Harmonic Analysis and the Trace Formula

Abstracts

Symmetry Breaking Operators for Orthogonal Groups $O(n, 1)$

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Given an irreducible representation $\pi$ of a group $G$ and a subgroup $G'$, we may think of $\pi$ as a representation of the subgroup $G'$ (the restriction $\pi|_{G'}$). A typical example is the tensor product representation $\pi_1 \otimes \pi_2$ of two representations $\pi_1$ and $\pi_2$ of a group $H$, which is obtained by the restriction of the outer tensor product $\pi_1 \otimes \pi_2$ of the direct product group $G := H \times H$ to its subgroup $G' := \text{diag}(H)$.

As branching problems, we wish to understand how the restriction $\pi|_{G'}$ behaves as a $G'$-module. For reductive groups, this is a difficult problem, partly because the restriction $\pi|_{G'}$ may not be well under control as a representation of $G'$ even when $G'$ is a maximal subgroup of $G$. Wild behavior such as infinite multiplicities may occur, for instance, already in the tensor product representation of $SL_3(\mathbb{R})$.

The author proposed in [3] to go on successively, further steps in the study of branching problems via the following three stages:

Stage A: Abstract feature of the restriction $\pi|_{G'}$

Stage B: Branching laws.

Stage C: Construction of symmetry breaking operators (Definition 0.1).

Here, branching laws in Stage B ask an explicit decomposition of the restriction into irreducible representations of the subgroup $G'$ when $\pi$ is a unitary representation, and also ask the multiplicity $m(\pi, \tau) := \dim \text{Hom}_{G'}(\pi|_{G'}, \tau)$ for irreducible representations $\tau$ of $G'$. The latter makes sense even when $\pi$ and $\tau$ are nonunitary. Stage C refines Stage B, by asking an explicit construction of SBOs when $\pi$ and $\tau$ are realized geometrically.

Definition 0.1. An element in $\text{Hom}_{G'}(\pi|_{G'}, \tau)$ is called a symmetry breaking operator, SBO for short.

Stage A includes a basic question whether spectrum is discrete or not, see [1]. Another fundamental question in Stage A is an estimate of multiplicities. In [2, 6], we discovered the following geometric criteria to control multiplicities:

Theorem 0.2 (geometric criteria for finite/bounded multiplicities).

1. The dimension of $\text{Hom}_{G'}(\pi|_{G'}, \tau)$ is finite for any irreducible representations $\pi$ of $G$ and any $\tau$ of $G'$ iff $G \times G'/\text{diag}(G')$ is real spherical.

2. The dimension of $\text{Hom}_{G'}(\pi|_{G'}, \tau)$ is uniformly bounded with respect to $\pi$ and $\tau$ iff $(G_C \times G'_C)/\text{diag}(G'_C)$ is spherical.

Here we recall

Definition 0.3. (1) A complex manifold $X_C$ with holomorphic action of a complex reductive group $G_C$ is spherical if a Borel subgroup of $G_C$ has an open orbit in $X_C$. 

A real manifold $X$ with continuous action of a real reductive group $G$ is
real spherical if a minimal parabolic subgroup of $G$ has an open orbit in
$X$.

The latter terminology was introduced in [2] in search for a broader framework
for global analysis on homogeneous spaces than the usual (e.g. group manifolds,
symmetric spaces). That is, the function space $C^\infty(G/H)$ (or $L^2(G/H)$ etc.)
should be under control by representation theory if
\begin{equation}
\dim \text{Hom}_G(\pi, C^\infty(G/H)) < 1 \quad \text{for all } \pi \in \hat{G}_{\text{adm}},
\end{equation}
and hence we could expect to develop global analysis on $G/H$ by using representa-
tion theory if (1) holds [2]. We discovered and proved that the geometric property
“real spherical” characterizes exactly the representation-theoretic property (1):

**Fact 0.4** ([2, 6]). Let $X = G/H$ where $G \supset H$ are algebraic real reductive groups.

1. $\dim \text{Hom}_G(\pi, C^\infty(X)) < \infty$ $\text{iff } X$ is real spherical.
2. $\dim \text{Hom}_G(\pi, C^\infty(X))$ is uniformly bounded $\text{iff } X_C$ is spherical.

Theorem 0.2 follows from Fact 0.4.

The classification of the real spherical spaces of the form $(G \times G')/\text{diag}(G')$
was accomplished in [5] when $(G, G')$ is a reductive symmetric pair. This a priori
estimate in Stage A singles out the settings which would be potentially promising
for Stages B and C of branching problems. One of such settings arises from a
different discipline, namely, from conformal geometry. The first complete solution
to Stage C obtained [7] is related to this geometric setting as below.

Given a Riemannian manifold $(X, g)$, we write $G = \text{Conf}(X, g)$ for the group of
conformal diffeomorphisms of $X$. Then there is a natural family of representations
$\pi_\lambda$ of $G$ on $C^\infty(X)$ for $\lambda \in \mathbb{C}$ given by
\begin{equation}
(\pi_\lambda(h)f)(x) = \Omega(h^{-1}, x)^{\lambda} f(h^{-1} \cdot x) \quad \text{for } h \in G, x \in X.
\end{equation}
We can extend this to a family of representations on the space $\mathcal{E}^i(X)$ of differential
$i$-forms, to be denoted by $\pi^{(i)}_\lambda$.

If $Y$ is a submanifold of $X$, then there is a natural morphism
\[ G' := \{ h \in G : h \cdot Y \subset Y \} \rightarrow \text{Conf}(Y, g|_Y). \]

Then we may compare two families of representations of the group $G'$:

- the restriction $\pi^{(i)}_\lambda|_{G'}$ acting on $\mathcal{E}^{(i)}(X)$,
- the representation $\pi^{(j)}_\nu$ acting on $\mathcal{E}^{(j)}(Y)$.

A conformally covariant SBO on differential forms is a linear map $\mathcal{E}^{(i)}(X) \rightarrow \mathcal{E}^{(j)}(Y)$
that intertwines $\pi^{(i)}_\lambda|_{G'}$ and $\pi^{(j)}_\nu$. Here is a basic question arising from conformal
geometry:

**Question 0.5.** Let $X$ be a Riemannian manifold $X$, and $Y$ a hypersurface. Construct and classify conformally covariant SBOs from $\mathcal{E}^i(X)$ to $\mathcal{E}^j(Y)$. 
We are interested in “natural operators” $D$ that persist for all pairs $(X, Y)$. The larger $\text{Conf}(X; Y)$ is, the more constraints are on $D$, and hence, we first focus on the model space with largest symmetries which is given by $(X, Y) = (S^n, S^{n-1})$. In this case the pair $(G, G')$ of conformal groups is locally isomorphic to $(O(n+1, 1), O(n, 1))$. It then turns out that the criterion in Theorem 0.2 (2) for Stage A is fulfilled. Then Question 0.5 is regarded as Stages B and C of branching problems. Recently, we have solved completely Question 0.5 in the model space:

- Continuous SBOs for $i = j = 0$ were constructed and classified in [7].
- Differential SBOs for general $i$ and $j$ were constructed and classified in [4].
- The final classification is announced in [8].

Here is a flavor of the complete classification:

**Theorem 0.6.** If $\text{Hom}_{G'}(\pi^{(i)}_{\lambda}(G), \pi^{(j)}_{\nu}(G')) \neq \{0\}$ for some $\lambda, \nu \in \mathbb{C}$, then $j \in \{i - 2, i - 1, i, i + 1\}$ or $i + j \in \{n - 2, n - 1, n, n + 1\}$.

In the talk, I gave briefly the methods of the complete solution [4, 7, 8], some of which are also applicable in a more general setting that Theorem 0.2 (an \textit{a priori} estimate for Stage A) suggests.

Finally, some applications of these results include:

- an evidence of a conjecture of Gross and Prasad for $O(n, 1)$, see [8];
- periods of irreducible unitary representations with nonzero cohomologies;
- a construction of discrete spectrum of the branching laws of complementary series [7, Chap. 15].

**References**


