π, V) is an irreducible admissible representation of G on a Banach space V , then the Fréchet representation (Π^∞, V^∞) is also irreducible, and called of *moderate growth* from the behavior of the matrix coefficients, see [127, Chap. 11]. We denote by $\text{Irr}(G)$ the set of equivalence classes of such irreducible admissible Fréchet representations (Π^∞, V^∞) .

Remark 2.14. (1) (Harish-Chandra) If (Π, V) is an irreducible representation realized on a Banach space V , then the following are equivalent.

(2.4) $\begin{aligned} & \text{The center of the universal enveloping algebra } U(\mathfrak{g}) \text{ acts on } V^\infty \text{ by scalar.} \\ & \iff (\Pi, V) \text{ is an admissible representation.} \end{aligned}$

(2) If (Π_1, V_1) and (Π_2, V_2) are two irreducible admissible representations of G on Banach spaces V_1 and V_2 . Then the underlying (\mathfrak{g}, K) -modules $(\Pi_1)_K$ and $(\Pi_2)_K$

are isomorphic to each other as (\mathfrak{g}, K) -modules if and only if $(\Pi_1)^\infty$ and $(\Pi_2)^\infty$ are isomorphic as Fréchet representations of G . We note that Π_1 and Π_2 are not necessarily isomorphic as Banach representations of G .

(3) For every irreducible (\mathfrak{g}, K) -module X , there exists $\Pi \in \text{Irr}(G)$ such that $\Pi_K \simeq X$, hence one has a natural bijection between $\text{Irr}(G)$ and the set of equivalence classes of irreducible (\mathfrak{g}, K) -modules.

(4) Via the correspondence $(\Pi, V) \rightsquigarrow (\Pi^\infty, V^\infty)$, the unitary dual \widehat{G} can be regarded as a subset of $\text{Irr}(G)$.

The statements (2) and (3) in Remark 2.14 are part of Casselman–Wallach’s globalization theory [127, Chap. 11].

The classification of irreducible unitary representations of reductive Lie groups: The classification problem of the unitary dual \widehat{G} of a reductive Lie group G has a history of over 70 years. It is completed for some special cases such as $SL(n, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) explained in Example 2.8, complex Lie groups $SO(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$ or some real reductive Lie groups of low rank. Moreover, recently, the “Atlas project” led by Adams, van Leeuwen, Trapa, Vogan, and others, which aims to give a “description” of \widehat{G} by finite algorithms, has been also developed [1]. However, the unitary duals of simple Lie groups remain to be fully elucidated even for classical cases such as indefinite orthogonal groups $O(p, q)$ with general p, q .

As we mention below in Example 2.5, the classification of $\widehat{G}_{\text{temp}}$ (tempered representations), which is a subset of the unitary dual \widehat{G} , is completed by Knapp–Zuckerman.

The classification of irreducible representations of reductive Lie groups (when no unitarity is imposed): In contrast to the long-standing problem on the classification of the unitary dual \widehat{G} , the classification of $\text{Irr}(G)$, which is larger than \widehat{G} , was completed from 1970s to the early 1980s. (In other words, the current status of the classification of \widehat{G} is that although the classification of $\text{Irr}(G)$ is completed, we do not fully understand the whole picture of the subset \widehat{G} of $\text{Irr}(G)$.)

The classification of the set of infinitesimal equivalence classes of irreducible admissible representations (equivalently, that of $\text{Irr}(G)$ as Fréchet modules, see Remark 2.14 (3)) is reduced to that of irreducible (\mathfrak{g}, K) -modules. There are three approaches to the classification of irreducible (\mathfrak{g}, K) -modules.

- (Langlands classification) This is an analytic approach that reduces to the classification of $\widehat{G}_{\text{temp}}$ by focusing on the asymptotic behavior of matrix coefficients. The classification of $\widehat{G}_{\text{temp}}$ (irreducible tempered representations) was accomplished by Knapp–Zuckerman.
- (Vogan’s classification) This employs a purely algebraic method that uses minimal K -type theory, Zuckerman’s derived functor modules (an algebraic generalization of the Borel–Weil–Bott theory), and the Lie algebra cohomology (a generalization of highest weights).
- (\mathcal{D} -module approach) This method has a geometric feature: it reduces the classification of irreducible (\mathfrak{g}, K) -modules (with regular infinitesimal characters) to that of irreducible modules of the ring of twisted differential operators over flag manifolds (Beilinson–Bernstein, Brylinski–Kashiwara) and to the geometry of $K_{\mathbb{C}}$ -orbits.

2.7. A role of the branching law on the classification of irreducible representations. The concept of the “smallest units” varies depending on the viewpoint. Taking subgroups changes the viewpoint in representation theory: irreducible representations of a group are no longer “smallest units” when the action is restricted to subgroups. The **theory of branching law** aims to elucidate structures of representations from the viewpoint of subgroups. Conversely, its idea plays a useful role in the classification theory of irreducible representations itself [59], as mentioned in Section 1. Indeed, the idea of branching laws is used to define “invariants” of irreducible representations, e.g., in Cartan–Weyl’s highest weight theory for the classification of irreducible finite-dimensional representations and also in Vogan’s theory for that of irreducible (\mathfrak{g}, K) -modules, where the restriction to a maximal torus and a maximal compact subgroups respectively, is discretely decomposable and the “edge” of the branching law (with respect to a certain partial order) gives “invariants” in the classification. Further, sometimes, “new irreducible unitary representations” have been discovered in the process of finding branching laws such as via the theta correspondence (Example 1.6).

3. FROM LOCAL TO GLOBAL—GLOBAL ANALYSIS ON MANIFOLDS WITH INDEFINITE METRIC

The theory of discontinuous groups beyond the Riemannian setting is another young field that has remarkably developed in the last 30 years. The ideas from discontinuous groups inspired the author at several turning points in inventing the theory of the restriction of infinite-dimensional representations. In this section we shall shed light on the parts in which both theories are related.

3.1. Mysterious phenomena on the global analysis for indefinite metrics.

A *pseudo-Riemannian manifold* is a manifold M equipped with a non-degenerate quadratic form g_x at the tangent space $T_x M$ depending smoothly on $x \in M$. When g is positive definite, (M, g) is a Riemannian manifold. When only one eigenvalue of g is negative, it is called a Lorentzian manifold, which appears as a geometric structure of spacetime in general relativity. The group of isometries of a pseudo-Riemannian manifold is automatically a Lie group. Semisimple symmetric spaces (in particular, irreducible symmetric spaces) are examples of pseudo-Riemannian manifolds, on which semisimple Lie groups act transitively as isometric transformations.

The motif

$$\text{local property} \rightsquigarrow \text{global form}$$

has been one of the main streams in geometry since the 20th century, especially in Riemannian geometry. The question that when a local structure is fixed, “how flexible the global form is, or conversely, what kind of limitations exists in the global form?” is a prototype of the motif, which would bring us to deformation theory and rigidity theorems, respectively.

The study of “local to global” interacts with various fields of mathematics, depending on the types of local properties of interest. If one highlights “local homogeneity” as a local property, Lie theory and number theory will enter naturally in this study through an algebraic structure called **discontinuous groups**, which controls the global nature of such manifolds. In the case of Riemannian manifolds (where the metric g is positive definite), the study of discontinuous groups has

been already entered its golden age since the 1950s, of which the interaction ranges from Riemannian symmetric spaces, Lie theory, and number theory to differential geometry and topology.

On the other hand, the study of pseudo-Riemannian geometry seemed to be behind the trend of “local to global”.

An early work is due to Calabi–Markus [13] in 1960s for de Sitter manifolds. A systematic study of discontinuous groups for pseudo-Riemannian homogeneous manifolds started with [46] in the late 1980s. As an introduction to this theme, we begin this section with three phenomena in pseudo-Riemannian (locally homogeneous) geometry that look strange from the “classical point of view” in Riemannian geometry, see also the research survey paper [57] for more details of the first two of them.

3.1.(1) Curvature and global form: Curvatures are typical examples of local invariants of (pseudo-)Riemannian manifolds. To see what kind of constraints the curvature (local property) gives to the global form of a manifold, let us compare classical results on Riemannian manifolds (Theorems 3.1 and 3.3) and some different features on pseudo-Riemannian manifolds (Theorems 3.2 and 3.4). A Riemannian manifold, or more generally, a pseudo-Riemannian manifold is called a *space form* if its sectional curvature κ is constant. In the Riemannian case, it is called a hyperbolic manifold when $\kappa < 0$. In the Lorentzian case, it is a **de Sitter manifold** or an **anti-de Sitter manifold** when $\kappa < 0$ or $\kappa > 0$, respectively. We highlight the contrast between Riemannian and Lorentzian cases: Theorems 3.1 and 3.2 are for positive curvatures, whereas Theorems 3.3 and 3.4 are for negative curvatures.

Theorem 3.1 (Myers). *Every complete Riemannian manifold with Ricci curvature $\geq \varepsilon (> 0)$ is compact.*

Theorem 3.2 (Calabi–Markus phenomenon [13]). *Every de Sitter manifold is non-compact.*

Theorem 3.3. *There exists a compact hyperbolic manifold for every dimension.*

Theorem 3.4. *Compact anti-de Sitter manifolds exist if and only if the dimension is odd.*

3.1.(2) Rigidity of discontinuous groups and “deformability”: Can we “deform” discontinuous groups for pseudo-Riemannian symmetric spaces G/H ? Since automorphisms of G induce “uninteresting” deformation of discontinuous groups, we consider “deformation” up to automorphisms of G . Let us compare a classical rigidity theorem (Theorem 3.5) when the metric tensor g is positive definite with a discovery on the “flexibility of discontinuous groups” (Theorem 3.6) when g is indefinite.

Theorem 3.5 (Selberg–Weil’s local rigidity [128]). *Any cocompact discontinuous group for an irreducible Riemannian symmetric space of dimension > 2 does not allow non-trivial continuous deformation.*

Theorem 3.6 (Kobayashi [56]). *There exists an irreducible symmetric space of arbitrarily higher dimension having a cocompact discontinuous group that allows non-trivial continuous deformation.*

The local rigidity theorem by Selberg and Weil in Theorem 3.5 has brought a sequence of revolutionary works on rigidity theorems by Mostow, Margulis, and Zimmer among others, whereas in deformation theory of discontinuous groups for the Poincaré disk (the two-dimensional case) the last century has seen the development of an abundant theory of **Teichmüller spaces** on moduli spaces of Riemann surfaces. Theorem 3.6 concerning indefinite metrics may be thought of as providing a new theme such as “higher dimensional Teichmüller theory” for locally semisimple symmetric spaces with indefinite metric.

3.1.(3) Rigidity of spectrum: So far we have seen distinguished aspects in the geometry of discontinuous groups for pseudo-Riemannian manifolds. We now address an analytic question: Suppose that a discontinuous group Γ for a (pseudo-)Riemannian manifold X admits a non-trivial continuous deformation. Do the eigenvalues of the Laplacian on the quotient space $\Gamma \backslash X$ vary according to deformation of Γ , or are there “stable eigenvalues”? We consider this problem in the setting where X is a space form, i.e., a pseudo-Riemannian manifold with sectional curvature -1 . Theorem 3.7 is a classical result for Riemannian manifolds of dimension two and Theorem 3.8 is a new phenomenon for the Lorentzian manifold of dimension three discovered in a joint work with F. Kassel [38]. See also [74].

Theorem 3.7 (Wolpert [130]). *There does not exist a “stable eigenvalue” ($> 1/4$) on closed Riemann surfaces.*

Theorem 3.8 (Kassel–Kobayashi [38]). *There exist infinitely many positive “stable eigenvalues” on three-dimensional compact anti-de Sitter manifolds.*

Here we say λ is a *stable eigenvalue* if there exists a non-zero L^2 -function f (depending on φ) on the quotient manifold $\varphi(\Gamma) \backslash X$ satisfying $\Delta f = \lambda f$ (weak sense) for every injective homomorphism $\varphi: \Gamma \rightarrow G$ that is sufficiently close to the original embedding $\iota: \Gamma \hookrightarrow G$.

Remark 3.9. The existence of “stable eigenvalues” in Theorem 3.8 is proved in [38] by constructing Γ -periodic eigenfunctions that are obtained as the “ T -average” of non-periodic eigenfunctions. For the proof of the convergence and the non-vanishing of the “ T -average”, one uses a geometric estimate such as the “counting” of the Γ -orbit in a pseudo-ball of a pseudo-Riemannian symmetric space (Section 3.3) and analytic estimate of eigenfunctions, see Section 4.1 for the non-commutative harmonic analysis. On the other hand, by using the branching law (Section 6) of infinite-dimensional representations, it can be shown that there also exist infinitely many eigenvalues (< 0) that “vary” according to deformation of a discontinuous group, see [40, 76] for a precise formulation.

3.2. Inspiration from the theory of discontinuous groups. Let us explain a loose idea that connects the “theory of discontinuous groups of pseudo-Riemannian manifolds” for which some mysterious phenomena are described in Section 3.1 with the “theory of the restriction of infinite-dimensional representations” which is the main theme of this article.

First, we review basic notion on group actions. Suppose a topological group Γ acts continuously on a locally compact space X . We define a subset $\Gamma_S \subset \Gamma$ for a given subset $S \subset X$ as

$$\Gamma_S := \{\gamma \in \Gamma : \gamma S \cap S \neq \emptyset\}.$$

When S is a singleton $\{x\}$, the subset Γ_S is a subgroup. Let us recall the following basic concepts.

- Definition 3.10.** (1) An action is **properly discontinuous** $\iff \#\Gamma_S < \infty$ for any compact set S .
 (2) An action is **proper** $\iff \Gamma_S$ is compact for any compact set S .
 (3) An action is **free** $\iff \Gamma_{\{x\}} = \{e\}$ for any $x \in X$.

One may think of each of the above three concepts as a kind of properties that Γ_S is reasonably “small” whenever S is “small”.

If Γ acts on a manifold X properly discontinuously and freely, then the quotient space $\Gamma \backslash X$ is a Hausdorff space with respect to quotient topology and, further, there exists a unique manifold structure in $\Gamma \backslash X$ for which the quotient map $X \rightarrow \Gamma \backslash X$ is a smooth covering. Conversely, the fundamental group $\pi_1(M)$ of a manifold M acts properly discontinuously and freely on the universal covering space \widetilde{M} as covering transformations and the original manifold M can be recovered as $\pi_1(M) \backslash \widetilde{M} \simeq M$.

Suppose that X is a pseudo-Riemannian manifold with metric tensor g , and that G is the group of isometries of X . Then G is always a Lie group. For a subgroup Γ of G , the following equivalence does *not* hold generally even when Γ acts freely on X (the implication \Leftarrow always holds):

$$\Gamma \text{ is discrete (in } G) \iff \text{the action of } \Gamma \text{ on } X \text{ is properly discontinuous.}$$

This is a significant difference from Riemannian manifolds with g positive definite, where the equivalence automatically holds, and the failure of the implication \Rightarrow in the pseudo-Riemannian case is one of the causes of “mysterious phenomena” described in Section 3.1. Thus it is crucial to gain a profound understanding (not a formal paraphrase of the definition of proper discontinuity) when a discrete subgroup of G acts on a non-Riemannian homogenous space. An explicit properness criterion is known for reductive Lie groups G , which we recall now.

Let $G = K \exp(\mathfrak{a})K$ be the Cartan decomposition of a reductive Lie group G and $\mu: G \rightarrow \mathfrak{a}/W$ the corresponding Cartan projection. Here W stands for the Weyl group of the restricted root system of the Lie algebra \mathfrak{g} with respect to the maximal abelian split subalgebra \mathfrak{a} . The next theorem extends the properness criterion given by the author [46] in the setting where both Γ and H are reductive subgroups.

Theorem 3.11 (Criteria for proper discontinuity; Benoist [4], Kobayashi [52]). *Let Γ be a discrete subgroup of a reductive Lie group G and H a closed subgroup of G . Then the following two conditions on (Γ, G, H) are equivalent:*

- (i) *the action of Γ on G/H is properly discontinuous;*
- (ii) *the set $\mu(\Gamma) \cap \mu(H)_\varepsilon$ is a finite set for any $\varepsilon > 0$, where $\mu(H)_\varepsilon$ is a tubular neighborhood of $\mu(H)$ in \mathfrak{a}/W .*

Theorem 3.11 plays a key role to prove the necessary and sufficient condition for the Calabi-Markus phenomenon for reductive homogeneous spaces G/H (cf. Theorem 3.2) in [46] and the existence problem of compact pseudo-Riemannian locally homogeneous spaces such as space forms (Theorem 3.4 for the Lorentzian case), which were introduced in Section 3.1 as “mysterious phenomena” for pseudo-Riemannian manifolds, and also Theorem 3.6 (the deformation of discontinuous groups in higher dimension).

The criterion for discrete decomposability for the branching law of unitary representations (Section 6) was inspired by the properness criterion [46] in its formulation. Although the techniques to prove Theorems 3.11 (topology) and 6.5 (decomposition of a Hilbert space, micro-local analysis) are quite different, there are some common characteristics in both cases, that is, one searches a setting in which non-compact subgroups behave **as if they were compact groups**.⁴

3.3. Application from the theory of discontinuous groups: from qualitative theorems to quantitative estimates. Properness (or proper discontinuity) of the action is initially a qualitative property. As a second step, we may deepen such qualitative properties to **quantitative** estimates, from which a further connection to another field of mathematics emerges. Let us give two such examples.

(1) Quantitative estimates of proper discontinuity

If a discrete group Γ acts on X properly discontinuously (Definition 3.10), then the following inequality holds for any compact subset $S \subset X$:

$$\#\Gamma_S < \infty.$$

Then we may consider a generalization of the classical counting of lattice points in non-Riemannian geometry as follows.

Problem 3.12. *When a compact set S is gradually increased, how does $\#\Gamma_S$ increase?*

For instance, if X is a Riemannian manifold and S is a ball $B(R)$ of radius R centered at $o \in X$, then $\Gamma_{B(R)}$ coincides with the set $\{\gamma \in \Gamma : \gamma \cdot o \in B(2R)\}$, and Problem 3.12 is a classical counting problem of a lattice which asks about the asymptotic behavior of $\#\Gamma_{B(R)}$ as the radius R tends to infinity. Such an estimate for a pseudo-Riemannian manifold is utilized in the proof of the existence (Theorem 3.8) of “stable eigenvalues” for the Laplacian on a locally pseudo-Riemannian symmetric space [38].

(2) Quantitative estimates of non-proper actions

We may also formalize “quantification” of “non-properness” in the opposite situation where the action of the group is not proper. Along this direction we propose a new approach in non-commutative harmonic analysis by using an idea of dynamical system instead of the traditional method of differential equations for the study of the unitary representation on $L^2(G/H)$, see Section 4.3.

4. PROGRAM FOR NON-COMMUTATIVE HARMONIC ANALYSIS

In contrast to local analysis, global analysis involves many difficult problems. Further, in order to establish global results, certain “appropriate structure” (e.g., curvature pinching) should be imposed on the manifold X unless otherwise easy “counterexamples” show up because of the non-compactness of X . Such a “structure on X ” may be provided by means of a non-compact transformation group G of X . This is an idea to investigate global analysis successfully in the following scheme:

$$\text{global analysis on } X \longleftrightarrow \text{representation theory of } G.$$

⁴Not just “apparent similarities”, a close relationship between the “proper action” and the “discretely decomposable restriction” is recently elucidated in [76] under the assumption of certain “hidden symmetry” on spherical varieties.

In this section we focus on global analysis via transformation groups. After giving a brief overview on some major achievements about non-commutative harmonic analysis in the 20th century in Section 4.1, we describe a program for a new line of studies in Sections 4.2–4.4. Its connection to the theory of branching laws will be dealt with in Sections 5–8.

4.1. Non-commutative harmonic analysis on semisimple symmetric spaces —perspectives and achievements in the 20th century. In global analysis on a manifold X , **non-commutative harmonic analysis** concerns the space of functions rather than individual functions, and uses the regular representation of the transformation group G defined on the space of all functions.

Let us review basic notions. When a group G acts on a manifold X , a linear action of G on the space $\Gamma(X)$ of functions ($\Gamma = C^\infty, C, \mathcal{D}', \dots$) is naturally defined. That is, for each $g \in G$, one transforms functions f on X to different functions $\pi(g)f := f(g^{-1}\cdot)$ by the pull-back of the geometric action. Since the resulting family of the linear map $\pi(g): \Gamma(X) \rightarrow \Gamma(X)$ satisfy $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ with respect to compositions of maps, π defines a representation of the group G on the function space $\Gamma(X)$, which is referred to as **regular representation**. Further, if there exists a G -invariant Radon measure on X , then $\pi(g)$ preserves the L^2 -norm and therefore one can define a unitary representation on the Hilbert space $L^2(X)$ of square integrable functions on X .

Remark 4.1. Even when a manifold X is too small to allow a non-trivial action of a group (no symmetry in geometry), it may be possible to define a representation of the group on the space of functions (symmetry in analysis). This viewpoint leads us to “global analysis with minimal representations as a motif” [65], where one utilizes “hidden symmetry” on the space of functions. This new direction of global analysis will be mentioned in Section 4.4.

Before explaining what “non-commutative harmonic analysis” means, we first review classical “commutative” harmonic analysis. The Fourier transform on the Euclidean space

$$(4.1) \quad \mathcal{F}: C_c(\mathbb{R}) \rightarrow C(\mathbb{R}), \quad (\mathcal{F}f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi} dx$$

initially defined, for instance, in the space $C_c(\mathbb{R})$ of compactly supported continuous functions, extends to a unitary operator on the Hilbert space $L^2(\mathbb{R})$ of square-integrable functions (**the Plancherel theorem**). On the other hand, if we put $\pi(t)f := f(\cdot - t)$, then the Fourier transform \mathcal{F} also satisfies the following algebraic relations

$$\mathcal{F}(\pi(t)f)(\xi) = e^{-it\xi}(\mathcal{F}f)(\xi) \quad (\forall t \in \mathbb{R}).$$

These properties of the Fourier transform \mathcal{F} may be reinterpreted from the standpoint of the representation theory of groups as follows. For later purpose, we separate the role of the transformation group G from that of the geometry X , and

consider G acts on X and $L^2(X)$, even though $G = X = \mathbb{R}$ here.

Algebraic relations ... The Fourier transform $f \mapsto \mathcal{F}f(\xi)$ gives a G -homomorphism for each $\xi \in \mathbb{C}$ from the regular representation π on $C_c(X)$ of the additive group G to the representation space \mathbb{C} of the irreducible representation $\chi_{-\xi}: G \rightarrow GL(1, \mathbb{C}), t \mapsto e^{-it\xi}$ of G .

L^2 -theory ... The Plancherel theorem decomposes the regular representation $L^2(X)$ into irreducible unitary representations of G .

When a transformation group G is the abelian group \mathbb{R} , the representation space of the irreducible representation χ_ξ of G is one-dimensional, hence the above group-theoretic interpretation may seem stress too much on formalism. However, thanks to this Weyl's point of view, the representation theory of transformation groups G of manifolds X have been incorporated with global analysis in a broad sense and led to a great development of its own. The irreducible decomposition of the unitary representation $L^2(X)$ of G is called the **Plancherel-type theorem**. The harmonic analysis on the Euclidean space is extended to that on abelian locally compact groups G ("commutative harmonic analysis") by Pontryagin in 1930s, where all irreducible unitary representations of G are one-dimensional. In contrast, this analysis is called "non-commutative harmonic analysis" if the transformation group G is non-commutative. When the transformation group G is compact, then all irreducible representations are finite-dimensional and $L^2(X)$ decomposes discretely as in the case of Fourier series expansions in the case $G = X = S^1$. When the transformation group G is non-abelian and non-compact, the theory of "infinite-dimensional irreducible representations" of G may be used for global analysis on the manifold X as a powerful tool. Here are successful cases for the Plancherel-type theorem for G -spaces X with emphasis on real reductive Lie groups G .

$X = G$ (group manifold)	
Peter-Weyl (1927)	G is a compact group.
Pontryagin (1934)	G is an abelian locally compact group.
Gelfand school, Harish-Chandra (1950s)	G is a complex semisimple Lie group.
Harish-Chandra (1976)	G is a semisimple Lie group.
$X = G/H$ (symmetric space)	
T. Oshima (1980s), Delorme,	
van den Ban-Schlichtkrull (late 1990s)	X is a semisimple symmetric space.

These great achievements in the 20th century were not limited to theorems in representation theory, but also served as the driving forces for the development of analysis such as functional analysis and algebraic analysis. On the other hand, the beautiful and successful theory on global analysis on symmetric spaces (including group manifolds) seemed to give an impression that these objects live in a "closed world". Here we note that group manifolds G may be seen as a special case of symmetric spaces $(G \times G)/\text{diag}(G)$. In fact, many of the techniques used in global analysis in there were based on the structure theory of these spaces (e.g. the proof for group manifolds or symmetric spaces can be often reduced to that for smaller groups

or smaller symmetric spaces), and thus it was not obvious to foresee a promising direction of global analysis beyond symmetric spaces in the 1980s around the time when the Plancherel-type theorem for reductive symmetric spaces was announced by T. Oshima [106].

4.2. “Grip strength” of representations on global analysis — new perspective, part 1. To find a nice framework beyond symmetric spaces, let us start from scratch, and consider the very basic question whether representation theory is “useful” for global analysis in the first place. As a guiding principle to explore this question, we propose the following perspective.

Basic Problem 4.2 (“Grip strength” of representations [51]). *When a Lie group G acts on X , can the space of functions on X be “sufficiently controlled” by the representation theory of G ?*

The vague words, “sufficiently controlled”, or conversely, “uncontrollable”, need to be formulated rigorously as mathematics. Suppose V is the space of functions of a G -manifold X . There are (sometimes uncountably many) inequivalent irreducible subrepresentations in the infinite-dimensional G -module V . Moreover the multiplicity of each irreducible representation can range from finite to infinite.

Confronting such a general situation, we emphasize the following principle:

- even though there are infinitely many inequivalent irreducible subrepresentations, the group action **can** distinguish the inequivalent parts;
- the group action **cannot** distinguish the parts where the same irreducible representations occur with multiplicities.

This observation suggests us to think of the multiplicity of irreducible representations as the quantity measuring the “grip strength of a group”. For an irreducible representation Π of a group G , the multiplicity of Π in $C^\infty(X)$ is defined by

$$(4.2) \quad \dim_{\mathbb{C}} \text{Hom}_G(\Pi, C^\infty(X)) \in \mathbb{N} \cup \{\infty\}.$$

We formulate Basic Problem 4.2 as follows.

Problem 4.3 (Grip strength of representations on global analysis). *Let X be a manifold on which a Lie group G acts.*

- (1) *Find a necessary and sufficient condition on the pair (G, X) for which the multiplicity of each irreducible representation Π of G in the regular representation $C^\infty(X)$ is always **finite**.*
- (2) *Determine a condition on the pair (G, X) for which the multiplicity is **uniformly bounded** with respect to all irreducible representations Π .*

Since the condition (1) concerns individual finiteness with no constraint on the dependence of irreducible representations Π , we may think that the group G has “stronger grip power” in (2). The case that the group action of G is transitive is essential in Problem 4.3, which we shall assume in the following. Then Problem 4.3 is completely solved by Kobayashi–Oshima [91] when G is a reductive algebraic Lie group. To state the necessary and sufficient condition, we prepare some terminology.

Definition 4.4 (Spherical variety and real spherical variety).

- (1) Suppose that a complex reductive Lie group $G_{\mathbb{C}}$ acts biholomorphically on a connected complex manifold $X_{\mathbb{C}}$. We say $X_{\mathbb{C}}$ is **spherical** or $G_{\mathbb{C}}$ -**spherical** if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.
- (2) Suppose that a reductive Lie group G acts continuously on a connected real manifold X . We say X is **real spherical** or G -**real spherical** if a minimal parabolic subgroup of G has an open orbit in X .

The terminology of “real sphericity” was introduced by the author [51] in the early 1990s for the study of Basic Problem 4.2. By definition, if $X_{\mathbb{C}}$ is $G_{\mathbb{C}}$ -spherical, then $X_{\mathbb{C}}$ is also $G_{\mathbb{C}}$ -real spherical as a real manifold.

Example 4.5. Let X be a homogeneous space of a reductive algebraic Lie group G and $X_{\mathbb{C}}$ its complexification.

- (1) The following implications hold (Aomoto, Wolf, and Kobayashi–Oshima [91, Prop. 4.3]).

$$\begin{array}{c}
 X \text{ is a symmetric space.} \\
 \Downarrow \text{Aomoto, Wolf} \\
 X_{\mathbb{C}} \text{ is } G_{\mathbb{C}}\text{-spherical.} \\
 \Downarrow \text{Kobayashi–Oshima} \\
 X \text{ is } G\text{-real spherical.} \\
 \Uparrow \text{obvious} \\
 G \text{ is compact.}
 \end{array}$$

When X admits a G -invariant Riemannian structure, then $X_{\mathbb{C}}$ is $G_{\mathbb{C}}$ -spherical if and only if X is a weakly symmetric space in the sense of Selberg, see Vinberg [124] and Wolf [129].

- (2) The irreducible symmetric spaces were classified by Berger [10] at the level of Lie algebras.
- (3) The classification theory of spherical varieties $X_{\mathbb{C}}$ is given by Krämer [100], Brion [12], and Mikityuk [103].
- (4) The homogeneous space $(G \times G \times G) / \text{diag}(G)$ is not a symmetric space. It was determined by the author [51, Ex. 2.8.6] when it becomes real spherical in the study of multiplicities when decomposing tensor product representations (Example 6.16). A generalization of this will be described in Example 6.10 (2) in connection with branching problems for symmetric pairs.

The solutions to Problem 4.3, which is a reformalization of Basic Problem 4.2, are given by the following two theorems. For simplicity of the exposition, we assume H to be reductive, see Remark 4.8 for more general cases.

Theorem 4.6 (Criterion for the finiteness of multiplicity). *Let G be a reductive algebraic Lie group and H a reductive algebraic subgroup of G . We set $X = G/H$. Then the following two conditions on the pair (G, H) are equivalent.*

- (i) (representation theory) $\dim_{\mathbb{C}} \text{Hom}_G(\Pi, C^{\infty}(X)) < \infty$ ($\forall \Pi \in \text{Irr}(G)$).
- (ii) (geometry) X is G -real spherical.

In [91], we have proved not only a qualitative result (Theorem 4.6) but also quantitative results, namely, an upper estimate of the multiplicity by using a boundary

problem of partial differential equations and a lower estimate by generalizing the classical Poisson transform [30, Ch. II], see [71, Sect. 6.1]. These estimates from the above and below yield a criterion of the uniform boundedness of multiplicity as in the following theorem, where the equivalence (ii) \iff (iii) is classically known, and the main theme here is a connection with the representation theoretic property (i).

Theorem 4.7 (Criterion for the uniform boundedness of multiplicity [91]). *Let G be a reductive algebraic Lie group, H a reductive algebraic subgroup of G , and $X = G/H$. Then the following three conditions on the pair (G, H) are equivalent.*

(i) (representation theory) *There exists a constant C such that*

$$\dim_{\mathbb{C}} \text{Hom}_G(\Pi, C^\infty(X)) \leq C \quad (\forall \Pi \in \text{Irr}(G)).$$

(ii) (complex geometry) *The complexification $X_{\mathbb{C}}$ of X is $G_{\mathbb{C}}$ -spherical.*

(iii) (ring theory) *The ring of G -invariant differential operators on X is commutative.*

Remark 4.8. Theorems 4.6 and 4.7 give solutions to Problem 4.3 (1) and (2), respectively. More generally, these theorems hold not only for the space $C^\infty(X)$ of smooth functions but also for the space of distributions/hyperfunctions and the space of sections of equivariant vector bundles. Furthermore, one can drop the assumption that the subgroup H is reductive, see [91, Thms. A and B] for precise formulation. For instance, the theory of the Whittaker model (Kostant–Lynch, H. Matumoto) considers the case where H is a maximal unipotent subgroup N . In this case, G/N is always G -real spherical, and $G_{\mathbb{C}}/N_{\mathbb{C}}$ is $G_{\mathbb{C}}$ -spherical if and only if G is quasi-split. Thus Theorems 4.6 and 4.7 (in a generalized form) can be applied.

Remark 4.9. Theorem 4.7 includes the new discovery that the property of the “uniform boundedness of multiplicity” is determined only by the complexification $(G_{\mathbb{C}}, X_{\mathbb{C}})$ and is independent of a real form (G, X) . This observation suggests that an analogous result should hold more generally for reductive algebraic groups over non-archimedean local fields as well. Recently, Sakellaridis–Venkatesh [112] has obtained some positive results in this direction. See also [79, 122] for an alternative approach to the proof (ii) \Rightarrow (i) by using holonomic \mathcal{D} -modules.

Theorems 4.6 and 4.7 provide nice settings of global analysis in which the “grip strength” of representation theory is “firm” on the space of functions. The existing successful theory such as the Whittaker model and the analysis on semisimple symmetric spaces (Section 4.1) mentioned above may be thought of as the global analysis in this framework (Example 4.5). There are also “new” settings suggested by Theorems 4.6 and 4.7, to which the global analysis has not been paid much attention, and as one of such settings, we shall discuss an application to branching problems in Section 6.2.

4.3. Spectrum of the regular representation $L^2(X)$: a geometric criterion for temperedness — new perspective, part 2. In the previous section 4.2, we focused on “multiplicity” from the perspective of the “grip strength” of a group on a function space and proposed (real) sphericity as “a nice geometric framework for **detailed** study of global analysis” beyond symmetric spaces. On the other hand, even when the “grip strength” of representation theory is not “firm”, we may still expect to analyze $L^2(X)$ in a “**coarse** standpoint”. In this subsection, including

non-spherical cases, let us focus on the support of the Plancherel measure and consider the following problem.

Basic Problem 4.10. *Suppose that a reductive Lie group G acts on a manifold X , and that there is a G -invariant Radon measure on X . Determine a necessary and sufficient condition on a pair (G, X) for which the regular representation of G on $L^2(X)$ is a tempered representation (Definition 2.3 (2)).*

Basic Problem 4.10 asks the condition that any irreducible non-tempered representation (e.g. a complementary series representation) is not allowed to contribute to the unitary representation $L^2(X)$.

Observation 4.11.

- (1) In the case where G/H is a semisimple symmetric space, the Plancherel-type theorem is known [18, 106], however, it seems that a necessary and sufficient condition on which $L^2(G/H)$ is tempered had not been found until the general theory [5] is proved. In fact, if one tried to apply the Plancherel-type formula to find an answer to Problem 4.10, one would encounter a problem to find a precise (non-)vanishing condition of discrete series representations for G/H with singular parameters, or equivalently, that of certain cohomologies (Zuckerman derived functor modules) after crossing a number of “walls” and such a condition is combinatorically complicated in many cases, see [47, Chaps. 4 and 5] and [123] for instance.
- (2) More generally, when $X_{\mathbb{C}}$ is not necessarily a spherical variety of $G_{\mathbb{C}}$, the ring $\mathbb{D}_G(X)$ of G -invariant differential operators on X is not commutative as seen in the equivalence (ii) \iff (iii) of Theorem 4.7, and so we cannot use effectively the existing method on non-commutative harmonic analysis based on an expansion of the functions on X into joint eigenfunctions with respect to the commutative ring $\mathbb{D}_G(X)$ as was the case of symmetric spaces.

As observed above, to tackle Basic Problem 4.10, we need to develop a completely new method itself. For this, we bring an idea of geometric group theory mentioned in Section 3. Let us start with an observation of an easy example. If H is a compact subgroup of G , then $L^2(G/H) \subset L^2(G)$ holds. The following can be readily drawn from this observation.

Example 4.12. If the action of a group G on X is proper (Definition 3.10), then the regular representation in $L^2(X)$ is tempered.

Therefore, in the study of Basic Problem 4.10, the non-trivial case is when the action of G on X is not proper. Properness of the action is **qualitative property**, namely, a non-proper action means that there exists a compact subset S of X such that the set $\{g \in G : gS \cap S \neq \emptyset\}$ is not compact. In order to **quantify** this property, we highlight the volume $\text{vol}(gS \cap S)$ with respect to a Radon measure on X . Viewed as a function of $g \in G$

$$(4.3) \quad G \ni g \mapsto \text{vol}(gS \cap S) \in \mathbb{R}$$

is a continuous function on G . Definition 3.10 tells us that the G -action on X is not proper if and only if the support of the function (4.3) is non-compact for some compact subset S of X . This suggests that the “decay” of the function (4.3) at infinity may be considered as a “quantification” of non-properness of the action. By pursuing this idea, Basic Problem 4.10 is settled in Benoist–Kobayashi

[5, 6] when X is a homogeneous space of a real reductive group G . To describe the solution, let us introduce a piecewise linear function associated to a finite-dimensional representation of a Lie algebra.

Definition 4.13 (Moment of a representation [6, Sect. 2.3]). For a representation $\sigma: \mathfrak{h} \rightarrow \text{End}_{\mathbb{R}}(V)$ of a Lie algebra \mathfrak{h} on a finite-dimensional real vector space V , the **moment** ρ_V is a function on \mathfrak{h} defined as

$$\rho_V: \mathfrak{h} \rightarrow \mathbb{R}, \quad Y \mapsto \text{the sum of the absolute values of the real parts} \\ \text{of the eigenvalues of } \sigma(Y) \text{ on } V \otimes_{\mathbb{R}} \mathbb{C}.$$

The function ρ_V is uniquely determined by the restriction to a maximal abelian split subalgebra \mathfrak{a} of \mathfrak{h} . Further, the restriction $\rho_V|_{\mathfrak{a}}$ is piecewise linear, in the sense that there exist finitely many convex polyhedral cones which cover \mathfrak{a} and on which ρ_V is linear. When (σ, V) is the adjoint representation of a semisimple Lie algebra \mathfrak{h} , the restriction $\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ can be computed by using a root system and coincides with a scalar multiple of the usual “ ρ ” on the dominant Weyl chamber.

With the use of the functions ρ_V for the adjoint actions of \mathfrak{h} on $V = \mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$, one can give a necessary and sufficient condition for Basic Problem 4.10.

Theorem 4.14 (Criterion on the temperedness of $L^2(X)$). *Let G be a real reductive Lie group and H a connected closed subgroup of G . Also let \mathfrak{g} and \mathfrak{h} be the Lie algebras of G and H , respectively. Then the following two conditions on a pair (G, H) are equivalent.*

- (i) (global analysis) *The regular representation $L^2(G/H)$ is tempered.*
- (ii) (combinatorial geometry) $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$.

The implication (i) \Rightarrow (ii) follows from a local estimate of the asymptotic behavior of the volume $\text{vol}(gS \cap S)$, and the converse implication (ii) \Rightarrow (i) is much more involved. We note that Basic Problem 4.10 makes sense even when there is no G -invariant Radon measure on X by replacing $L^2(X)$ with the Hilbert space of L^2 -sections for the half-density bundle over X . Theorem 4.14 holds in this generality. See [5, 6] for details, [7] for the classification of the pairs (G, H) of real reductive groups for which $L^2(G/H)$ is non-tempered, and [8] for some connection with other disciplines such as the orbit philosophy and the limit algebras.

Remark 4.15. If G is an algebraic group and X is an algebraic G -variety, then, even when X is not a homogeneous space of G , one can give an answer to Basic Problem 4.10 by applying Theorem 4.14 to generic G -orbits [6].

4.4. The sizes of a group G and a manifold in global analysis. Let us mention yet another new perspective in global analysis via representation theory. To give its flavor, we begin with “coarse comparison” of the “size” of the transformation group G with that of the geometry X . Suppose that a Lie group G acts on a manifold X . If there are at most finitely many G -orbits in X (in particular, if G acts transitively), one may regard that the size of G is “comparable” with X which we write symbolically as

$$\text{group } G \approx \text{manifold } X.$$

Homogeneous spaces $X = G/H$ are typical examples for the relation $G \approx X$. We may think of the main results of Sections 4.2 and 4.3 as a refinement of the “relation $G \approx X$ ” by introducing some kind of “smallness of $X = G/H$ relative to G ” or “largeness of H ” from the following points of view:

- Grip strength \cdots multiplicity (Theorems 4.6 and 4.7): The “larger” H is (or the “smaller” X is), the better G controls the function space on X .
- Spectrum \cdots temperedness criterion (Theorem 4.14): The “larger” H is (or the “smaller” X is), the less likely the regular representation of G on $L^2(X)$ becomes tempered.

As we have seen, the two notions of “smallness of X ” are alike in appearance but quite different in nature among the case $G \approx X$. In the rest of this section, we discuss new directions of representation theory and global analysis beyond the setting $G \approx X$:

- (1) group $G \gg$ manifold X (G is “too large” to act on X non-trivially.)
- (2) group $G \ll$ manifold X (The dimensions of all G -orbits are smaller than that of X .)

Global analysis of minimal representations — an example for the case that the size of $G \gg$ the size of a manifold X

Let $e(G)$ be the smallest value of the codimensions of proper subgroups of a Lie group G . For example, if $G = GL(n, \mathbb{R})$, then $e(G) = n - 1$. This means that if the dimension of a manifold X is less than $e(G)$, then any (continuous) action of G on X must be trivial. In this way, if a Lie group G is “too large” compared to a manifold X , for instance, if $e(G) > \dim X$, then G cannot act on X geometrically. We write group $G \gg$ manifold X in this case. Even when $G \gg X$, we may perform global analysis on X from a different perspective if one can define a natural representation of the group G on the space of functions on X although the representation does not arise from a geometric action of G on X . In such a case the “grip strength” of G on the space of functions will be extremely firm, hence one may expect that the representation theory plays a powerful role in the global analysis, even more powerful than in the analysis on homogeneous spaces [65]. One of the examples is that X is a Lagrangian submanifold of the minimal nilpotent coadjoint orbit (Section 2.4) and in this point of view, “global analysis with minimal representations as a motif” has emerged [65]. Global analysis in new directions based on the Schrödinger model of a minimal representation beyond the Segel–Shale–Weil representation [33, 82, 89] has been developed rapidly in recent years. It includes the construction and the unitarization of a minimal representation by the use of conformal geometry [87, 88], the theory of special functions associated with fourth-order differential equations [32, 72] and the deformation theory of the Fourier transform such as the (k, a) -generalized Fourier transform [9, 17].

Visible action — an example for the case that the size of $G \ll$ the size of a manifold X

We consider the opposite extremal case where group $G \ll$ manifold X in the sense that there is no open G -orbit X , and in particular, G has continuously many orbits in X . Even in such a case, there is still a possibility to develop global analysis with a reasonable “control” by group representations if one imposes further constraints such as geometrically defined differential equations. The theory of visible actions on complex manifolds and the propagation of multiplicity-free property [60, 69] is a new attempt in the direction (see Section 6.3).

5. PROGRAM FOR BRANCHING PROBLEMS IN REPRESENTATION THEORY

In the latter part of this article we discuss branching problems that ask how the restrictions of irreducible representations behave when restricted to subgroups, e.g., finding their irreducible decompositions (the *branching laws*) for infinite-dimensional representations of Lie groups. In spite of its potential importance, the study of the restriction of irreducible representations of a reductive Lie group to non-compact subgroups was still underdeveloped in the 1980s, except for some specific cases such as the theta correspondence [34], highest weight modules, the compact case, or the SL_2 -cases [104, 111]. The main difficulty seemed to be the lack of promising perspectives for the general study of the restrictions of representations. In fact, if one tries to find a branching law for a group larger than SL_2 , then bad phenomena appear such as infinite multiplicity in the branching laws, which we express as “the grip strength of the subgroup is not firm enough”. For instance, when $n \geq 3$, the tensor product of two principal series representations of $SL_n(\mathbb{R})$ “contains” the same irreducible representation with infinite multiplicity. In the late 1980s, inspired by the new theory of discontinuous groups beyond the Riemannian setting, the author discovered a new example of “good branching laws” in the sense that an infinite-dimensional irreducible representation Π decomposes discretely and multiplicity-freely when restricted to *non-compact* subgroups in the setting that Π is *not* a highest weight module.⁵ This type of branching law had not been previously known and an attempt to elucidate this example more generally became a trigger of the general study of the restriction of representations which had been in a kind of a chaotic state. Enough tools had been accumulated, it was poised to take off. In the following three steps let us give an overview of some of the progress of the study of branching problems over around 30 years from the discovery:

Stage A: Abstract feature of the restriction of representations (Section 6).

Stage B: Branching laws (Section 7).

Stage C: Construction of symmetry breaking operators (Section 8).

The name of each stage comes from their initials [73]. We shall look into the roles of Stages A, B, and C in Sections 8.1 and 8.2 later. For more details of the program, see [73].

6. PROGRAM FOR BRANCHING PROBLEMS: STAGE A

The aim of Stage A is to develop an abstract theory on the restriction of representations in the setting as general as possible. Stage A will provide a bird’s-eye view to the problem of the “restriction of representations” in which various new directions of research may open up. In particular, this stage will play a role to single out a nice setting where one could develop a detailed study of branching laws in Stages B and C. For instance, in Stage A, we aim to construct a general theory to elucidate the following properties.

- (Existence of the continuous spectrum [49]) Does a continuous spectrum appear in the branching law of the restriction $\Pi|_{G'}$ of a unitary representation Π of a group G to non-compact subgroups G' ? Or, does it decompose discretely? (Section 6.1)

⁵As its geometric background there was an open problem of spectral geometry proposed by Toshikazu Sunada at the time. For the details see [86] on Sunada’s conjecture and also [64].

- (Finiteness/boundedness of multiplicity [51, 91]) For irreducible representations of a group G , is the multiplicity (i.e. the number of times that irreducible representations of a subgroup G' appears) finite or infinite? In case each multiplicity is finite, we may also ask, even strongly, if it has uniform boundedness? Even further, under what assumptions does a multiplicity-free theorem hold? We may formulate these problems without assuming that the representations are unitary, see Section 6.2.
- (Support of branching laws [5]) Properties on irreducible representations that appear in a branching law. For instance, as an analogue of temperedness criterion in Section 4.3, one may ask if the restriction $\Pi|_{G'}$ is tempered as a representation of a subgroup G' when Π is a non-tempered irreducible unitary representations of a group G .

6.1. Existence problem of the continuous spectrum in branching laws. If G' is a reductive Lie group, then with the use of the notion of the direct integral (Section 2.3), the restriction of $\Pi|_{G'}$ of a unitary representation Π is decomposed uniquely into irreducible representations of the subgroup G' as

$$(6.1) \quad \Pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} n_{\Pi}(\pi) \pi d\mu(\pi)$$

where μ is a Borel measure on the unitary dual $\widehat{G'}$ (Theorem 2.2). As one of the problems in Stage A, first consider whether a continuous spectrum appears in the branching law (6.1). When a continuous spectrum appears in a branching law, analytic approaches will be natural for a detailed study of the restriction $\Pi|_{G'}$. On the other hand, if one knows *a priori* the branching law (6.1) is discrete, one may study the restriction $\Pi|_{G'}$ also by purely algebraic approach, and expect to develop even more combinatorial techniques and algorithms that compute branching laws. Thus we address the following problem.

Basic Problem 6.1 ([48]). *Suppose that Π is an irreducible unitary representation of a group G and that G' is a subgroup of G . Find a criterion on a triple (G, G', Π) to decompose the restriction $\Pi|_{G'}$ into the discrete direct sum of irreducible representations of G' . Moreover, find a criterion that the multiplicity of each irreducible representation is finite in the discrete branching law of the restriction $\Pi|_{G'}$.*

In the latter case, we say the restriction $\Pi|_{G'}$ is **G' -admissible** ([49, Sect. 1]). When G' is a maximal compact subgroup K of G , the K -admissibility is nothing but Harish-Chandra's admissibility (Definition 2.12).

In the following let G be a real reductive Lie group, Π an irreducible unitary representation of G , and G' a reductive subgroup of G .

Let us give three elementary examples of discretely decomposable restrictions that we ask in Basic Problem 6.1.

Example 6.2. If Π is finite-dimensional, then the restriction $\Pi|_{G'}$ is completely reducible, and hence decomposes discretely.

Example 6.3. If G' is compact, then the restriction $\Pi|_{G'}$ decomposes discretely.

Example 6.4 ([53, 55]). If Π is a highest weight representation and also a pair $G \supset G'$ is of holomorphic type, then the restriction $\Pi|_{G'}$ decomposes discretely.

Let us recall the terminology in Example 6.4. An irreducible representation Π of a reductive Lie group G is called a **highest weight representation** if its differential representation contains a nontrivial subspace invariant under some Borel subalgebra of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. This subspace is automatically one-dimensional because Π is irreducible. All finite-dimensional irreducible representations are highest weight representations, whereas highest weight representations are “rare” among irreducible infinite-dimensional representations. Moreover, a simple Lie group G has an infinite-dimensional highest weight representation if and only if G is of Hermitian type, that is, $X = G/K$ has the structure of a Hermitian symmetric space. Here K is the fixed point subgroup of a Cartan involution θ of G . For a reductive Lie group G , we say G is of Hermitian type if it is locally isomorphic to the direct product of a compact Lie group and simple Lie groups of Hermitian type. Now, let G' be a reductive subgroup of G . Without loss of generality, we assume the Cartan involution θ leaves G' invariant. We set $K' := G' \cap K$. We say a pair $G \supset G'$ is of **holomorphic type** if both G and G' are of Hermitian type and if the natural embedding $G'/K' \hookrightarrow G/K$ is holomorphic when we choose appropriate G' -invariant and G -invariant complex structures on G'/K' and G/K , respectively. See [55, 61] for the list of symmetric pairs (G, G') of holomorphic type.

Highest weight representations and finite-dimensional representations are very much alike and they share many common properties, whereas irreducible infinite-dimensional representations that are not highest weight representations to which we refer as “truly infinite-dimensional representations”, do not. In contrast to Example 6.4 for highest weight representations, experts seemed to have believed for a long time that it would not be plausible for “truly infinite-dimensional” representations to decompose discretely when they are restricted to a non-compact subgroup. This “common sense” was reversed through a study of discontinuous groups for the indefinite Kähler manifold $X = SU(2, 2)/U(1, 2)$: the trigger was a discovery that any irreducible representation π of $SU(2, 2)$ in $L^2(X)$ is discretely decomposable with respect to the restriction to the subgroup $Sp(1, 1) \simeq Spin(4, 1)$ though π is a “truly infinite-dimensional representation” (1988). We refer to [64] for the details. The general theory of discretely decomposable branching laws has emerged in the 1990s from the attempts that elucidate this particular example as a general principle. The series of the three papers [49, 53, 54] answer Basic Problem 6.1 from perspectives in geometry, analysis, and algebra, respectively. We provide a brief introduction to the main theorem in [53] here, which was obtained by techniques of microlocal analysis. An alternative proof based on symplectic geometry is given in [80].

Theorem 6.5 (Criterion for the discrete decomposability of unitary representations). *Let Π be any irreducible unitary representation of a reductive Lie group G and G' a reductive subgroup of G . Then the implication (ii) \Rightarrow (i) on a triple (Π, G, G') holds.*

- (i) *The restriction $\Pi|_{G'}$ is G' -admissible, namely, it decomposes discretely and also the multiplicity is finite.*
- (ii) $AS(\Pi) \cap Cone(G') = \{0\}$.

Here $AS(\Pi)$ denotes the asymptotic cone of the K -types of the representation Π [37] and $Cone(G')$ is the cone determined from the subgroup G' (or more precisely, determined only by its maximal compact subgroup K'), which is defined by the

structure theory of Lie algebras [53] or by the moment map for the Hamiltonian K -action on $T^*(K/K')$ [59, Thm 6.4.3].

Remark 6.6. (1) When $G' = K$, one has $\text{Cone}(K) = \{0\}$ hence Theorem 6.5 means Harish-Chandra’s basic theorem, see [127, Thm. 3.4.10] for instance, that $\Pi|_K$ is K -admissible for any irreducible unitary representation Π of G .

(2) When G' is compact, the equivalence (i) \iff (ii) in Theorem 6.5 holds [59, 80].

(3) For a discrete series representation Π of G , the equivalence (i) \iff (ii) in Theorem 6.5 holds [21, 59, 132].

(4) Even for nonunitary representations, one may formulate algebraically the notion of “discrete decomposability” of the restriction of representations [54]. Then (ii) \implies (i) in Theorem 6.5 still holds. Moreover the necessary condition for (algebraic) discrete decomposability is also given in [54, Cor. 3.4] for the category of (\mathfrak{g}, K) -modules and in [67, Thm. 4.1] for the category \mathcal{O} .

Classification theory of discretely decomposable branching laws: For reductive symmetric pairs (G, G') , the triple (G, G', Π) , for which the underlying (\mathfrak{g}, K) -module of an irreducible unitary representation Π is discretely decomposable when restricted to the subgroup G' in the algebraic sense (Remark 6.6 (4)), has been classified in the following settings by carrying out the combinatorial computations for the condition (ii) in Theorem 6.5 and for the criterion by using the associated variety in [54] detecting the opposite direction:

- Π is a “geometric quantization” of an elliptic orbit (\iff the underlying (\mathfrak{g}, K) -module Π_K is a Zuckerman derived functor module $A_q(\lambda)$) [92],
- Π is a “geometric quantization” of the minimal nilpotent orbit (\iff Π is a minimal representation) [93], and
- the tensor product representation of two irreducible representations (\iff (G, G') is of the form $(H \times H, \text{diag}(H))$) [93].

These classification results for discretely decomposable restrictions for symmetric pairs have been recently extended to *non-symmetric pairs* (G, G') by several authors, see [20, 29, 101] for example.

6.2. Multiplicity of branching laws. Next let us discuss the multiplicity in branching laws. In the following, let $G \supset G'$ be a pair of reductive Lie groups. Here we allow representations to be nonunitary, and $\Pi \in \text{Irr}(G)$ and $\pi \in \text{Irr}(G')$, that is, Π and π be irreducible admissible representations of moderate growth of G and G' , respectively (Definition 2.13). We say

$$(6.2) \quad m(\Pi, \pi) := \dim_{\mathbb{C}} \text{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}$$

is the **multiplicity** of π in the restriction $\Pi|_{G'}$.⁶ Unlike the case that Π is finite-dimensional, the second equation of (1.3) does not hold generally when Π is infinite dimensional. Moreover, the multiplicity $m(\Pi, \pi)$ could be infinite even in a natural situation that G' is a maximal subgroup of G . For instance, the multiplicity on the tensor product of infinite-dimensional irreducible representations is often infinite, see Example 6.11.

⁶When Π is a unitary representation, the multiplicity $m(\Pi, \pi)$ for the spaces of C^∞ vectors could be larger than the multiplicity $n_\Pi(\pi)$ in the direct integral (6.1).

In view of these observations, let us illustrate schematically the hierarchy of the multiplicity in branching laws.

- a. Some multiplicity is infinite.
- b. All the multiplicity is finite.
- c. The multiplicity is uniformly bounded.
- d. multiplicity-free.

The “grip strength” of the subgroup is larger in order of $a \prec b \prec c \prec d$ and it is expected that one could do more detailed analysis of branching laws accordingly.

By applying the criterion for the “grip strength of representations in global analysis” on homogeneous spaces (Theorem 4.6) discussed in Section 4.2 to the homogeneous space $(G \times G')/\text{diag}(G')$, one obtains a necessary and sufficient condition that the multiplicity in the branching law is always finite ([51, 71, 91]):

Theorem 6.7 (Finiteness of the multiplicity of branching laws). *The following two conditions on reductive Lie groups $G \supset G'$ are equivalent.*

- (i) (representation theory) *For any $\Pi \in \text{Irr}(G)$ and $\pi \in \text{Irr}(G')$, $m(\Pi, \pi) < \infty$.*
- (ii) (geometry) *$(G \times G')/\text{diag}(G')$ is a real spherical variety (Definition 4.4).*

If one requires the “uniform boundedness” of the multiplicity of branching laws, then the following characterization holds [51, 71, 91].

Theorem 6.8 (Uniform boundedness of multiplicity). *The following three conditions on a pair (G, G') of reductive Lie groups are equivalent.*

- (i) (representation theory) *There exists a constant $C > 0$ such that*

$$m(\Pi, \pi) \leq C \quad (\forall \Pi \in \text{Irr}(G), \forall \pi \in \text{Irr}(G')).$$

- (ii) (complex geometry) *$(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\text{diag}(G'_{\mathbb{C}})$ is a spherical variety.*
- (iii) (ring theory) *The subalgebra $U(\mathfrak{g})^{G'}$ of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra \mathfrak{g} is commutative.*

As variants of Theorems 6.7 and 6.8, one may fix $\Pi \in \text{Irr}(G)$ and also ask a criterion for the triple (G, G', Π) with uniformly bounded multiplicity $m(\Pi, \pi)$ where $\pi \in \text{Irr}(G')$ varies [73, Prob. 6.2]. See [78, Thm. 7.6] and [79, Thm. 4.2] for necessary and sufficient conditions on the triple (G, G', Π) with uniformly bounded multiplicity property in the setting where Π is H -distinguished for symmetric pairs (G, H) or where Π is a (degenerate) principal series representation of G , respectively.

Remark 6.9. Similar to the criterion (Theorem 4.7) for uniform boundedness in global analysis, Theorem 6.8 includes the discovery that the uniform boundedness of the multiplicity of branching laws is determined only by the complexifications $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ of the Lie algebras and independent of the real forms. This suggests that similar results may also hold for reductive algebraic groups over other local fields. In fact, the assertion corresponding to (ii) \Rightarrow (i) in Theorem 6.8 (more strongly $C = 1$) is shown by Aizenbud–Gourevitch–Rallis–Schiffmann [3] over a non-Archimedean local field.

Classification theory of pairs of reductive Lie groups for which the branching laws have always finite multiplicity: The easy-to-check geometric condition in

Theorems 6.7 and 6.8 allow us to extract settings in which branching laws behave nicely in terms of multiplicity.

(1) Since the criterion (ii) in Theorem 6.8 is determined by the complexifications $(G_{\mathbb{C}}, G'_{\mathbb{C}})$, one may check when G is a compact group, and thus the criterion (ii) in Theorem 6.8 coincides with the one which already appeared in finite-dimensional representation theory. In fact, the complexified pairs $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ satisfying (ii) were classified in the 1970s, that is, such pairs $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ are locally isomorphic to the direct product of the pairs $(SL(n, \mathbb{C}), GL(n-1, \mathbb{C}))$, $(SO(n, \mathbb{C}), SO(n-1, \mathbb{C}))$, or pairs of abelian Lie algebras (Kostant and Krämer [100]). The following example on a pair of reductive Lie groups is their real forms.

Example 6.10. The constant C in Theorem 6.8 (i) can be taken to be $C = 1$ for many of real forms (G, G') of $(SL(n, \mathbb{C}), GL(n-1, \mathbb{C}))$ or $(SO(n, \mathbb{C}), SO(n-1, \mathbb{C}))$ such as $(G, G') = (SL(n, \mathbb{R}), GL(n-1, \mathbb{R}))$, $(SO(p, q), SO(p-1, q))$, see Aizenbud–Gourevitch [2] and Sun–Zhu [118].

(2) The symmetric pairs (G, G') that satisfy the criterion (ii) in Theorem 6.7 or in other words, for which $(G \times G')/\text{diag } G'$ is real spherical were classified in 2013 (Kobayashi–Matsuki [83]). This is also a generalization of the following earlier example in 1995 by the author [51], see also [71, Cor. 4.2].

Example 6.11. For a simple Lie group G , the following four conditions are equivalent.

- (i) (invariant trilinear form) For any $\pi_1, \pi_2, \pi_3 \in \text{Irr}(G)$, the space of invariant trilinear forms $\text{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C})$ is finite-dimensional.
- (ii) (tensor product representation) For any $\pi_1, \pi_2, \pi_3 \in \text{Irr}(G)$, $m(\pi_1 \otimes \pi_2, \pi_3) < \infty$.
- (iii) (geometric condition) $(G \times G \times G)/\text{diag}(G)$ is real spherical.
- (iv) (classification) $\mathfrak{g} \simeq \mathfrak{o}(n, 1)$ ($n \geq 2$) or G is compact.

See also [73, 79] for finer classification results of the triples (G, G', Π) rather than the pairs (G, G') for the uniformly bounded multiplicity restriction $\Pi|_{G'}$.

Example 6.11 suggests that invariant trilinear forms could be investigated explicitly when $G = O(n, 1)$. Indeed, invariant trilinear forms have been studied in detail for this group in recent years, not only algebraically but also analytically (Stage C) (for $n = 2$, Bernstein–Reznikov [11]; for general n , Deitmar, Clerc–Kobayashi–Ørsted–Pevzner [15], Clerc [14], etc.).

In recent years, rapid progress in “Stage C” for the branching problems has been made in the construction and classification problems of symmetry breaking operators [23, 36, 81, 97, 99] in the “good framework” suggested by Theorem 6.7 or more strongly by Theorem 6.8. Some of them interact with parabolic geometry such as conformal geometry and also with the theory of automorphic forms. These topics will be discussed in Section 8.3.

Case of infinite multiplicity: When the multiplicity $m(\Pi, \pi) = \infty$ for irreducible representations Π and π of G and its subgroup G' , respectively, we have viewed “the grip strength of the subgroup G' in the restriction $\Pi|_{G'}$ is weak”. Apparently, the branching problems are “uncontrollable”. However, even in this case, if we find an external algebraic structure to control the (infinite-dimensional) space $\text{Hom}_{G'}(\Pi|_{G'}, \pi)$ of symmetry breaking operators (Section 8), then, by using the

structure as a good clue, it would be still possible to investigate the restriction of representations. A plausible candidate of such a structure is the algebra $U(\mathfrak{g})^{G'}$ (see Theorem 6.8 (iii)), which acts on $\text{Hom}_{G'}(\Pi|_{G'}, \pi)$ naturally. Loosely speaking, the property $m(\Pi, \pi) = \infty$ can be described (in the level of the actions on the representation space) as follows.

G' is relatively small. \iff The ring $U(\mathfrak{g})^{G'}$ is large.

One of the motivations for Kitagawa's thesis [43] is to understand $\text{Hom}_{G'}(\Pi|_{G'}, \pi)$ algebraically as a $U(\mathfrak{g})^{G'}$ -module beyond the case where $\text{Hom}_{G'}(\Pi|_{G'}, \pi)$ is finite-dimensional.

6.3. Visible actions and multiplicity-freeness. In Section 6.2, we discussed mainly the multiplicity $m(\Pi, \pi)$ for all irreducible representations Π of a group G and for all π of its subgroup G' , and gave geometric criteria for the pairs $G \supset G'$ of groups that guarantee the finiteness of the multiplicity (Theorem 6.7) and the uniform boundedness (Theorem 6.8). In this section we discuss an estimate of the multiplicity of branching laws for the individual irreducible representations Π in more detail, see also [73, Problems 6.1 and 6.2].

Basic Problem 6.12. *Classify triples (G, G', Π) for which the restriction of an irreducible representation Π of a group G to its subgroup G' is multiplicity-free.*

The following two formulations are possible for Basic Problem 6.12 based on the definitions of the ‘‘multiplicity’’.

- the case of unitary representations: the multiplicity $n_{\Pi}(\pi)$ in the irreducible decomposition by using the direct integral, see (6.1).
- the case of representations that are not necessarily unitary: the multiplicity $m(\Pi, \pi)$ as the dimension of the space of symmetry breaking operators, see (6.2).

In this subsection we consider the former case and introduce a new geometric principle that gives the multiplicity-freeness of branching laws.

Definition 6.13 ([60, Def. 3.3.1]). Let X be a connected complex manifold. A biholomorphic action of a Lie group G on X is said to be **strongly visible** if there exist a non-empty G -invariant open set X' , an anti-holomorphic diffeomorphism σ of X' , and real submanifold S (*slice*) such that

$$\sigma|_S = \text{id} \quad \text{and} \quad G \cdot S = X'.$$

A strongly visible action is visible [60, Thm. 4], where we recall that the G -action on a complex manifold is **visible** if there exist a non-empty G' -invariant open set X' and a totally real submanifold S in X such that $G \cdot S = X'$ and $J_x(T_x S) \subset T_x(G \cdot x)$ for all $x \in X$ [58, Def. 2.3].

On the multiplicity-freeness of representations, the strong visibility gives a new ‘‘mechanism’’ that produces systematically from simple examples (for instance, one-dimensional representations, which are clearly multiplicity-free) to more complicated examples (for instance, multiplicity-free infinite-dimensional representations). This mechanism is formulated as a ‘‘propagation theorem of multiplicity-freeness’’, for which we describe in a slightly simplified way by omitting some small technical compatibility conditions about the isotropy action on fibers (that are automatically satisfied in many situations) so the next theorem mentions only the main assumptions and conclusions. For precise statements, see [69].

Theorem 6.14 (Propagation theorem of multiplicity-freeness). *Suppose that a group G acts on a holomorphic vector bundle \mathcal{V} over a complex manifold X and that the action of G on the base space X is strongly visible. If the isotropy representation of the isotropy subgroup at a generic point on the fiber is multiplicity-free, then any unitary representation Π of G realized on subspaces of the space $\mathcal{O}(X, \mathcal{V})$ of holomorphic sections is multiplicity-free.*

Here are two examples of the applications.

Example 6.15 (Highest weight representations). Suppose that (G, G') is a symmetric pair and Π is a unitary highest weight representation of G . If the minimal K -type of Π is one-dimensional, then the irreducible decomposition of the restriction $\Pi|_{G'}$ is multiplicity-free (Kobayashi [60]). This theorem can be derived from the fact that the subgroup G' acts on the Hermitian symmetric space G/K strongly visibly [62] and from Theorem 6.14. The restriction $\Pi|_{G'}$ may or may not contain continuous spectrum, cf [54, Sect. 5]. See Theorem 7.1 for an explicit branching law in the discretely decomposable setting.

Example 6.16 (Tensor product representations). Consider the pairs (π_1, π_2) of irreducible finite-dimensional representations of the unitary group $U(n)$, for which the tensor product representation $\pi_1 \otimes \pi_2$ decomposes multiplicity-freely. Such pairs (π_1, π_2) include the classical cases when $\pi_2 \simeq \bigwedge^j(\mathbb{C}^n)$ or $S^j(\mathbb{C}^n)$ (Pieri's rule) and were classified by a combinatorial method (Stembridge [116], 2001). All such pairs (π_1, π_2) can be reconstructed geometrically from a $(G \times G)$ -equivariant holomorphic vector bundle over a double flag variety $X = G/P_1 \times G/P_2$ with strongly visible action of G via the diagonal action (Kobayashi [58]). This geometric interpretation based on the theory of visible actions is extended to a reconstruction of all multiplicity-free tensor product representations of $SO(n)$ (Tanaka [119]).

Classification theory of visible actions: Whereas (strongly) visible actions are defined on complex manifolds, there are analogous notions in other geometries: polar actions of isometry groups on Riemannian manifolds and coisotropic actions on symplectic manifolds (Guillemin–Sternberg and Huckleberry–Wurzbacher [35]). For Kähler manifolds on which complex, symplectic and Riemannian structures are defined in a compatible fashion, these three notions are close to each other to some extent, see Podestà–Thorbergsson [109] and the author [60, Thms. 7 and 8] for precise formulation. For polar actions of compact groups on Riemannian manifolds, the classification theory has been developed over decades. On the other hand, the classification theory for visible actions has started only recently (see Kobayashi [62, 63], Sasaki [113], Tanaka [119], and the references therein), and it would be interesting to pursue further developments and discoveries in this “young” area. Whereas the classification theory of polar actions has focused mainly on the compact setting from topological viewpoints historically, the classification theory of visible actions will be useful also for non-compact transformation groups G because it will yield new families of infinite-dimensional multiplicity-free representations of non-compact Lie groups. We note that for a compact Lie group G , strongly visible action of the group G is essentially equivalent to sphericity of the complexified group $G_{\mathbb{C}}$, see Tanaka [120], and thus the classification theory of strongly visible actions of a **compact** G is essentially the same with that of spherical varieties for the complexified reductive group $G_{\mathbb{C}}$. In the expository paper [60], we have

presented various classification results of visible actions including those of **non-compact** Lie groups, which have led us, via Theorem 6.14, to new multiplicity-free theorems as well as a unified and geometric proof of the existing multiplicity-free theorems for some families of representations that were found in the past by individual arguments.

7. BRANCHING LAWS: STAGE B

In Stage B of the program, we aim to find explicitly the irreducible decomposition of the restriction of representations (**branching law**). Here Stage A (Section 6) serves as a guideline to single out a “nice setting” in which we could expect a simple and detailed study of branching laws through a priori estimate. In this section we describe typical examples for branching laws with focus on the following two “nice settings”: multiplicity-free cases (Section 7.1) and discretely decomposable cases (Section 7.2).

7.1. Multiplicity-free representations. Multiplicity-free representations are often hidden in classical analysis, even though we usually do not notice that there are representations behind. For example, the Fourier expansion, Taylor expansion, and spherical harmonics expansion were already useful tools in analysis, historically much before the notion of groups and representations emerged. From representation-theoretic viewpoints, these expansions may be regarded as irreducible decompositions of multiplicity-free representations. We observe here the “multiplicity-freeness” is the underlying algebraic structure of these expansions in the sense that multiplicity-freeness assures that the irreducible decomposition is canonical. The expansion by the Gelfand–Tsetlin basis is also defined by this principle. More generally, we may utilize multiplicity-free representations as a driving force not only for studying the explicit formulas of branching laws (Stage B) but also for pursuing global analysis through canonical expansions of functions via representation theory (Stage C).

In this subsection we consider the setting of Example 6.15 as a case in which we know in advance that the branching law is multiplicity-free. Moreover, we assume that (G, G') is of holomorphic type, by which we know a priori that the branching law is further discretely decomposable (Example 6.4). In this case the explicit formulas of branching laws (Stage B) should take a simple form, which we describe now.

First we set up some notation. Let G be a real simple Lie group of Hermitian type (Example 6.4) and (G, G') a symmetric pair of holomorphic type defined by an involution σ of G . For simplicity, we take G' to be the connected component G_0^σ containing the identity element of $G^\sigma := \{g \in G : \sigma g = g\}$. Take a Cartan involution θ of G commuting with σ and we write K for the fixed point group of the Cartan involution θ . Then the complexification $\mathfrak{g}_\mathbb{C}$ of the Lie algebra of G can be decomposed into the direct sum $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} + \mathfrak{p}_+ + \mathfrak{p}_-$ as a K -module, and G/K carries a structure of a Hermitian symmetric space with holomorphic tangent space $T_{eK}(G/K) \simeq \mathfrak{p}_+$. We take a Cartan subalgebra \mathfrak{j} of \mathfrak{k} such that \mathfrak{j}^σ is also a Cartan subalgebra of $\mathfrak{k}^\sigma = \{X \in \mathfrak{k} : \sigma X = X\}$. We fix compatible positive systems $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ and $\Delta^+(\mathfrak{k}_\mathbb{C}^\sigma, \mathfrak{j}_\mathbb{C}^\sigma)$. Since $\sigma\theta = \theta\sigma$, we have $(\sigma\theta)^2 = \text{id}$, hence $\mathfrak{g}_\mathbb{C}^{\sigma\theta}$ is a reductive Lie algebra. We decompose $\mathfrak{g}_\mathbb{C}^{\sigma\theta}$ into the direct sum of the simple Lie algebras $\mathfrak{g}_\mathbb{C}^{(i)}$ ($1 \leq i \leq N$) and the abelian ideal $\mathfrak{g}_\mathbb{C}^{(0)}$. For each i

($\neq 0$), we write $\{\nu_1^{(i)}, \dots, \nu_{k_i}^{(i)}\}$ for a maximal set of strongly orthogonal roots of $\Delta(\mathfrak{p}_+^{-\sigma} \cap \mathfrak{g}_{\mathbb{C}}^{(i)}) \subset (\mathfrak{j}_{\mathbb{C}}^{\sigma})^*$ with $\nu_k^{(i)}$ the lowest among the elements in $\Delta(\mathfrak{p}_+^{-\sigma} \cap \mathfrak{g}_{\mathbb{C}}^{(i)})$ that are strongly orthogonal to $\nu_1^{(i)}, \dots, \nu_{k-1}^{(i)}$.

We parametrize holomorphic discrete series representations as follows. Any holomorphic discrete series representation Π of G is determined by its unique minimal K -type. We denote Π by $\Pi^G(\lambda)$ if $\lambda \in \mathfrak{j}_{\mathbb{C}}^*$ is the highest weight of the minimal K -type of Π . Similarly, holomorphic discrete series representations of the subgroup $G' = G_0^{\sigma}$ are expressed as $\pi^{G'}(\mu) \in \widehat{G'}$ if $\mu \in (\mathfrak{j}_{\mathbb{C}}^{\sigma})^*$ is the highest weight of the minimal K' -type with respect to $\Delta^+(\mathfrak{k}_{\mathbb{C}}^{\sigma}, \mathfrak{j}_{\mathbb{C}}^{\sigma})$, where $K' = K \cap G' = K_0^{\sigma}$.

Theorem 7.1 (Hua–Kostant–Schmid–Kobayashi). *Let (G, G') be a symmetric pair of holomorphic type. For any holomorphic discrete series representation $\Pi^G(\lambda)$ of scalar type, the following multiplicity-free decomposition holds.*

$$(7.1) \quad \Pi^G(\lambda)|_{G'} \simeq \bigoplus_{i=1}^N \sum_{a_1^{(i)} \geq \dots \geq a_{k_i}^{(i)} \geq 0} \pi^{G'}(\lambda|_{\mathfrak{j}_{\mathbb{C}}^{\sigma}} - \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)}) \quad (\text{Hilbert direct sum}).$$

Remark 7.2. If G' is a maximal compact subgroup of G , then $\mathfrak{g}_{\mathbb{C}}^{(0)} = \{0\}$, $N = 1$, $\mathfrak{g}^{\sigma\theta} = \mathfrak{g}$, and any irreducible summand $\pi^{G'}(\mu)$ is finite-dimensional. In this case, the formula (7.1) is due to Hua (classical groups), Kostant (unpublished), and Schmid [114]. The proof for the branching law (7.1) in the general setting where G' is non-compact can be found in [61, Thm. 8.3].

Remark 7.3. The “geometric quantization” in Section 2.4 commutes with the reduction in this case, and the “classical limit” of the branching law (7.1) is given by the Corwin–Greenleaf function for coadjoint orbits [84, 85, 108].

Yet another important multiplicity-free results in the branching laws are for more special pairs (G, G') , but for more general representations. Typical cases are those which we have seen in Section 6.2, that is, the real forms (G, G') of $(GL(n, \mathbb{C}), GL(n-1, \mathbb{C}))$ or $(SO(n, \mathbb{C}), SO(n-1, \mathbb{C}))$, such as $(GL(n, \mathbb{R}), GL(n-1, \mathbb{R}))$ or $(SO(p, q), SO(p-1, q))$, have the property that the multiplicity $m(\Pi, \pi) = \dim_{\mathbb{C}} \text{Hom}_{G'}(\Pi|_{G'}, \pi)$ is either 0 or 1 for any $\Pi \in \text{Irr}(G)$ and $\pi \in \text{Irr}(G')$ even when Π is neither a unitarizable representation nor a highest weight representation (Example 6.10). For these pairs, when Π and π are both tempered representations, the description of $m(\Pi, \pi)$ is predicted by the (local) Gan–Gross–Prasad conjecture [27, 28]. In Section 8.3, we shall consider the case in which some aspect of this conjecture (and more generally, non-tempered cases) is connected with a particular problem in conformal geometry.

7.2. Discretely decomposable branching laws. When the restriction is discretely decomposable, it is expected that algebraic approaches would be useful in finding explicit branching laws. The first general theory of discretely decomposable restriction was established in 1990s in [49, 53, 54], see e.g. Theorem 6.5 for the criterion of the triples (G, G', Π) with discretely decomposable restriction $\Pi|_{G'}$. In parallel, there have been various attempts to find concrete branching laws in the framework of discretely decomposable restriction. Among them, especially important is the case that Π is the geometric quantization of an elliptic orbit (see Section 2.4), namely, the case that the (\mathfrak{g}, K) -module Π_K is a Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda)$, where \mathfrak{q} is a θ -stable parabolic subalgebra, see Knapp–Vogan

[44] for instance, for the definition of $A_q(\lambda)$. The first explicit branching laws for $\Pi_K = A_q(\lambda)$ in the general setting where Π_K do not have highest weights were given for any of the adjacent pairs (G, G') in the following diagram (Kobayashi [48, 49]).

$$\begin{array}{ccc} O(4p, 4q) & \supset & O(4k) \times O(4p - 4k, 4q) \\ \cup & & \cup \\ U(2p, 2q) & \supset & U(2k) \times U(2p - 2k, 2q) \\ \cup & & \cup \\ Sp(p, q) & \supset & Sp(k) \times Sp(p - k, q) \end{array}$$

The method in obtaining the branching laws [48, 49] was to use a structural theorem of the ring of invariant differential operators on homogeneous spaces X with “overgroups” and to realize $A_q(\lambda)$ in the space of functions on X , see [64] for a related geometric problem and [39, 76] for the method in full generality. Ever since [48, 49], various methods have been developed for finding explicit formulas of discretely decomposable branching laws (Stage B) in the other settings by Gross–Wallach [28], Loke, J.-S. Li, Huang–Pandžić–Savin, Ørsted–Speh [105], Duflo–Vargas [21], Sekiguchi [115], Y. Oshima [107], Kobayashi [61, 76], and so on. Among them, Duflo–Vargas [21] develops the idea of the orbit method and symplectic geometry when Π is a discrete series representation, and Y. Oshima [107] makes use of the theory of \mathcal{D} -modules for Zuckerman derived functor modules $\Pi_K = A_q(\lambda)$.

8. PROGRAM FOR THE THEORY OF BRANCHING LAWS: STAGE C

In Stage C, we consider not only abstract branching laws (the decomposition of representations) but also how they decompose (the decomposition of vectors). For the latter purpose, a crucial step is to construct G' -intertwining operators from Π to π (**symmetry breaking operators**) in a geometric model of irreducible representations Π and π of a group G and its subgroup G' , respectively. One can also consider G' -intertwining operators in the opposite direction, namely, from π to Π (**holographic operators** [78, 96]) as a dual notion, but we do not discuss holographic operators in this article. Let us start with an elementary example.

8.1. The regular representation of \mathbb{R} and Fourier transform. By using the regular representation $L^2(\mathbb{R})$ of the additive group \mathbb{R} , we illustrate Stages A, B, and C in Section 5.

Stage A. The irreducible decomposition of the regular representation $L^2(\mathbb{R})$ is multiplicity-free and has only continuous spectrum.

Stage B. The regular representation $L^2(\mathbb{R})$ is decomposed into the direct integral (Theorem 2.2) of one-dimensional Hilbert spaces $\mathbb{C}e^{ix\xi}$.

Stage C. The irreducible decomposition of $L^2(\mathbb{R})$ is realized by the Fourier transform (4.1) concretely.

Apparently, this example on the classical harmonic analysis looks nothing to do with branching laws, however, one may interpret it as an example of restriction problems. For example, let Π be a unitary principal series representation of $G = SL(2, \mathbb{R})$, and $N(\simeq \mathbb{R})$ a maximal unipotent subgroup. Then the restriction $\Pi|_N$ is unitarily equivalent to the regular representation $L^2(\mathbb{R})$ of the additive group \mathbb{R}

(see, for instance, [59, Prop. 3.3.2]). Thus the above case may be interpreted as Stages A–C for the restriction problem of $SL(2, \mathbb{R}) \downarrow \mathbb{R}$. Needless to say, since we already know the theory of the Fourier transform on $L^2(\mathbb{R})$ which corresponds to Stage C for this particular example $\Pi|_N$, Stages A and B stay in the background.

8.2. Tensor product representation of $SL(2, \mathbb{R})$. Next, we illustrate Stages A–C of branching laws by another example where the groups are “highly non-commutative” this time rather than abelian subgroups as in the previous case.

Let $\mathcal{O}(\mathcal{H})$ denote the space of holomorphic functions on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$. Then, there are a family of linear actions of the group $G = SL(2, \mathbb{R})$ on $\mathcal{O}(\mathcal{H})$ with integer parameters $\lambda \in \mathbb{Z}$ as follows.

$$(\pi_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} f)(z) := (-cz + a)^{-\lambda} f\left(\frac{dz - b}{-cz + a}\right).$$

Moreover, if $\lambda > 1$, then $V_\lambda := \mathcal{O}(\mathcal{H}) \cap L^2(\mathcal{H}, y^{\lambda-2} dx dy)$ is an infinite-dimensional Hilbert space and the action π_λ on V_λ defines an irreducible and unitary representation of $G = SL(2, \mathbb{R})$, referred to as the holomorphic discrete series representation. We consider the tensor product $\pi_{\lambda'} \otimes \pi_{\lambda''}$ of two such representations, which may be interpreted as an example of the restriction from the direct product group $G \times G$ to the diagonal subgroup $\text{diag}(G) \simeq G$.

- In Stage A, we have the following a priori estimate on “abstract features” of the branching law:

For $\lambda', \lambda'' > 1$, the tensor product representation $\pi_{\lambda'} \otimes \pi_{\lambda''}$ decomposes discretely and multiplicity-freely into irreducible unitary representations of G . Here the symbol for the tensor product \otimes of two Hilbert spaces means taking the Hilbert completion of the algebraic tensor product. (Each property “discrete decomposability” and “multiplicity-freeness” of the tensor product representation $\pi_{\lambda'} \otimes \pi_{\lambda''}$ is a special case of the general results the restriction in Stage A as we have seen in Theorems 6.5 and 6.14, respectively.)

- In Stage B, we determine a concrete branching law:

If $\lambda', \lambda'' > 1$, then one has the following irreducible decomposition

$$(8.1) \quad \pi_{\lambda'} \otimes \pi_{\lambda''} \simeq \sum_{a \in \mathbb{N}}^{\oplus} \pi_{\lambda' + \lambda'' + 2a} \quad (\text{Hilbert direct sum}).$$

The formula (8.1) in the $SL(2, \mathbb{R})$ case is due to Molchanov [104] and Repka [111]. Theorem 7.1 is a generalization of the formula (8.1) to arbitrary semisimple symmetric pairs of holomorphic type.

- In Stage C, we construct **symmetry breaking operators**.

For given $\lambda', \lambda'', \lambda''' \in \mathbb{Z}$, a linear map $R: \mathcal{O}(\mathcal{H}) \otimes \mathcal{O}(\mathcal{H}) \rightarrow \mathcal{O}(\mathcal{H})$ satisfying

$$(8.2) \quad R(\pi_{\lambda'}(g)f_1 \otimes \pi_{\lambda''}(g)f_2) = \pi_{\lambda'''}(g)R(f_1 \otimes f_2) \quad (\forall g \in G)$$

is a symmetry breaking operator from the tensor product representation $\pi_{\lambda'} \otimes \pi_{\lambda''}$ to $\pi_{\lambda'''}$ with respect to the restriction $G \times G \downarrow G$ where $G = SL(2, \mathbb{R})$. The next theorem interprets the classical Rankin–Cohen bidifferential operator [16, 110], which was originally used to construct modular forms of higher weight from those of lower weight, as a symmetry breaking operator in branching problems of representation theory.

Theorem 8.1 (Rankin–Cohen bidifferential operator [16, 95, 110]). *Suppose that $\lambda''' - \lambda' - \lambda''$ is a non-negative even integer. If we set $2a := \lambda''' - \lambda' - \lambda''$ (a is a natural number), then the linear map $RC_{\lambda', \lambda''}^{\lambda'''} : \mathcal{O}(\mathcal{H}) \otimes \mathcal{O}(\mathcal{H}) \rightarrow \mathcal{O}(\mathcal{H})$ defined by*

$$(8.3) \quad RC_{\lambda', \lambda''}^{\lambda'''}(f_1 \otimes f_2)(z) := \sum_{\ell=0}^a \frac{(-1)^\ell \Gamma(\lambda' + a) \Gamma(\lambda'' + a)}{\ell!(a - \ell)! \Gamma(\lambda' + a - \ell) \Gamma(\lambda'' + \ell)} \frac{\partial^{a-\ell} f_1}{\partial z^{a-\ell}} \frac{\partial^\ell f_2}{\partial z^\ell}$$

satisfies (8.2), hence is a symmetry breaking operator from $\pi_{\lambda'} \otimes \pi_{\lambda''}$ to $\pi_{\lambda'''}$.

Remark 8.2. (1) It turns out that the coefficients appeared in the finite sum (8.3) coincide with those of a Jacobi polynomial. This fact can be checked if we know the formula (8.3), which is found, for example, by recurrence relations that reflect the intertwining property (8.2). But more intrinsically, the “F-method”, a method that the author and his collaborators introduced [68, 70, 94], reveals directly why the Jacobi polynomial shows up, see [95, Sect. 9] for example.

(2) Theorem 8.1 does not assert the uniqueness of the Rankin–Cohen bidifferential operators. In fact, it turns out quite recently that there exist symmetry breaking operators other than the Rankin–Cohen operator $RC_{\lambda', \lambda''}^{\lambda'''}$ for exceptional (negative) parameters $(\lambda', \lambda'', \lambda''')$, and we gave the complete classification in [95, Cor. 9.3] (2015). The main machinery of the proof is the “F-method”, which connects the following different topics:

- the dimension of the polynomial solutions to the hypergeometric differential equation,
- the determination of the composition series of the tensor product of two reducible Verma modules.

The name “F-method” originated from the fact that it utilizes the “algebraic Fourier transform of Verma modules” [68, 94]. This method is also applied to the construction of differential symmetry breaking operators for groups of higher rank [22, 81, 90, 95].

8.3. Classification theory of symmetry breaking operators in conformal geometry. We end this article with yet another example for interactions of the theory of branching laws with different fields of mathematics, this time with conformal geometry.

Consider the following problem: given a Riemannian manifold X and its submanifold Y , find “conformally covariant” operators from the space of functions on X to that on the submanifold Y . We may also consider a generalization of this problem, e.g. from “functions” to “differential forms” or from “Riemannian manifolds” to “pseudo-Riemannian manifolds”. To formulate the problem rigorously, we introduce the following notation.

$G := \text{Conf}(X)$: the group of conformal transformations of X ,

$G' := \text{Conf}(X; Y)$: the subgroup of G consisting of elements that preserve Y .

For any Riemannian manifold X , one can form a family of $\text{Conf}(X)$ -equivariant line bundles \mathcal{L}_λ ($\lambda \in \mathbb{C}$) over X , hence obtain a natural family of representations Π_λ of the conformal group G on the space $\Gamma(X, \mathcal{L}_\lambda)$ of smooth sections for the line bundle \mathcal{L}_λ . Since these line bundles are trivial topologically, one may realize the family Π_λ of representations of G as multiplier representations on the vector space $C^\infty(X)$ [87, Sect. 2]. Since the subgroup G' acts conformally on the submanifold Y equipped

with the induced metric, a similar family π_ν ($\nu \in \mathbb{C}$) of representations of G' can be defined on $C^\infty(Y)$ via the canonical group homomorphism $\text{Conf}(X; Y) \rightarrow \text{Conf}(Y)$. These representations are extended to the representations $\Pi_\lambda^{(i)}$ and $\pi_\nu^{(j)}$ on the spaces $\mathcal{E}^i(X)$ and $\mathcal{E}^j(Y)$ of differential forms, respectively.

Basic Problem 8.3 (Symmetry breaking operators in conformal geometry [22, 81, 99]). *Let X be a Riemannian manifold and Y a submanifold of X . For what parameters (i, j, λ, ν) , does a non-zero continuous operator $T: \mathcal{E}^i(X) \rightarrow \mathcal{E}^j(Y)$ satisfying*

$$\pi_\nu^{(j)}(h) \circ T = T \circ \Pi_\lambda^{(i)}(h) \quad \forall h \in \text{Conf}(X; Y)$$

exist? Further, find an explicit formula for such T .

For Basic Problem 8.3, if there exists a “universal construction” of such an operator T for any pair (X, Y) of Riemannian manifolds, then the “solution” must survive in the model space $(X, Y) = (S^n, S^{n-1})$, which has “large symmetry” in the sense that the dimension of the group $\text{Conf}(X; Y)$ attains its maximum among all pairs (X, Y) of Riemannian manifolds with $\dim Y = n - 1 (\geq 3)$. Recently, the construction and classification theory of conformally covariant symmetry breaking operators for the case $(X, Y) = (S^n, S^{n-1})$ have been rapidly developed as follows and completed in [99].

- ($i = j = 0$; T : differential operator) Juhl constructed all conformally covariant, differential symmetry breaking operators T in the flat model by determining the coefficients of T using recurrence relations (Book [36], 2009). Afterwards, a short proof on the construction and classification of T was given by a different approach (F-method) (Kobayashi–Ørsted–Somberg–Souček [90]).
- ($i = j = 0$; T : general) In general, there is much more possibility of having symmetry breaking operators if we allow integral operators or singular integral operators other than just differential operators [70]. For the scalar-valued case ($i = j = 0$), all the symmetry breaking operators (including integral operators and singular integral operators) were classified with explicit construction of the distribution kernels of operators T by Kobayashi–Speh (Book [97], 2015).
- (i, j : general ; T : differential operator) By enhancing the F-method to the matrix-valued case, the construction and classification of T in the matrix-valued case (i, j : general) were shown by Kobayashi–Kubo–Pevzner (Book [81], 2016). See also Fischmann–Juhl–Somberg (Book [22], 2020) which uses the F-method, too.
- (i, j : general; T : general) The classification was completed by Kobayashi–Speh (Book [99], 2018). For $j = i + 1$ or $i - 2$, all the symmetry breaking operators are differential operators [99, Thm. 3.6], whereas all the differential symmetry breaking operators for $j = i$, or $i - 1$ are obtained as the “residues” of integral symmetry breaking operators with meromorphic parameter [77].

In this way, Basic Problem 8.3 (construction and classification of symmetry breaking operators) for the model space $(X, Y) = (S^n, S^{n-1})$ was completely solved in [99] based on the results of [81] and [97]. The three books [81, 97, 99] over 600

pages long in total develop a general idea for the construction and the classification of symmetry breaking operators for a pair $G \supset G'$ of reductive Lie groups, and then apply the idea to the particular case $(G, G') = (O(n+1, 1), O(n, 1))$, which is also important in conformal geometry, giving a proof of the construction and classification of those operators. Functional equations of such operators yield some new results to the (local) Gross–Prasad conjecture in number theory and its generalization to the non-tempered case (Kobayashi–Speh [98]). It is not easy to explain the methods in [81, 97, 99] in a few lines, but we try to give its flavor here at the last part of this article.

The pair $(X, Y) = (S^n, S^{n-1})$ is regarded as a pair $(G/P, G'/P')$ of real flag varieties for the pair $(G, G') = (O(n+1, 1), O(n, 1))$ of real reductive Lie groups. We have chosen this specific pair (G, G') in the article [97], as the first test case of an explicit construction and complete classification of (non-local) symmetry breaking operators between principal series representations. The pair (G, G') satisfies the sphericity condition (ii) in Theorem 6.8, hence the uniform boundedness of multiplicity in the branching laws is a priori guaranteed (Stage A). In this sense, various results of [81, 97, 99] can be interpreted as a step forward to branching problems in Stages B and C in the general setting with uniformly bounded (in particular, finite) multiplicity property. With this in mind, we explain the ideas and methods for the proofs given in the three books in this general setting. First, it follows from Theorem 6.7 that $(G \times G')/\text{diag}(G')$ is real spherical; in particular, the number of G' -orbits on the real flag variety $(G \times G')/(P \times P')$ under the diagonal action is finite, where P and P' are minimal parabolic subgroups of G and G' , respectively. Second, the support of the distribution kernel for any symmetry breaking operator from a principal series representation of G to that of the subgroup G' is a closed $\text{diag}(G')$ -invariant subset of the real flag variety $(G \times G')/(P \times P')$. Thus there are only finitely many possibilities of the support. Third, we construct distribution kernels of symmetry breaking operators for each orbit and show meromorphic continuation and functional equations. The residue of the meromorphic family of symmetry breaking operators are symmetry breaking operators with distribution kernels of *smaller support*. Then we proceed by induction on the stratification for the closure relations in $\text{diag}(G') \backslash (G \times G')/(P \times P') \simeq P' \backslash G/P$. The first step of the induction is the symmetry breaking operators that can be described as “differential operators”, which correspond to the smallest orbit, namely, the unique closed orbit [94, Lem. 2.3]. Symmetry breaking operators that can be described as differential operators could appear not only as a “series” but also “sporadically” [95, 99]. The former one may be obtained as the “residues” of the meromorphic continuation of other operators such as integral symmetry operators (e.g. [77]), but the latter is more involved. By the F-method, the construction of all such operators can be reduced to a problem of determining some polynomials (“special polynomials”) that satisfy a system of differential equations. The classification for differential symmetry breaking operators is completed by solving such a system of equations [81, 90]. The support of other symmetry breaking operators is strictly larger than the closed orbit. Thus an inductive argument with respect to the closure relation of $\text{diag}(G')$ -orbits on the real flag variety $(G \times G')/(P \times P')$ exhausts all symmetry breaking operators, with the last one being “regular symmetry breaking operators” that are obtained by the analytic continuation of integral symmetry breaking operators. See [99, Chap. 3, Sect. 3] for further details.

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