

# Conformal symmetry breaking on differential forms and some applications

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**Abstract.** Rapid progress has been made recently on symmetry breaking operators for real reductive groups. Based on Program A–C for branching problems (T. Kobayashi [Progr. Math. 2015]), we illustrate a scheme of the classification of (local and nonlocal) symmetry breaking operators by an example of conformal representations on differential forms on the model space  $(X, Y) = (S^n, S^{n-1})$ , which generalizes the scalar case (Kobayashi–Speh [Mem. Amer. Math. Soc. 2015]) and the case of local operators (Kobayashi–Kubo–Pevzner [Lect. Notes Math. 2016]). Some applications to automorphic form theory, motivations from conformal geometry, and the methods of proofs are also discussed.

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## 1. Branching problems—Stages A to C

Suppose  $\Pi$  is an irreducible representation of a group  $G$ . We may regard  $\Pi$  as a representation of its subgroup  $G'$  by restriction, which we denote by  $\Pi|_{G'}$ . The restriction  $\Pi|_{G'}$  is not irreducible in general. In case it can be given as the direct sum of irreducible  $G'$ -modules, the decomposition is called the *branching law* of the restriction  $\Pi|_{G'}$ .

*Example 1* (fusion rule). Let  $\pi_1$  and  $\pi_2$  be representations of a group  $H$ . The outer tensor product  $\Pi := \pi_1 \boxtimes \pi_2$  is a representation of the product group  $G := H \times H$ , and its restriction  $\Pi|_{G'}$  to the subgroup  $G' := \text{diag}(H)$  is nothing but the tensor product representation  $\pi_1 \otimes \pi_2$ . In this case, the branching law is called the *fusion rule*.

For real reductive Lie groups such as  $G = GL(n, \mathbb{R})$  or  $O(p, q)$ , irreducible representations  $\Pi$  are usually infinite-dimensional and do not always

possess highest weight vectors, consequently, the restriction  $\Pi|_{G'}$  to subgroups  $G'$  may involve various (sometimes “wild”) aspects:

*Example 2.* The fusion rule of two irreducible unitary principal series representations of  $GL(n, \mathbb{R})$  ( $n \geq 3$ ) involve continuous spectrum and infinite multiplicities in the direct integral of irreducible unitary representations.

By the *branching problem* (in a wider sense than the usual), we mean the problem of understanding how the restriction  $\Pi|_{G'}$  behaves as a representation of the subgroup  $G'$ . We treat non-unitary representations  $\Pi$  as well. In this case, instead of considering the irreducible decomposition of the restriction  $\Pi|_{G'}$ , we may investigate continuous  $G'$ -homomorphisms

$$T: \Pi|_{G'} \rightarrow \pi$$

to irreducible representations  $\pi$  of the subgroup  $G'$ . We call  $T$  a *symmetry breaking operator* (SBO, for short). The dimension of the space of symmetry breaking operators

$$m(\Pi, \pi) := \dim_{\mathbb{C}} \text{Hom}_{G'}(\Pi|_{G'}, \pi)$$

may be thought of as a variant of the “multiplicity”. Finding a formula of  $m(\Pi, \pi)$  is a substitute of the branching law  $\Pi|_{G'}$  when  $\Pi$  is not a unitary representation.

The author proposed in [19] a program for branching problems in the following three stages:

**Stage A.** Abstract feature of the restriction  $\Pi|_{G'}$ .

**Stage B.** Branching laws.

**Stage C.** Construction of symmetry breaking operators.

Loosely speaking, Stage B concerns a decomposition of representations, whereas Stage C asks for a decomposition of vectors.

For “abstract features” of the restriction in Stage A, we may think of the following aspects:

**A.1.** Spectrum of the restriction  $\Pi|_{G'}$ :

- (discretely decomposable case, [12, 14, 15]) branching problems could be studied purely algebraic and combinatorial approaches;
- (continuous spectrum) branching problems may be of analytic feature (*e.g.*, Example 2).

**A.2.** Estimate of multiplicities for the restriction  $\Pi|_{G'}$ :

- multiplicities may be infinite (see Example 2);
- multiplicities may be at most one in special settings (*e.g.*, theta correspondence [7], Gross–Prasad conjecture [6], real forms of strong Gelfand pairs [35], visible actions [17], *etc.*).

The goal of Stage A in branching problems is to analyze aspects such as A.1 and A.2 in complete generality. If multiplicities of the restriction  $\Pi|_{G'}$  are known *a priori* to be bounded in Stage A, one might be tempted to

find irreducible decompositions (Stage B), and moreover to construct explicit symmetry breaking operators (Stage C). Thus, results in Stage A might also serve as a foundation for further detailed study of the restriction  $\Pi|_{G'}$  (Stages B and C).

This article is divided into three parts. First, we discuss Stage A in Section 3 with focus on multiplicities in both regular representations on homogeneous spaces and branching problems based on a joint work [26] with T. Oshima, and give some perspectives of the subject through the classification theory [23] joint with T. Matsuki about the pairs  $(G, G')$  for which multiplicities in branching laws are always finite.

Second, we take  $(G, G')$  to be  $(O(n+1, 1), O(n, 1))$  as an example of such pairs, and explain the first test case for the classification problem of symmetry breaking operators (Stages B and C). The choice of our setting is motivated by conformal geometry, and is also related to the local Gross–Prasad conjecture [6, 31]. We survey the classification theory of conformally covariant SBO for differential forms on the model space  $(X, Y) = (S^n, S^{n-1})$ : for local operators based on a recent book [21] with T. Kubo and M. Pevzner in Section 5 and for nonlocal operators based on a recent monograph [29] with B. Speh and its generalization [30] in Section 6.

In Section 7, we discuss an ongoing work with Speh on some applications of these results to a question from automorphic form theory, in particular, about the periods of irreducible representations with nonzero  $(\mathfrak{g}, K)$ -cohomologies. The resulting condition to admit periods is compared with a recent  $L^2$ -theory [1] joint with Y. Benoist.

Detailed proofs of the new results in Sections 6 and 7 will be given in separate papers [20, 30].

**Notation.**  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

## 2. Preliminaries: smooth representations

We would like to treat non-unitary representations as well for the study of branching problems. For this we recall some standard concepts of continuous representations of Lie groups.

Suppose  $\Pi$  is a continuous representation of  $G$  on a Banach space  $V$ . A vector  $v \in V$  is said to be *smooth* if the map  $G \rightarrow V$ ,  $g \mapsto \Pi(g)v$  is of  $C^\infty$ -class. Let  $V^\infty$  denote the space of smooth vectors of the representation  $(\Pi, V)$ . Then  $V^\infty$  is a  $G$ -invariant dense subspace of  $V$ , and  $V^\infty$  carries a Fréchet topology with a family of semi-norms  $\|v\|_{i_1 \dots i_k} := \|d\Pi(X_{i_1}) \cdots d\Pi(X_{i_k})v\|$ , where  $\{X_1, \dots, X_n\}$  is a basis of the Lie algebra  $\mathfrak{g}_0$  of  $G$ . Thus we obtain a continuous Fréchet representation  $(\Pi^\infty, V^\infty)$  of  $G$ .

Suppose now that  $G$  is a real reductive linear Lie group,  $K$  a maximal compact subgroup of  $G$ , and  $\mathfrak{g}$  the complexification of the Lie algebra  $\mathfrak{g}_0$  of  $G$ . Let  $\mathcal{HC}$  denote the category of Harish-Chandra modules whose objects and

morphisms are  $(\mathfrak{g}, K)$ -modules of finite length and  $(\mathfrak{g}, K)$ -homomorphisms, respectively. Let  $\Pi$  be a continuous representation of  $G$  on a complete locally convex topological vector space  $V$ . Assume that the  $G$ -module  $\Pi$  is of finite length. We say  $\Pi$  is *admissible* if

$$\dim_{\mathbb{C}} \text{Hom}_K(\tau, \Pi|_K) < \infty$$

for all irreducible finite-dimensional representations  $\tau$  of  $K$ . We denote by  $V_K$  the space of  $K$ -finite vectors. Then  $V_K \subset V^\infty$  and the Lie algebra  $\mathfrak{g}$  leaves  $V_K$  invariant. The resulting  $(\mathfrak{g}, K)$ -module on  $V_K$  is called the underlying  $(\mathfrak{g}, K)$ -module of  $\Pi$ , and will be denoted by  $\Pi_K$ .

For any admissible representation  $\Pi$  on a Banach space  $V$ , the smooth representation  $(\Pi^\infty, V^\infty)$  depends only on the underlying  $(\mathfrak{g}, K)$ -module. We say  $(\Pi^\infty, V^\infty)$  is an *admissible smooth representation*. By the Casselman–Wallach globalization theory,  $(\Pi^\infty, V^\infty)$  has moderate growth, and there is a canonical equivalence of categories between the category  $\mathcal{HC}$  of Harish-Chandra modules and the category of admissible smooth representations of  $G$  ([37, Chap. 11]). In particular, the Fréchet representation  $\Pi^\infty$  is uniquely determined by its underlying  $(\mathfrak{g}, K)$ -module. We say  $\Pi^\infty$  is the *smooth globalization* of  $\Pi_K \in \mathcal{HC}$ .

For simplicity, by an *irreducible smooth representation*, we shall mean an irreducible admissible smooth representation of  $G$ . We denote by  $\widehat{G}_{\text{smooth}}$  the set of equivalence classes of irreducible smooth representations of  $G$ . Via the underlying  $(\mathfrak{g}, K)$ -modules, we may regard the unitary dual  $\widehat{G}$  as a subset of  $\widehat{G}_{\text{smooth}}$ .

### 3. Multiplicities in symmetry breaking

Let  $G \supset G'$  be a pair of real reductive groups. For  $\Pi \in \widehat{G}_{\text{smooth}}$  and  $\pi \in \widehat{G}'_{\text{smooth}}$ , we denote by  $\text{Hom}_{G'}(\Pi|_{G'}, \pi)$  the space of symmetry breaking operators, and define the *multiplicity* (for smooth representations) by

$$m(\Pi, \pi) := \dim_{\mathbb{C}} \text{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}. \quad (1)$$

Note that  $m(\Pi, \pi)$  is well-defined without the unitarity assumption on  $\Pi$  and  $\pi$ .

We established a geometric criterion for multiplicities to be finite (more strongly, to be bounded) as follows:

**Theorem 3** ([26], see also [13, 18]). *Let  $G \supset G'$  be a pair of real reductive algebraic Lie groups.*

- (1) *The following two conditions on the pair  $(G, G')$  are equivalent:*
  - (FM) (finite multiplicities)  $m(\Pi, \pi) < \infty$  for all  $\Pi \in \widehat{G}_{\text{smooth}}$  and  $\pi \in \widehat{G}'_{\text{smooth}}$ ;
  - (PP) (geometry)  $(G \times G')/\text{diag}(G')$  is real spherical.
- (2) *The following two conditions on the pair  $(G, G')$  are equivalent:*

(BM) (bounded multiplicities) *There exists  $C > 0$  such that*

$$m(\Pi, \pi) \leq C \quad \text{for all } \Pi \in \widehat{G}_{\text{smooth}} \text{ and } \pi \in \widehat{G}'_{\text{smooth}};$$

(BB) (complex geometry)  *$(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\text{diag}(G'_{\mathbb{C}})$  is spherical.*

Here we recall that a connected complex manifold  $X_{\mathbb{C}}$  with holomorphic action of a complex reductive group  $G_{\mathbb{C}}$  is called *spherical* if a Borel subgroup of  $G_{\mathbb{C}}$  has an open orbit in  $X_{\mathbb{C}}$ . There has been an extensive study of spherical varieties in algebraic geometry and finite-dimensional representation theory. In contrast, concerning the real setting, in search of a good framework for global analysis on homogeneous spaces which are broader than the usual (*e.g.*, reductive symmetric spaces), the author proposed:

**Definition 4** ([13]). Let  $G$  be a real reductive Lie group. We say a connected smooth manifold  $X$  with smooth  $G$ -action is *real spherical* if a minimal parabolic subgroup  $P$  of  $G$  has an open orbit in  $X$ .

We discovered in [13, 26] that these geometric properties (spherical/real spherical) are exactly the conditions that a reductive group  $G$  has a “strong grip” of the space of functions on  $X$  in the context of multiplicities of (infinite-dimensional) irreducible representations occurring in the regular representation of  $G$  on  $C^{\infty}(X)$ :

**Theorem 5** ([26, Thms. A and C]). *Suppose  $G$  is a real reductive linear Lie group,  $H$  is an algebraic reductive subgroup, and  $X = G/H$ .*

(1) *The homogeneous space  $X$  is real spherical if and only if*

$$\dim_{\mathbb{C}} \text{Hom}_G(\pi, C^{\infty}(X)) < \infty \text{ for all } \pi \in \widehat{G}_{\text{smooth}}.$$

(2) *The complexification  $X_{\mathbb{C}}$  is spherical if and only if*

$$\sup_{\pi \in \widehat{G}_{\text{smooth}}} \dim_{\mathbb{C}} \text{Hom}_G(\pi, C^{\infty}(X)) < \infty.$$

**Methods of proof.** In [26], we obtained not only the equivalences in Theorem 5 but also quantitative estimates of the dimension. The proof for the upper estimate in [26] uses the theory of regular singularities of a system of partial differential equations by taking an appropriate compactification with normal crossing boundaries, whereas the proof for the lower estimate uses the construction of a “generalized Poisson transform”. Furthermore, these estimates hold for the representations of  $G$  on the space of smooth sections for equivariant vector bundles over  $X = G/H$  without assuming that  $H$  is reductive. For instance, this applies also to the case where  $H$  is a maximal unipotent subgroup of  $G$ , giving a Kostant–Lynch estimate to the dimension of the space of Whittaker vectors ([26, Ex. 1.4 (3)]).

Back to Theorem 3 on branching problems, the geometric estimates of multiplicities is proved by applying Theorem 5 to the pair  $(G \times G', \text{diag}(G'))$  together with some careful arguments on topological vector spaces ([18, Thm. 4.1]).

**Classification theory.** Theorem 3 serves Stage A in branching problems, and singles out nice settings in which we could expect to go further on Stages B and C of the detailed study of symmetry breaking.

So it would be useful to develop a classification theory of pairs  $(G, G')$  for which the geometric criteria (PP) or (BB) in Theorem 3 are satisfied.

- The geometric criterion (BB) in Theorem 3 appeared in the context of *finite-dimensional* representations already in 1970s, and such pairs  $(G_{\mathbb{C}}, G'_{\mathbb{C}})$  were classified infinitesimally, see [32]. The classification of real forms  $(G, G')$  satisfying the condition (BB) follows readily from that of complex pairs  $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ , see [23]. Sun–Zhu [35] proved that the constant  $C$  in Theorem 3 can be taken to be one (*multiplicity-free theorem*) in many of real forms  $(G, G')$ , see [31, Rem. 2.2] for multiplicity-two results for some other real forms.
- The pairs  $(\backslash G \times \backslash G, \text{diag}(\backslash G))$  for real reductive groups  $\backslash G$  satisfying the geometric criterion (PP) in Theorem 3 were classified in [13].
- More generally, symmetric pairs  $(G, G')$  satisfying the geometric criterion (PP) in Theorem 3 was classified by the author and Matsuki [23]. The methods are a linearization technique and invariants of quivers.

In turn, these classification results give an *a priori* estimate of multiplicities in branching problems by Theorem 3.

*Example 6* (finite multiplicities for the fusion rule, [13, Ex. 2.8.6], see also [18, Cor. 4.2]). Suppose  $G$  is a simple Lie group. Then the following two conditions are equivalent:

- (i)  $\dim_{\mathbb{C}} \text{Hom}_G(\pi_1 \otimes \pi_2, \pi_3) < \infty$  for all  $\pi_1, \pi_2, \pi_3 \in \widehat{G}_{\text{smooth}}$ ;
- (ii)  $G$  is either compact or locally isomorphic to  $SO(n, 1)$ .

*Example 7.* Let  $(G, G') = (O(p+r, q), O(r) \times O(p, q))$ .

- (1)  $m(\Pi, \pi) < \infty$  for all  $\Pi \in \widehat{G}_{\text{smooth}}$  and  $\pi \in \widehat{G}'_{\text{smooth}}$ .
- (2)  $m(\Pi, \pi) \leq 1$  for all  $\Pi \in \widehat{G}_{\text{smooth}}$  and  $\pi \in \widehat{G}'_{\text{smooth}}$  if and only if  $p+q+r \leq 4$  or  $r = 1$ .

See [18] for the further classification theory of symmetric pairs  $(G, G')$  that guarantee finite multiplicity properties for symmetry breaking.

## 4. Conformally covariant SBOs

This section discusses a question on symmetry breaking with respect to a pair of conformal manifolds  $X \supset Y$ .

Let  $(X, g)$  be a Riemannian manifold. Suppose that a Lie group  $G$  acts conformally on  $X$ . This means that there exists a positive-valued function  $\Omega \in C^\infty(G \times X)$  (*conformal factor*) such that

$$L_{h^*}^* g_{h \cdot x} = \Omega(h, x)^2 g_x \quad \text{for all } h \in G, x \in X,$$

where we write  $L_h: X \rightarrow X, x \mapsto h \cdot x$  for the action of  $G$  on  $X$ . When  $X$  is oriented, we define a locally constant function

$$or: G \times X \longrightarrow \{\pm 1\}$$

by  $or(h)(x) = 1$  if  $(L_h)_{*x}: T_x X \longrightarrow T_{L_h x} X$  is orientation-preserving, and  $= -1$  if it is orientation-reversing.

Since both the conformal factor  $\Omega$  and the orientation map  $or$  satisfy cocycle conditions, we can form a family of representations  $\varpi_{\lambda, \delta}^{(i)}$  of  $G$  with parameters  $\lambda \in \mathbb{C}$  and  $\delta \in \mathbb{Z}/2\mathbb{Z}$  on the space  $\mathcal{E}^i(X)$  of differential  $i$ -forms on  $X$  ( $0 \leq i \leq \dim X$ ) defined by

$$\varpi_{\lambda, \delta}^{(i)}(h)\alpha := or(h)^\delta \Omega(h^{-1}, \cdot)^\lambda L_{h^{-1}}^* \alpha, \quad (h \in G). \quad (2)$$

The representation  $\varpi_{\lambda, \delta}^{(i)}$  of the conformal group  $G$  on  $\mathcal{E}^i(X)$  will be simply denoted by  $\mathcal{E}^i(X)_{\lambda, \delta}$ , and referred to as the *conformal representation* on differential  $i$ -forms.

Suppose that  $Y$  is an orientable submanifold. Then  $Y$  is endowed with a Riemannian structure  $g|_Y$  by restriction, and we can define in a similar way a family of representations  $\mathcal{E}^j(Y)_{\nu, \varepsilon}$  ( $\nu \in \mathbb{C}, \varepsilon \in \mathbb{Z}/2\mathbb{Z}, 0 \leq j \leq \dim Y$ ) of the conformal group of  $(Y, g|_Y)$ .

We consider the full group of conformal diffeomorphisms and its subgroup defined as

$$\begin{aligned} \text{Conf}(X) &:= \{\text{conformal diffeomorphisms of } (X, g)\}, \\ \text{Conf}(X; Y) &:= \{\varphi \in \text{Conf}(X) : \varphi(Y) = Y\}. \end{aligned} \quad (3)$$

Then there is a natural group homomorphism

$$\text{Conf}(X; Y) \rightarrow \text{Conf}(Y), \quad \varphi \mapsto \varphi|_Y. \quad (4)$$

**Definition 8.** A linear map  $T: \mathcal{E}^i(X)_{\lambda, \delta} \rightarrow \mathcal{E}^j(Y)_{\nu, \varepsilon}$  is a *conformally covariant symmetry breaking operator* (conformally covariant SBO, for short) if  $T$  intertwines the actions of the group  $\text{Conf}(X; Y)$ .

We shall write

$$H \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) := \text{Hom}_{\text{Conf}(X; Y)}(\mathcal{E}^i(X)_{\lambda, \delta} |_{\text{Conf}(X; Y)}, \mathcal{E}^j(Y)_{\nu, \varepsilon}) \quad (5)$$

\(\cup\)

$$D \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) := \text{Diff}_{\text{Conf}(X; Y)}(\mathcal{E}^i(X)_{\lambda, \delta} |_{\text{Conf}(X; Y)}, \mathcal{E}^j(Y)_{\nu, \varepsilon}) \quad (6)$$

for the space of continuous conformally covariant SBOs and its subspace of differential SBOs, namely, those operators  $T$  satisfying the local property:  $\text{Supp}(T\alpha) \subset \text{Supp}(\alpha)$  for all  $\alpha \in \mathcal{E}^i(X)_{\lambda, \delta}$ . This support condition is a generalization of Peetre's characterization [34] of differential operators in the  $X = Y$  case ([27, Def. 2.1], for instance).

We address a general problem motivated by conformal geometry:

**Problem 9 (conformally covariant symmetry breaking operators).** *Let  $X \supset Y$  are orientable Riemannian manifolds.*

- (1) *Determine when  $H \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) \neq \{0\}$ .*
- (2) *Determine when  $D \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) \neq \{0\}$ .*
- (3) *Construct an explicit basis of  $H \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right)$  and  $D \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right)$ .*

Problem 9 (1) and (2) may be thought of as Stage B of branching problems in Section 1, while Problem 9 (3) as Stage C.

In the case where  $X = Y$  and  $i = j = 0$ , a classical prototype of such operators is a second order differential operator called the Yamabe operator

$$\Delta + \frac{n-2}{4(n-1)}\kappa \in \text{Diff}_{\text{Conf}(X)}(\mathcal{E}^0(X)_{\frac{n}{2}-1, \delta}, \mathcal{E}^0(X)_{\frac{n}{2}+1, \delta}),$$

where  $n$  is the dimension of the manifold  $X$ ,  $\Delta$  is the Laplacian, and  $\kappa$  is the scalar curvature, see [24, Thm. A], for instance. Conformally covariant differential operators of higher order are also known: the Paneitz operator (fourth order) [33], or more generally, the so-called GJMS operators [5] are such operators. Turning to operators acting on differential forms, we observe that the exterior derivative  $d$ , the codifferential  $d^*$ , and the Hodge  $*$  operator are also examples of conformally covariant operators on differential forms, namely,  $j = i + 1$ ,  $i - 1$ , and  $n - i$ , respectively, with an appropriate choice of the parameter  $(\lambda, \nu, \delta, \varepsilon)$ . As is well-known, Maxwell's equations in four-dimension can be expressed in terms of conformally covariant operators on differential forms.

Let us consider the general case where  $X \neq Y$ . From the viewpoint of conformal geometry, we are interested in “natural operators”  $T$  that persist for all pairs of Riemannian manifolds  $X \supset Y$  of fixed dimension. We note that Problem 9 is trivial for individual pairs  $X \supset Y$  such that  $\text{Conf}(X; Y) = \{e\}$ , because any linear operator becomes automatically an SBO. In contrast, the larger  $\text{Conf}(X; Y)$  is, the more constraints on  $T$  will be imposed. Thus we highlight the case of large conformal groups as the first step to attack Problem 9.

In general, the conformal group cannot be so large. We recall from [10, Thms. 6.1 and 6.2] the upper estimate of the dimension of the conformal group:

**Fact 10.** *Let  $X$  be a compact Riemannian manifold of dimension  $n \geq 3$ . Then  $\dim \text{Conf}(X) \leq \frac{1}{2}(n+1)(n+2)$ . The equality holds if and only if  $(\text{Conf}(X), X)$  is locally isomorphic to  $(O(n+1, 1), S^n)$ .*

Concerning a pair  $(X, Y)$  of Riemannian manifolds, we obtain the following.



**Proposition 11.** *Let  $X \supset Y$  be Riemannian manifolds of dimension  $n$  and  $m$ , respectively. Then  $\dim \text{Conf}(X; Y) \leq \frac{1}{2}(m+1)(m+2)$ . The equality holds if  $X = S^n$  and  $Y$  is a totally geodesic submanifold which is isomorphic to  $S^m$ .*

*Proof.* The first inequality follows from Fact 10 via the group homomorphism (4). If  $(X, Y) = (S^n, S^m)$ , then  $\text{Conf}(X)$  and  $\text{Conf}(X; Y)$  are locally isomorphic to  $O(n+1, 1)$  and  $O(m+1, 1)$ , respectively, whence the second assertion.  $\square$

From now on, we shall consider the pair

$$(X, Y) = (S^n, S^{n-1}), \tag{7}$$

as a model case with largest symmetries, where  $Y = S^{n-1}$  is embedded as a totally geodesic submanifold of  $X = S^n$ . As mentioned, the pair  $(\text{Conf}(X), \text{Conf}(X; Y))$  is locally isomorphic to the pair

$$(G, G') = (O(n+1, 1), O(n, 1)). \tag{8}$$

We remind that this pair appeared in Section 3 on branching problems, see the case where  $r = 1$  in Example 7. As an *a priori* estimate in Stage A, see Theorem 3 (2), Example 7, [21, Thm. 2.6], and [35], we have

$$\dim_{\mathbb{C}} H \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) \leq 4 \quad \text{for any } (i, j, \lambda, \nu, \delta, \varepsilon). \tag{9}$$

In turn, the estimate (9) gives an upper bound for the dimension of the space of “natural” conformal covariant SBOs,  $\mathcal{E}^i(X)_{\lambda, \delta} \rightarrow \mathcal{E}^j(Y)_{\nu, \varepsilon}$  that persist for all pairs  $X \supset Y$  of codimension one. In the next two sections, we explain briefly a solution to Problem 9 (Stages B and C) in the model case (7).

## 5. Classification theory of conformally covariant differential SBOs

In the case where symmetry breaking operators are given as differential operators, Problem 9 in the model space (7) was solved in a joint work [21] with Kubo and Pevzner. In this section, we introduce its flavors briefly. First of all, the solution to Problem 9 (2), a question in Stage B of branching problems, may be stated as follows.

**Theorem 12.** *Suppose  $n \geq 3$ ,  $0 \leq i \leq n$ ,  $0 \leq j \leq n-1$ ,  $\lambda, \nu \in \mathbb{C}$ , and  $\delta, \varepsilon \in \{\pm\}$ . Then the following three conditions on 6-tuple  $(i, j, \lambda, \nu, \delta, \varepsilon)$  are equivalent:*

- (i)  $D \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) \neq \{0\}$ .
- (ii)  $\dim_{\mathbb{C}} D \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) = 1$ .

(iii) *The parameter  $(i, j, \lambda, \nu, \delta, \varepsilon)$  satisfies*

$$\{j, n - j - 1\} \cap \{i - 2, i - 2, i, i + 1\} \neq \emptyset, \quad (10)$$

$$\nu - \lambda \in \mathbb{N},$$

$$\text{a certain condition } Q \equiv Q_{i,j} \text{ on } (\lambda, \nu, \delta, \varepsilon). \quad (11)$$

The first condition (10) concerns the degrees  $i$  and  $j$  of differential forms. Loosely speaking, conformally covariant differential SBOs exist only if the degrees  $i$  and  $j$  are close to each other or the sum  $i + j$  is close to  $n$ . The last “additional” condition  $Q_{i,j}$  depends on  $(i, j)$ . We give the condition  $Q_{i,j}$  explicitly in the following two cases:

- Case  $j = i$ .  $Q_{i,i}$  amounts to  $\nu \in \mathbb{C}$  and  $\delta \equiv \varepsilon \equiv \nu - \lambda \pmod{2}$ .
- Case  $j = i + 1$ . For  $1 \leq i \leq n - 2$ ,  $Q_{i,i+1}$  amounts to  $(\lambda, \nu) = (0, 0)$  and  $\delta \equiv \varepsilon \equiv 0 \pmod{2}$ ; for  $i = 0$ ,  $Q_{0,1}$  amounts to  $\lambda \in -\mathbb{N}$ ,  $\nu = 0$ , and  $\delta \equiv \varepsilon \equiv \lambda \pmod{2}$ .

See [21, Thm. 1.1] for the precise conditions in the other remaining six cases.

Second, we go on with Problem 9 (3) (Stage C) about the construction of symmetry breaking operators. For this we work with the pair  $(\mathbb{R}^n, \mathbb{R}^{n-1})$  of the flat Riemannian manifolds which are conformal to  $(S^n \setminus \{\text{pt}\}, S^{n-1} \setminus \{\text{pt}\})$  via the stereographic projection.

We begin with a *scalar-valued* operator (Juhl’s operator, [8]). Suppose that our hyperplane  $Y = \mathbb{R}^{n-1}$  of  $X = \mathbb{R}^n$  is defined by  $x_n = 0$  in the coordinates  $(x_1, \dots, x_n)$ . For  $\mu \in \mathbb{C}$  and  $k \in \mathbb{N}$ , we define a homogeneous differential operator of order  $k$  by

$$\mathcal{D}_k^\mu := \sum_{0 \leq i \leq [\frac{k}{2}]} a_i(\mu) (-\Delta_{\mathbb{R}^{n-1}})^i \frac{\partial^{k-2i}}{\partial x_n^{k-2i}} : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n),$$

where  $\{a_i(\mu)\}$  are the coefficients of the Gegenbauer polynomial:

$$C_k^\mu(t) = \sum_{0 \leq i \leq [\frac{k}{2}]} a_i(\mu) t^{k-2i}.$$

Building on the scalar-valued operators, we introduced in [21] *matrix-valued* differential symmetry breaking operators

$$\mathcal{D}_{\lambda,k}^{i \rightarrow j} : \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$$

for each pair  $(i, j)$  satisfying (10). We illustrate a concrete formula when  $j = i$ . We set

$$\mathcal{D}_{\lambda,k}^{i \rightarrow i} := \text{Rest}_{x_n=0} \circ (\mathcal{D}_{k-2}^{\mu+1} dd^* + a \mathcal{D}_{k-1}^\mu dt_{\frac{\partial}{\partial x_n}} + b \mathcal{D}_k^\mu),$$

where  $d^*$  is the codifferential,  $\iota_{\frac{\partial}{\partial x_n}} : \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$  is the inner multiplication of the vector field  $\frac{\partial}{\partial x_n}$ , and

$$a := \begin{cases} 1 & (k: \text{odd}) \\ \lambda + i - \frac{n}{2} + k & (k: \text{even}) \end{cases}, \quad b := \frac{\lambda + k}{2}, \quad \mu := \lambda + i - \frac{n-1}{2}.$$

Thus the operator  $\mathcal{D}_{\lambda,k}^{i \rightarrow i}$  is obtained as the composition of a  $\text{Hom}_{\mathbb{C}}(\wedge^i(\mathbb{C}^n), \wedge^i(\mathbb{C}^{n-1}))$ -valued homogeneous differential operator on  $\mathbb{R}^n$  of order  $k$  with the restriction map to the hyperplane  $\mathbb{R}^{n-1}$ .

The matrix-valued differential operators  $\mathcal{D}_{\lambda,k}^{i \rightarrow j}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$  were defined in [21, Chap. 1] also for the other seven cases when the condition (iii) in Theorem 12 is fulfilled.

**Methods of proof** in finding the formulæ for  $\mathcal{D}_{\lambda,k}^{i \rightarrow j}$ . The approach in [21] is based on the *F-method* [16], which reduces a problem of finding the operators  $\mathcal{D}_{\lambda,k}^{i \rightarrow j}$  to another problem of finding polynomial solutions to a system of ordinary differential equations (*F-system*). An alternative approach for  $j = i - 1, i$  is given in [20] by taking the residues of the regular symmetry breaking operators (see also Section 6 below).

With the aforementioned operators  $\mathcal{D}_{\lambda,k}^{i \rightarrow j}$ , Problem 9 (3) for differential operators were solved in [21, Thms. 1.4–1.8], which may be thought of as an answer to Stage C of branching problems. We illustrate the results with the following two theorems in the case where  $j = i$  and  $i + 1$ .

**Theorem 13 ( $j = i$  case).** *Suppose  $\nu \in \mathbb{C}$ ,  $k := \nu - \lambda \in \mathbb{N}$ , and  $\delta \equiv \varepsilon \equiv k \pmod{2}$ .*

- (1) *The linear map  $\mathcal{D}_{\lambda,k}^{i \rightarrow i}$  extends to a conformally covariant symmetry breaking operator from  $\mathcal{E}^i(S^n)_{\lambda,\delta}$  to  $\mathcal{E}^i(S^{n-1})_{\nu,\varepsilon}$ .*
- (2) *Conversely, any conformally covariant differential symmetry breaking operator from  $\mathcal{E}^i(S^n)_{\lambda,\delta}$  to  $\mathcal{E}^i(S^{n-1})_{\nu,\varepsilon}$  is proportional to  $\mathcal{D}_{\lambda,k}^{i \rightarrow i}$ , or its renormalization ([21, (1.10)]).*

**Theorem 14 ( $j = i + 1$  case).** (1) *Suppose  $1 \leq i \leq n - 2$ ,  $(\lambda, \nu) = (n - 2i, n - 2i + 3)$ , and  $\delta \equiv \varepsilon \equiv 1 \pmod{2}$ . Then the linear map*

$$\text{Rest} \circ d: \mathcal{E}^i(S^n)_{\lambda,\delta} \rightarrow \mathcal{E}^{i+1}(S^{n-1})_{\nu,\varepsilon}$$

*is a conformally covariant SBO. Conversely, a nonzero conformally covariant differential SBO from  $\mathcal{E}^i(S^n)_{\lambda,\delta}$  to  $\mathcal{E}^{i+1}(S^{n-1})_{\nu,\varepsilon}$  exists only for the above parameters, and such an operator is proportional to  $\text{Rest} \circ d$ .*

- (2) *Suppose  $i = 0$ ,  $\lambda \in \{0, -1, -2, \dots\}$ ,  $\nu = 0$ , and  $\delta \equiv \varepsilon \equiv \lambda \pmod{2}$ . Then the linear map*

$$\text{Rest}_{x_n=0} \circ \mathcal{D}_{-\lambda}^{\lambda - \frac{n-1}{2}} \circ d: \mathcal{E}^0(\mathbb{R}^n) \rightarrow \mathcal{E}^1(\mathbb{R}^{n-1})$$

*extends to a conformally covariant SBO from  $\mathcal{E}^0(S^n)_{\lambda,\delta}$  to  $\mathcal{E}^1(S^{n-1})_{0,\varepsilon}$ . Conversely, a nonzero conformally covariant differential SBO from  $\mathcal{E}^0(S^n)_{\lambda,\delta}$  to  $\mathcal{E}^1(S^{n-1})_{\nu,\varepsilon}$  exists only for the above parameters, and such an operator is proportional to the above operator.*

*Remark 15.*

- (1) By using the Hodge  $*$  operator on  $X$  or its submanifold  $Y$ , the other six cases can be reduced to either the  $j = i$  case (Theorem 13) or the  $j = i + 1$  case (Theorem 14). The construction and classification of

differential symmetry breaking operators in the model space (7) is thus completed. Its generalization to the pseudo-Riemannian case is proved in [22].

- (2) Special cases of Theorem 13 were known earlier. The case  $j = i = 0$  (scalar-valued case) was discovered by A. Juhl [8]. Different approaches have been proposed by Fefferman–Graham [4], Kobayashi–Ørsted–Souček–Somberg [25], and Clerc [3] among others. Our approach uses an algebraic Fourier transform of Verma modules (*F-method*), see [16, 27].
- (3) The case  $n = 2$  is closely related to the celebrated Rankin–Cohen bidifferential operator via holomorphic continuation [28].

## 6. Classification theory: nonlocal conformally covariant SBOs

In this section we consider nonlocal operators such as integral operators as well, and thus complete the classification problem (Problem 9) for the model space  $(X, Y) = (S^n, S^{n-1})$ .

Building on the classification results on  $D \left( \begin{smallmatrix} i & j \\ \lambda, \delta & \nu, \varepsilon \end{smallmatrix} \right)$  in Section 5, we want to

- find  $\dim_{\mathbb{C}} H \left( \begin{smallmatrix} i & j \\ \lambda, \delta & \nu, \varepsilon \end{smallmatrix} \right) / D \left( \begin{smallmatrix} i & j \\ \lambda, \delta & \nu, \varepsilon \end{smallmatrix} \right)$ ;
- find a basis in  $H \left( \begin{smallmatrix} i & j \\ \lambda, \delta & \nu, \varepsilon \end{smallmatrix} \right)$  modulo  $D \left( \begin{smallmatrix} i & j \\ \lambda, \delta & \nu, \varepsilon \end{smallmatrix} \right)$ .

This idea fits well with the general strategy to understand the whole space of symmetry breaking operators between principal series representations of a reductive group and its subgroup  $G'$  by using the filtration given by the support of distribution kernels [29, Chap. 11, Sec. 2]. Thus we start with the general setting where  $(G, G')$  is a pair of real reductive Lie groups. Let  $P = MAN$  and  $P' = M'A'N'$  be Langlands decompositions of minimal parabolic subgroups of  $G$  and  $G'$ , respectively. For an irreducible representation  $(\sigma, V)$  of  $M$  and a one-dimensional representation  $\mathbb{C}_\lambda$  of  $A$ , we define a principal series representation of  $G$  by unnormalized parabolic induction

$$I(\sigma, \lambda) := \text{Ind}_P^G(\sigma \otimes \mathbb{C}_\lambda \otimes \mathbf{1}).$$

Similarly, we define that of the subgroup  $G'$ , to be denoted by

$$J(\tau, \nu) := \text{Ind}_{P'}^{G'}(\tau \otimes \mathbb{C}_\nu \otimes \mathbf{1})$$

for an irreducible representation  $(\tau, W)$  of  $M'$  and a one-dimensional representation  $\mathbb{C}_\nu$  of  $A'$ .

By abuse of notation, we identify a representation with its representations space, and set  $V_\lambda := V \otimes \mathbb{C}_\lambda$  and  $W_\nu := W \otimes \mathbb{C}_\nu$ . Let  $\mathcal{V}_\lambda^*$  be the dualizing bundle of the  $G$ -homogeneous bundle  $G \times_P V_\lambda$  over the real flag manifold  $G/P$ . Then there is a natural linear bijection between the space of symmetry breaking operators and the space of invariant distributions (see [29, Prop. 3.2]):

$$\text{Hom}_{G'}(I_\delta(\sigma, \lambda)|_{G'}, J_\varepsilon(\tau, \nu)) \xrightarrow{\sim} (\mathcal{D}'(G/P, \mathcal{V}_\lambda^*) \otimes W_\nu)^{\Delta(P')}, \quad T \mapsto K_T, \quad (12)$$

Suppose now that the condition (PP) in Theorem 3 is fulfilled. Then this implies that  $\#(P' \backslash G/P) < \infty$ , see [26, Rem. 2.5 (4)]. We denote by  $\{Z_\alpha\}$  the totality of  $P'$ -orbits on  $G/P$ . We define a partial order  $\alpha \prec \beta$  by  $Z_\alpha \subset \overline{Z_\beta}$ , the closure of  $Z_\beta$  in  $G/P$ . Then there is the unique minimal index  $\alpha_{\min}$  corresponding to the closed  $P'$ -orbit in  $G/P$ , and maximal ones  $\beta_1, \dots, \beta_N$  corresponding to open  $P'$ -orbits in  $G/P$ .

We observe that the support  $\text{Supp}(K_T)$  of the distribution kernel  $K_T$  is a closed  $P'$ -invariant subset of  $G/P$ , and accordingly, define

$$H(\alpha) \equiv H_{\tau, \nu}^{\sigma, \lambda}(\alpha) := \{T \in \text{Hom}_{G'}(I(\sigma, \lambda)|_{G'}, J(\tau, \nu)) : \text{Supp}(K_T) \subset \overline{Z_\alpha}\}$$

via the isomorphism (12). Clearly,  $H(\alpha) \subset H(\beta)$  if  $\alpha \prec \beta$ . It follows from [27, Lem. 2.3] that

$$H(\alpha_{\min}) = \text{Diff}_{G'}(I(\sigma, \lambda)|_{G'}, J(\tau, \nu)).$$

In contrast to the smallest support  $Z_{\alpha_{\min}}$ , a symmetry breaking operator  $T$  is called *regular* ([29, Def. 3.3]) if  $\text{Supp}(K_T)$  contains  $Z_{\beta_j}$  for some  $1 \leq j \leq N$ .

We now return to the special setting (8). Then the Levi subgroup  $MA$  of the minimal parabolic subgroup  $P = MAN$  of  $G = O(n+1, 1)$  is given by  $(O(n) \times O(1)) \times \mathbb{R}$ . For  $0 \leq i \leq n$ ,  $\delta \in \{\pm\}$ , and  $\lambda \in \mathbb{C}$ , we consider the outer tensor product representation  $\Lambda^i(\mathbb{C}^n) \boxtimes \delta \boxtimes \mathbb{C}_\lambda$  of  $MA$ , and extend it to  $P$  by letting  $N$  act trivially. The resulting  $P$ -module is denoted simply by  $\Lambda^i(\mathbb{C}^n) \otimes \delta \otimes \mathbb{C}_\lambda$ . We define an unnormalized principal series representation of  $G = O(n+1, 1)$  by

$$I_\delta(i, \lambda) \equiv I(\Lambda^i(\mathbb{C}^n) \otimes \delta, \lambda) := \text{Ind}_P^G(\Lambda^i(\mathbb{C}^n) \otimes \delta \otimes \mathbb{C}_\lambda).$$

**Lemma 16.** *Let  $0 \leq i \leq n$ ,  $\delta \in \{\pm\}$ ,  $\lambda \in \mathbb{C}$ .*

- (1) *The  $G$ -module  $I_\delta(i, \lambda)$  is irreducible if  $\lambda \notin \mathbb{Z}$ .*
- (2) *There is a natural isomorphism  $\mathcal{E}^i(S^n)_{\lambda, \delta} \simeq I_{(-1)^i \delta}(i, \lambda + i)$  as  $G$ -modules.*

For the proof of Lemma 16 (2), see [21, Prop. 2.3].

Lemma 16 (2) suggests that we can reformulate Problem 9 about differential forms on the pair of conformal manifolds (7) into a question of symmetry breaking operators between principal series representations for the pair (8) of reductive groups. We write  $\tilde{D}$  and  $\tilde{H}$  if we use  $I_\delta(i, \lambda)$  and  $J_\varepsilon(j, \nu) = \text{Ind}_{P'}^G(\Lambda^j(\mathbb{C}^{n-1}) \otimes \varepsilon \otimes \mathbb{C}_\nu)$  instead of  $D$  and  $H$  in (6) and (5), respectively. By Lemma 16 (2), we have

$$H \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) = \tilde{H} \left( \begin{array}{c|c} i & j \\ \lambda + i, (-1)^i \delta & \nu + j, (-1)^j \varepsilon \end{array} \right)$$

and similarly for  $D$  and  $\tilde{D}$ . Thus we want to

- find  $\dim_{\mathbb{C}} \tilde{H} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) / \tilde{D} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right)$ ;
- find a basis in  $\tilde{H} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right)$  modulo  $\tilde{D} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right)$ .

First, we obtain:

**Theorem 17 (localness theorem).** *If  $j \neq i - 1$  or  $i$ , then*

$$\tilde{H} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) = \tilde{D} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right).$$

In the setting (8), there exists a unique open  $P'$ -orbit in  $G/P$ , and accordingly, there exists at most one family of (generically) regular symmetry breaking operators from the  $G$ -modules  $I_\delta(i, \lambda)$  to the  $G'$ -modules  $J_\varepsilon(j, \nu)$ . We prove that such a family exists if and only if  $j = i - 1$  or  $i$ , and it plays a crucial role in the classification problem of SBOs modulo the space  $\tilde{D} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right)$  of differential SBOs as follows. We introduce the set of “special parameters” by

$$\Psi_{\text{sp}} := \{(\lambda, \nu, \delta, \varepsilon) \in \mathbb{C}^2 \times \{\pm\}^2 : \nu - \lambda \in 2\mathbb{N} \text{ when } \delta\varepsilon = 1 \\ \text{or } \nu - \lambda \in 2\mathbb{N} + 1 \text{ when } \delta\varepsilon = -1\}. \quad (13)$$

**Theorem 18.** *Suppose  $j = i - 1$  or  $i$ , and  $\delta, \varepsilon \in \{\pm\}$ . Then there exists a family of continuous  $G'$ -homomorphism*

$$\tilde{\mathbb{A}}_{\lambda, \nu, \delta\varepsilon}^{i, j} : I_\delta(i, \lambda) \rightarrow J_\varepsilon(j, \nu)$$

such that  $\tilde{\mathbb{A}}_{\lambda, \nu, \delta\varepsilon}^{i, j}$  depends holomorphically on  $(\lambda, \nu) \in \mathbb{C}^2$  and that the set of the zeros of  $\tilde{\mathbb{A}}_{\lambda, \nu, \delta\varepsilon}^{i, j}$  is discrete in  $(\lambda, \nu) \in \mathbb{C}^2$ .

(1) *If  $(\lambda, \nu, \delta, \varepsilon) \notin \Psi_{\text{sp}}$  then  $\tilde{\mathbb{A}}_{\lambda, \nu, \delta\varepsilon}^{i, j} \neq 0$  and*

$$\tilde{H} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) = \mathbb{C}\tilde{\mathbb{A}}_{\lambda, \nu, \delta\varepsilon}^{i, j} \supsetneq \tilde{D} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) = \{0\}.$$

(2) *If  $(\lambda, \nu, \delta, \varepsilon) \in \Psi_{\text{sp}}$  and  $\tilde{\mathbb{A}}_{\lambda, \nu, \delta\varepsilon}^{i, j} \neq 0$ , then*

$$\tilde{H} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) = \tilde{D} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right).$$

(3) *If  $(\lambda, \nu, \delta, \varepsilon) \in \Psi_{\text{sp}}$  and  $\tilde{\mathbb{A}}_{\lambda, \nu, \delta\varepsilon}^{i, j} = 0$ , then*

$$\dim_{\mathbb{C}} \tilde{H} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) = \dim_{\mathbb{C}} \tilde{D} \left( \begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array} \right) + 1.$$

The discrete set  $\{(i, j, \lambda, \nu, \delta, \varepsilon) : \tilde{\mathbb{A}}_{\lambda, \nu, \delta\varepsilon}^{i, j} = 0\}$  has been determined in [20], and thus the classification of conformally covariant symmetry breaking operators

$$\mathcal{E}^i(X)_{\lambda, \delta} \rightarrow \mathcal{E}^j(Y)_{\nu, \varepsilon}$$

for the model space  $(X, Y) = (S^n, S^{n-1})$  is accomplished. A detailed proof for the classification together with some important properties of symmetry breaking operators (Stage C) such as

- $(K, K')$ -spectrum (a generalized eigenvalue),
- functional equations,
- residue formulæ,

will be given in separate papers (see [20] for the residue formulæ, and [30] for the classification).

## 7. Application to periods and automorphic form theory

Let  $G$  be a reductive group, and  $H$  a reductive subgroup.

**Definition 19.** An irreducible admissible smooth representation  $\Pi$  of  $G$  is  $H$ -*distinguished* if  $\text{Hom}_H(\Pi|_H, \mathbb{C}) \neq \{0\}$ . In this case, it is also said that  $\Pi$  has an  $H$ -*period*. By the Frobenius reciprocity theorem, the condition is equivalent to  $\text{Hom}_G(\Pi, C^\infty(G/H)) \neq \{0\}$ .

In this section, we discuss an application of symmetry breaking operators to find periods (Definition 19) of irreducible unitary representations. We highlight the case when  $\Pi$  has nonzero  $(\mathfrak{g}, K)$ -cohomologies. The motivation comes from automorphic form theory, of which we now recall a prototype.

**Fact 20 (Matsushima–Murakami, [2]).** *Let  $\Gamma$  be a cocompact discrete subgroup of  $G$ . Then we have*

$$H^*(\Gamma \backslash G/K; \mathbb{C}) \simeq \bigoplus_{\Pi \in \widehat{G}} \text{Hom}_G(\Pi, L^2(\Gamma \backslash G)) \otimes H^*(\mathfrak{g}, K; \Pi_K).$$

The left-hand side gives topological invariants of the locally symmetric space  $M = \Gamma \backslash G/K$ , whereas the right-hand side is described in terms of the representation theory. We note that  $\text{Hom}_G(\Pi, L^2(\Gamma \backslash G))$  is finite-dimensional for all  $\Pi \in \widehat{G}$  by a theorem of Gelfand–Piatetski-Shapiro, and the sum is taken over the following finite set

$$\widehat{G}_{\text{cohom}} := \{\Pi \in \widehat{G} : H^*(\mathfrak{g}, K; \Pi_K) \neq \{0\}\},$$

which was classified by Vogan and Zuckerman [36].

In the case where  $G = O(n+1, 1)$ , there are  $2(n+1)$  elements in  $\widehat{G}_{\text{cohom}}$ . Following the notation in [21, Thm. 2.6], we label them as

$$\{\Pi_{\ell, \delta} : 0 \leq \ell \leq n+1, \delta \in \{\pm\}\},$$

and we define

$$\begin{aligned} \text{Index} &\equiv \text{Index}_G : \widehat{G}_{\text{cohom}} \rightarrow \{0, 1, \dots, n+1\}, & \Pi_{\ell, \delta} &\mapsto \ell, \\ \text{sgn} &\equiv \text{sgn}_G : \widehat{G}_{\text{cohom}} \rightarrow \{\pm\}, & \Pi_{\ell, \delta} &\mapsto \delta. \end{aligned}$$

We illustrate the labeling by two examples:

*Example 21* (one-dimensional representations). There are four one-dimensional representations of  $G$ , which are given as

$$\{\Pi_{0,+} \simeq \mathbf{1}, \Pi_{0,-}, \Pi_{n+1,+}, \Pi_{n+1,-} \simeq \det\}.$$

*Example 22* (tempered representations). For  $n$  odd  $\Pi$  is the smooth representation of a discrete series representation of  $G$  iff  $\text{Index}(\Pi) = \frac{1}{2}(n+1)$ , whereas for  $n$  even  $\Pi$  is that of tempered representation of  $G$  iff  $\text{Index}(\Pi) \in \{\frac{n}{2}, \frac{n}{2}+1\}$ .

We give a necessary and sufficient condition for the existence of symmetry breaking operators between irreducible representations of  $G$  and those of the subgroup  $G'$  with nonzero  $(\mathfrak{g}, K)$ -cohomologies:

**Theorem 23 ([30]).** *Let  $(G, G') = (O(n+1, 1), O(n, 1))$ , and  $(\Pi, \pi) \in \widehat{G}_{\text{cohom}} \times \widehat{G}'_{\text{cohom}}$ . Then the following three conditions on  $(\Pi, \pi)$  are equivalent.*

- (i)  $\text{Hom}_{G'}(\Pi^\infty|_{G'}, \pi^\infty) \neq \{0\}$ .
- (ii) *The outer tensor product representation  $\Pi^\infty \boxtimes \pi^\infty$  is  $\text{diag}(G')$ -distinguished.*
- (iii)  $\text{Index}_G(\Pi) - 1 \leq \text{Index}_{G'}(\pi) \leq \text{Index}_G(\Pi)$  and  $\text{sgn}(\Pi) = \text{sgn}(\pi)$ .

The proof uses the symmetry breaking operators that are discussed in Section 6 and the relationship between  $\widehat{G}_{\text{cohom}}$  and conformal representations on differential forms on the sphere  $S^n$  summarized as below.

**Lemma 24 ([21, Thm. 2.6]).** *If  $\Pi \in \widehat{G}_{\text{cohom}}$ , then  $\Pi^\infty$  can be realized as a subrepresentation of  $\mathcal{E}^i(S^n)_{0,\delta}$  with  $i = \text{Index}_G(\Pi)$  and  $\delta = (-1)^i \text{sgn}_G(\Pi)$  if  $\text{Index}_G(\Pi) \neq n+1$ , and also as a quotient of  $\mathcal{E}^i(S^n)_{0,\delta}$  with  $i = \text{Index}_G(\Pi) - 1$  and  $\delta = (-1)^i \text{sgn}_G(\Pi)$  if  $\text{Index}_G(\Pi) \neq 0$ .*

To end this section, we consider a tower of subgroups of a reductive group  $G$ :

$$\{e\} = G^{(0)} \subset G^{(1)} \subset \dots \subset G^{(n)} \subset G^{(n+1)} = G.$$

Accordingly, there is a family of homogeneous spaces with  $G$ -equivariant quotient maps:

$$G = G/G^{(0)} \rightarrow G/G^{(1)} \rightarrow \dots \rightarrow G/G^{(n+1)} = \{\text{pt}\}.$$

In turn, we have natural inclusions of  $G$ -modules:

$$C^\infty(G) = C^\infty(G/G^{(0)}) \supset C^\infty(G/G^{(1)}) \supset \dots \supset C^\infty(G/G^{(n+1)}) = \mathbb{C}.$$

A general question is:

**Problem 25.** *Let  $\Pi \in \widehat{G}_{\text{smooth}}$ . Find  $k$  as large as possible such that  $\Pi$  is  $G^{(k)}$ -distinguished, or equivalently, such that the smooth representation  $\Pi^\infty$  can be realized in  $C^\infty(G/G^{(k)})$ .*

Any irreducible admissible smooth representation of  $G$  can be realized in the regular representation on  $C^\infty(G/G^{(0)}) \simeq C^\infty(G)$  via matrix coefficients, whereas irreducible representations that can be realized in  $C^\infty(G/G^{(0)}) = \mathbb{C}$  is the trivial one-dimensional representation  $\mathbf{1}$ .

Suppose that  $G = O(n+1, 1)$ , and consider a chain of subgroups of  $G$  by

$$G^{(k)} := O(k, 1) \quad (0 \leq k \leq n+1).$$

Then  $G^{(n+1)} = G$ , however,  $G^{(0)}$  is not exactly  $\{e\}$  but  $G^{(0)} = O(1)$  is a finite group of order two. Accordingly, we consider  $\Pi \in \widehat{G}_{\text{cohom}}$  with  $\text{sgn}(\Pi) = +$  below.



**Theorem 26.** *Suppose  $\Pi \in \widehat{G}_{\text{cohom}}$  with  $\text{sgn}(\Pi) = +$ . Then*

$$\text{Hom}_G(\Pi^\infty, C^\infty(G/G^{(k)})) \neq \{0\} \quad \text{for all } k \leq n + 1 - \text{Index}_G(\Pi).$$

*Example 27* (one-dimensional representations). Suppose  $\Pi \in \widehat{G}_{\text{cohom}}$  with  $\text{sgn}_G(\Pi) = +$ . We consider two opposite extremal cases, *i.e.*,  $\text{Index}_G(\Pi) = 0$  and  $= n + 1$ . If  $\text{Index}_G(\Pi) = 0$ , then  $\Pi$  is isomorphic to the trivial one-dimensional representation  $\mathbf{1}$ , and can be realized in  $C^\infty(G/G^{(k)})$  for all  $0 \leq k \leq n + 1$  as in Theorem 26. On the other hand, if  $\text{Index}_G(\Pi) = n + 1$ , then  $\Pi$  is another one-dimensional representation of  $G$  ( $\Pi_{n+1,+} \simeq \chi_{-+}$  with the notation [21, (2.9)]). In this case,  $\Pi$  can be realized in  $C^\infty(G/G^{(k)})$  iff  $k = 0$ , namely, iff  $G^{(k)} = O(1)$ .

*Remark 28.* The size of an (infinite-dimensional) representation could be measured by its Gelfand–Kirillov dimension, or more precisely, by its associated variety or by the partial flag variety for which its localization can be realized as a  $\mathcal{D}$ -module. Then one might expect the following assertion:

*the larger the isotropy subgroup  $G^{(k)}$  is (i.e., the larger  $k$  is),*  
*the “smaller” irreducible subrepresentations of  $C^\infty(G/G^{(k)})$  become.* (14)

This is reflected partially in Theorem 26, however, Theorem 26 asserts even sharper results. To see this, we set

$$r := \min(\text{Index}_G(\Pi), n + 1 - \text{Index}_G(\Pi)).$$

Then the underlying  $(\mathfrak{g}, K)$ -module  $\Pi_K$  can be expressed as a cohomological parabolic induction from a  $\theta$ -stable parabolic subalgebra  $\mathfrak{q}_r$  with Levi subgroup  $N_G(\mathfrak{q}_r) \simeq SO(2)^r \times O(n+1-2r, 1)$  ([9], see also [11, Thm. 3]). Theorem 26 tells that if  $n + 1 \leq 2k$ , then the larger  $k$  is, the smaller  $r = \text{Index}_G(\Pi)$  becomes, namely, the smaller the  $(\mathfrak{g}, K)$ -modules that are cohomologically parabolic induced modules from  $\mathfrak{q}_r$  become. This matches (14). On the other hand, if  $2k \leq n + 1$ , then the constraints in Theorem 26 provide an interesting phenomenon which is opposite to (14) because  $r = n + 1 - \text{Index}_G(\Pi)$ , and thus suggest sharper estimates than (14). For instance, the representation  $\Pi_{n+1,+} (\simeq \chi_{-+})$  is “small” because it is one-dimensional, but it can be realized in  $C^\infty(G/G^{(k)})$  only for  $k = 0$  as we saw in Example 27.

*Remark 29* (comparison with  $L^2$ -theory). Theorem 26 implies that the smooth representation  $\Pi^\infty$  of a tempered representation  $\Pi$  with nonzero  $(\mathfrak{g}, K)$ -cohomologies (see Example 22) occurs in  $C^\infty(G/G^{(k)})$  if  $k \leq \frac{n}{2} + 1$ . On the other hand, for a reductive homogeneous space  $G/H$ , a general criterion for the unitary representation  $L^2(G/H)$  to be tempered was proved in a joint work [1] with Y. Benoist by a geometric method. In particular, the unitary representation  $L^2(G/G^{(k)})$  is tempered if and only if  $k \leq \frac{n}{2} + 1$ , see [1, Ex. 5.10].

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