Conformal symmetry breaking operators for anti-de Sitter spaces

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Abstract. For a pseudo-Riemannian manifold X and a totally geodesic hypersurface Y, we consider the problem of constructing and classifying all linear differential operators $\mathcal{E}^i(X) \to \mathcal{E}^j(Y)$ between the spaces of differential forms that intertwine multiplier representations of the Lie algebra of conformal vector fields. Extending the recent results in the Riemannian setting by Kobayashi–Kubo–Pevzner [Lecture Notes in Math. 2170, (2016)], we construct such differential operators and give a classification of them in the pseudo-Riemannian setting where both X and Y are of constant sectional curvature, illustrated by the examples of anti-de Sitter spaces and hyperbolic spaces.

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1. Introduction

Let X be a manifold endowed with a pseudo-Riemannian metric g. A vector field Z on X is called *conformal* if there exists $\rho(Z, \cdot) \in C^\infty(X)$ (*conformal factor*) such that

$$L_Z g = \rho(Z, \cdot) g,$$

where L_Z stands for the Lie derivative with respect to the vector field Z. We denote by $\mathfrak{conf}(X)$ the Lie algebra of conformal vector fields on X.

Let $\mathcal{E}^i(X)$ be the space of (complex-valued) smooth i-forms on X. We define a family of multiplier representations of the Lie algebra $\mathfrak{conf}(X)$ on $\mathcal{E}^i(X)$ $(0 \leq i \leq \dim X)$ with parameter $u \in \mathbb{C}$ by

$$\Pi_u^{(i)}(Z)\alpha := L_Z \alpha + \frac{1}{2} u \rho(Z, \cdot)\alpha \quad \text{for } \alpha \in \mathcal{E}^i(X). \tag{1.1}$$

For simplicity, we write $\mathcal{E}^i(X)_u$ for the representation $\Pi_u^{(i)}$ of $\mathfrak{conf}(X)$ on $\mathcal{E}^i(X)$.

For a submanifold Y of X, conformal vector fields along Y form a subalgebra

$$\mathfrak{conf}(X;Y) := \{Z \in \mathfrak{conf}(X) : Z_y \in T_yY \text{ for all } y \in Y\}.$$

If the metric tensor g is nondegenerate when restricted to the submanifold Y, then Y carries a pseudo-Riemannian metric $g|_Y$ and there is a natural Lie algebra homomorphism $\mathfrak{conf}(X;Y) \to \mathfrak{conf}(Y)$, $Z \mapsto Z|_Y$. In this case we compare the representation $\Pi_u^{(i)}$ of the Lie algebra $\mathfrak{conf}(X)$ on $\mathcal{E}^i(X)$ with an analogous representation denoted by the lowercase letter $\pi_v^{(j)}$ of the Lie algebra $\mathfrak{conf}(Y)$ on $\mathcal{E}^j(Y)$ for $u, v \in \mathbb{C}$. For this, we analyze *conformal symmetry breaking operators*, that is, linear maps $T\colon \mathcal{E}^i(X) \to \mathcal{E}^j(Y)$ satisfying

$$\pi_v^{(j)}(Z|_Y) \circ T = T \circ \Pi_u^{(i)}(Z) \quad \text{for all } Z \in \mathfrak{conf}(X;Y). \tag{1.2}$$

Some of such operators are given as differential operators (*e.g.* [3, 6, 12, 14, 15]), and others are integral operators and their analytic continuation (*e.g.* [16]). We denote by $\mathrm{Diff}_{\mathfrak{conf}(X;Y)}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$ the space of differential operators satisfying (1.2).

In the case $X = Y$ and $i = j = 0$, the Yamabe operator, the Paneitz operator [18], which appears in four-dimensional supergravity [4], or more generally, the so-called GJMS operators [5] are such differential operators. Branson and Gover [1, 2] extended such operators to differential forms when $i = j$. The exterior derivative d and the codifferential d^* also give examples of such operators for $j = i + 1$ and $i - 1$, respectively. Maxwell's equations in a vacuum can be expressed in terms of conformally covariant operators on 2-forms in the Minkowski space $\mathbb{R}^{1,3}$ (see [17] for a bibliography). All these classical examples concern the case where $X = Y$. On the other hand, the more general setting where $X \supsetneqq Y$ is closely related to branching laws of infinite-dimensional representations (cf. "Stage C" of branching problems in [11]). In recent years, for $(X, Y) = (\mathrm{S}^n, \mathrm{S}^{n-1})$, such operators in the scalar-valued case ($i = j = 0$) were classified by Juhl [6], see also [3, 10, 14] for different approaches. More generally, such operators have been constructed and classified also in the matrix-valued case (i, j arbitrary) by the authors [12]. In this paper, we give a variant of [12] by extending the framework as follows:

the group of

conformal diffeomorphisms	$\Longrightarrow$	the Lie algebra of conformal vector fields;
homogeneous spaces	$\Longrightarrow$	locally homogeneous spaces;
Riemannian setting	$\Longrightarrow$	pseudo-Riemannian setting.

Let $\mathbb{R}^{p,q}$ denote the space $\mathbb{R}^{p+q}$ endowed with the flat pseudo-Riemannian metric:

$$g_{\mathbb{R}^{p,q}} = dx_1^2 + \cdots + dx_p^2 - dy_{p+1}^2 - \cdots - dy_{p+q}^2. \tag{1.3}$$

For $p, q \in \mathbb{N}$, we define a hypersurface $\mathrm{S}^{p,q}$ of $\mathbb{R}^{1+p+q}$ by

$$\mathrm{S}^{p,q} := \begin{cases} \{(\omega_0, \omega, \eta) \in \mathbb{R}^{1+p+q} : \omega_0^2 + |\omega|^2 - |\eta|^2 = 1\} & (p > 0), \\ \{(\omega_0, \eta) \in \mathbb{R}^{1+q} : \omega_0 > 0,\ \omega_0^2 - |\eta|^2 = 1\} & (p = 0). \end{cases} \tag{1.4}$$

Then, the metric $g_{\mathbb{R}^{1+p,q}}$ on the ambient space $\mathbb{R}^{1+p+q}$ induces a pseudo-Riemannian structure on the hypersurface $\mathrm{S}^{p,q}$ of signature (p, q) with constant sectional curvature $+1$, which is sometimes referred to as the (positively curved) *space form* of a pseudo-Riemannian manifold. We may regard $\mathrm{S}^{p,q}$ also as a pseudo-Riemannian manifold of signature (q, p) with constant curvature -1 by using $-g_{\mathbb{R}^{1+p,q}}$ instead, giving rise to the negatively curved space form.

Example 1.1 (Riemannian and Lorentzian cases).

$\mathrm{S}^{n,0}$	$= \mathrm{S}^n$	(sphere),	$\mathrm{S}^{0,n}$	$= \mathrm{H}^n$	(hyperbolic space),
$\mathrm{S}^{n-1,1}$	$= \mathrm{dS}^n$	(de Sitter space),	$\mathrm{S}^{1,n-1}$	$= \mathrm{AdS}^n$	(anti-de Sitter space).

In Theorems A–C below, we assume $n = p + q \geq 3$ and consider

$$(X, Y) = (\mathrm{S}^{p,q}, \mathrm{S}^{p-1,q}),\ (\mathrm{S}^{p,q}, \mathrm{S}^{p,q-1}),\ (\mathbb{R}^{p,q}, \mathbb{R}^{p-1,q}),\ \text{or}\ (\mathbb{R}^{p,q}, \mathbb{R}^{p,q-1}). \tag{1.5}$$

Example 1.2. $\mathfrak{conf}(X; Y) \simeq \mathfrak{o}(p, q{+}1)$ *if* $(X, Y) = (\mathrm{S}^{p,q}, \mathrm{S}^{p-1,q})$ *or* $(\mathbb{R}^{p,q}, \mathbb{R}^{p-1,q})$.

Theorem A below addresses the question if any conformal symmetry breaking operator defined locally can be extended globally.

Theorem A (automatic continuity). *Let V be any open set of X such that $V \cap Y$ is connected and nonempty. Suppose $u, v \in \mathbb{C}$. Then the map taking the restriction to V induces a bijection:*

$$\mathrm{Diff}_{\mathfrak{conf}(X;Y)}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v) \xrightarrow{\sim} \mathrm{Diff}_{\mathfrak{conf}(V, V\cap Y)}(\mathcal{E}^i(V)_u, \mathcal{E}^j(V \cap Y)_v).$$

We recall from [19, Chap. II] that the pseudo-Riemannian manifolds $\mathbb{R}^{p,q}$ and $\mathrm{S}^{p,q}$ have a common conformal compactification:

$$\mathbb{R}^{p,q} \hookrightarrow (\mathrm{S}^p \times \mathrm{S}^q)/\mathbb{Z}_2 \hookleftarrow \mathrm{S}^{p,q}$$

where $(\mathrm{S}^p \times \mathrm{S}^q)/\mathbb{Z}_2$ denotes the direct product of p- and q-spheres equipped with the pseudo-Riemannian metric $g_{\mathrm{S}^p} \oplus (-g_{\mathrm{S}^q})$, modulo the direct product of antipodal maps, see also [13, II, Lem. 6.2 and III, Sect. 2.8]. For $X = \mathbb{R}^{p,q}$ or $\mathrm{S}^{p,q}$, we denote by $\overline{X}$ this conformal compactification of X.

Theorem B. *(1)* (Automatic continuity to the conformal compactification). *Suppose $u, v \in \mathbb{C}$ and $0 \leq i \leq n$, $0 \leq j \leq n-1$. Then the map taking the restriction to X is a bijection*

$$\mathrm{Diff}_{\mathfrak{conf}(\overline{X};\overline{Y})}(\mathcal{E}^i(\overline{X})_u, \mathcal{E}^j(\overline{Y})_v) \xrightarrow{\sim} \mathrm{Diff}_{\mathfrak{conf}(X;Y)}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v).$$

(2) If $n \geq 3$, all these spaces are isomorphic to each other for (X, Y) in (1.5) *as far as (p, q) satisfies $p + q = n$.*

By Theorems A and B, we see that all conformal symmetry breaking operators given locally in some open sets in the pseudo-Riemannian case (1.5) are derived from the Riemannian case (*i.e.* $p = 0$ or $q = 0$). We note that our representation (1.1) is normalized in a way that $\Pi_u^{(i)}$ coincides with the differential of the representation $\varpi_{u,\delta}^{(i)}$ ($\delta \in \mathbb{Z}/2\mathbb{Z}$) of the conformal group $\mathrm{Conf}(X)$ introduced in [12, (1.1)]. In particular, we can read from [12, Thms. 1.1 and 2.10] the dimension of $\mathrm{Diff}_{\mathfrak{conf}(X;Y)}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$ for any i, j, u, v. For simplicity of exposition, we present a coarse feature as follows.

Theorem C. *Suppose (X, Y) is as in* (1.5)*, and V any open set of X such that $V \cap Y$ is connected and nonempty. Let $u, v \in \mathbb{C}$, $0 \leq i \leq n$, and $0 \leq j \leq n-1$.*

(1) *For any $u, v \in \mathbb{C}$ and $0 \leq i \leq n$, $0 \leq j \leq n-1$,*

$$\dim_{\mathbb{C}} \mathrm{Diff}_{\mathfrak{conf}(V;V\cap Y)}(\mathcal{E}^i(V)_u, \mathcal{E}^j(V \cap Y)_v) \leq 2.$$

(2) $\mathrm{Diff}_{\mathfrak{conf}(V;V\cap Y)}(\mathcal{E}^i(V)_u, \mathcal{E}^j(V \cap Y)_v) \neq \{0\}$ *only if u, v, i, j satisfy*

$$(v+j)-(u+i) \in \mathbb{N} \quad \text{and} \quad \bigl(-1 \leq i-j \leq 2 \text{ or } n-2 \leq i+j \leq n+1\bigr). \tag{1.6}$$

A precise condition when the equality holds in Theorem C (1) will be explained in Section 7 in the case $n = 4$. We shall give explicit formulæ of generators of $\mathrm{Diff}_{\mathfrak{conf}(X;Y)}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$ in Theorem D in Section 2 for the flat pseudo-Riemannian manifolds, and in Theorem E in Section 3 for positively (or negatively) curved space forms. These operators (with "renormalization") and their compositions by the Hodge star operators with respect to the pseudo-Riemannian metric exhaust all differential symmetry breaking operators (Remark 2.2). The proof of Theorems A–C will be given in Section 5.

Notation. $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}_+ = \{1, 2, \cdots\}$.

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2. Conformally covariant symmetry breaking operators—flat case

In this section, we give explicit formulæ of conformal symmetry breaking operators in the flat pseudo-Riemannian case $(X, Y) = (\mathbb{R}^{p,q}, \mathbb{R}^{p-1,q})$ or $(\mathbb{R}^{p,q}, \mathbb{R}^{p,q-1})$. This extends the results in [12] that dealt with the Riemannian case $(X, Y) = (\mathbb{R}^n, \mathbb{R}^{n-1})$.

We note that the signature of the metric restricted to nondegenerate hyperplanes of $\mathbb{R}^{p,q}$ is either $(p-1, q)$ or $(p, q-1)$. Thus it is convenient to introduce two types of coordinates in $\mathbb{R}^{p+q}$ accordingly. We set

$$\begin{aligned} \mathbb{R}^{p,q}_+ &= \{(y, x) \in \mathbb{R}^{q+p}\} \quad \text{with} \quad -dy_1^2 - \cdots - dy_q^2 + dx_{q+1}^2 + \cdots + dx_{p+q}^2, \\ \mathbb{R}^{p,q}_- &= \{(x, y) \in \mathbb{R}^{p+q}\} \quad \text{with} \quad dx_1^2 + \cdots + dx_p^2 - dy_{p+1}^2 - \cdots - dy_{p+q}^2. \end{aligned}$$

Then by letting the last coordinate to be zero, we get hypersurfaces of $\mathbb{R}^{p,q}$ of two types:

$$\mathbb{R}^{p-1,q}_+ \subset \mathbb{R}^{p,q}_+ \ (p \geq 1), \quad \mathbb{R}^{p,q-1}_- \subset \mathbb{R}^{p,q}_- \ (q \geq 1).$$

For $\ell \in \mathbb{N}$ and $\mu \in \mathbb{C}$, we define a family of differential operators on $\mathbb{R}^{p+q}$ by using the above coordinates:

$$(\mathcal{D}_\ell^\mu)_+ \equiv (\mathcal{D}_\ell^\mu)_{\mathbb{R}_+^{p,q}} := \sum_{k=0}^{[\frac{\ell}{2}]} a_k(\mu,\ell) \left(\sum_{j=1}^{q} \frac{\partial^2}{\partial y_j^2} - \sum_{j=q+1}^{n-1} \frac{\partial^2}{\partial x_j^2} \right)^k \left(\frac{\partial}{\partial x_n} \right)^{\ell-2k} \text{ on } \mathbb{R}_+^{p,q},$$

$$(\mathcal{D}_\ell^\mu)_- \equiv (\mathcal{D}_\ell^\mu)_{\mathbb{R}_-^{p,q}} := \sum_{k=0}^{[\frac{\ell}{2}]} a_k(\mu,\ell) \left(\sum_{j=1}^{p} \frac{\partial^2}{\partial x_j^2} - \sum_{j=p+1}^{n-1} \frac{\partial^2}{\partial y_j^2} \right)^k \left(\frac{\partial}{\partial y_n} \right)^{\ell-2k} \text{ on } \mathbb{R}_-^{p,q},$$

where we set for $k \in \mathbb{N}$ with $0 \le 2k \le \ell$

$$a_k(\mu,\ell) := \frac{(-1)^k 2^{\ell-2k} \Gamma(\ell-k+\mu)}{\Gamma(\mu+[\frac{\ell+1}{2}]) k! (\ell-2k)!}. \tag{2.1}$$

In the case $(p,q,\varepsilon) = (n,0,+)$, $(\mathcal{D}_\ell^\mu)_{\mathbb{R}_\varepsilon^{p,q}}$ coincides with the differential operator $\mathcal{D}_\ell^\mu$ in [12, (1.2)], which was originally introduced in [6] (up to scalar).

The coefficients $a_k(\mu,\ell)$ arise from a hypergeometric polynomial

$$\widetilde{C}_\ell^\mu(t) := \sum_{k=0}^{[\frac{\ell}{2}]} a_k(\mu,\ell) t^{\ell-2k}.$$

This is a "renormalized" Gegenbauer polynomial [15, II, (11.16)] in the sense that $\widetilde{C}_\ell^\mu(t)$ is nonzero for all $\mu \in \mathbb{C}$ and $\ell \in \mathbb{N}$ and satisfies the Gegenbauer differential equation:

$$\left((1-t^2)\frac{d^2}{dt^2} - (2\mu+1)t\frac{d}{dt} + \ell(\ell+2\mu) \right) f(t) = 0.$$

We set $\mu =: u + i - \frac{1}{2}(n-1)$ and $\gamma(\mu,a) := 1$ (a: odd), $\mu + \frac{a}{2}$ (a: even).

For parameters $u \in \mathbb{C}$ and $\ell \in \mathbb{N}$, we define a family of linear operators

$$(\mathcal{D}_{u,\ell}^{i\to j})_- \colon \mathcal{E}^i(\mathbb{R}^{p,q}) \to \mathcal{E}^j(\mathbb{R}^{p,q-1})$$

in the coordinates $(x_1, \cdots, x_p, y_{p+1}, \cdots, y_{p+q})$ of $\mathbb{R}_-^{p,q}$ as follows: For $j = i-1$ or i,

$$(\mathcal{D}_{u,\ell}^{i\to i-1})_{\mathbb{R}_-^{p,q}} := \operatorname{Rest}_{y_n=0} \circ \left((\mathcal{D}_{\ell-2}^{\mu+1})_- \, dd^* \iota_{\frac{\partial}{\partial y_n}} + \gamma(\mu,a)(\mathcal{D}_{\ell-1}^{\mu+1})_- \, d^* \right.$$
$$\left. + \frac{u+2i-n}{2} (\mathcal{D}_\ell^\mu)_- \, \iota_{\frac{\partial}{\partial y_n}} \right),$$

$$(\mathcal{D}_{u,\ell}^{i\to i})_{\mathbb{R}_-^{p,q}} := \operatorname{Rest}_{y_n=0} \circ \left(-(\mathcal{D}_{\ell-2}^{\mu+1})_- \, dd^* - \gamma(\mu - \frac{1}{2}, \ell)(\mathcal{D}_{\ell-1}^\mu)_- \, d\iota_{\frac{\partial}{\partial y_n}} \right.$$
$$\left. + \frac{u+\ell}{2} (\mathcal{D}_\ell^\mu)_- \right).$$

Here $d^* \colon \mathcal{E}^i(\mathbb{R}_-^{p,q}) \to \mathcal{E}^{i-1}(\mathbb{R}_-^{p,q})$ is the codifferential $d^*_{\mathbb{R}_-^{p,q}} = (-1)^i *^{-1} d*$, where $* \equiv *_{\mathbb{R}_-^{p,q}}$ is the Hodge operator with respect to the pseudo-Riemannian structure on $\mathbb{R}^{p,q}_-$, $\iota_{\frac{\partial}{\partial y_n}}$ is the interior multiplication by the vector field $\frac{\partial}{\partial y_n}$, and $(\mathcal{D}^\mu_\ell)_-$ acts on $\mathcal{E}^i(\mathbb{R}^{p,q}_-)$ as a scalar differential operator.

In contrast to the case $j = i-1$ or i where the family of operators $\mathcal{D}^{i\to j}_{u,\ell}$ contains a continuous parameter $u \in \mathbb{C}$ and discrete one $\ell \in \mathbb{N}$, it turns out that the remaining case where $j \notin \{i-1, i\}$ or its Hodge dual $j \notin \{n-i+1, n-i\}$ is not abundant in conformal symmetry breaking operators. Actually, for $j \in \{i-2, i+1\}$, we define $(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{p,q}_-}$ only for special values of (i, u, ℓ) as follows:

$$
\begin{aligned}
(\mathcal{D}^{i\to i-2}_{n-2i,1})_{\mathbb{R}^{p,q}_-} &:= -\mathrm{Rest}_{y_n=0} \circ \iota_{\frac{\partial}{\partial y_n}} d^* && (2 \leq i \leq n-1),\\
(\mathcal{D}^{n\to n-2}_{1-n-\ell,\ell})_{\mathbb{R}^{p,q}_-} &:= -\mathrm{Rest}_{y_n=0} \circ \left(\mathcal{D}^{\frac{3-n}{2}-\ell}_{\ell-1}\right)_- \iota_{\frac{\partial}{\partial y_n}} d^* && (\ell \in \mathbb{N}_+),\\
(\mathcal{D}^{i\to i+1}_{0,1})_{\mathbb{R}^{p,q}_-} &:= \mathrm{Rest}_{y_n=0} \circ d && (1 \leq i \leq n-2),\\
(\mathcal{D}^{0\to 1}_{1-\ell,\ell})_{\mathbb{R}^{p,q}_-} &:= \mathrm{Rest}_{y_n=0} \circ \left(\mathcal{D}^{\frac{3-n}{2}-\ell}_{\ell-1}\right)_- d && (\ell \in \mathbb{N}_+).
\end{aligned}
$$

Likewise, for $\mathbb{R}^{p,q}_+$, we define a family of linear operators

$$
(\mathcal{D}^{i\to j}_{u,\ell})_+ \colon \mathcal{E}^i(\mathbb{R}^{p,q}) \to \mathcal{E}^j(\mathbb{R}^{p-1,q})
$$

in the coordinates $(y_1, \cdots, y_q, x_{q+1}, \cdots, x_{p+q})$ of $\mathbb{R}^{p,q}_+$ with parameters $u \in \mathbb{C}$ and $\ell \in \mathbb{N}$. In this case, the formulæ are essentially the same as those in the Riemannian case ($q = 0$) which were introduced in [12, (1.4)–(1.12)]. (The changes from $(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{p,q}_-}$ to $(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{p,q}_+}$ are made by replacing $y_n = 0$ with $x_n = 0$, $\frac{\partial}{\partial y_n}$ with $\frac{\partial}{\partial x_n}$, and $d^*_{\mathbb{R}^{p,q}_-}$ with $-d^*_{\mathbb{R}^{p,q}_+}$.) For the convenience of the reader, we give formulæ for $j = i-1$ or i and omit the case $j = i-2$ and $i+1$:

$$
\begin{aligned}
(\mathcal{D}^{i\to i-1}_{u,\ell})_+ := \mathrm{Rest}_{x_n=0} \circ \Big(&-(\mathcal{D}^{\mu+1}_{\ell-2})_+ dd^* \iota_{\frac{\partial}{\partial x_n}} - \gamma(\mu,\ell)(\mathcal{D}^{\mu+1}_{\ell-1})_+ d^* \\
&+ \frac{u+2i-n}{2}(\mathcal{D}^\mu_\ell)_+ \iota_{\frac{\partial}{\partial x_n}} \Big),\\
(\mathcal{D}^{i\to i}_{u,\ell})_+ := \mathrm{Rest}_{x_n=0} \circ \Big(&(\mathcal{D}^{\mu+1}_{\ell-2})_+ dd^* - \gamma(\mu - \frac{1}{2}, \ell)(\mathcal{D}^\mu_{\ell-1})_+ d\iota_{\frac{\partial}{\partial x_n}} \\
&+ \frac{u+\ell}{2}(\mathcal{D}^\mu_\ell)_+ \Big).
\end{aligned}
$$

If $i = j = 0$, the operators $(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{p,q}_+}$ reduce to scalar-valued differential operators that are proportional to $(\mathcal{D}^\mu_\ell)_+$ because d^* and $\iota_{\frac{\partial}{\partial x_n}}$ are identically zero on $\mathcal{E}^0(X) = C^\infty(X)$.

Theorem D below gives conformal symmetry breaking operators on the flat pseudo-Riemannian manifolds:

Theorem D. *Let* $p+q \geq 3$, $0 \leq i \leq p+q$, $0 \leq j \leq p+q-1$, *and* $u, v \in \mathbb{C}$. *Assume* $j \in \{i-2, i-1, i, i+1\}$ *and* $\ell := (v+j)-(u+i) \in \mathbb{N}$. *(For* $j \in \{i-2, i+1\}$*, we need an additional condition on the quadruple* (i,j,u,v)*, or equivalently, on* (i,j,u,ℓ) *as indicated in the* $\mathbb{R}^{p,q}_{-}$ *case.) Then*

$$(\mathcal{D}^{i\to j}_{u,\ell})_{+} \in \mathrm{Diff}_{\mathfrak{conf}(\mathbb{R}^{p,q};\mathbb{R}^{p-1,q})}(\mathcal{E}^i(\mathbb{R}^{p,q})_u, \mathcal{E}^j(\mathbb{R}^{p-1,q})_v) \quad \text{for } p \geq 1,$$
$$(\mathcal{D}^{i\to j}_{u,\ell})_{-} \in \mathrm{Diff}_{\mathfrak{conf}(\mathbb{R}^{p,q};\mathbb{R}^{p,q-1})}(\mathcal{E}^i(\mathbb{R}^{p,q})_u, \mathcal{E}^j(\mathbb{R}^{p,q-1})_v) \quad \text{for } q \geq 1.$$

Remark 2.1. In recent years, special cases of Theorem D have been obtained as below.

1. $i=j=0$, $\varepsilon=+$, $q=0$: [6], see also [3, 10, 14] for different approaches.
2. $i=j=0$, $\varepsilon=+$, p and q are arbitrary: [14, Thm. 4.3].
3. i and j are arbitrary, $\varepsilon=+$, $q=0$: [12, Thms. 1.5, 1.6, 1.7 and 1.8].

The main machinery of finding symmetry breaking operators in various geometric gettings in [12], [14], and [15, II] is the "algebraic Fourier transform of generalized Verma modules" (*F-method* [9]), see [15, I] for a detailed exposition of the F-method.

The proof of Theorem D will be given in Section 6.

Remark 2.2. There are a few values of parameters (u,ℓ,i,j) for which $(\mathcal{D}^{i\to j}_{u,\ell})_{+}$ or $(\mathcal{D}^{i\to j}_{u,\ell})_{-}$ vanishes, but we can define nonzero conformal symmetry breaking operators for such values by "renormalization" as in [12, (1.9), (1.10)]. The "renormalized" operators $(\widetilde{\mathcal{D}}^{i\to j}_{u,\ell})_{\pm}$ and the compositions $*\circ(\widetilde{\mathcal{D}}^{i\to j}_{u,\ell})_{\pm}$ by the Hodge operator $*$ for $\mathbb{R}^{p-1,q}$ or $\mathbb{R}^{p,q-1}$ exhaust all conformal differential symmetry breaking operators in our framework, as is followed from Theorem B (2) and from the classification theorem [12, Thms. 1.1 and 2.10] in the Riemannian setting.

3. Symmetry breaking operators in the space forms

In this section we explain how to transfer the formulæ for symmetry breaking operators in the flat case (Theorem D) to the ones in the space form $\mathrm{S}^{p,q}$ (see Theorem E). In particular, Theorem E gives conformal symmetry breaking operators in the anti-de Sitter space (Example 3.2).

We consider the following open dense subsets of the flat space $\mathbb{R}^{p,q}$ and the space form $\mathrm{S}^{p,q}$ (see (1.4)), respectively:

$$(\mathbb{R}^{p,q}_-)' := \{(x,y) \in \mathbb{R}^{p+q} : |x|^2 - |y|^2 \neq -4\},$$
$$(\mathrm{S}^{p,q})' := \{(\omega_0, \omega, \eta) \in \mathrm{S}^{p,q} : \omega_0 \neq -1\} \qquad \subset \mathbb{R}^{1+p+q}.$$

We define a variant of the stereographic projection and its inverse by

$$\Psi\colon (\mathrm{S}^{p,q})' \longrightarrow (\mathbb{R}^{p,q}_-)', \quad (\omega_0, \omega, \eta) \mapsto \frac{2}{1+\omega_0}(\omega, \eta),$$
$$\Phi\colon (\mathbb{R}^{p,q}_-)' \longrightarrow (\mathrm{S}^{p,q})', \quad (x,y) \mapsto \frac{1}{|x|^2 - |y|^2 + 4}(4 - |x|^2 + |y|^2, 4x, 4y).$$

Lemma 3.1. *The map Φ is a conformal diffeomorphism from $(\mathbb{R}^{p,q}_-)'$ onto $(\mathrm{S}^{p,q})'$ with its inverse Ψ, and the conformal factor is given by*

$$\Phi^* g_{\mathrm{S}^{p,q}} = \frac{16}{(|x|^2 - |y|^2 + 4)^2} g_{\mathbb{R}^{p,q}_-}, \qquad \Psi^* g_{\mathbb{R}^{p,q}_-} = \frac{4}{(1+\omega_0)^2} g_{\mathrm{S}^{p,q}}.$$

Proof. See [13, I, Lem. 3.3], for instance. ☐

The pseudo-Riemannian spaces $\mathbb{R}^{p,q}_+$ and $\mathbb{R}^{p,q}_-$ are obviously isomorphic to each other by switch of the coordinates

$$s\colon \mathbb{R}^{p,q}_+ \xrightarrow{\sim} \mathbb{R}^{p,q}_- \quad (y,x) \mapsto (x,y).$$

We set

$$\Phi_- := \Phi, \quad \Phi_+ := \Phi \circ s, \quad \Psi_- := \Psi, \quad \Psi_+ := s \circ \Psi.$$

For $v \in \mathbb{C}$, we define the "twisted pull-back" of differential forms as defined in [13, I, (2.3.2)]:

$$(\Phi_\pm)^*_v\colon \mathcal{E}^j\left((\mathbb{R}^{p,q}_\pm)'\right) \longrightarrow \mathcal{E}^j((\mathrm{S}^{p,q})'), \quad \alpha \mapsto \left(\frac{1+\omega_0}{2}\right)^{-v} \Phi^*\alpha, \tag{3.1}$$

$$(\Psi_\pm)^*_v\colon \mathcal{E}^j((\mathrm{S}^{p,q})') \longrightarrow \mathcal{E}^j((\mathbb{R}^{p,q}_\pm)'), \quad \beta \mapsto \left(\frac{|x|^2 - |y|^2 + 4}{4}\right)^{-v} \Psi^*\beta. \tag{3.2}$$

Then $(\Psi_\pm)^*_v$ is the inverse of $(\Phi_\pm)^*_v$ in accordance with $\Psi_\pm = (\Phi_\pm)^{-1}$.

We realize the space forms $\mathrm{S}^{p-1,q}$ $(p \geq 1)$ and $\mathrm{S}^{p,q-1}$ $(q \geq 1)$ as totally geodesic hypersurfaces of $\mathrm{S}^{p,q}$ by letting $\omega_p = 0$ and $\eta_q = 0$, respectively. Then $\Phi_\pm$ induce the following diffeomorphisms between hypersurfaces.

$$\begin{array}{ccc} (\mathbb{R}^{p,q}_-)' & \xrightarrow[\Phi_-]{\sim} & (\mathrm{S}^{p,q})' \\ \cup & & \cup \\ (\mathbb{R}^{p,q-1}_-)' & \xrightarrow{\sim} & (\mathrm{S}^{p,q-1})', \end{array} \qquad \begin{array}{ccc} (\mathbb{R}^{p,q}_+)' & \xrightarrow[\Phi_+]{\sim} & (\mathrm{S}^{p,q})' \\ \cup & & \cup \\ (\mathbb{R}^{p-1,q}_+)' & \xrightarrow{\sim} & (\mathrm{S}^{p-1,q})' \end{array}$$

We are ready to transfer the formulæ of conformal symmetry breaking operators for the flat case (Theorem D) to those for negatively (or positively) curved spaces:

Theorem E. *For $\varepsilon = \pm$, let $(\mathcal{D}^{i\to j}_{u,\ell})_\varepsilon$ be as in Theorem D. Then the operator $(\Phi_\varepsilon)^*_v \circ (\mathcal{D}^{i\to j}_{u,\ell})_\varepsilon \circ (\Psi_\varepsilon)^*_u$, originally defined in the open dense set $(\mathrm{S}^{p,q})'$ of the space form $\mathrm{S}^{p,q}$, extends uniquely to the whole $\mathrm{S}^{p,q}$ and gives an element in $\mathrm{Diff}_{\mathfrak{conf}(X;Y)}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$ where $(X,Y) = (\mathrm{S}^{p,q}, \mathrm{S}^{p,q-1})$ for $\varepsilon = -$ and $(X,Y) = (\mathrm{S}^{p,q}, \mathrm{S}^{p-1,q})$ for $\varepsilon = +$.*

Here, by a little abuse of notation, we have used the symbol $(\Phi_\varepsilon)^*_v$ to denote the operator in the $(n-1)$-dimensional case.

Admitting Theorem A, we give a proof of Theorem E.

Proof of Theorem E. Similarly to [12, Prop. 11.3] in the Riemannian case ($q = 0$ and $\varepsilon = +$), the composition $(\Phi_\varepsilon)^*_v \circ (\mathcal{D}^{i\to j}_{u,\ell})_\varepsilon \circ (\Phi_\varepsilon)^*_u$ gives an element in $\mathrm{Diff}_{\mathfrak{conf}(V;V\cap Y)}(\mathcal{E}^i(V)_u, \mathcal{E}^j(V\cap Y)_v)$ for $V = (\mathrm{S}^{p,q})'$. Then this operator extends to the whole $X = \mathrm{S}^{p,q}$ by Theorem A. $\square$

The n-dimensional anti-de Sitter space $\mathrm{AdS}^n (= \mathrm{S}^{1,n-1})$ contains the hyperbolic space $\mathrm{H}^{n-1} (= \mathrm{S}^{0,n-1})$ and the anti-de Sitter space $\mathrm{AdS}^{n-1} (= \mathrm{S}^{1,n-2})$ as totally geodesic hypersurfaces.

Example 3.2 (hypersurfaces in the anti-de Sitter space). *For $(p,q) = (1, n-1)$, the formulæ in Theorem E give conformal symmetry breaking operators as follows.*

$$\begin{aligned} \mathcal{E}^i(\mathrm{AdS}^n)_u &\longrightarrow \mathcal{E}^j(\mathrm{H}^{n-1})_v && \text{for } \varepsilon = +, \\ \mathcal{E}^i(\mathrm{AdS}^n)_u &\longrightarrow \mathcal{E}^j(\mathrm{AdS}^{n-1})_v && \text{for } \varepsilon = -. \end{aligned}$$

4. Idea of holomorphic continuation

In this section we explain an idea of holomorphic continuation that will bridge between differential symmetry breaking operators in the Riemannian setting and those in the non-Riemannian setting.

We begin with an observation from Example 1.2 that, for any p, q with $p \geq 1$, the Lie algebras

$$\mathfrak{conf}(\mathrm{S}^{p,q}; \mathrm{S}^{p-1,q}) \simeq \mathfrak{conf}(\mathbb{R}^{p,q}; \mathbb{R}^{p-1,q}) \simeq \mathfrak{o}(p, q+1)$$

have the same complexification $\mathfrak{o}(n+1, \mathbb{C})$ as far as $p+q = n$. In turn to geometry, we shall compare (real) conformal vector fields on pseudo-Riemannian manifolds $\mathrm{S}^{p,q}$ or $\mathbb{R}^{p,q}$ of various signatures (p,q) via holomorphic vector fields on a complex manifold which contains $\mathrm{S}^{p,q}$ or $\mathbb{R}^{p,q}$ as totally real submanifolds.

Let $X_{\mathbb{C}}$ be a connected complex manifold, and $\Omega^i(X_{\mathbb{C}})$ the space of holomorphic i-forms on $X_{\mathbb{C}}$. If X is a totally real submanifold, then the restriction map

$$\mathrm{Rest}_X \colon \Omega^i(X_{\mathbb{C}}) \longrightarrow \mathcal{E}^i(X)$$

is obviously injective.

Definition-Lemma 4.1. Suppose $D_{\mathbb{C}} \colon \Omega^i(X_{\mathbb{C}}) \to \Omega^j(X_{\mathbb{C}})$ is a holomorphic differential operator. Then there is a unique differential operator $E \colon \mathcal{E}^i(X) \to \mathcal{E}^j(X)$, such that

$$E|_{V\cap X} \circ \mathrm{Rest}_{V\cap X}\alpha = \mathrm{Rest}_{V\cap X} \circ D_{\mathbb{C}}|_V \alpha$$

for any open set V of $X_{\mathbb{C}}$ with $V \cap X \neq \emptyset$ and for any $\alpha \in \Omega^i(V)$. We say that $D_{\mathbb{C}}$ is the *holomorphic extension* of E. We write $(\mathrm{Rest}_X)_* D_{\mathbb{C}}$ for E.

If X is a real analytic, pseudo-Riemannian manifold with complexification $X_{\mathbb{C}}$, then a holomorphic analogue of the action (1.1) makes sense by analytic continuation for $Z \in \mathfrak{conf}(X) \otimes_{\mathbb{R}} \mathbb{C}$: L_Z being understood as the holomorphic Lie derivative with respect to a holomorphic extension of the vector field Z in a complex neighbourhood U of X, which acts on $\alpha \in \Omega^i(U)$; and the conformal factor $\rho(\cdot,\cdot)$ being understood as its holomorphic extension (complex linear in the first argument). Likewise for the pair $X \supset Y$ of pseudo-Riemannian manifolds with complexification $X_{\mathbb{C}} \supset Y_{\mathbb{C}}$, we may consider a holomorphic analogue of the covariance condition (1.2). Then we have:

Lemma 4.2. *Suppose $D_{\mathbb{C}} \colon \Omega^i(X_{\mathbb{C}}) \to \Omega^j(Y_{\mathbb{C}})$ is a holomorphic differential operator, and $D = (\mathrm{Rest}_X)_*(D_{\mathbb{C}})$. Then $D \colon \mathcal{E}^i(X) \to \mathcal{E}^j(Y)$ satisfies the conformal covariance* (1.2) *if and only if*

$$\pi_v^{(j)}(Z|_{Y_{\mathbb{C}}}) \circ D_{\mathbb{C}}\,\alpha = D_{\mathbb{C}} \circ \Pi_u^{(i)}(Z)\alpha$$

for any $Z \in \mathfrak{conf}(X;Y) \otimes_{\mathbb{R}} \mathbb{C}$, any open subset U of $X_{\mathbb{C}}$ with $U \cap Y_{\mathbb{C}} \neq \emptyset$ and any $\alpha \in \Omega^i(U)$.

We define a family of totally real vector spaces of $\mathbb{C}^n$ by embedding the space $\mathbb{R}^n = \mathbb{R}^p_x \oplus \mathbb{R}^q_y$ as

$$\iota_+ \colon \mathbb{R}^q_y \oplus \mathbb{R}^p_x \overset{\sim}{\to} \sqrt{-1}\mathbb{R}^q \oplus \mathbb{R}^p = \left\{(\sqrt{-1}y_1, \cdots, \sqrt{-1}y_q, x_{q+1}, \cdots x_{p+q}) : x_j, y_j \in \mathbb{R}\right\},$$
$$\iota_- \colon \mathbb{R}^p_x \oplus \mathbb{R}^q_y \overset{\sim}{\to} \mathbb{R}^p \oplus \sqrt{-1}\mathbb{R}^q = \left\{(x_1, \cdots, x_p, \sqrt{-1}y_{p+1}, \cdots, \sqrt{-1}y_{p+q}) : x_j, y_j \in \mathbb{R}\right\}.$$

Let us apply Lemma 4.2 to the following setting where $n = p+q$.

$$\begin{array}{ccccccc}
\mathbb{R}^n & \simeq \mathbb{R}^{p,q}_+ & \underset{\iota_+}{\hookrightarrow} & X_{\mathbb{C}} = \mathbb{C}^n & \underset{\iota_-}{\hookleftarrow} & \mathbb{R}^{p,q}_- & \simeq \mathbb{R}^n \\
& \cup & & \cup & & \cup & \\
\mathbb{R}^{n-1} & \simeq \mathbb{R}^{p-1,q}_+ & \hookrightarrow & Y_{\mathbb{C}} = \mathbb{C}^{n-1} & \hookleftarrow & \mathbb{R}^{p,q-1}_- & \simeq \mathbb{R}^{n-1}
\end{array}$$

The holomorphic symmetric 2-tensor

$$ds^2 = dz_1^2 + \cdots + dz_n^2$$

on $\mathbb{C}^n$ induces a flat pseudo-Riemannian structure on $\mathbb{R}^n$ of signature (p,q) by restriction via $\iota_\pm$. The resulting pseudo-Riemannian structures (and coordinates) on $\mathbb{R}^n$ are nothing but those of $\mathbb{R}^{p,q}_+$ and $\mathbb{R}^{p,q}_-$ given in Section 2.

5. Proof of Theorems A, B, and C

This section gives a proof of Theorems A, B, and C. The key machinery for differential symmetry breaking operators (SBOs for short) is in threefold:

(1) holomorphic extension of differential SBOs (Section 4);
(2) duality theorem between differential SBOs and homomorphisms for generalized Verma modules that encode branching laws [15, I, Thm. 2.9];
(3) automatic continuity theorem of differential SBOs in the Hermitian symmetric setting [15, I, Thm. 5.3].

We note that both (1) and (2) indicate the independence of real forms as formulated in Theorem B (2), whereas (3) appeals to the theory of admissible restrictions of real reductive groups [7] for a specific choice of real forms of complex reductive Lie groups.

Let G be $SO_0(p+1, q+1)$, the identity component of the indefinite orthogonal group $O(p+1,q+1)$, $P = LN$ a maximal prabolic subgroup of G with Levi subalgebra $\mathrm{Lie}(L) \simeq \mathfrak{so}(p,q) + \mathbb{R}$, and H the identity component of P. Then G acts conformally on $G/H \simeq \mathrm{S}^p \times \mathrm{S}^q$ equipped with the pseudo-Riemannian structure $g_{\mathrm{S}^p} \oplus (-g_{\mathrm{S}^q})$. Similarly, H' is defined by taking $G' := SO_0(p, q+1)$ $(\varepsilon = +)$ or $SO_0(p+1, q)$ $(\varepsilon = -)$.

Applying the duality theorem [15, I, Thm. 2.9] to the quadruple (G, H, G', H'), we see that any element in

$$\mathrm{Hom}_{\mathfrak{g}'_{\mathbb{C}}}(U(\mathfrak{g}'_{\mathbb{C}})\otimes_{U(\mathfrak{p}'_{\mathbb{C}})}(\textstyle\bigwedge^{n-1-j}(\mathbb{C}^{n-1})\otimes\mathbb{C}_{-v-j}), U(\mathfrak{g}_{\mathbb{C}})\otimes_{U(\mathfrak{p}_{\mathbb{C}})}(\bigwedge^{n-i}(\mathbb{C}^n)\otimes\mathbb{C}_{-u-i})) \tag{5.1}$$

with notation as in [12, Sect. 2.6] induces a differential symmetry breaking operator $\overline{D} \in \mathrm{Diff}_{\mathfrak{conf}(\overline{X};\overline{Y})}(\mathcal{E}^i(\overline{X})_u, \mathcal{E}^j(\overline{Y})_v)$ on the conformal compactification $\overline{X}$, and hence the one on any open subset V of X with $V \cap Y \neq \emptyset$ by restriction. In order to prove Theorem A and Theorem B (1), it is then sufficient to show the following converse statement.

Claim 5.1. *Any* $D \in \mathrm{Diff}_{\mathfrak{conf}(V;V\cap Y)}(\mathcal{E}^i(V)_u, \mathcal{E}^j(V \cap Y)_v)$ *is derived from an element in* (5.1).

Let us prove Claim 5.1.

- Step 1. Reduction to the flat case

By using the twisted pull-backs $(\Phi_\pm)_v^*$ and $(\Psi_\pm)_v^*$ (see (3.1)), we may and do assume that $X = \mathbb{R}^{p,q}$ $(\simeq \mathbb{R}^n)$ and that Y is the hypersurface $\mathbb{R}^{n-1}$ given by the condition that the last coordinate is zero. By replacing V with an open subset V' of $\mathbb{R}^n$ with $V \cap \mathbb{R}^{n-1} = V' \cap \mathbb{R}^{n-1}$ if necessary, we may further assume that V is a convex neighbourhood of $V \cap \mathbb{R}^{n-1}$ in $\mathbb{R}^n$.

- Step 2. Holomorphic extension

With the coordinates $x = (x', x_n) \equiv (x_1, \cdots, x_{n-1}, x_n)$ of $X = \mathbb{R}^n$, any differential operator $D\colon \mathcal{E}^i(\mathbb{R}^n) \to \mathcal{E}^j(\mathbb{R}^{n-1})$ takes the form

$$D = \mathrm{Rest}_{x_n=0} \circ \sum_{\alpha\in\mathbb{N}^n} a_\alpha(x') \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where $a_\alpha \in C^\infty(\mathbb{R}^{n-1}) \otimes \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1})\right)$ (see [15, I, Ex. 2.4]). Since $Z_k := \frac{\partial}{\partial x_k}$ $(1 \leq k \leq n-1)$ is a Killing vector filed, namely, $Z_k \in \mathfrak{conf}(X;Y)$ with $\rho(Z_k, \cdot) \equiv 0$, the conformal covariance (1.2) reduces to $L_{\frac{\partial}{\partial x_k}} \circ D = D \circ L_{\frac{\partial}{\partial x_k}}$, which implies that the matrix-valued function $a_\alpha(x')$ is independent of x' for every α. We shall denote $a_\alpha(x')$ simply by a_α. Then D extends to a holomorphic differential operator $D_{\mathbb{C}}\colon \Omega^i(\mathbb{C}^n) \to \Omega^j(\mathbb{C}^{n-1})$, by setting

$$D_{\mathbb{C}} := \mathrm{Rest}_{z_n=0} \circ \sum_{\alpha\in\mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$

If D satisfies the conformal covariance condition (1.2) on $\mathcal{E}^i(V)$ for all $Z \in \mathfrak{conf}(V; V \cap Y) \simeq \mathfrak{o}(p, q+1)$ or $\mathfrak{o}(p+1, q)$, then by Lemma 4.2, $D_{\mathbb{C}}$ satisfies the holomorphic extension of the condition (1.2) on $\Omega^i(\mathbb{C}^n)$ for all $Z \in \mathfrak{conf}(V; V \cap Y) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{o}(n+1, \mathbb{C})$.

• Step 3. Automatic continuity in the Hermitian symmetric spaces $G_{\mathbb{R}}/K_{\mathbb{R}} \supset G'_{\mathbb{R}}/K'_{\mathbb{R}}$

Automatic continuity theorem is known for holomorphic differential SBOs in the Hermitian symmetric setting [15, I, Thm. 5.3]. Then our strategy to prove Claim 5.1 is to utilize the automatic continuity theorem in the Hermitian symmetric setting by embedding a pair $(G_{\mathbb{R}}/K_{\mathbb{R}}, G'_{\mathbb{R}}/K'_{\mathbb{R}})$ of Hermitian symmetric spaces into the pair $(\mathbb{C}^n, \mathbb{C}^{n-1})$ of the affine spaces as in Step 2. For this, we shall choose a specific real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}} := SO(n+2, \mathbb{C})$ such that $G_{\mathbb{R}}$ is the group of biholomorphic transformations of a bounded symmetric domain in $\mathbb{C}^n$ as below.

Let $Q(\tilde{x}) := -x_0^2 + x_1^2 + \cdots + x_n^2 - x_{n+1}^2$ be the quadratic form on $\mathbb{R}^{n+2}$, and $G_{\mathbb{R}}$ the identity component of the isotropy group

$$\{h \in GL(n+2, \mathbb{R}) : Q(h \cdot \tilde{x}) = Q(\tilde{x}) \text{ for all } \tilde{x} \in \mathbb{R}^{n+2}\}.$$

Then $K_{\mathbb{R}} := G_{\mathbb{R}} \cap SO(n+2)$ is a maximal compact subgroup of $G_{\mathbb{R}} \simeq SO_0(n, 2)$ such that $G_{\mathbb{R}}/K_{\mathbb{R}}$ is the Hermitian symmetric space of type IV in the É. Cartan classification. We take $G'_{\mathbb{R}}$ to be the stabilizer of x_n. Then $G'_{\mathbb{R}} \simeq SO_0(n-1, 2)$.

We use the notation as in [15, II, Sect. 6], and identify $\mathbb{C}^n$ with the open Bruhat cell of the complex quadric

$$\mathbb{Q}^n\mathbb{C} = \{\tilde{z} \in \mathbb{C}^{n+2} \setminus \{0\} : Q(\tilde{z}) = 0\}/\mathbb{C}^\times \simeq G_{\mathbb{C}}/P_{\mathbb{C}}.$$

Then $G_{\mathbb{R}}/K_{\mathbb{R}}$ is realized as the Lie ball

$$U = \{z \in \mathbb{C}^n : |z\,{}^t z|^2 + 1 - 2\bar{z}\,{}^t z > 0,\ |z\,{}^t z| < 1\}.$$

We compare the real form G of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{conf}(X) \simeq \mathfrak{o}(p+1, q+1)$ in Step 1 and another real form $G_{\mathbb{R}} \simeq SO_0(n, 2)$ in Step 3 ($n = p+q$). The point here is that the G-orbit $G \cdot o \simeq G/P$ through the origin $o = eP_{\mathbb{C}} \in G_{\mathbb{C}}/P_{\mathbb{C}}$ is closed in $G_{\mathbb{C}}/P_{\mathbb{C}}$, while the $G_{\mathbb{R}}$-orbit $G_{\mathbb{R}} \cdot o \simeq G_{\mathbb{R}}/K_{\mathbb{R}}$ is open in $G_{\mathbb{C}}/P_{\mathbb{C}}$, as is summarized in the figure below.

$$\begin{array}{ccccccc}
G_{\mathbb{R}}/K_{\mathbb{R}} \simeq U & \underset{\text{open}}{\subset} & \mathbb{C}^n & \underset{\text{Bruhat cell}}{\subset} & \mathbb{Q}^n\mathbb{C} & \simeq & G_{\mathbb{C}}/P_{\mathbb{C}} \\
 & & \cup & & \cup & & \cup \ \text{totally real} \\
 & & \mathbb{R}^{p,q} & \underset{\text{conformal compactfication}}{\subset} & (S^p \times S^q)/\mathbb{Z}_2 & \simeq & G/P
\end{array}$$

We note that the $G'_{\mathbb{R}}$-orbit $G'_{\mathbb{R}} \cdot o \simeq G'_{\mathbb{R}}/K'_{\mathbb{R}}$ is realized as the subsymmetric domain $U \cap \{z_n = 0\} \simeq \mathbb{C}^{n-1}$. Since the holomorphic differential operator $D_{\mathbb{C}}$ is defined on $\Omega^i(\mathbb{C}^n)$, $D_{\mathbb{C}}$ induces a holomorphic differential operator

$$D_{\mathbb{C}}|_{G_{\mathbb{R}}/K_{\mathbb{R}}} \colon \Omega^i(G_{\mathbb{R}}/K_{\mathbb{R}}) \longrightarrow \Omega^j(G'_{\mathbb{R}}/K'_{\mathbb{R}}) \tag{5.2}$$

via the inclusion $G_{\mathbb{R}}/K_{\mathbb{R}} \simeq U \subset \mathbb{C}^n$.

Then the automatic continuity theorem [15, I, Thm. 5.3 (2)] (and its proof), applied to (5.2) implies that $D_{\mathbb{C}}|_{G_{\mathbb{R}}/K_{\mathbb{R}}}$ is derived from an element of (5.1) via the duality theorem in the holomorphic setting (see [15, I, Thm. 2.12]). Thus the proof of Claim 5.1 is completed. Therefore Theorem A and Theorem B (1) follow from [15, I, Thm. 2.9].

Since (5.1) is independent of the choice of real forms, Theorem B (2) is now clear.

Proof of Theorem C. Owing to Theorems A and B, Theorem C is reduced to the Riemannian case $p = 0$ and $\varepsilon = -$ or $q = 0$ and $\varepsilon = +$. Then the assertion follows from the classification results [12, Thm. 1.1] for the (disconnected) conformal group and from a discussion on the connected group case (see [12, Thm. 2.10]). □

6. Proof of Theorem D

In this section, we give a proof of Theorem D in Section 2 by reducing it to the Riemannian case $(p, q, \varepsilon) = (n, 0, +)$ or $(0, n, -)$ which was established in [12, Thms. 1.5, 1.6, 1.7 and 1.8]. For this, we apply Definition-Lemma 4.1 to the totally real embedding $\iota_{\pm} \colon \mathbb{R}^{p,q}_{\pm} \hookrightarrow \mathbb{C}^{p+q}$.

With the coefficients $a_k(\mu, \ell)$ given in (2.1), we define a family of (scalar-valued) holomorphic differential operators on $\mathbb{C}^n$ by

$$(\mathcal{D}^{\mu}_{\ell})_{\mathbb{C}} := \sum_{k=0}^{[\frac{\ell}{2}]} a_k(\mu, \ell) \left(- \sum_{j=1}^{n-1} \frac{\partial^2}{\partial z_j^2} \right)^k \left(\frac{\partial}{\partial z_n} \right)^{\ell - 2k},$$

which are the holomorphic extensions of the operators $(\mathcal{D}^{\mu}_{\ell})_{\mathbb{R}^{n,0}_{+}}$ defined in the Riemannian case, that is, $(\mathrm{Rest}_{\mathbb{R}^{n,0}_{+}})_{*}\big((\mathcal{D}^{\mu}_{\ell})_{\mathbb{C}}\big) = (\mathcal{D}^{\mu}_{\ell})_{\mathbb{R}^{n,0}_{+}}$. Likewise, we extend $(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{n,0}_+}$ to a (matrix-valued) holomorphic differential operator

$$(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{C}} \colon \Omega^i(\mathbb{C}^n) \longrightarrow \Omega^j(\mathbb{C}^{n-1})$$

in such a way that $(\mathrm{Rest}_{\mathbb{R}^{n,0}_+})_*(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{C}}$ coincides with $(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{n,0}_+}$. By definition of $(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{p,q}_+}$, it is readily seen that $(\mathrm{Rest}_{\mathbb{R}^{p,q}_+})_*(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{C}} = (\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{p,q}_+}$ for all (p,q) with $p+q=n$. Concerning the other real form $\mathbb{R}^{p,q}_-$, we have the following.

Lemma 6.1. $(\mathrm{Rest}_{\mathbb{R}^{p,q}_-})_*(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{C}} = e^{-\frac{\pi\sqrt{-1}(\ell+i-j)}{2}}(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{p,q}_-}$.

Proof. The assertion is deduced from the formulæ of $(\mathrm{Rest}_{\mathbb{R}^{p,q}_-})_*$ for the following basic operators. (For the convenience of the reader, we also list the cases $\mathbb{R}^{p,q}_+$ as well.)

	$d_{\mathbb{C}}$	$d^*_{\mathbb{C}}$	$\frac{\partial}{\partial z_n}$	$\iota_{\frac{\partial}{\partial z_n}}$	$(\mathcal{D}^{\mu}_{\ell})_{\mathbb{C}}$
$(\mathrm{Rest}_{\mathbb{R}^{p,q}_+})_*$	$d_{\mathbb{R}^{p,q}_+}$	$d^*_{\mathbb{R}^{p,q}_+}$	$\frac{\partial}{\partial x_n}$	$\iota_{\frac{\partial}{\partial x_n}}$	$(\mathcal{D}^{\mu}_{\ell})_{\mathbb{R}^{p,q}_+}$
$(\mathrm{Rest}_{\mathbb{R}^{p,q}_-})_*$	$d_{\mathbb{R}^{p,q}_-}$	$d^*_{\mathbb{R}^{p,q}_-}$	$\frac{1}{\sqrt{-1}}\frac{\partial}{\partial y_n}$	$\frac{1}{\sqrt{-1}}\iota_{\frac{\partial}{\partial y_n}}$	$e^{-\frac{\pi\sqrt{-1}\ell}{2}}(\mathcal{D}^{\mu}_{\ell})_{\mathbb{R}^{p,q}_-}$

□

We are ready to complete the proof of Theorem D.

Proof of Theorem D. Since $(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{R}^{n,0}_+} \in \mathrm{Diff}_{\mathfrak{conf}(\mathbb{R}^n;\mathbb{R}^{n-1})}(\mathcal{E}^i(\mathbb{R}^n)_u, \mathcal{E}^j(\mathbb{R}^{n-1})_v)$ by [12, Thms. 1.5, 1.6, 1.7, 1.8], the holomorphic differential operator $(\mathcal{D}^{i\to j}_{u,\ell})_{\mathbb{C}}$ satisfies the holomorphic covariance condition by Lemma 4.2. In turn, we conclude Theorem D by Lemmas 6.1 and 4.2.

□

7. Four-dimensional example

In contrast to the multiplicity-free theorem ([12, Thm. 1.1]) for differential SBOs for (disconnected) conformal *groups* $(\mathrm{Conf}(X), \mathrm{Conf}(X;Y))$ when $(X,Y) = (\mathrm{S}^n, \mathrm{S}^{n-1})$ $(n \geq 3)$, it may happen that an analogous statement for the *Lie algebras* $(\mathfrak{conf}(X), \mathfrak{conf}(X;Y))$ does not hold anymore. In fact, for some u, v, i, j, one has

$$\dim_{\mathbb{C}} \mathrm{Diff}_{\mathfrak{conf}(X;Y)}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v) > 1 \qquad (7.1)$$

(or equivalently, $= 2$).

In this section we first address the question when and how (7.1) happens and then describe the corresponding generators when $(X,Y) = (\mathbb{R}^{p,q}, \mathbb{R}^{p-1,q})$ with $p+q\ (=n) = 4$.

As we have seen in Theorems C and D, there are two types of conditions on (i,j), namely,

$$-1 \le i-j \le 2 \quad \text{or} \quad n-2 \le i+j \le n+1,$$

for which nontrivial differential symmetry breaking operators $\mathcal{E}^i(X)_u \to \mathcal{E}^j(Y)_v$ exist for some $u, v \in \mathbb{C}$. (The latter inequality arises from the composition of the Hodge star operator with respect to the pseudo-Riemannian metric.) It turns out that (7.1) happens only if these two conditions are simultaneously fulfilled, that is, only if

$$-1 \le i-j \le n \quad \text{and} \quad n-2 \le i+j \le n+1.$$

The four-dimensional case is illustrative to understand (7.1) for the arbitrary dimension n. We give a complete list of parameters (i,j,u,v) for which (7.1) happens together with explicit generators of $\mathrm{Diff}_{\mathfrak{conf}(X;Y)}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$.

Let $X = \mathbb{R}^{p,q}_+$ and $Y = \mathbb{R}^{p-1,q}_+$ with $n = p+q = 4$. We shall simply write as $(X,Y) = (\mathbb{R}^{p,q}, \mathbb{R}^{p-1,q})$. (The case $(X,Y) = (\mathbb{R}^{p,q}_-, \mathbb{R}^{p,q-1}_-)$ is essentially the same and we omit it.) We set

$$\begin{aligned} A &:= \mathrm{Rest}_{x_4=0} \circ d, & B &:= \mathrm{Rest}_{x_4=0} \circ d^*, \\ C &:= \mathrm{Rest}_{x_4=0} \circ \iota_{\frac{\partial}{\partial x_4}} d, & D &:= \mathrm{Rest}_{x_4=0} \circ \iota_{\frac{\partial}{\partial x_4}} d^*. \end{aligned}$$

By using the formulæ in [12, Ch. 8. Sect. 5], we readily see that

$$D \circ *_{\mathbb{R}^{p,q}} = \pm *_{\mathbb{R}^{p-1,q}} \circ A, \quad C \circ *_{\mathbb{R}^{p,q}} = \pm *_{\mathbb{R}^{p-1,q}} \circ B. \tag{7.2}$$

Theorem F. *Suppose* $(X,Y) = (\mathbb{R}^{p,q}, \mathbb{R}^{p-1,q})$ *with* $p+q = 4$ *and* $p \ge 1$. *Then* (7.1) *occurs if and only if* (i,j,u,v) *appears in the nonempty boxes in the table below. Moreover, the pairs of operators in the table provide generators of* $\mathrm{Diff}_{\mathfrak{conf}(X;Y)}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$.

$i \backslash j$	0	1	2	3
0				
1		$u=0,\ v=1,$ $\ell=1$ - - - - - $*_{\mathbb{R}^{p-1,q}} \circ A,$ C	$u=v=0,$ $\ell=1$ - - - - - $A,$ $*_{\mathbb{R}^{p-1,q}} \circ C$	
2	$u=0,\ v=3,$ $\ell=1$ - - - - - $D,$ $*_{\mathbb{R}^{p-1,q}} \circ A$	$v-u \in \mathbb{N}_+,$ $\ell=v-u-1$ - - - - - $(\mathcal{D}^{2\to 1}_{u,\ell})_+,$ $*_{\mathbb{R}^{p-1,q}} \circ (\mathcal{D}^{2\to 2}_{u,\ell})_+$	$v-u \in \mathbb{N},$ $\ell=v-u$ - - - - - $(\mathcal{D}^{2\to 2}_{u,\ell})_+,$ $*_{\mathbb{R}^{p-1,q}} \circ (\mathcal{D}^{2\to 1}_{u,\ell})_+$	$u=v=0,$ $\ell=1$ - - - - - $A,$ $*_{\mathbb{R}^{p-1,q}} \circ D$
3		$u=-2,\ v=1,$ $\ell=1$ - - - - - $D,$ $*_{\mathbb{R}^{p-1,q}} \circ B$	$u=-2,\ v=0,$ $\ell=1$ - - - - - $B,$ $*_{\mathbb{R}^{p-1,q}} \circ D$	
4				

Remark 7.1. (1) By the duality theorem [15, I, Thm. 9], the multiplicity in the branching laws of the generalized Verma modules, given as the dimension of (5.1) is also equal to 2 for the parameters in Theorem F (cf. [8]).
(2) For $(p,q) = (1,3)$, Maxwell's equations are expressed as $d\alpha = 0$ and $d^*\alpha = 0$ for $\alpha \in \mathcal{E}^2(\mathbb{R}^{1,3})$, see [17] for instance.
(3) $\mathcal{D}^{2\to 1}_{u,\ell}$ reduces to $-\mathrm{Rest}_{x_4=0} \circ d^*$ if $(u,\ell) = (0,1)$.

Proof of Theorem F. The assertions follow from [12, Thms. 1.1 and 2.10] owing to Theorems B and D. □

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