Admissible restrictions of irreducible representations of reductive Lie groups: symplectic geometry and discretely decomposability

— Dedicated to Bertram Kostant with admiration to his deep and vast perspectives and with sincere gratitude to his constant encouragement for many years

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Abstract

Let G be a real reductive Lie group, L a compact subgroup, and π an irreducible admissible representation of G. This paper proves a necessary and sufficient condition for the finiteness of the multiplicities of L-types occurring in π based on symplectic techniques. This leads us to a new proof of the criterion for the discrete decomposability of the restriction of unitary representations with respect to noncompact subgroups (the author, Ann. Math. 1998). A number of examples are presented in connection with Kostant's convexity theorem and also with non-Riemannian locally symmetric spaces.

§1 Introduction and Statement of Main Results

This article is a continuation of [11, 12, 13], where we studied the restriction of an irreducible unitary representation π of a real reductive Lie group G with respect to a reductive subgroup G'. There, we highlight branching laws without continuous spectrum. A key to this property is K'-admissibility of π ([11, Thm. 1.2]), that is,

$$\dim \operatorname{Hom}_{K'}(\tau, \pi|_{K'}) < \infty \quad \text{for any} \quad \tau \in \widehat{K'}, \tag{1.1}$$

where K' is a maximal compact subgroup of G'.

In this article we prove a necessary and sufficient condition for the K'-admissibility, of irreducible $(\mathfrak{g}, K)$-modules X.

1.1. Two closed cones $\mathrm{AS}_K(X)$ and $C_K(K')$

In order to state our main results, let us fix some notation.

Let G be a connected linear reductive Lie group, K a maximal compact subgroup of G, and T a maximal torus of K. Their Lie algebras will be denoted by the lowercase German letters. Fix a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and we write $\mathfrak{t}^*_+$ $(\subset \sqrt{-1}\mathfrak{t}^*)$ for the dominant Weyl chamber. The set of dominant weights which lift to the torus T is denoted by Λ_+. It is a submonoid of $\mathfrak{t}^*_+$ (that is, it contains 0 and is invariant under addition). The Cartan–Weyl highest weight theory for the group version establishes a bijection between $\widehat{K}$ with Λ_+. We shall denote by V_μ the irreducible representation of K with highest weight $\mu \in \Lambda_+$.

For a subset S in a Euclidean space E, the limit cone $S\infty$ is the set of E consisting of all elements of the form $\lim_{j\to\infty} \epsilon_j \mu_j$ for some sequence $(\mu_j, \varepsilon_j) \in S \times \mathbb{R}_+$ with $\lim_{j\to\infty} \epsilon_j = 0$ ([7, Def. 2.4.2]). The asymptotic K-support $\mathrm{AS}_K(X)$ of a K-module X is defined to be the limit cone of the K-support of X (Kashiwara–Vergne [8]):

$$\mathrm{Supp}_K(X) := \{\mu \in \Lambda_+ : \mathrm{Hom}_K(V_\mu, X) \neq \{0\}\} \ \subset \Lambda_+, \tag{1.2}$$

$$\mathrm{AS}_K(X) := \mathrm{Supp}_K(X)\infty \qquad \subset \mathfrak{t}^*_+. \tag{1.3}$$

Let K' be a closed subgroup of K, and set $(\mathfrak{k}')^\perp := \{\lambda \in \mathfrak{k}^* : \lambda|_{\mathfrak{k}'} \equiv 0\}$. We regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{k}^*$ via a K-invariant inner product on $\mathfrak{k}$, and define a closed cone in $\sqrt{-1}\mathfrak{t}^*$ by

$$C_K(K') := \mathfrak{t}^*_+ \cap \sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}')^\perp. \tag{1.4}$$

These two closed cones $\mathrm{AS}_K(X)$ and $C_K(K')$ are a finite union of convex polyhedral cones (Propositions 2.6 and 2.3, respectively).

1.2. Criterion for finite multiplicities

Here is our main theorem:

Theorem 1.1. *Let X be a $(\mathfrak{g}, K)$-module of finite length, and K' a closed subgroup of K. Then the following two conditions are equivalent:*

(i) *X is K'-admissible;*

(ii) $\mathrm{AS}_K(X) \cap C_K(K') = \{0\}$.

Some remarks are in order.

(1) The main result of [12] was a discovery of the criterion (ii) in Theorem 1.1, and the implication (ii) $\Rightarrow$ (i) was proved in [12, Thm. 2.8] based on micro-local study: the asymptotic K-support $\mathrm{AS}_K(X)$ played a role in an estimate of the singularity spectrum of the hyperfunction character of $X|_K$. In this article we give a new and simpler proof for the implication (ii) $\Rightarrow$ (i) based on symplectic geometry: the cone $C_K(K')$ is interpreted as the momentum set for the natural Hamiltonian action on the cotangent bundle $T^*(K/K')$, see Section 2.3.

(2) The implication (i) $\Rightarrow$ (ii) was announced in [16, Chap. 6].

(3) Theorem 1.1 still holds for disconnected groups, namely, we may allow K to have finitely many connected components. In this case, we use the asymptotic K_0-support for $\mathrm{AS}_K(X)$, where K_0 is the identity component of K.

1.3. Admissible restriction to noncompact subgroups

Let π be a unitary representation of G, and G' a subgroup. By the general theory of unitary representations of locally compact groups [25], the restriction $\pi|_{G'}$ is decomposed into the direct integral of irreducible unitary representations of G', uniquely up to isomorphisms when G' is reductive [5], as follows:

$$\pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} m_\pi(\tau) d\mu(\tau) \quad \text{(direct integral)}, \tag{1.5}$$

where $\widehat{G'}$ denotes the unitary dual of G', that is, the set of equivalence classes of irreducible unitary representations of G', $d\mu$ is a Borel measure of $\widehat{G'}$, and $m_\pi \colon \widehat{G'} \to \mathbb{N} \cup \{\infty\}$ is a measurable function. The irreducible decomposition (1.5) is called the *branching law* of the restriction $\pi|_{G'}$, and m_π is the *multiplicity*. In general the branching law may involve continuous spectrum, and the multiplicity m_π may take infinite values. The following definition singles out a framework in which we could expect a simple and detailed algebraic study of the restriction $\pi|_{G'}$ (*symmetry breaking*).

Definition 1.2 ([11]). A unitary representation π of G is G'-*admissible* if π splits into a direct sum of irreducible unitary representations of G'

$$\pi|_{G'} \simeq \sum_{\tau \in \widehat{G'}}^{\oplus} m(\tau)\tau \quad \text{(Hilbert direct sum)}$$

with multiplicity $m(\tau) < \infty$ for all $\tau \in \widehat{G'}$.

If G' itself is compact, then the decomposition (1.5) is automatically discrete, and thus, G'-admissibility is nothing but the finiteness of the multiplicity $m_\pi(\tau)$ for all τ. In the general case where G' is noncompact, we take a maximal compact subgroup K' of G'. Then K'-admissibility implies G'-admissibility ([11, Thm. 1.2]). Therefore, as an immediate corollary of Theorem 1.1, we recover:

Corollary 1.3 ([12, Thm. 2.9])**.** *Let $\pi \in \widehat{G}$, and G' a reductive subgroup of G. If $\mathrm{AS}_K(\pi) \cap \mathrm{Ad}^*(K)(\mathfrak{k}')^\perp = \{0\}$, then the restriction $\pi|_{G'}$ splits into a* **discrete** *sum of irreducible unitary representations of G' with* **finite multiplicities***.*

1.4. Restriction of discrete series representations

It is plausible that G'-admissibility is equivalent to K'-admissibility if the representation arises as the restriction of an irreducible unitary representation of a real reductive linear Lie group G to its reductive subgroup G' [15, Conj. D]. See [2, 18, 31] for some affirmative results. If this is affirmative, then the criterion in Theorem 1.1 will give a necessary and sufficient condition for the restriction $\pi|_{G'}$ to be G'-admissible. In this section we discuss such an example.

An irreducible unitary representation π of G is called a *square-integrable representation* if it is realized in a closed invariant subspace of the regular representation on the Hilbert space $L^2(G)$. The isomorphism classes of all such irreducible, square integrable representations constitute a subset $\mathrm{Disc}(G) \subset \widehat{G}$, the *discrete series* of G. By Theorem 1.1, we can detect whether π is G'-admissible or not when restricted to a reductive subgroup G':

Corollary 1.4. *Let π be a square-integrable representation of G, and G' a closed reductive subgroup of G. Then the following four conditions on the triple (G, G', π) are equivalent:*

(i) *The restriction $\pi|_{G'}$ is G'-admissible.*

(i)$'$ *There is a map $m\colon \mathrm{Disc}(G') \to \mathbb{N}$ such that*

$$\pi|_{G'} \simeq \sum_{\tau \in \mathrm{Disc}(G')}^{\oplus} m(\tau)\tau \qquad \text{(Hilbert direct sum).}$$

(ii) *The restriction $\pi|_{K'}$ is K'-admissible.*

(iii) $\mathrm{AS}_K(\pi) \cap \sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}')^\perp = \{0\}$.

Remark 1.5. When (G, G') is an irreducible symmetric pair, the triple (G, G', π) satisfying the criterion (iii) was classified in [20]. The case $G' = SL(2, \mathbb{R})$ was studied in Duflo–Galina–Vargas [2].

The proof of Theorem 1.1 and Corollary 1.4 is given in Section 2. Applications of Theorem 1.1 are given in connection with Kostant's convexity theorem for momentum maps and with the boundaries of semisimple symmetric spaces in Sections 3 and 4, respectively.

Notation: $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{Q}_{\geq 0} := \mathbb{Q} \cap \mathbb{R}_{\geq 0}$ and $\mathbb{N}_{\geq 0} := \mathbb{N} \cap \mathbb{R}_{\geq 0}$.

§2 Proof of Main Results

In this section, we give an interpretation of the two invariants $\mathrm{AS}_K(\pi)$ and $C_K(K')$ from a viewpoint of symplectic geometry, and prove Theorem 1.1.

2.1. Rational convex polyhedral cones

Let E be a finite-dimensional vector space over $\mathbb{Q}$, and S a finite subset of E. The *convex polyhedral cone* spanned by S is the smallest convex cone in E, that is,

$$\mathbb{Q}_{\geq 0}\text{-span}\, S = \{\sum_{j=1}^{k} a_j S_j : a_1, \cdots, a_k \in \mathbb{Q}_{\geq 0}, s_1, \cdots, s_k \in S\}.$$

Similarly, we can define $\mathbb{Z}_{\geq 0}\text{-span}\, S$ ($\subset E$) and $\mathbb{R}_{\geq 0}\text{-span}\, S$ ($\subset E \otimes_{\mathbb{Q}} \mathbb{R}$).

Here is an elementary observation of the intersections of two such polyhedral cones.

Lemma 2.1. *Let S, T be finite subsets of $\mathbb{Q}^n$. Then the following four conditions on S and T are equivalent:*

(i) $\mathbb{Z}_{\geq 0}\text{-span}\, S \cap \mathbb{Z}_{\geq 0}\text{-span}\, T \neq \{0\}$;

(ii) $\mathbb{Q}_{\geq 0}\text{-span}\, S \cap \mathbb{Q}_{\geq 0}\text{-span}\, T \neq \{0\}$;

(iii) $\mathbb{R}_{\geq 0}\text{-span}\, S \cap \mathbb{R}_{\geq 0}\text{-span}\, T \neq \{0\}$;

(iv) (δ-neighbourhood of $\mathbb{R}_{\geq 0}\text{-span}\, T$) $\cap\, \mathbb{R}_{\geq 0}\text{-span}\, T$ *is unbounded for some* $\delta > 0$.

Proof. The implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. The implication (iv) $\Rightarrow$ (iii) is immediate by taking the limit cone. For the remaining implication (iii) $\Rightarrow$ (ii), we observe that the condition (iii) holds if and only if $\mathbb{R}_{\geq 0}\text{-span}\, S \cap \mathbb{R}_{\geq 0}\text{-span}\, T$ contains a face of positive dimension, say W'. We extend W' to the equi-dimensional subspace W in $\mathbb{R}^n$. Then W is defined over $\mathbb{Q}$, hence $\mathbb{Q}_{\geq 0}\text{-span}\, S \cap \mathbb{Q}_{\geq 0}\text{-span}\, T \supset W' \cap \mathbb{Q}^n \neq \{0\}$. Thus we have proved (iii) $\Rightarrow$ (ii). $\square$

2.2. Regular functions on affine $K_{\mathbb{C}}$-varieties

Let $\mathcal{V}$ be an irreducible affine $K_{\mathbb{C}}$-variety over $\mathbb{C}$. Then the ring $\mathbb{C}[\mathcal{V}]$ of regular functions is finitely generated. We need some basic fact on the $K_{\mathbb{C}}$-module structure of $\mathbb{C}[\mathcal{V}]$.

Lemma 2.2. *The K-support* $\mathrm{Supp}_K(\mathbb{C}[\mathcal{V}])$ *is a finitely generated submonoid of* Λ_+, *that is, there exist finitely many* $\lambda_1, \dots, \lambda_k \in \Lambda_+$ *such that*

$$\mathrm{Supp}_K(\mathbb{C}[\mathcal{V}]) = \mathbb{Z}_{\geq 0}\text{-span}\,\{\lambda_1, \dots, \lambda_k\}.$$

For the convenience of the reader, we review quickly its proof, see [1, 27].

Proof. We write $N(K_{\mathbb{C}})$ for the maximal unipotent subgroup of $K_{\mathbb{C}}$ corresponding to the positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then the ring $\mathbb{C}[K_{\mathbb{C}}/N(K_{\mathbb{C}})] \simeq \bigoplus_{\lambda \in \Lambda_+} V_\lambda$ is finitely generated since $V_\lambda V_\mu = V_{\lambda+\mu}$. Then the left-hand side of the isomorphism:

$$(\mathbb{C}[K_{\mathbb{C}}/N(K_{\mathbb{C}})] \otimes \mathbb{C}[\mathcal{V}])^{K_{\mathbb{C}}} \simeq \mathbb{C}[\mathcal{V}]^{N(K_{\mathbb{C}})}$$

is finitely generated because $K_{\mathbb{C}}$ is reductive. Thus the ring $\mathbb{C}[\mathcal{V}]^{N(K_{\mathbb{C}})}$ is finitely generated, whence the K-support $\mathrm{Supp}_K(\mathbb{C}[\mathcal{V}])$ is finitely generated as a monoid. □

2.3. Hamiltonian actions and cotangent bundles

Let (M, ω) be a symplectic manifold, and K a Lie group acting on M as symplectic diffeomorphisms. The action is called *Hamiltonian* if there exists a *momentum map* $\Phi\colon M \to \mathfrak{k}^*$ with the property that $d\Phi^Z = \iota(Z_M)\omega$ for all $Z \in \mathfrak{k}$, where Z_M denotes the vector field on M induced by Z, and Φ^Z is the function on M defined by $\Phi^Z(m) = \Phi(m)(Z)$. The *momentum set* $\Delta(M)$ is defined by

$$\Delta(M) := \sqrt{-1}\Phi(M) \cap \mathfrak{t}_+^* \tag{2.1}$$

Let K' be a connected closed subgroup of K. The cotangent bundle $T^*(K/K')$ of the homogeneous space K/K' is given as a homogeneous vector bundle $K \times_{K'} (\mathfrak{k}')^\perp$. Thus the symplectic manifold $T^*(K/K')$ is a Hamiltonian K-space with moment map

$$\Psi\colon T^*(K/K') \to \mathfrak{k}^*, \quad (k, X) \mapsto \mathrm{Ad}^*(k)X. \tag{2.2}$$

Let $K'_{\mathbb{C}} \subset K_{\mathbb{C}}$ be the complexifications of $K' \subset K$. For the affine variety $K_{\mathbb{C}}/K'_{\mathbb{C}}$, we take $\lambda_1, \dots, \lambda_k \in \Lambda_+$ as in Lemma 2.2 such that

$$\mathbb{C}[K_{\mathbb{C}}/K'_{\mathbb{C}}] = \mathbb{Z}_{\geq 0}\text{-span}\,\{\lambda_1, \dots, \lambda_k\}. \tag{2.3}$$

Proposition 2.3. (1) *The momentum set* $\Delta(T^*(K/K'))$ *is equal to* $C_K(K')$.

(2) $C_K(K') = \mathrm{AS}_K(C^\infty(K/K'))$. *In particular,* $C_K(K') = \mathbb{R}_{\geq 0}\text{-span}\,\{\lambda_1, \dots, \lambda_k\}$.

Proof. (1) It follows from the definitions (2.2) and (1.4) that

$$\Delta(T^*(K/K')) = \sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}')^\perp \cap \mathfrak{t}^*_+ = C_K(K'). \tag{2.4}$$

(2) By Sjamaar [27, Thms. 4.9 and 7.6], we have

$$\Delta(T^*(K/K')) = \Delta(K_{\mathbb{C}}/K'_{\mathbb{C}}) = \mathbb{R}_{\geq 0}\text{-span}\,\{\lambda_1, \dots, \lambda_k\}.$$

Combining this with (2.4), we get the second statement. □

2.4. Associated varieties

The associated varieties $\mathcal{V}(X)$ are coarse approximation of $\mathfrak{g}$-modules X, which we brought in [13] into the study of discretely decomposable restrictions of Harish-Chandra modules. In this section we collect some important properties of associated varieties, and reduce the K'-admissibility of a Harish-Chandra module on $\mathcal{V}(X)$ to that of the space of regular functions on $\mathcal{V}(X)$.

Let $\{U_j(\mathfrak{g}_{\mathbb{C}})\}_{j\in\mathbb{N}}$ be the standard increasing filtration of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$. Suppose X is a finitely generated $\mathfrak{g}$-module. Let F be a finite set of generators, and we set $X_j := U_j(\mathfrak{g}_{\mathbb{C}})F$. The graded algebra $\mathrm{gr}\,U(\mathfrak{g}_{\mathbb{C}}) := \bigoplus_{j\in\mathbb{N}} U_j(\mathfrak{g}_{\mathbb{C}})/U_{j-1}(\mathfrak{g}_{\mathbb{C}})$ is isomorphic to the symmetric algebra $S(\mathfrak{g}_{\mathbb{C}})$ by the Poincaré–Birkhoff–Witt theorem and we regard the graded module $\mathrm{gr}\,X := \bigoplus_{j\in\mathbb{N}} X_j/X_{j-1}$ as a $S(\mathfrak{g}_{\mathbb{C}})$-module. Define

$$\begin{aligned}\mathrm{Ann}_{S(\mathfrak{g}_{\mathbb{C}})}(\mathrm{gr}\,X) :=& \{f \in S(\mathfrak{g}_{\mathbb{C}}) : fv = 0 \quad \text{for any } v \in \mathrm{gr}\,X\},\\ \mathcal{V}(X) :=& \{x \in \mathfrak{g}^*_{\mathbb{C}} : f(x) = 0 \quad \text{for any } f \in \mathrm{Ann}_{S(\mathfrak{g}_{\mathbb{C}})}(\mathrm{gr}\,X)\}.\end{aligned}$$

Then $\mathcal{V}(X)$ does not depend on the choice of F, and is called the *associated variety* of X. If X is a Harish-Chandra module, that is, a $(\mathfrak{g}, K)$-module of finite length, then the associated variety $\mathcal{V}(X)$ is a $K_{\mathbb{C}}$-stable closed subvariety of $\mathcal{N}(\mathfrak{p}^*_{\mathbb{C}})$, see [30].

For two K-modules X_1, X_2, we use the notation from [11], and write $X_1 \leq_K X_2$ if

$$\dim \mathrm{Hom}_K(\tau, X_1) \leq \dim \mathrm{Hom}_K(\tau, X_2) \quad \text{for any } \tau \in \widehat{K}.$$

Lemma 2.4 ([21, Prop. 3.3])**.** *Let X be a $(\mathfrak{g}, K)$-module of finite length, and $\mathcal{V}(X)$ the associated variety. We write $\mathcal{V}(X) = \overline{\mathcal{O}_1} \cup \cdots \overline{\mathcal{O}_N}$ for the decomposition into irreducible components. Then there exist finite-dimensional representations F_j $(1 \leq j \leq N)$ of K such that*

$$X \leq_K \bigoplus_{j=1}^{N} \mathbb{C}[\overline{\mathcal{O}_j}] \otimes F_j, \tag{2.5}$$

$$X \otimes F_j^* \geq_K \mathbb{C}[\overline{\mathcal{O}_j}] \quad \textit{for any } j \quad (1 \leq j \leq N). \tag{2.6}$$

2.5. Basic properties of asymptotic K-support

We recall some basic properties of asymptotic K-support defined in (1.3).

Lemma 2.5. *Let X and Y be K-modules.*

(1) *If $Y \leq_K X$ then $\mathrm{AS}_K(Y) \subset \mathrm{AS}_K(X)$.*

(2) *$\mathrm{AS}_K(X) = \mathrm{AS}_K(X \otimes F)$ for any finite-dimensional representation F of K.*

(3) *$\mathrm{AS}_K(X \oplus Y) = \mathrm{AS}_K(X) \cup \mathrm{AS}_K(Y)$.*

Proof. (1) Clear from $\mathrm{Supp}_K(Y) \subset \mathrm{Supp}_K(X)$.
(2) See [12, Lem. 3.1].
(3) Immediate from $(S \cup T)\infty = S\infty \cup T\infty$ for any subsets S and T. □

2.6. Asymptotic K-supports of Harish-Chandra modules

The asymptotic K-support $\mathrm{AS}_K(X)$ of a Harish-Chandra module X is determined by its associated variety $\mathcal{V}(X)$, and is a finite union of convex polyhedral cones. These properties will be used in the proof of Theorem 1.1.

Suppose we are in the setting of Lemma 2.4. For each irreducible component $\overline{\mathcal{O}_j}$ of the associated variety $\mathcal{V}(X)$, we take a finite set $S_j := \{\beta_1, \ldots, \beta_{k_j}\}$ so that $\mathrm{Supp}_K(\mathbb{C}[\overline{\mathcal{O}_j}]) = \mathbb{Z}_{\geq 0}\text{-span}\, S_j$ as in Lemma 2.2. Taking the limit cone, we have:

$$\mathrm{AS}_K(\mathbb{C}[\overline{\mathcal{O}_j}]) = \mathbb{R}_{\geq 0}\text{-span}\, S_j. \tag{2.7}$$

Proposition 2.6. *Let X be a $(\mathfrak{g}, K)$-module of finite length, and S_j $(1 \leq j \leq N)$ finite subsets of Λ_+ as above. Then, $\mathrm{AS}_K(X) = \mathrm{AS}_K(\mathbb{C}[\mathcal{V}(X)]) = \bigcup_{j=1}^{N} \mathbb{R}_{\geq 0}\text{-span}\, S_j$.*

Proof. By Lemmas 2.4 and 2.5, we have

$$\mathrm{AS}_K(X) \subset \bigcup_{j=1}^{N} \mathrm{AS}_K(\mathbb{C}[\overline{\mathcal{O}_j}] \otimes F_j) = \bigcup_{j=1}^{N} \mathrm{AS}_K(\mathbb{C}[\overline{\mathcal{O}_j}]).$$

Again, by Lemmas 2.4 and 2.5, we get the reverse inclusion:

$$\mathrm{AS}_K(X) = \mathrm{AS}_K(X \otimes F_j^*) \supset \mathrm{AS}_K(\mathbb{C}[\overline{\mathcal{O}_j}]).$$

By (2.7), we obtain Proposition 2.6. $\square$

We note that $\mathrm{AS}_K(X) = \{0\}$ if and only if $\mathrm{Supp}_K(X)$ is a finite set. When X is a $(\mathfrak{g}, K)$-module of finite length, this is equivalent to the condition $\mathcal{V}(X) = \{0\}$, or equivalently, $\dim X < \infty$.

2.7. Transversality of the K-supports of two K-modules

In this section we formulate the "stability of the transversality" of the K-supports of two K-modules under taking the tensor product with finite-dimensional representations.

Lemma 2.7. *Let X and Y be K-modules.*

(1) *For any finite-dimensional K-module F, we have*

$$\sharp\,(\mathrm{Supp}_K(X) \cap \mathrm{Supp}_K(Y \otimes F)) \leq \dim F\ \sharp\,(\mathrm{Supp}_K(X \otimes F^*) \cap \mathrm{Supp}_K(Y))\,.$$

(2) *The following two conditions are equivalent:*

(i) $\sharp\,(\mathrm{Supp}_K(X \otimes F^*) \cap \mathrm{Supp}_K(Y)) < \infty$ *for any finite-dimensional representation F of K.*

(ii) $\sharp\,(\mathrm{Supp}_K(X \otimes F_1) \cap \mathrm{Supp}_K(Y \otimes F_2)) < \infty$ *for any finite-dimensional representations F_1 and F_2 of K.*

Proof. (1) Suppose $\mu \in \mathrm{Supp}_K(X) \cap \mathrm{Supp}_K(Y \otimes F)$. Since V_μ occurs in $V_\nu \otimes F$ for some $\nu \in \mathrm{Supp}_K(Y)$, one finds a weight υ of F such that

$$\mu = \nu + \upsilon. \tag{2.8}$$

Then we have $\mathrm{Hom}_K(V_\nu, X \otimes F^*) = \mathrm{Hom}_K(V_\nu \otimes F, X) \supset \mathrm{Hom}_K(V_\mu, V_\mu) \neq \{0\}$. Hence $\nu \in \mathrm{Supp}_K(X \otimes F^*)$. The above consideration yields to a (non-canonical) map

$$\mathrm{Supp}_K(X) \cap \mathrm{Supp}_K(Y \otimes F) \to \mathrm{Supp}_K(X \otimes F^*) \cap \mathrm{Supp}_K(Y), \quad \mu \mapsto \nu \tag{2.9}$$

with constraints (2.8). The cardinality of each fiber of the map (2.9) bounded by $\dim F$. Hence (1) is proved.

(2) The second assertion is a direct consequence of (1) by setting $F = F_1 \otimes F_2^*$. $\square$

2.8. Admissible restriction and regular functions on $K_{\mathbb{C}}/K'_{\mathbb{C}}$

Let K' be a closed subgroup of a compact Lie group K, and $K'_{\mathbb{C}} \subset K_{\mathbb{C}}$ be their complexifications. In this section we relate K'-admissibility of the restriction of a K-module with the K-support of the space $\mathbb{C}[K_{\mathbb{C}}/K'_{\mathbb{C}}]$ of regular functions on $K_{\mathbb{C}}/K'_{\mathbb{C}}$.

Lemma 2.8. *The following three conditions on a K-module X are equivalent:*

(i) *X is K'-admissible.*

(ii) *$X \otimes F'$ is K'-admissible for any finite-dimensional representation F' of K'.*

(iii) *X is K-admissible, and for any finite-dimensional representation F of K,*

$$\sharp\,(\mathrm{Supp}_K(X \otimes F) \cap \mathrm{Supp}_K(\mathbb{C}[K_{\mathbb{C}}/K'_{\mathbb{C}}])) < \infty. \tag{2.10}$$

Proof. The implication (i) $\Leftarrow$ (ii) is obvious.
(i) $\Rightarrow$ (ii): Suppose (i) holds. Then for any $\tau \in \widehat{K'}$, we have

$$\dim \mathrm{Hom}_{K'}(\tau, X \otimes F') = \dim \mathrm{Hom}_{K'}(\tau \otimes (F')^*, X) < \infty$$

because $\tau \otimes (F')^*$ is a finite direct sum of irreducible K'-modules. Hence (ii) is proved.
(ii) $\Rightarrow$ (iii): The K-admissibility is obvious from the K'-admissibility. Let us verify (2.10). Let $\mathbf{1}$ denote the one-dimensional trivial representation of K. Then we have

$$\sharp\{\mu \in \mathrm{Supp}_K(X \otimes F) : \mathrm{Hom}_{K'}(\mathbf{1}, \mu|_{K'}) \neq \{0\}\} \leq \dim \mathrm{Hom}_{K'}(\mathbf{1}, X \otimes F),$$

which is finite by the condition (ii). Hence (2.10) holds.
(iii) $\Rightarrow$ (ii): Fix any $\tau \in \widehat{K'}$, and any finite-dimensional representation F of K. Let $\mathrm{Ind}_{K'}^{K}\tau$ be an (algebraically) induced representation. We define a subset of $\widehat{K}$ by

$$\mathcal{P} := \mathrm{Supp}_K(\mathrm{Ind}_{K'}^{K}\tau) \cap \mathrm{Supp}_K(X \otimes F). \tag{2.11}$$

We claim $\mathcal{P}$ is a finite set. To see this, we take a finite-dimensional K-module F_1 such that $\mathrm{Hom}_{K'}(\tau, F_1|_{K'}) \neq \{0\}$. Then, we have

$$\mathrm{Ind}_{K'}^{K}\tau \leq_K \mathrm{Ind}_{K'}^{K}(F_1|_{K'}) \simeq \mathbb{C}[K_{\mathbb{C}}/K'_{\mathbb{C}}] \otimes F_1$$

as K-modules. In particular, we have

$$\mathcal{P} \subset \mathrm{Supp}_K(\mathbb{C}[K_{\mathbb{C}}/K'_{\mathbb{C}}] \otimes F_1) \cap \mathrm{Supp}_K(X \otimes F) \tag{2.12}$$

The right-hand side of (2.12) is a finite set by the assumption (iii) and Lemma 2.7 (2). Therefore, $\mathcal{P}$ is a finite set.

Next, let us consider the following equation:

$$\dim \operatorname{Hom}_{K'}(\tau, X \otimes F) = \sum_{\mu \in \widehat{K}} \dim \operatorname{Hom}_{K'}(\tau, \mu) \ \dim \operatorname{Hom}_K(\mu, X \otimes F) \tag{2.13}$$

The summation in (2.13) is actually taken over the finite set $\mathcal{P}$. Furthermore, each summand is finite because $X \otimes F$ is K-admissible. Hence, (2.13) is finite. This means that $X \otimes F$ is K'-admissible. Since F is an arbitrary finite-dimensional representation of K, (ii) follows. □

2.9. Proof of Theorem 1.1

We are ready to complete the proof of the main result of this article.

Proof of Theorem 1.1. Let $\mathcal{V}(X)$ be the associated variety of a $(\mathfrak{g}, K)$-module X, and $\mathcal{V}(X) = \overline{\mathcal{O}_1} \cup \cdots \cup \overline{\mathcal{O}_N}$ the decomposition into irreducible components. By Lemma 2.2, there are finite subsets $S_1, \cdots, S_N$ and T such that

$$\operatorname{Supp}_K(\mathbb{C}[\overline{\mathcal{O}_j}]) = \mathbb{Z}_{\geq 0}\text{-span}\, S_j \quad (1 \leq j \leq N), \qquad \operatorname{Supp}_K(\mathbb{C}[K_{\mathbb{C}}/K'_{\mathbb{C}}]) = \mathbb{Z}_{\geq 0}\text{-span}\, T.$$

In place of the conditions (i) and (ii) in Theorem 1.1, we consider the following conditions:
(i)$'$: $\sharp\,(\operatorname{Supp}_K(X \otimes F) \cap \operatorname{Supp}_K(\mathbb{C}[K_{\mathbb{C}}/K'_{\mathbb{C}}])) < \infty$ for any finite-dimensional representation F.
(ii)$'$: $\mathbb{R}_{\geq 0}\text{-span}\, S_j \cap \mathbb{R}_{\geq 0}\text{-span}\, T = \{0\}$ for any $j = 1, \ldots, N$.

We already know the equivalence (i) $\Leftrightarrow$ (i)$'$ from Lemma 2.8, and the equivalence (ii) $\Leftrightarrow$ (ii)$''$ from Propositions 2.3 and 2.6. Thus, the proof of Theorem 1.1 will be completed if we show the equivalence (i)$'$ $\Leftrightarrow$ (ii)$'$.
(i)$'$ $\Rightarrow$ (ii)$'$: If (i)$'$ holds, then Lemma 2.4 implies

$$\sharp \left(\operatorname{Supp}_K(\mathbb{C}[\overline{O_j}]) \cap \operatorname{Supp}_K(\mathbb{C}[K_{\mathbb{C}}/K'_{\mathbb{C}}])\right) < \infty,$$

or equivalently, $\sharp\,(\mathbb{Z}_{\geq 0}\text{-span}\, S_j \cap \mathbb{Z}_{\geq 0}\text{-span}\, T) < \infty$, whence the condition (ii)$'$ follows from Lemma 2.1.
(ii)$'$ $\Rightarrow$ (i)$'$: Let F_j be as in Lemma 2.4. It follows from (2.5) that

$$\operatorname{Supp}_K(X \otimes F) \subset \bigcup_{j=1}^{N} \operatorname{Supp}_K(\mathbb{C}[\overline{\mathcal{O}_j}] \otimes F_j \otimes F).$$

Take $\delta := \max\{\|\nu\| : \nu \text{ is a weight of } F_j \otimes F \text{ for some } j\}$. Then,

$$\begin{aligned} &\subset \bigcup_{j=1}^{N} \delta\text{-neighborhood of } \operatorname{Supp}_K(\mathbb{C}[\overline{\mathcal{O}_j}]) \\ &\subset \bigcup_{j=1}^{N} \delta\text{-neighborhood of } \mathbb{R}_{\geq 0}\text{-span}\, S_j. \end{aligned}$$

Since the condition (i)$'$ implies that the intersection of $\mathbb{R}_{\geq 0}$-span T with any δ-neighborhood of $\mathbb{R}_{\geq 0}$-span S_j is relatively compact (Lemma 2.1), we get

$$\sharp\,(\operatorname{Supp}_K(X \otimes F) \cap \mathbb{Z}_{\geq 0}\text{-span}\, T) < \infty.$$

This shows the implication (ii)$'$ $\Rightarrow$ (i)$'$. Hence Theorem 1.1 is proved. □

2.10. Proof of Corollary 1.4

Proof of Corollary 1.4. The implication (i)$'$ $\Rightarrow$ (i) is obvious, and the reverse implication (i) $\Rightarrow$ (i)$'$ follows from the fact that any discrete summand in the restriction $\pi|_{G'}$ for $\pi \in \operatorname{Disc}(G)$ belongs to $\operatorname{Disc}(G')$, see [14, Cor. 8.7]. Then the implication (i)$'$ $\Rightarrow$ (ii) follows from the fact that for every $\mu \in \widehat{K'}$ there are at most finitely many elements in $\operatorname{Disc}(G')$ having μ as a K'-type, whereas the implication (ii) $\Rightarrow$ (i) is proved in [11, Thm. 1.2]. Since the equivalence (ii) $\Leftrightarrow$ (iii) holds by Theorem 1.1, Corollary 1.4 is proved. □

§3 $(\mathfrak{g}, K)$-modules with finite weight multiplicities

In this section, we relate weight multiplicities for $(\mathfrak{g}, K)$-modules with Kostant's convexity theorem [23].

3.1. Simple Lie groups of (non)Hermitian type

Let G be a real reductive linear Lie group, K a maximal compact subgroup, Z_K the center of K, and T^s a maximal torus of the derived group $K^s := [K, K]$. Then $T := T^s Z_K$ is a maximal torus of K. When G is a simple Lie group, Z_K is at most one-dimensional.

A simple Lie group G (or its Lie algebra $\mathfrak{g}$) is called *of Hermitian type*, if Z_K is one-dimensional, or equivalently, if the associated Riemannian symmetric space G/K

is a Hermitian symmetric space. It is the case when the Lie algebra $\mathfrak{g}$ is $\mathfrak{su}(p,q)$, $\mathfrak{so}(2n)$, $\mathfrak{so}^*(2n)$, $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{e}_{6(-14)}$, or $\mathfrak{e}_{7(-25)}$, whereas $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ $(n \neq 2)$, $\mathfrak{so}(p,q)$ $(p,q \neq 2)$, $\mathfrak{su}^*(2n)$, $\mathfrak{sp}(p,q)$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, or $\mathfrak{sp}(n,\mathbb{C})$ are not of Hermitian type.

3.2. Admissibility for the restriction to toral subgroups

In contrast to $\mathfrak{g}$-modules in the BGG category $\mathcal{O}$, there are not many $(\mathfrak{g},K)$-modules with finite weight multiplicities. We formulate this feature as follows.

Theorem 3.1. *Suppose that X is $(\mathfrak{g},K)$-module of finite length. If $\dim X = \infty$ then $\dim \operatorname{Hom}_{T^s}(\chi, X) = \infty$ for some $\chi \in \widehat{T^s}$.*

We shall see that Theorem 3.1 is derived from Kostant's convexity theorem (Fact 3.6) and from Theorem 1.1. The following two corollaries for simple Lie groups G are immediate consequence of Theorem 3.1 and its proof (Section 3.3).

Corollary 3.2. *Suppose G is not of Hermitian type. Then for any infinite-dimensional irreducible $(\mathfrak{g},K)$-module X, there exists $\chi \in \widehat{T}$ such that $\dim \operatorname{Hom}_T(\chi, X) = \infty$.*

Corollary 3.3. *Suppose G is of Hermitian type, and X a $(\mathfrak{g},K)$-module of finite length. Then X is T-admissible if and only if X is Z_K-admissible.*

Remark 3.4. An irreducible $(\mathfrak{g},K)$-module X is called a highest weight module if X is $\mathfrak{b}$-finite for some Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. There exist infinite-dimensional irreducible highest weight $(\mathfrak{g},K)$-modules if and only if G is of Hermitian type. In this case any such X is Z_K-admissible (see [12, Rem. 3.5 (3)]), hence X is also T-admissible.

Corollary 3.3 fits well into the Kirillov–Kostant–Duflo orbit philosophy (see [3, 10, 22] for instance):

Proposition 3.5 ([19])**.** *Suppose G is a simple Lie group of Hermitian type, and $\mathcal{O}$ a coadjoint orbit in $\mathfrak{g}^*$. Then the following two conditions are equivalent:*

(i) *The momentum map $\mathcal{O} \to \mathfrak{t}^*$ is proper.*

(ii) *The momentum map $\mathcal{O} \to \mathfrak{z}_{\mathfrak{k}}^*$ is proper.*

3.3. An application of Kostant convexity theorem

Suppose K is a connected compact Lie group, and T is a maximal torus of K. Let W_K be the Weyl group for the root system $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. By a K-invariant inner product $\langle\, ,\rangle$ on $\mathfrak{k}$, we identify $\mathfrak{t}^{\perp}$ $(\subset \mathfrak{k}^*)$ with the orthogonal complementary subspace of $\mathfrak{k}$, and write $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{t}} \colon \mathfrak{k} \to \mathfrak{t}$ for the projection with respect to the direct sum decomposition $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{t}^{\perp}$.

For a finite subset $S = \{s_1, \cdots, s_k\}$ of $\mathfrak{t}$, the convex hull of S is the smallest convex set containing S, which is expressed as:

$$\mathrm{Conv}(S) := \left\{ \sum_{i=1}^{k} a_i s_i : a_1, \cdots, a_k \geq 0, \quad a_1 + \cdots + a_k = 1 \right\}.$$

We recall Kostant's convexity theorem:

Fact 3.6 ([23, Thm. 8.2])**.** *For any $Y \in \mathfrak{t}$, we have* $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{t}}(\mathrm{Ad}(K)Y) = \mathrm{Conv}(W_K Y)$.

Fact 3.6 determines the momentum set $\Delta(T^*(K/T))$ of the cotangent bundle of the flag manifold K/T as follows:

Proposition 3.7. *Suppose that K is a connected semisimple compact Lie group. Then*

$$\Delta(T^*(K/T)) = C_K(T) = \mathfrak{t}^*_+.$$

Proof. Fix a nonzero element $Y \in \mathfrak{t}$. Then Kostant's convexity theorem shows that $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{t}}(\mathrm{Ad}(K)Y)$ contains the origin 0. In particular, there exists $k \in K$ such that $Y' := \mathrm{Ad}(k)Y \in \mathfrak{t}^{\perp}$. This means that $Y \in \mathrm{Ad}(K)\mathfrak{t}^{\perp}$, hence $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{t}}(\mathrm{Ad}(K)\mathfrak{t}^{\perp}) = \mathfrak{t}$. By (2.4), we get Proposition 3.7. $\square$

Proof of Theorem 3.1. Applying Proposition 3.7 to K^s/T^s, we obtain $C_K(T^s) = \mathfrak{t}^*_+$ because $K = K^s Z_K$. In turn, Theorem 1.1 tells that X is T^s-admissible if and only if $\mathrm{AS}_K(X) = \{0\}$, or equivalently, $\dim X < \infty$. $\square$

Proof of Corollary 3.2. Immediate from Theorem 3.1 because $T = T^s$. $\square$

Proof of Corollary 3.3. We regard $(\mathfrak{t}^s)^*$ as a subspace of $\mathfrak{t}^*$ via the direct sum decomposition $\mathfrak{t} = \mathfrak{t}^s \oplus \mathfrak{z}_{\mathfrak{k}}$. By Proposition 3.7, we have $C_K(T) = \mathfrak{t}^*_+ \cap (\mathfrak{t}^s)^* = C_K(Z_K)$, whence Corollary 3.3. $\square$

§4 Admissible restriction of degenerate principal series representations

In the orbit philosophy due to Kirillov–Kostant, the Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$ are supposed to be attached to elliptic coadjoint orbits, whereas parabolically induced representations $\mathrm{Ind}_Q^G(\mathbb{C}_\lambda)$ are to hyperbolic coadjoint orbits. Classification theory of admissible restrictions has been developed mainly for $A_{\mathfrak{q}}(\lambda)$, see [2, 11, 13, 16, 20] for example. In this section we apply Theorem 1.1 to induced representations from a parabolic subgroup Q of G and to their subquotient modules (Q-*series*).

4.1. Irreducible representations in the Q-series

Suppose that Q is a parabolic subgroup of a reductive Lie group G.

Definition 4.1. An irreducible admissible representation π of G is said to be in the *Q-series* if π occurs as a subquotient of the induced representation $\mathrm{Ind}_Q^G \tau$ from a finite-dimensional representation τ of Q.

Example 4.2. When $Q = G$, π is in the Q-series if and only if $\dim \pi < \infty$.

Example 4.3. When Q is a minimal parabolic subgroup P, any irreducible admissible representation of G belongs to the Q-series by Harish-Chandra's subquotient theorem.

The next example is a generalization of Example 4.3.

Example 4.4. Let G/H be a reductive symmetric space, that is, H is an open subgroup of $G^\sigma = \{g \in G : \sigma g = g\}$ for some involutive automorphism σ of a real reductive Lie group G. Take a Cartan involution θ of G commuting with σ, and a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{g}^{-\sigma,-\theta} = \{X \in \mathfrak{g} : \sigma X = \theta X = -X\}$. Let Q be a parabolic subgroup of G defined by a generic element $X \in \mathfrak{a}$, that is, Q is the normalizer of the real parabolic subalgebra:

$$\mathfrak{q} = \text{the sum of the eigenspaces of } \mathrm{ad}(X) \text{ with nonnegative eigenvalues.}$$

Such Q is uniquely determined up to conjugation by an element of G. We say that Q is a *minimal parabolic subgroup for G/H*.

Then any irreducible representation that can be realized as a subquotient in the regular representation on $C^\infty(G/H)$ belongs to the Q-series.

4.2. Restriction of representations in the Q-series

We give a necessary and sufficient condition for all irreducible representations in the Q-series to be K'-admissible where K' is a (not necessarily, maximal) compact subgroup.

Theorem 4.5. *Let G be a real reductive linear Lie group, K a maximal compact subgroup, K' a closed subgroup of K, and Q a parabolic subgroup of G. Then the following two conditions are equivalent:*

(i) *for any irreducible representation π of G in the Q-series, $\pi|_{K'}$ is K'-admissible;*

(ii) $C_K(Q \cap K) \cap C_K(K') = \{0\}$.

Proof. Since the induced representation $\mathrm{Ind}_Q^G(\tau)$ is of finite length as a G-module, the condition (i) is equivalent to the following condition:
(i)$'$ $\mathrm{Ind}_Q^G(\tau)$ is K'-admissible for any finite-dimensional representation τ of Q.
By Proposition 2.3 and Lemma 2.5, the asymptotic K-support of $\mathrm{Ind}_Q^G(\tau)$ is given by

$$\mathrm{AS}_K(\mathrm{Ind}_Q^G(\tau)) = \mathrm{AS}_K(\mathrm{Ind}_{Q\cap K}^K(\tau|_{Q\cap K})) = \mathrm{AS}_K(\mathrm{Ind}_{Q\cap K}^K(\mathbf{1})) = C_K(Q\cap K). \tag{4.1}$$

Hence Theorem 4.5 is derived from Theorem 1.1. $\square$

Let $P = MAN$ be a minimal parabolic subgroup of G. Applying Theorem 4.5 to the case $Q = P$, we obtain from Example 4.3 the following:

Corollary 4.6. *Let K' be a closed subgroup of K. Then the following two conditions are equivalent:*

(i) *any irreducible admissible representation of G is K'-admissible;*

(ii) $C_K(M) \cap C_K(K') = \{0\}$.

Remark 4.7. When G is of real rank one, then K/M is isomorphic to a sphere. In this case, Vargas [28] classified all subgroups K' satisfying the condition in Corollary 4.6.

Example 4.8. Let $G = SO(2p, 2q)$, and $K' = U(p) \times U(q)$. Suppose Q is a parabolic subgroup of G with Levi subgroup $L \simeq SO(2p-1, 2q-1) \times GL(1, \mathbb{R})$. Then $Q \cap K = L \cap K$, and via the standard basis of $\mathfrak{t}^* \simeq \mathbb{R}^{p+q}$,

$$\begin{aligned} C_K(Q\cap K) =& \{(a, 0, \cdots, 0; b, 0, \cdots, 0) : a, b \geq 0\}, \\ C_K(K') =& \{(x_1, x_1, \cdots, x_{[\frac{p}{2}]}, x_{[\frac{p}{2}]}, (0); y_1, y_1, \cdots) : x_1 \geq x_2 \geq \cdots, y_1 \geq y_2 \geq \cdots\}, \end{aligned}$$

hence $C_K(Q\cap K) \cap C_K(K') = \{0\}$. Thus the criterion (ii) in Theorem 4.5 is fulfilled. Let $G' = U(p, q)$ be the natural subgroup of G containing K'. Then for any irreducible unitary representation π of G in the Q-series is G'-admissible when restricted to the subgroup G' because it is K'-admissible. See [6] and [11] for branching laws of representations π in the Q-series with respect to the pair $(G, G') = (SO(2p, 2q), U(p, q))$.

In Example 4.8, the two polyhedral cones $C_K(Q \cap K)$ and $C_K(K')$ are easy to compute because both $(K, Q \cap K)$ and (K, K') are symmetric pairs. In the next section, we recall some useful general facts for this.

4.3. Momentum set $\Delta(T^*(K/K'))$ for symmetric pair

Suppose that σ is an involutive automorphism of K. We use the same letter σ to denote its differential, and write $\mathfrak{k} = \mathfrak{k}^\sigma + \mathfrak{k}^{-\sigma}$ for the eigenspace decomposition of σ

with eigenvalues $+1$ and -1. We take a σ-stable Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{k}$ such that $\mathfrak{j}^{-\sigma}$ is a maximal abelian subspace of $\mathfrak{k}^{-\sigma}$, and fix a positive system $\Sigma^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}^{-\sigma})$ of the restricted root system $\Sigma(\mathfrak{k}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}^{-\sigma})$. Choose a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ compatible with $\Sigma^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}^{-\sigma})$ in the following sense:

$$\{\alpha|_{\mathfrak{j}_{\mathbb{C}}^{-\sigma}} : \alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})\} \setminus \{0\} = \Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}^{-\sigma}).$$

Let $(\mathfrak{j}^{-\sigma})^*_+$ and $\mathfrak{j}^*_+$ be the dominant chamber for $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}^{-\sigma})$ and $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$, respectively. We may regard $(\mathfrak{j}^{-\sigma})^*_+ \subset \mathfrak{j}^*_+$ according to the direct decomposition $\mathfrak{j} = \mathfrak{j}^\sigma \oplus \mathfrak{j}^{-\sigma}$. When a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is given independently of σ, we choose an inner automorphism of $\mathfrak{k}$ which induces bijections $\iota \colon \mathfrak{t} \xrightarrow{\sim} \mathfrak{j}$ and $\iota^* \colon \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}) \xrightarrow{\sim} \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and set

$$(\mathfrak{t}^{-\sigma})^*_+ := \iota^*((\mathfrak{j}^{-\sigma})^*_+) \subset \sqrt{-1}\mathfrak{t}^*.$$

Proposition 4.9. *Suppose (K, K') is a symmetric pair defined by an involutive automorphism σ. Then $\Delta(T^*(K/K')) = C_K(K') = (\mathfrak{t}^{-\sigma})^*_+$.*

Remark 4.10. When the unipotent radical of Q is abelian, then $(K, Q \cap K)$ forms a symmetric pair, and therefore we can apply also Proposition 4.9 to the computation of $C_K(Q \cap K)$ in Theorem 4.5.

4.4. Boundaries of spherical varieties with hidden symmetries

As typical examples of Theorem 4.5, we formulate the following theorem motivated by analysis on standard pseudo-Riemannian locally symmetric spaces $\Gamma \backslash G/H$ ([9]):

Theorem 4.11. *Let G/H be a symmetric space with G simple Lie group, and Q a minimal parabolic subgroup for G/H. Let G' be a reductive subgroup of G acting properly on G/H, such that $G_{\mathbb{C}}/H_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical. Then any irreducible admissible representation π of G in the Q-series is K'-admissible. In particular, the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable in the sense of [16, Def. 4.2.3].*

Such triples (G, H, G') are classified (cf. [17]). In the setting of Theorem 4.11, the symmetric space G/H admits a compact Clifford–Klein form $\Gamma \backslash G/H$ as the quotient by a torsion-free cocompact subgroup Γ in G'. Applications of Theorem 4.11 will be discussed in subsequent papers. In this article, we illustrate Theorem 4.11 only by some examples:

Example 4.12. The triple $(G, H, G') = (SO(2p, 2q), SO(2p-1, 2q), U(p, q))$ satisfies the assumptions of Theorem 4.5. In this case, Example 4.8 is recovered.

Example 4.13. The triple $(G, H, G') = (SO(8,8), SO(7,8), Spin(1,8))$ satisfies the assumption of Theorem 4.5. Via the standard basis of $\mathfrak{t}^* \simeq \mathbb{R}^8$, we may write as

$$C_K(Q \cap K) = \{(a,0,0,0;b,0,0,0) : a,b \geq 0\},$$
$$C_K(K') = \{((x_1,x_2,x_3,x_4);\zeta(x_1,x_2,x_3,-x_4)) : x_1 \geq x_2 \geq x_3 \geq |x_4|\},$$

where ζ is an outer automorphism of order 3 for the root system D_4. Thus the criterion (ii) in Theorem 4.5 is fulfilled, and Theorem 4.11 is verified in this case. Explicit branching laws of irreducible square-integrable representations in the Q-series with respect to $(G, G') = (SO(8,8), Spin(1,8))$ are obtained in [17, Thm. 5.5] and in [26].

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