Spectral analysis on pseudo-Riemannian locally symmetric spaces

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Abstract: We summarize and announce some recent results initiating spectral analysis on pseudo-Riemannian locally symmetric spaces $\Gamma\backslash G/H$, beyond the classical setting where H is compact (e.g. theory of automorphic forms for arithmetic Γ) or Γ is trivial (e.g. Plancherel-type formula for semisimple symmetric spaces).

Key words: Locally symmetric space; pseudo-Riemannian manifold; discontinuous group; Laplacian; invariant differential operator; branching law; spherical variety.

1. Introduction A *pseudo-Riemannian manifold* is a smooth manifold M equipped with a smooth, nondegenerate symmetric bilinear tensor g of signature (p, q). It is called Riemannian if $q = 0$, and Lorenzian if $q = 1$. As in the Riemannian case, the metric g induces a Radon measure on M and a second-order differential operator

$$\square_M = \operatorname{div} \operatorname{grad}$$

called the *Laplacian.* It is a symmetric operator on the Hilbert space $L^2(M)$. The Laplacian $\square_M$ is not an elliptic differential operator if $p, q > 0$.

A *semisimple symmetric space* X is a homogeneous space G/H where G is a semisimple Lie group and H an open subgroup of the group of fixed points of G under some involutive automorphism. The manifold X carries a G-invariant pseudo-Riemannian metric induced by the Killing form of the Lie algebra $\mathfrak{g}$ of G. The group G acts on X by isometries, and the $\mathbb{C}$-algebra $\mathbb{D}_G(X)$ of G-invariant differential operators on X is commutative.

In this note we consider quotients $X_\Gamma = \Gamma\backslash X$ of a semisimple symmetric space $X = G/H$ by discrete subgroups Γ of G acting properly discontinuously and freely on X ("discontinuous groups for X"). Such quotients are called *pseudo-Riemannian locally symmetric spaces.* They are complete (G, X)-manifolds in the sense of Ehresmann and Thurston, and they inherit a pseudo-Riemannian structure from X. Any G-invariant differential operator D on X induces a differential operator D_Γ on X_Γ via the covering map $p_\Gamma : X \to X_\Gamma$. E.g. the Laplacian $\square_X$ on X is G-invariant, and $(\square_X)_\Gamma = \square_{X_\Gamma}$. As in [7, 8], we think of

$$\mathcal{P} := \{D_\Gamma : D \in \mathbb{D}_G(X)\}$$

as the set of "intrinsic differential operators" on the locally symmetric space X_Γ. It is a subalgebra of the $\mathbb{C}$-algebra $\mathbb{D}(X_\Gamma)$ of differential operators on X_Γ:

$$\text{(1.1)} \qquad \mathbb{D}_G(X) \xrightarrow{\sim} \mathcal{P} \subset \mathbb{D}(X_\Gamma), \quad D \mapsto D_\Gamma.$$

For a $\mathbb{C}$-algebra homomorphism $\lambda : \mathbb{D}_G(X) \to \mathbb{C}$, we denote by $C^\infty(X_\Gamma; \mathcal{M}_\lambda)$ the space of smooth functions f on X_Γ (*joint eigenfunctions*) satisfying the following system of partial differential equations:

$$(\mathcal{M}_\lambda) \qquad D_\Gamma f = \lambda(D) f \quad \text{for all } D \in \mathbb{D}_G(X).$$

Let $L^2(X_\Gamma; \mathcal{M}_\lambda)$ be the space of square-integrable functions on X_Γ satisfying $(\mathcal{M}_\lambda)$ in the weak sense. It is a closed subspace of the Hilbert space $L^2(X_\Gamma)$.

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We are interested in the following problems.

Problems 1. *For intrinsic differential operators on $X_\Gamma = \Gamma\backslash G/H$,*
(1) *construct joint eigenfunctions on X_Γ;*
(2) *find a spectral theory on $L^2(X_\Gamma)$.*

In the classical setting where H is a maximal compact subgroup K of G, i.e. X_Γ is a *Riemannian* locally symmetric space, a rich and deep theory has been developed over several decades, in particular, in connection with automorphic forms when Γ is arithmetic. For compact H, the spectral decomposition of $L^2(X_\Gamma)$ is closely related to a disintegration of the regular representation of G on $L^2(\Gamma\backslash G)$:

$$L^2(\Gamma\backslash G) \simeq \int_{\widehat{G}}^{\oplus} m_\Gamma(\pi)\,\pi\,\mathrm{d}\sigma(\pi), \tag{1.2}$$

where $\mathrm{d}\sigma$ is a Borel measure on the unitary dual $\widehat{G}$ and $m_\Gamma\colon \widehat{G} \to \mathbb{N}\cup\{\infty\}$ a measurable function called *multiplicity.* There is a natural isomorphism

$$L^2(X_\Gamma) \simeq L^2(\Gamma\backslash G)^H \tag{1.3}$$

and the Hilbert space $L^2(X_\Gamma)$ is decomposed as

$$L^2(X_\Gamma) \simeq \int_{(\widehat{G})_H} m_\Gamma(\pi)\,\pi^H\,\mathrm{d}\sigma(\pi), \tag{1.4}$$

where π^H denotes the space of H-invariant vectors in the representation space of π and

$$(\widehat{G})_H := \{\pi \in \widehat{G} : \pi^H \neq \{0\}\}.$$

Since the center $\mathfrak{Z}(\mathfrak{g}_\mathbb{C})$ of the enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ acts on the space of smooth vectors of π as scalars for every $\pi \in \widehat{G}$, the decomposition (1.4) respects the actions of $\mathbb{D}_G(X)$ and $\mathfrak{Z}(\mathfrak{g}_\mathbb{C})$ via the natural $\mathbb{C}$-algebra homomorphism $\mathrm{d}\ell\colon \mathfrak{Z}(\mathfrak{g}_\mathbb{C}) \to \mathbb{D}_G(X)$. This homomorphism is surjective e.g. if G is a classical group.

The situation changes drastically beyond the aforementioned classical setting, namely, when H is not compact anymore. New difficulties include:

(1) (Representation theory) If H is noncompact, then $L^2(\Gamma\backslash G)^H = \{0\}$ (because the fact that Γ acts properly on $X = G/H$ implies that H acts properly on $\Gamma\backslash G$), and so (1.3) fails:

$$L^2(X_\Gamma) \not\simeq L^2(\Gamma\backslash G)^H \tag{1.5}$$

and the irreducible decomposition (1.2) of the regular representation $L^2(\Gamma\backslash G)$ of G does not yield a spectral decomposition of $L^2(X_\Gamma)$.

(2) (Analysis) In contrast to the usual Riemannian case (see [22]), the Laplacian $\Box_{X_\Gamma}$ is not elliptic anymore, and thus even the following subproblems of Problem 1.(2) are open in general for $X_\Gamma = \Gamma\backslash G/H$ with H noncompact.

Questions 2.
(1) *Does the Laplacian $\Box_{X_\Gamma}$, defined on $C_c^\infty(X_\Gamma)$, extend to a self-adjoint operator on $L^2(X_\Gamma)$?*
(2) *Does $L^2(X_\Gamma;\mathcal{M}_\lambda)$ contain real analytic functions as a dense subspace?*
(3) *Does $L^2(X_\Gamma)$ decompose discretely into a sum of subspaces $L^2(X_\Gamma;\mathcal{M}_\lambda)$ when X_Γ is compact?*

2. Standard quotients

We observe that a discrete group of isometries on a pseudo-Riemannian manifold X does not always act properly discontinuously on X, and the quotient space $X_\Gamma = \Gamma\backslash X$ is not necessarily Hausdorff. In fact, some semisimple symmetric spaces X do not admit infinite discontinuous groups of isometries (Calabi–Markus phenomenon [2, 11]), and thus it is not obvious a priori whether there are interesting examples of pseudo-Riemannian locally symmetric spaces X_Γ beyond the classical Riemannian case.

Fortunately, there exist semisimple symmetric spaces $X = G/H$ admitting "large" discontinuous groups Γ such that X_Γ is compact or of finite volume. Let us recall a useful idea for finding such X and Γ. Suppose a Lie subgroup L of G acts properly on X. Then the action of any discrete subgroup Γ of L on X is automatically properly discontinuous, and this action is free whenever Γ is torsion-free. Moreover, if L acts cocompactly (e.g. transitively) on X, then $\mathrm{vol}(X_\Gamma) < +\infty$ if and only if $\mathrm{vol}(\Gamma\backslash L) < +\infty$.

Definition 3 (Standard quotient X_Γ, see [8, Def. 1.4])**.** A quotient $X_\Gamma = \Gamma\backslash X$ of $X = G/H$ by a discrete subgroup of G is called *standard* if Γ is contained in a reductive subgroup L of G acting properly on X.

A criterion on triples (G, L, H) of reductive Lie groups for L to act properly on $X = G/H$ was established in [11], and a list of irreducible symmetric spaces G/H admitting proper and cocompact actions of reductive subgroups L was given in [18]. Recently, Tojo [23] announced that the list in [18] exhausts all such triples (L, G, H) with L maximal.

3. Construction of discrete spectrum Let $X = G/H$ be a semisimple symmetric space. Let $\mathfrak{j}$ be a maximal semisimple abelian subspace in the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form, and W the Weyl group for the root system $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. The Harish-Chandra isomorphism $\Psi\colon S(\mathfrak{j}_{\mathbb{C}})^W \xrightarrow{\sim} \mathbb{D}_G(X)$ (see [6]) induces a bijection

$$\Psi^*\colon \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}) \xrightarrow{\sim} \mathfrak{j}_{\mathbb{C}}^*/W. \tag{3.1}$$

The dimension of $\mathfrak{j}$ is called the *rank* of the symmetric space $X = G/H$. Let K be a maximal compact subgroup of G such that $H \cap K$ is a maximal compact subgroup of H. Assume that G is connected without compact factor and that the following rank condition is satisfied:

$$\mathrm{rank}\, G/H = \mathrm{rank}\, K/(H \cap K). \tag{3.2}$$

Then we can take $\mathfrak{j}$ as a subspace of $\mathfrak{k}$. We fix compatible positive systems $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ and $\Sigma^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$, denote by ρ and ρ_c the corresponding half sums of positive roots counted with multiplicities, and set

$$\Lambda := 2\rho_c - \rho + \mathbb{Z}\text{-span}\left\{\text{highest weights of } (\widehat{K})_{H\cap K}\right\}.$$

For $C \geq 0$, we consider the countable set

$$\Lambda_C := \left\{\lambda \in \Lambda : \langle \lambda, \alpha\rangle > C \text{ for all } \alpha \in \Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})\right\}.$$

Fact 4 (Flensted-Jensen [5]). *If the rank condition* (3.2) *holds, then there exists $C > 0$ such that*

$$L^2(X; \mathcal{M}_\lambda) \neq \{0\} \quad \textit{for all } \lambda \in \Lambda_C.$$

In fact one can take $C = 0$ [19]. We now turn to locally symmetric spaces X_Γ:

Theorem 5 ([7], [8, Th. 1.5]). *Under the rank condition* (3.2)*, for any standard quotient X_Γ with Γ torsion-free, there exists $C_\Gamma > 0$ such that*

$$L^2(X_\Gamma; \mathcal{M}_\lambda) \neq \{0\} \quad \textit{for all } \lambda \in \Lambda_{C_\Gamma}.$$

Thus the *discrete spectrum* $\mathrm{Spec}_d(X_\Gamma)$, which is by definition the set of $\lambda \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C})$ such that $L^2(X_\Gamma; \mathcal{M}_\lambda) \neq \{0\}$, is infinite.

Theorem 5 applied to $(G \times \{1\}, G \times G, \mathrm{Diag}\, G)$ instead of (L, G, H) (group manifold case) implies:

Example 6. Suppose $\mathrm{rank}\, G = \mathrm{rank}\, K$. For *any* torsion-free discrete subgroup Γ and *any* discrete series representation π_λ of G with sufficiently regular Harish-Chandra parameter λ,

$$\mathrm{Hom}_G(\pi_\lambda, L^2(\Gamma\backslash G)) \neq \{0\}. \tag{3.3}$$

This sharpens and generalizes the known results asserting that if Γ is an *arithmetic* subgroup of G, then (3.3) holds after replacing Γ by a finite-index subgroup Γ' (possibly depending on π_λ), see Borel–Wallach [1], Clozel [3], DeGeorge–Wallach [4], Kazhdan [10], Rohlfs–Speh [20], and Savin [21].

Remark 7. (1) Theorem 5 extends to a more general setting where X_Γ is not necessarily standard: namely, the conclusion still holds as long as the action of Γ on X satisfies a strong properness condition called *sharpness* [8, Th. 3.8].
(2) The rank condition (3.2) is necessary for $\mathrm{Spec}_d(X)$ to be nonempty (see Matsuki–Oshima [19]), in which case Fact 4 applies. On the other hand, $\mathrm{Spec}_d(X_\Gamma)$ may be nonempty even if (3.2) fails. This leads us to the notion of discrete spectrum of type **I** and **II**, see Definition 12 below.

4. Spectral decomposition of $L^2(X_\Gamma)$ In this section, we discuss spectral decomposition on standard quotients X_Γ. We do not impose the rank condition (3.2), but require that $L_{\mathbb{C}}$ act spherically on $X_{\mathbb{C}}$, i.e. a Borel subgroup of $L_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. To be precise, our setting is as follows:

Setting 8. *We consider a symmetric space $X = G/H$ with G noncompact and simple, a reductive subgroup L of G acting properly on X such that $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ is $L_{\mathbb{C}}$-spherical, and a torsion-free discrete subgroup Γ of L.*

For compact H, we can take $L = G$. However, our main interest is for *noncompact* H, in which case the proper action of L in the setting 8 forces $L \neq G$ (see [11, Th. 4.1] for a properness criterion).

In Theorems 9 and 10 below, we allow the case where $\mathrm{vol}(X_\Gamma) = +\infty$.

Theorem 9 (Spectral decomposition). *In the setting 8, there exist a measure* $\mathrm{d}\mu$ *on* $\mathrm{Hom} := \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C})$ *and a measurable family* $(\mathcal{F}_\lambda)_{\lambda \in \mathrm{Hom}}$ *of linear maps, with*

$$\mathcal{F}_\lambda : C_c^\infty(X_\Gamma) \longrightarrow C^\infty(X_\Gamma; \mathcal{M}_\lambda),$$

such that any $f \in C_c^\infty(X_\Gamma)$ *can be expanded into joint eigenfunctions on* X_Γ *as*

$$f = \int_{\mathrm{Hom}} \mathcal{F}_\lambda f \ \mathrm{d}\mu(\lambda), \tag{4.1}$$

with a Parseval–Plancherel type formula

$$\|f\|^2_{L^2(X_\Gamma)} = \int_{\mathrm{Hom}} \|\mathcal{F}_\lambda f\|^2_{L^2(X_\Gamma)} \ \mathrm{d}\mu(\lambda).$$

The measure $\mathrm{d}\mu$ can be described via a "transfer map" discussed in Section 5, see (5.4). In particular, we see that (4.1) is a discrete sum if X_Γ is compact, answering Question 2.(3) in our setting. The proof of Theorem 9 gives an answer to Questions 2.(1)–(2):

Theorem 10. *In the setting 8,*
(1) *the pseudo-Riemannian Laplacian* $\square_{X_\Gamma}$ *defined on* $C_c^\infty(X_\Gamma)$ *is essentially self-adjoint on* $L^2(X_\Gamma)$*;*
(2) *any* L^2*-eigenfunction of the Laplacian* $\square_{X_\Gamma}$ *can be approximated by real analytic* L^2*-eigenfunctions.*

Theorem 11. *In the setting 8, the discrete spectrum* $\mathrm{Spec}_d(X_\Gamma)$ *is infinite whenever* Γ *is cocompact or arithmetic in the subgroup* L.

Let $\mathcal{D}'(X)$ be the space of distributions on X, endowed with its standard topology. Let $p_\Gamma^* \colon L^2(X_\Gamma) \to \mathcal{D}'(X)$ be the pull-back by the projection $p_\Gamma \colon X \to X_\Gamma$. For $\lambda \in \mathrm{Spec}_d(X_\Gamma)$, we denote by $L^2(X_\Gamma; \mathcal{M}_\lambda)_{\mathbf{I}}$ the preimage under p_Γ^* of the closure in $\mathcal{D}'(X)$ of $L^2(X_\Gamma; \mathcal{M}_\lambda)$, and by $L^2(X_\Gamma; \mathcal{M}_\lambda)_{\mathbf{II}}$ its orthogonal complement in $L^2(X_\Gamma; \mathcal{M}_\lambda)$.

Definition 12. For $i = \mathbf{I}$ or $\mathbf{II}$, the *discrete spectrum of type* i of X_Γ is the subset $\mathrm{Spec}_d(X_\Gamma)_i$ of $\mathrm{Spec}_d(X_\Gamma)$ consisting of those elements λ such that $L^2(X_\Gamma; \mathcal{M}_\lambda)_i \neq \{0\}$.

By construction, $\mathrm{Spec}_d(X_\Gamma)_{\mathbf{I}}$ is contained in $\mathrm{Spec}_d(X)$, hence it is nonempty only if (3.2) holds (Remark 7.(2)); in this case $\mathrm{Spec}_d(X_\Gamma)_{\mathbf{I}}$ is actually infinite for standard X_Γ by Theorem 5. On the other hand, Theorem 11 has the following refinement.

Theorem 13. *In the setting 8,* $\mathrm{Spec}_d(X_\Gamma)_{\mathbf{II}}$ *is infinite whenever* Γ *is cocompact or arithmetic in* L.

Example 14. For any compact standard anti-de Sitter 3-manifold $M = \Gamma\backslash \mathrm{SO}(2,2)/\mathrm{SO}(2,1)$, both $\mathrm{Spec}_d(X_\Gamma)_{\mathbf{I}}$ and $\mathrm{Spec}_d(X_\Gamma)_{\mathbf{II}}$ are infinite, and

$$\mathrm{Spec}_d(X_\Gamma)_{\mathbf{I}} \subset [0, +\infty), \quad \mathrm{Spec}_d(X_\Gamma)_{\mathbf{II}} \subset (-\infty, 0].$$

5. Transfer maps Let L be a reductive subgroup of G acting properly on $X = G/H$ and Γ a discrete subgroup of L. In Section 1 we considered spectral analysis on the standard locally symmetric space X_Γ through the algebra $\mathcal{P}$ ($\simeq \mathbb{D}_G(X)$) of intrinsic differential operators on X_Γ. Another $\mathbb{C}$-algebra $\mathcal{Q}$ of differential operators on X_Γ is obtained from the center $\mathfrak{Z}(\mathfrak{l}_\mathbb{C})$ of the enveloping algebra $U(\mathfrak{l}_\mathbb{C})$: indeed, $\mathfrak{Z}(\mathfrak{l}_\mathbb{C})$ acts on smooth functions on X by differentiation, yielding a $\mathbb{C}$-algebra of L-invariant differential operators on X, hence a $\mathbb{C}$-algebra of differential operators on $X_\Gamma = \Gamma\backslash X$ since $\Gamma \subset L$. In general, there is no inclusion relation between $\mathcal{P}$ and $\mathcal{Q}$. In order to compare the roles of $\mathcal{P}$ and $\mathcal{Q}$, we highlight a natural homomorphism $\mathfrak{Z}(\mathfrak{g}_\mathbb{C}) \to \mathcal{P}$ and a surjective one $\mathrm{d}\ell \colon \mathfrak{Z}(\mathfrak{l}_\mathbb{C}) \to \mathcal{Q}$. Loosely speaking, the algebras $\mathfrak{Z}(\mathfrak{g}_\mathbb{C})$ and $\mathfrak{Z}(\mathfrak{l}_\mathbb{C})$ separate irreducible representations of the groups G and L, respectively, hence it is important to understand how irreducible representations of G behave when restricted to the subgroup L (*branching problem*) in order to utilize the algebra $\mathcal{Q}$ for the spectral analysis on X_Γ via the algebra $\mathcal{P}$ (see [15, 16]). We shall return to this point in Theorem 15 below.

Now assume the proper action of L on $X = G/H$ is also transitive, so that $X \simeq L/L_H$ where $L_H := L \cap H$ is compact. Up to conjugation, we may assume that $L_K := L \cap K$ is a maximal compact subgroup of L containing L_H. Then the *pseudo-Riemannian* symmetric space X fibers over the *Riemannian* symmetric space $Y = L/L_K$ with fiber $F := L_K/L_H$, and this induces a fibration for the quotients by Γ:

$$F \longrightarrow X_\Gamma \simeq \Gamma\backslash L/L_H \longrightarrow Y_\Gamma = \Gamma\backslash L/L_K. \tag{5.1}$$

To expand functions on X_Γ along the fiber F, we define an endomorphism p_τ of $C^\infty(X_\Gamma)$ by

$$(p_\tau f)(\cdot) := \frac{1}{\dim \tau} \int_K f(\cdot\, k)\,\mathrm{Trace}\,\tau(k)\ \mathrm{d}k$$

for every $\tau \in \widehat{L_K}$. Then p_τ is an idempotent, namely, $p_\tau^2 = p_\tau$. The τ-component of $C^\infty(X_\Gamma)$ is defined by

$$C^\infty(X_\Gamma)_\tau := \mathrm{Image}(p_\tau) = \mathrm{Ker}(p_\tau - \mathrm{id}).$$

We note that $C^\infty(X_\Gamma)_\tau \neq \{0\}$ if and only if τ has a nonzero L_H-invariant vector, i.e. $\tau \in (\widehat{L_K})_{L_H}$. It is easy to see that the projection p_τ commutes with any element in $\mathcal{Q}$ ($\simeq \mathrm{d}\ell(\mathfrak{Z}(\mathfrak{l}_\mathbb{C}))$), but not always with "intrinsic differential operators" $D_\Gamma \in \mathcal{P}$ ($\simeq \mathbb{D}_G(X)$), and consequently it may well happen that

$$p_\tau(C^\infty(X_\Gamma; \mathcal{M}_\lambda)) \not\subset C^\infty(X_\Gamma; \mathcal{M}_\lambda).$$

To make a connection between the two subalgebras $\mathcal{P}$ and $\mathcal{Q}$, we introduce a third subalgebra $\mathcal{R}$ of $\mathbb{D}(X_\Gamma)$, coming from the fiber F in (5.1). Namely, $\mathcal{R}$ is isomorphic to the $\mathbb{C}$-algebra $\mathbb{D}_{L_K}(F)$ of L_K-invariant differential operators D on F, and obtained by extending elements of $\mathbb{D}_{L_K}(F)$ to L-invariant differential operators on X, yielding differential operators on the quotient X_Γ.

Suppose now that we are in the setting 8. The subgroup L acts transitively on X by [17, Lem. 4.2] and [12, Lem. 5.1]. Moreover, we can prove [9] that

$$\mathcal{Q} \subset \langle \mathcal{P}, \mathcal{R} \rangle \tag{5.2}$$

where $\langle \mathcal{P}, \mathcal{R} \rangle$ denotes the subalgebra of $\mathbb{D}(X_\Gamma)$ generated by $\mathcal{P}$ and $\mathcal{R}$. This implies the following strong constraints on the restriction of representations:

Theorem 15. *In the setting 8, any irreducible $(\mathfrak{g}, K)$-module occurring in $C^\infty(X)$ is discretely decomposable as an $(\mathfrak{l}, L \cap K)$-module.*

See [12, 13, 14] for a general theory of discretely decomposable restrictions of representations. See also [16] for a discussion on Theorem 15 when dropping the assumption that L acts properly on X.

In addition to (5.2), the quotient fields of $\mathcal{P}$ and $\langle \mathcal{Q}, \mathcal{R} \rangle$ coincide [9, Th. 1.3 & § 6.9], and we obtain:

Theorem 16 (Transfer map)**.** *In the setting 8, for any $\tau \in (\widehat{L_K})_{L_H}$ there is an injective map*

$$\boldsymbol{\nu}(\cdot, \tau)\colon\ \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{l}_\mathbb{C}), \mathbb{C})$$

such that for any $\lambda \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C})$, any $f \in C^\infty(X_\Gamma; \mathcal{M}_\lambda)$, and any $z \in \mathfrak{Z}(\mathfrak{l}_\mathbb{C})$,

$$\mathrm{d}\ell(z)(p_\tau f) = \boldsymbol{\nu}(\lambda, \tau)(z)\ p_\tau f.$$

We write $\boldsymbol{\lambda}(\cdot, \tau)$ for the inverse map of $\boldsymbol{\nu}(\cdot, \tau)$ on its image. We call $\boldsymbol{\nu}(\cdot, \tau)$ and $\boldsymbol{\lambda}(\cdot, \tau)$ *transfer maps*, as they "transfer" eigenfunctions for $\mathcal{P}$ to those for $\mathcal{Q}$, and vice versa, on the τ-component $C^\infty(X_\Gamma)_\tau$.

For an explicit description of transfer maps, let

$$\Phi^*\colon\ \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{l}_\mathbb{C}), \mathbb{C}) \overset{\sim}{\to} \mathfrak{t}_\mathbb{C}^*/W(\mathfrak{l}_\mathbb{C})$$

be the Harish-Chandra isomorphism as in (3.1), where $W(\mathfrak{l}_\mathbb{C})$ denotes the Weyl group of the root system $\Delta(\mathfrak{l}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ with respect to a Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{l}_\mathbb{C}$. We note that there is no natural inclusion relation between $\mathfrak{j}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$.

For each $\tau \in (\widehat{L_K})_{L_H}$, we find an affine map $S_\tau\colon \mathfrak{j}_\mathbb{C}^* \to \mathfrak{t}_\mathbb{C}^*$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{j}_\mathbb{C}^* & \xrightarrow{\ S_\tau\ } & \mathfrak{t}_\mathbb{C}^* \\
\downarrow & & \downarrow \\
\mathfrak{j}_\mathbb{C}^*/W & & \mathfrak{t}_\mathbb{C}^*/W(\mathfrak{l}_\mathbb{C}) \\
\Psi^* \uparrow \wr & & \wr \uparrow \Phi^* \\
\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}) & \xrightarrow{\ \boldsymbol{\nu}(\cdot,\tau)\ } & \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{l}_\mathbb{C}), \mathbb{C})
\end{array}$$

Then a closed formula for the transfer map $\boldsymbol{\nu}(\cdot, \tau)$ is derived from that of the affine map S_τ which was determined explicitly in [9, § 6–7] for the complexifications of the triples (L, G, H) in the setting 8.

Via the transfer maps, we can utilize representations of the subgroup L efficiently for the spectral analysis on X_Γ, as follows. As in (1.2), let

$$L^2(\Gamma\backslash L) \simeq \int_{\widehat{L}}^{\oplus} m_\Gamma(\vartheta)\, \vartheta\, \mathrm{d}\sigma(\vartheta) \tag{5.3}$$

be a disintegration of the regular representation

$L^2(\Gamma\backslash L)$ of the subgroup L. Then the transform $\mathcal{F}_\lambda$ in Theorem 9 can be built naturally by using (5.3) and the expansion of $C_c^\infty(X_\Gamma)$ along the fiber F in (5.1). Consider the map

$$\Lambda\colon (\widehat{L})_{L_H} \times (\widehat{L_K})_{L_H} \to \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}),$$

$(\vartheta, \tau) \mapsto \boldsymbol{\lambda}(\chi_\vartheta, \tau)$, where $\chi_\vartheta \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{l}_\mathbb{C}), \mathbb{C})$ is the infinitesimal character of $\vartheta \in \widehat{L}$. Then the Plancherel measure $\mathrm{d}\mu$ on $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C})$ in Theorem 9 can be defined by

$$\mathrm{d}\mu = \Lambda_*(\mathrm{d}\sigma|_{(\widehat{L})_{L_H}} \times (\widehat{L_K})_{L_H}) \tag{5.4}$$

Detailed proofs of Theorems 9, 10, 11, 15, and 16 will appear elsewhere.

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