Temperedness criterion of the tensor product of parabolic induction for GL_n

Yves Benoist, Yui Inoue and Toshiyuki Kobayashi

Abstract

We give a necessary and sufficient condition for a pair of parabolic subgroups P and Q of $G = GL_n(\mathbb{R})$ such that the tensor product of any two unitarily induced representations from P and Q are tempered.

Key words and phrases: tempered representation, reductive group, tensor product, unitary representation, degenerate principal series representation

1 Statement of main results

Let Π_1 and Π_2 be unitarily induced representations from parabolic subgroups P and Q of $G = GL_n$. In this paper we address the following:

Problem 1.1. When is the tensor product representation $\Pi_1 \otimes \Pi_2$ tempered?

Let us explain some background of this problem.

Problem 1.1 asks a coarse information of the spectrum of the tensor product $\Pi_1 \otimes \Pi_2$. We note that the disintegration of $\Pi_1 \otimes \Pi_2$ is far from being understood in general. For the very special case where P is a maximal parabolic subgroup and Q is its opposite parabolic subgroup, the tensor product $\mathrm{Ind}_P^G(\mathbf{1}) \otimes \mathrm{Ind}_Q^G(\mathbf{1})$ is unitarily equivalent to the regular representation for a reductive symmetric space, for which the Plancherel-type theorem is known up to some complicated vanishing condition of cohomologically induced representations with singular parameters which may affect an answer to Problem 1.1 ([3, Sect. 1] and references therein). Slightly more generally, when both P and Q are maximal parabolic subgroups which are not necessarily opposite to each other, Problem 1.1 was solved in [4, Prop. 5.9]. If P or Q is a Borel subgroup, then Problem 1.1 has an affirmative answer by the general theory (Remark 2.3). However, for the general P and Q, an explicit answer to Problem 1.1 is not known.

Tempered representations of a locally compact group G are unitary representations that are weakly contained in $L^2(G)$ (Definition 2.1). For real reductive Lie groups G, irreducible ones were classified by Knapp and Zuckerman [8] and play a role as building blocks both in Harish-Chandra's theory for the Plancherel formula of $L^2(G)$ and in Langlands' classification theory of irreducible admissible representations, whereas the Selberg's $1/4$ conjecture for congruence subgroups can be reformulated as the temperedness of certain unitary representations of $SL_2(\mathbb{R})$ and the Gan–Gross–Prasad conjecture is formulated as a branching problem for tempered representations. A complete description of pairs (G, H) of real reductive algebraic groups for which $L^2(G/H)$ is not tempered was accomplished in [5], but such a description is not known for non-reductive subgroups H except for a few cases [4, Cor. 5.8].

In this note, we give a solution to Problem 1.1. We shall prove that the solution depends only on the G-conjugacy classes of Levi parts of parabolic subgroups P and Q. Thus it is convenient to introduce the following notation: for a parabolic subgroup P of GL_n with the Levi subgroup $GL_{n_1} \times \cdots \times GL_{n_r}$ $(n_1 + \cdots + n_r = n)$, we set

$$d(P) := \max_{1 \leq j \leq r} n_j.$$

Then $1 \leq d(P) \leq n$ with two extreme cases: $d(P) = 1 \iff P$ is a Borel subgroup, and $d(P) = n \iff P = G$. We prove:

Theorem 1.2. Let P and Q be parabolic subgroups of $G = GL_n(\mathbb{R})$. Then the following three conditions are equivalent:

(i) The tensor product representation $\mathrm{Ind}_P^G(\sigma) \otimes \mathrm{Ind}_Q^G(\tau)$ is tempered for all unitary representations σ of P and τ of Q.

(ii) The tensor product representation $\mathrm{Ind}_P^G(\mathbf{1}) \otimes \mathrm{Ind}_Q^G(\mathbf{1})$ is tempered.

(iii) $d(P) + d(Q) \leq n + 1$.

An analogous statement holds also for $G = GL_n(\mathbb{C})$.

Theorem 1.2 is derived from the following results about the regular representation on $L^2(G/H)$ where H is of maximal rank but is not necessarily reductive:

Theorem 1.3. Let H be a closed subgroup of $G = GL_n(\mathbb{R})$ with Lie algebra $\mathfrak{h}$. Assume that $\mathfrak{h}$ is stable by a split Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathfrak{a}^*$ such that $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$, and $\{E_1, \cdots, E_n\}$ the dual basis of $\mathfrak{a}$. Then the following three conditions are equivalent:

(i) $\operatorname{Ind}_H^G(\sigma)$ is tempered for any unitary representation σ of H.

(ii) $L^2(G/H)$ is tempered.

(iii) $\dim \operatorname{Image}(\operatorname{ad}(E_i) \colon \mathfrak{h} \to \mathfrak{h}) \leq n - 1$ for all i $(1 \leq i \leq n)$.

In general $0 \leq \dim \operatorname{Image}(\operatorname{ad}(E_i) \colon \mathfrak{h} \to \mathfrak{h}) \leq 2n - 1$ for any $\mathfrak{a}$-stable Lie algebra $\mathfrak{h}$ and any i $(1 \leq i \leq n)$. Theorem 1.3 justifies the "heuristic recipe" in [4, Rem. 5.7] for subgroups of three-by-three block matrix form.

Our proof relies on the temperedness criterion (Fact 2.5), which was established in [3, 4] by an analytic and dynamical approach in the general case. The criterion can be reduced to finitely many inequalities arising from the edges of convex polyhedral cones, actually 2^n inequalities in our setting. To solve Problem 1.1, we still need to analyze the 2^n inequalities. A number of combinatorial techniques were proposed in [4, 5], and among them, Theorem 1.3 was proved in the special setting where $\mathfrak{h}$ is a subgroup of three-by-three block matrix form ([4, Cor. 5.8]). The new technical ingredients here include yet another combinatorial trick which reduces them to very simple n inequalities (the third condition in Theorem 1.3). The same technique also yields an L^p-estimate of the matrix coefficients of the regular representation $L^2(G/H)$ when H is reductive, see Proposition 4.4.

This note is organized as follows. In Section 2, we review the Herz majoration principle and the temperedness criterion in a general setting. Section 3 provides a proof of Theorems 1.2 and 1.3, postponing a combinatorial proof of Lemma 3.2 until Section 4. In Section 5, we discuss Problem 1.1 for any simple groups under the assumption that Q is the opposite parabolic subgroup of P.

2 Preliminaries

In this section we fix some notations and recall the previous results on unitary representations that will be needed later.

2.1 Regular representations

For an m-dimensional manifold X, we denote by $\mathcal{L}_{vol} \equiv \mathcal{L}_{X,vol} := |\bigwedge^m(T^*X)|$ the density bundle of X, and by $L^2(X)$ the Hilbert space of square integrable sections for the half-density bundle $\mathcal{L}_{vol}^{1/2}$. Suppose a Lie group G acts continuously on X. Then G acts equivariantly on the half-density bundle $\mathcal{L}_{vol}^{1/2}$, and one has naturally a unitary representation λ_X of G on $L^2(X)$, referred to as the *regular representation*. Associated to a unitary representation (σ, W) of a closed subgroup H of G, the unitary induction $\mathrm{Ind}_H^G(W)$ is defined as a unitary representation of G on the Hilbert space of square integrable sections for the G-equivariant Hilbert bundle $(G \times_H W) \otimes \mathcal{L}_{vol}^{1/2}$ over the homogeneous space G/H. By definition, $\mathrm{Ind}_H^G(\mathbf{1})$ is the regular representation $\lambda_{G/H}$ on $L^2(G/H)$, where $\mathbf{1}$ denotes the trivial one-dimensional representation of H.

2.2 Tempered representations

Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be unitary representations of a locally compact group G. We say π is *weakly contained* in π', to be denoted by $\pi \prec \pi'$ if for every $v \in \mathcal{H}$ the matrix coefficient $(\pi(g)v, v)$ can be approximated uniformly on compact subsets of G by a sequence of finite sums of functions $(\pi'(g)u_j, u_j)$ with $u_1, \cdots, u_k \in \mathcal{H}'$.

Definition 2.1. A unitary representation π of G is called *tempered* if π is weakly contained in the (left) regular representation λ_G on $L^2(G)$.

When G is a semisimple Lie group, π is tempered if and only if π is almost L^2 ([7]). Here we recall:

Definition 2.2. Let $p \geq 2$. A unitary representation $(\pi, \mathcal{H})$ of G is said to be *almost* L^p if there exists a dense subset D of $\mathcal{H}$ for which the matrix coefficients $g \mapsto (\pi(g)u, v)$ are in $L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$ and all $u, v \in D$.

Remark 2.3. Temperedness is closed under induction and restrictions of unitary representations. Moreover, if π is tempered, then the tensor product representation $\pi \otimes \sigma$ is tempered for any unitary representation σ of G. In fact, if $\pi \prec \lambda_G$, then $\pi \otimes \sigma \prec \lambda_G \otimes \sigma$. Since $\lambda_G \otimes \sigma$ is a multiple of λ_G ([2, Cor. E.2.6]), one concludes $\pi \otimes \sigma \prec \lambda_G$.

We recall a classical lemma called "Herz majoration principle", see [1, Sect. 6]:

Lemma 2.4 ([4, Lem. 3.2])**.** Let G be a linear semisimple Lie group, and H a closed subgroup of G. If the regular representation $\lambda_{G/H}$ is tempered, then the induced representation $\mathrm{Ind}_H^G(\sigma)$ is tempered for any unitary representation σ of H.

2.3 Temperedness criterion for $L^2(G/H)$

Let $\mathfrak{a}$ be a maximal split abelian subalgebra in an (algebraic) Lie algebra $\mathfrak{h}$. Such $\mathfrak{a}$ is unique up to conjugation, and we denote by $\mathrm{rank}_{\mathbb{R}}\,\mathfrak{h}$ its dimension.

Let V be a finite-dimensional representation of $\mathfrak{h}$. Following [3, 4], we define a non-negative function ρ_V on $\mathfrak{a}$ by

$$\rho_V(Y) := \frac{1}{2} \sum_{\lambda \in \Delta(V,\mathfrak{a})} m_\lambda |\lambda(Y)| \quad \text{for } Y \in \mathfrak{a},$$

where $\Delta(V, \mathfrak{a})$ is the set of weights of $\mathfrak{a}$ in V and m_λ denotes the dimension of the corresponding weight space V_λ. The function ρ_V is continuous and is piecewise linear i.e. there exist finitely many convex polyhedral cones which covers $\mathfrak{a}$ and on which ρ_V is linear, see [3, Sect. 4.7]. We set

$$p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}. \tag{2.1}$$

Fact 2.5. Let G be a linear semisimple Lie group with finite center, and H an algebraic subgroup.

(1) ([4, Thm. 2.9]) One has the equivalence:

$$L^2(G/H) \text{ is tempered} \iff 2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}} \text{ on } \mathfrak{a}.$$

(2) ([3, Thm. 4.1]) Let p be a positive even integer. If H is reductive, one has the equivalence:

$$L^2(G/H) \text{ is almost } L^p \iff p_{\mathfrak{g}/\mathfrak{h}} \le p - 1.$$

The inequality in Fact 2.5 can be checked only at finitely many points in $\mathfrak{a}$, namely, at the generators of the edges of the convex polyhedral cones, as we shall see in Lemma 3.1 below in the setting we need.

3 Proof of Theorems 1.2 and 1.3

In this section, we show the main results by using the temperedness criterion (Fact 2.5) and some combinatorial lemmas. We postpone the proof of Lemma 3.2 until Section 4.

Suppose $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{h}$ is an $\mathfrak{a}$-invariant subalgebra as in the setting of Theorem 1.3. Since split Cartan subalgebras $\mathfrak{a}$ are conjugate to each other by inner automorphisms, we may and do assume $\mathfrak{a} = \bigoplus_{i=1}^n \mathbb{R}E_{ii}$, where E_{ij} denotes the matrix unit.

For $1 \leq i, j \leq n$, we set

$$\varepsilon_{ij} \equiv \varepsilon_{ij}(\mathfrak{h}) := \dim_{\mathbb{R}}(\mathfrak{h} \cap \mathbb{R}E_{ij}) \in \{0, 1\}. \tag{3.1}$$

By the weight decomposition of $\mathfrak{h}$ with respect to $\mathfrak{a}$, one sees

$$\dim \operatorname{Image}(\operatorname{ad}(E_{ii})\colon \mathfrak{h} \to \mathfrak{h}) = \sum_{j \in \{1,\dots,n\} \setminus \{i\}} (\varepsilon_{ij} + \varepsilon_{ji}) = 2\rho_{\mathfrak{h}}(E_{ii}). \tag{3.2}$$

Since $\rho_{\mathfrak{g}}(E_{ii}) = n - 1$, the condition (iii) in Theorem 1.3 amounts to

$$2\rho_{\mathfrak{h}}(E_{ii}) \leq \rho_{\mathfrak{g}}(E_{ii}) \quad \text{for all } i \ (1 \leq i \leq n).$$

3.1 Reduction to finite inequalities

The temperedness criterion (Fact 2.5) is given by the inequality on $\mathfrak{a}$, which reduces to a finite number of inequalities on the generators of convex polyhedral cones. This is Lemma 3.1 below which reduces to 2^n inequalities. A further combinatorial argument reduces to n inequalities (Lemma 3.2).

For a non-empty subset $I \subset \{1, \dots, n\}$, we set $E_I := \sum_{i \in I} E_{ii}$. Then $E_I = E_{ii}$ if $I = \{i\}$; E_I generates the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ if $I = \{1, 2, \dots, n\}$.

Lemma 3.1. The condition (ii) in Theorem 1.3 is equivalent to

$$2\rho_{\mathfrak{h}}(E_I) \leq \rho_{\mathfrak{g}}(E_I) \quad \text{for all } I \subset \{1, \dots, n\}. \tag{3.3}$$

Proof. By the temperedness criterion (Fact 2.5), the condition (ii) in Theorem 1.3 is given by $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ on $\mathfrak{a}/\mathfrak{z}(\mathfrak{g})$. Thus it suffices to show

$$\max_{0 \neq Y \in \mathfrak{a}/\mathfrak{z}(\mathfrak{g})} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_{\mathfrak{g}}(Y)} = \max_{I \subsetneq \{1,\dots,n\}} \frac{\rho_{\mathfrak{h}}(E_I)}{\rho_{\mathfrak{g}}(E_I)}. \tag{3.4}$$

To see the non-trivial inequality $\leq$, we begin with the dominant chamber $\overline{\mathfrak{a}_+} = \{\mathrm{diag}(x_1, \dots, x_n) \colon x_1 \geq \cdots \geq x_n\}$. Since both $\rho_{\mathfrak{h}}$ and $\rho_{\mathfrak{g}}$ are linear on $\overline{\mathfrak{a}_+}$, the restriction of the function $\rho_{\mathfrak{h}}/\rho_{\mathfrak{g}}$ to the line segment $tY + (1-t)Z$ $(Y, Z \in \overline{\mathfrak{a}_+} \setminus \mathfrak{z}(\mathfrak{g}))$ is a linear fractional function of t $(0 \leq t \leq 1)$, which attains its maximum either at $t = 0$ or $t = 1$. An iteration of the argument tells that the maximum of $\rho_{\mathfrak{h}}/\rho_{\mathfrak{g}}$ on $(\overline{\mathfrak{a}_+}/\mathfrak{z}(\mathfrak{g})) \setminus \{0\}$ is attained at one of the edges of the convex polyhedral cone $\overline{\mathfrak{a}_+}/\mathfrak{z}(\mathfrak{g})$, namely, at $\mathbb{R}_+ E_I$ for some $I = \{1, 2, \dots, k\}$ with $1 \leq k \leq n-1$.

Similar argument applies to the other Weyl chambers. $\square$

The following lemma tells that it suffices to use E_I with $\#I = 1$ for "witness vectors" ([5]) in our setting, and will be proved in Section 4.

Lemma 3.2. If $2\rho_{\mathfrak{h}}(E_{ii}) \leq \rho_{\mathfrak{g}}(E_{ii})$ for all i $(1 \leq i \leq n)$, then (3.3) holds.

3.2 Proof of Theorem 1.3

The equivalence (i) $\iff$ (ii) in Theorem 1.3 follows from the Herz majoration principle (Lemma 2.4). Let us verify the equivalence (ii) $\iff$ (iii). We may and do assume that $\mathfrak{h}$ contains $\mathfrak{a} = \sum_{i=1}^n \mathbb{R} E_{ii}$. In fact, if $\mathfrak{h}$ is stable by $\mathfrak{a}$, then $\tilde{\mathfrak{h}} := \mathfrak{h} + \mathfrak{a}$ is a Lie subalgebra containing $\mathfrak{a}$. We write $\tilde{H}$ for the connected subgroup of G with Lie algebra $\tilde{\mathfrak{h}}$. Then $L^2(G/H)$ is tempered if and only if $L^2(G/\tilde{H})$ is tempered by [4, Cor. 3.3]. Moreover, $\mathrm{Image}(\mathrm{ad}(E_{ii}) \colon \mathfrak{h} \to \mathfrak{h})$ remains the same if we replace $\mathfrak{h}$ with $\tilde{\mathfrak{h}}$, hence the conditions (ii) and (iii) in Theorem 1.3 are unchanged. Now one has the equivalences:

$$\begin{aligned}
\text{(ii)} &\iff 2\rho_{\mathfrak{h}}(Y) \leq \rho_{\mathfrak{g}}(Y) \quad ({}^\forall Y \in \mathfrak{a}) && \text{by Fact 2.5} \\
&\iff 2\rho_{\mathfrak{h}}(E_I) \leq \rho_{\mathfrak{g}}(E_I) \quad ({}^\forall I \subset \{1, \dots, n\}) && \text{by Lemma 3.1} \\
&\iff 2\rho_{\mathfrak{h}}(E_{ii}) \leq \rho_{\mathfrak{g}}(E_{ii}) \quad (1 \leq {}^\forall i \leq n) && \text{by Lemma 3.2,}
\end{aligned}$$

which is equivalent to (iii). Thus Theorem 1.3 is proved.

3.3 Proof of Theorem 1.2

Without loss of generality, we may and do assume that P and Q are standard parabolic subgroups with Levi subgroups $GL_{n_1} \times \cdots \times GL_{n_r}$ and $GL_{m_1} \times \cdots \times GL_{m_s}$, respectively. Let $w := \sum_{i=1}^n E_{i\ n+1-i} \in G$, a representative of the longest element of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$. Then $Q^o := w^{-1}Qw$ is a parabolic subgroup of G with Levi subgroup $GL_{m_s} \times \cdots \times GL_{m_1}$, and PQ^o is open dense in G, hence the diagonal map $G \to G \times G$, $g \mapsto (g, g)$ induces an open dense embedding $\iota\colon G/H \hookrightarrow G/P \times G/Q^o$, where $H := P \cap Q^o$. Thus the tensor product representation $\mathrm{Ind}_P^G(\mathbf{1}) \otimes \mathrm{Ind}_Q^G(\mathbf{1}) \simeq \mathrm{Ind}_P^G(\mathbf{1}) \otimes \mathrm{Ind}_{Q^o}^G(\mathbf{1})$ is unitarily equivalent to $L^2(G/H)$ via the G-isomorphism of the equivariant line bundles $\iota^*(\mathcal{L}_{G/P,vol} \otimes \mathcal{L}_{G/Q^o,vol}) \simeq \mathcal{L}_{G/H,vol}$.

We define integers $N(a)$ $(0 \le a \le r)$ and $M(b)$ $(0 \le b \le s)$ by

$$N(a) := \sum_{j=1}^{a} n_j \quad (1 \le a \le r), \;\; M(b) := \sum_{j=1}^{b} m_{s+1-j} \quad (1 \le b \le s),$$

and set $N(0) = M(0) = 0$. We note $N(r) = M(s) = n$. By definition, for each $1 \le i \le n$, there exist uniquely $a(i) \in \{1, \ldots, r\}$ and $b(i) \in \{1, \ldots, s\}$ such that

$$N(a(i) - 1) < i \le N(a(i)) \text{ and } M(b(i) - 1) < i \le M(b(i)). \tag{3.5}$$

By definition, one has for $1 \le i, j \le n$,

$$\begin{aligned} E_{ij} \in \mathfrak{p} &\iff N(a(i) - 1) < j, \qquad & E_{ij} \in \mathfrak{q}^o &\iff j \le M(b(i)), \\ E_{ji} \in \mathfrak{p} &\iff j \le N(a(i)), & E_{ji} \in \mathfrak{q}^o &\iff M(b(i) - 1) < j. \end{aligned}$$

Since the Lie algebra $\mathfrak{h}$ of H is equal to $\mathfrak{p} \cap \mathfrak{q}^o$, (3.2) shows

$$\begin{aligned} 2\rho_{\mathfrak{h}}(E_{ii}) =& (M(b(i)) - N(a(i) - 1) - 1) + (N(a(i)) - M(b(i) - 1) - 1) \\ =& n_{a(i)} + m_{s-b(i)+1} - 2. \end{aligned}$$

Since $\mathfrak{h}$ contains $\mathfrak{a}$, we can apply Theorem 1.3, and conclude that $L^2(G/H)$ is tempered if and only if

$$n_{a(i)} + m_{s-b(i)+1} \le n + 1 \quad \text{for all } i \ (1 \le i \le n). \tag{3.6}$$

We claim (3.6) holds if and only if

$$d(P) + d(Q) \le n + 1. \tag{3.7}$$

The implication (3.7) $\Rightarrow$ (3.6) is obvious. To see the converse implication, we take $a \in \{1, \ldots, r\}$ and $b \in \{1, \ldots, s\}$ such that $n_a = d(P)$ and $m_{s+1-b} = d(Q)$. Then the subsets $\{N(a-1)+1, \ldots, N(a)\}$ and $\{M(b-1)+1, \ldots, M(b)\}$ of $\{1, 2, \ldots, n\}$ have $d(P)$ and $d(Q)$ elements, respectively. If (3.7) fails, then one finds a common element, say i. By (3.5), $a = a(i)$ and $b = b(i)$, hence (3.6) fails. Thus Theorem 1.3 is proved.

4 Proof of Lemma 3.2

In this section, we show Lemma 3.2, hence complete the proof of Theorems 1.2 and 1.3. Actually, we prove a generalization of Lemma 3.2 (see Lemma 4.1 below) which will be used also in an L^p estimate of matrix coefficients (Proposition 4.4).

4.1 Reduction to quadratic inequalities

We recall that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ containing the Lie algebra $\mathfrak{a}$ of diagonal matrices. We also recall the notation $E_I = \sum_{i \in I} E_{ii} \in \mathfrak{a}$ for a subset I of $\{1, \ldots, n\}$. We prove the following.

Lemma 4.1. Suppose p is an even integer ≥ 2. Then the inequality

$$p\rho_{\mathfrak{h}}(E_I) \leq (p-1)\rho_{\mathfrak{g}}(E_I)$$

is true for all subsets I as soon as it is true when I is a singleton.

Remark 4.2. An analogous statement to Lemma 4.1 fails for $p = 3$, for instance, when $n = 4$ and $\mathfrak{h}$ is a maximal parabolic subalgebra of dimension 12.

First note that there exists a direct sum decomposition $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_r$ such that the Lie algebra $\mathfrak{l} := \mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_r)$ is a maximal reductive subalgebra of $\mathfrak{h}$ containing $\mathfrak{a}$. We set

$$n_k := \dim V_k \text{ and } m_k \equiv m_k(I) := \#\{i \in I \mid e_i \in V_k\} \text{ so that}$$

$$n_1 + \cdots + n_r = n, \ m_1 + \cdots + m_r = \#I \text{ and } 0 \leq m_k \leq n_k, \text{ for all } k \leq r.$$

Similarly to (3.1), we set $\varepsilon_{k\ell} := 1$ if $V_\ell^* \otimes V_k \subset \mathfrak{h}$, and $\varepsilon_{k\ell} := 0$ otherwise. By the maximality of ℓ, one has $\varepsilon_{kk} = 1$ $(1 \leq k \leq r)$ and $\varepsilon_{k\ell} + \varepsilon_{\ell k} \in \{0, 1\}$. We compute

$$\rho_{\mathfrak{g}}(E_I) = \sum_{1 \leq k, \ell \leq r} m_k(n_\ell - m_\ell),$$

$$2\rho_{\mathfrak{h}}(E_I) = \sum_{1 \leq k, \ell \leq r} \varepsilon_{k\ell}(m_k(n_\ell - m_\ell) + m_\ell(n_k - m_k)) = \sum_{1 \leq k, \ell \leq r} b_{k\ell} m_k(n_\ell - m_\ell),$$

where $b_{kk} = 2$ and $b_{k\ell} = \varepsilon_{k\ell} + \varepsilon_{\ell k}$ $(k \neq \ell)$. Hence, setting $a_{kk} = 1$ and $a_{k\ell} = 1 + \frac{p}{2}(\varepsilon_{k\ell} + \varepsilon_{\ell k} - 2)$, one has

$$p\rho_{\mathfrak{h}}(E_I) - (p-1)\rho_{\mathfrak{g}}(E_I) = \sum_{1 \leq k,\ell \leq r} a_{k\ell} m_k (n_\ell - m_\ell).$$

Since $\varepsilon_{k\ell} + \varepsilon_{\ell k} \in \{0, 1\}$, $a_{k\ell} \in \{1-p, 1-\frac{p}{2}\}$ for all $k \neq \ell$, in particular, $a_{k\ell}$ are non-negative integers when p is even. Hence Lemma 4.1 follows from Lemma 4.3 below.

4.2 Quadratic inequalities

This section is independent of the previous one. We forget about Lie algebras. We fix integers $r \geq 1$, $n_1, \ldots, n_r \geq 1$ and $(a_{k\ell})_{1 \leq k,\ell \leq r}$ a symmetric matrix with integer coefficients which are equal to 1 on the diagonal and are non-positive outside the diagonal:

$$a_{k\ell} = a_{\ell k} \in -\mathbb{N} \text{ for all } k \neq \ell \text{ and } a_{\ell\ell} = 1 \text{ for all } \ell.$$

Here, we used the notation $\mathbb{N} = \{0, 1, 2, \ldots\}$. We denote by $\mathbf{e}_\ell \in \mathbb{N}^r$ the r-tuple $\mathbf{e}_\ell = (\delta_{k,\ell})_{1 \leq k \leq r}$. We fix $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, and set

$$f(\mathbf{m}) = \sum_{1 \leq k,\ell \leq r} a_{k\ell} m_k (n_\ell - m_\ell) \text{ for } \mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r.$$

For instance, one has $f(\mathbf{e}_{\ell_0}) = n_{\ell_0} - 1 + \sum_{\ell \neq \ell_0} a_{\ell_0 \ell} n_\ell$.

Lemma 4.3. Assume that $f(\mathbf{e}_\ell) \leq 0$ for all $1 \leq \ell \leq r$. Then one has $f(\mathbf{m}) \leq 0$ for all $\mathbf{m}$ in $\mathbb{N}^r$ with $\mathbf{n} - \mathbf{m} \in \mathbb{N}^r$.

Proof. We argue by induction on $s := \sum_k m_k$. Our assumption tells us that the conclusion is true for $s \leq 1$. We assume $s \geq 2$ and distinguish two cases.

Case 1 : there exists $1 \leq \ell \leq r$ such that $\sum_k a_{k\ell} m_k \geq 1$.

In this case, we fix such an ℓ. Since $a_{k\ell} \leq 0$ for all $k \neq \ell$ and $a_{\ell\ell} > 0$, we can write $\mathbf{m} = \mathbf{m}' + \mathbf{e}_\ell$ with $\mathbf{m}' \in \mathbb{N}^r$. Since $a_{\ell\ell} = 1$ and $a_{\ell k} = a_{k\ell}$, one has

$$f(\mathbf{m}) = f(\mathbf{m}') + f(\mathbf{e}_\ell) + 2 - 2\sum_k a_{k\ell} m_k.$$

Using our assumptions and the induction hypothesis, we get $f(\mathbf{m}) \leq 0$.

Case2 : For all $1 \leq \ell \leq r$, one has $\sum_k a_{k\ell} m_k \leq 0$.

In this case, since $n_\ell - m_\ell \geq 0$ for all ℓ, the inequality $f(\mathbf{m}) \leq 0$ follows directly from the definition of $f(\mathbf{m})$.

Since the coefficients $a_{k\ell}$ are integers, these two cases are the only possibilities and this ends the proof of Lemma 4.3 and hence of Lemma 4.1. $\square$

4.3 L^p-estimate of matrix coefficients

When H is reductive, Lemma 4.1 determines an explicit bound of p such that $L^2(G/H)$ is almost L^p. We end this section with the following:

Proposition 4.4. Let $n_1 + \cdots + n_r \leq n$ and $p \in 2\mathbb{N}$. We set $m := \max(n_1, \ldots, n_r)$. Then one has the equivalence:

(i) $L^2(GL_n(\mathbb{R})/(GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_r}(\mathbb{R})))$ is almost L^p.

(ii) $m \leq n - \frac{n-1}{p}$.

The case $p = 2$ was proved in [5, Thms. 1.4 and 3.1].

Proof. For $\mathfrak{h} = \mathfrak{gl}_{n_1}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{gl}_{n_r}(\mathbb{R})$, we set

$$c \equiv c(\mathfrak{h}) := \min_{1 \leq i \leq n} \frac{\rho_{\mathfrak{g}}(E_{ii})}{\rho_{\mathfrak{h}}(E_{ii})} = \frac{2(n-1)}{\max_{1 \leq i \leq n} \dim \operatorname{Image}(\operatorname{ad}(E_{ii}) \colon \mathfrak{h} \to \mathfrak{h})}. \tag{4.1}$$

By definition, $c(\mathfrak{h}) = \frac{n-1}{m-1}$, and therefore $p_{\mathfrak{g}/\mathfrak{h}} = \frac{1}{c(\mathfrak{h})-1} = \frac{m-1}{n-m}$ by (3.4) and Lemma 4.1. Then Proposition 4.4 follows from the criterion given in Fact 2.5 (2). $\square$

5 Appendix — the opposite parabolic case

Suppose Π_1 and Π_2 are unitarily induced representations from parabolic subgroups P and Q, respectively, of a real reductive Lie group G. So far we have discussed Problem 1.1 for general P and Q when $G = GL_n$. In this appendix, we discuss other reductive groups G under the assumption that Q is the opposite parabolic subgroup of P. In this case we can use the list of pairs (G, H) of real reductive algebraic groups for which $L^2(G/H)$ is non-tempered [5], and obtain the following classification:

Theorem 5.1. Let G be a non-compact real simple Lie group, P a proper parabolic subgroup, and Q the opposite parabolic. Let $\mathfrak{l}$ be the Lie algebra of a Levi subgroup L of P. Then the following three conditions on the pair (G, P) are equivalent:

(i) The tensor product representation $\mathrm{Ind}_P^G(\sigma) \otimes \mathrm{Ind}_Q^G(\tau)$ is tempered for all unitary representations σ of P and τ of Q.

(ii) The tensor product representation $\mathrm{Ind}_P^G(\mathbf{1}) \otimes \mathrm{Ind}_Q^G(\mathbf{1})$ is tempered.

(iii) One of the following conditions holds:

Case (a). P is any proper parabolic subgroup when $\mathrm{rank}_{\mathbb{R}}\, \mathfrak{g} = 1$.

Case (b). P is any proper parabolic subgroup when $\mathfrak{g} = \mathfrak{su}(p, q)$ $(p+q \leq 5)$, $\mathfrak{so}(p, q)$ $(p+q \leq 6)$, $\mathfrak{sp}(p, q)$ $(p+q \leq 4)$, $\mathfrak{e}_{6(2)}, \mathfrak{e}_{6(-14)}, \mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(4)}, \mathfrak{f}_{4,\mathbb{C}}, \mathfrak{g}_{2(2)}$, or $\mathfrak{g}_{2,\mathbb{C}}$.

Case (c). $\mathfrak{g}$ is complex simple or split. The semisimple part $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$ is not in the list of Table 1.

Case (d). $\mathfrak{g}$ is neither complex nor split. The semisimple part $[\mathfrak{l}, \mathfrak{l}]$ or its non-compact semisimple part $\mathfrak{l}_{ns}$ is not in the list of Table 2.

Table 1: $\mathfrak{g}$ is complex or split

$\mathfrak{g}$	$[\mathfrak{l}, \mathfrak{l}]$	
$\mathfrak{a}_n$	$\mathfrak{a}_{n_1} \oplus \cdots \oplus \mathfrak{a}_{n_k}$	$2 \max_{1 \leq j \leq k} n_j \geq n+1$
$\mathfrak{b}_n$	$\mathfrak{a}_{n_1} \oplus \cdots \oplus \mathfrak{a}_{n_k} \oplus \mathfrak{b}_m$	$2m \geq n+1$
$\mathfrak{c}_n$	$\mathfrak{a}_{n_1} \oplus \cdots \oplus \mathfrak{a}_{n_k} \oplus \mathfrak{c}_m$	$2m \geq n+1$
$\mathfrak{d}_n$	$\mathfrak{a}_{n_1} \oplus \cdots \oplus \mathfrak{a}_{n_k} \oplus \mathfrak{d}_m$	$2m \geq n+2$
$\mathfrak{d}_n$	$\mathfrak{a}_{n-1}$	$n \geq 3$
$\mathfrak{e}_6$	$\mathfrak{d}_5$	
$\mathfrak{e}_7$	$\mathfrak{d}_6$ or $\mathfrak{e}_6$	
$\mathfrak{e}_8$	$\mathfrak{e}_7$	

Table 2: $\mathfrak{g}$ is neither complex nor split

$\mathfrak{g}$	$\mathfrak{l}_{ns}$	
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p-k,q-k)$	$1 \leq k \leq \min\left(p-1, q-1, \frac{p+q-2}{4}\right)$
$\mathfrak{so}(p,q)$	$\mathfrak{so}(p-k,q-k)$	$1 \leq k \leq \min\left(p-1, q-1, \frac{p+q-3}{4}\right)$
$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(p-k,q-k)$	$1 \leq k \leq \min\left(p-1, q-1, \frac{p+q-1}{4}\right)$
$\mathfrak{g}$	$[\mathfrak{l},\mathfrak{l}]$	
$\mathfrak{su}^*(2n)$	$\bigoplus_{j=1}^{k} \mathfrak{su}^*(2m_j)$	$2\max_{1\leq j\leq k} m_j \geq n+2$
$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n)$	$n \geq 2$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2m) \oplus \bigoplus_{j=1}^{k} \mathfrak{su}^*(2m_j)$	$m \geq n+2$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}^*(12)$	
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(2,10)$ or $\mathfrak{e}_{6(-26)}$	
$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-25)}$	

Proof. The equivalence (i) $\iff$ (ii) follows from the Herz majoration principle (Lemma 2.4) as in Theorem 1.2. Let us prove the equivalence (ii) $\iff$ (iii). Since the diagonal map $G \to G \times G$ induces an open dense embedding $G/L \hookrightarrow G/P \times G/Q$ where $L = P \cap Q$, the tensor product representation $\mathrm{Ind}_P^G(\mathbf{1}) \otimes \mathrm{Ind}_Q^G(\mathbf{1})$ is unitarily equivalent to $L^2(G/L)$. Then one can read the list of the pairs (G, L) such that $L^2(G/L)$ is non-tempered from the classification results in [5, Thms. 3.1 and 4.1] when G is complex or split. For non-split G, we use the Satake diagram to describe Levi parts $\mathfrak{l}$ and their non-compact semisimple factors $\mathfrak{l}_{ns}$. Since $L^2(G/L)$ is tempered if and only if $L^2(G/L_{ns})$ is tempered by [4, Prop. 3.1], we apply [5, Thm. 1.4] to the pair $(\mathfrak{g}, \mathfrak{l}_{ns})$. In most cases $L^2(G/L_{ns})$ is tempered if and only if $L^2(G_{\mathbb{C}}/(L_{ns})_{\mathbb{C}})$ is tempered, and we can simply use the list in [5, Thms. 3.1 and 4.1] again. There are a few exceptions: it may happen that $L^2(G/L_{ns})$ is tempered but $L^2(G_{\mathbb{C}}/(L_{ns})_{\mathbb{C}})$ is not tempered when $\mathfrak{g} = \mathfrak{su}^*(4m-2), \mathfrak{e}_{6(-26)}$, or $\mathfrak{e}_{6(-14)}$. In this case we apply [5, Thm. 1.4 (ii)-(iv)]. This completes the proof of Theorem 5.1. □

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Y. Benoist : CNRS-Université Paris-Saclay, Orsay, France, e-mail : `yves.benoist@math.u-psud.fr`

Y. Inoue : Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Japan, e-mail : `inoue-78u@gl.pen-kanagawa.ed.jp`

T. Kobayashi : Graduate School of Mathematical Sciences and Kavli IPMU (WPI), The University of Tokyo, Komaba, Japan, e-mail : `toshi@ms.u-tokyo.ac.jp`