A program for branching problems in the representation theory of real reductive groups

Toshiyuki Kobayashi

Dedicated to David Vogan on the occasion of his 60th birthday, with admiration of his epoch-making contributions to the field

Abstract We wish to understand how irreducible representations of a group G behave when restricted to a subgroup G' (the *branching problem*). Our primary concern is with representations of reductive Lie groups, which involve both algebraic and analytic approaches. We divide branching problems into three stages: (A) abstract features of the restriction; (B) branching laws (irreducible decompositions of the restriction); and (C) construction of symmetry breaking operators on geometric models. We could expect a simple and detailed study of branching problems in Stages B and C in the settings that are *a priori* known to be "nice" in Stage A, and conversely, new results and methods in Stage C that might open another fruitful direction of branching problems including Stage A. The aim of this article is to give new perspectives on the subjects, to explain the methods based on some recent progress, and to raise some conjectures and open questions.

Key words: branching law, symmetry breaking operator, unitary representation, Zuckerman–Vogan's $A_{\mathfrak{q}}(\lambda)$ module, reductive group, spherical variety, multiplicity-free representation

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1 Program — ABC for branching problems

From the viewpoint of analysis and synthesis, one of the fundamental problems in representation theory is to classify the smallest objects (e.g., irreducible representations), and another is to understand how a given representation can be built

Toshiyuki Kobayashi
Kavli IPMU (WPI) and Graduate School of Mathematical Sciences, The University of Tokyo
e-mail: toshi@ms.u-tokyo.ac.jp

up from the smallest objects (e.g., irreducible decomposition). A typical example of the latter is the *branching problem*, by which we mean the problem of understanding how irreducible representations π of a group G behave when restricted to subgroups G'. We write $\pi|_{G'}$ for a representation π regarded as a representation of G'. Our primary concern is with real reductive Lie groups. We propose a program for branching problems in the following three stages:

Stage A. Abstract features of the restriction $\pi|_{G'}$.
Stage B. Branching laws.
Stage C. Construction of symmetry breaking operators.

Here, by a *symmetry breaking operator* we mean a continuous G'-homomorphism from the representation space of π to that of an irreducible representation τ of the subgroup G'.

Branching problems for infinite-dimensional representations of real reductive groups involve various aspects. Stage A involves several aspects of the branching problem, among which we highlight that of multiplicity and spectrum here:

A.1. Estimates of multiplicities of irreducible representations of G' occurring in the restriction $\pi|_{G'}$ of an irreducible representation π of G. (There are several "natural" but inequivalent definitions of multiplicities, see Sections 3.1 and 4.2.) Note that:

- multiplicities of the restriction $\pi|_{G'}$ may be infinite even when G' is a maximal subgroup in G;
- multiplicities may be at most one (e.g., Howe's theta correspondence [18], Gross–Prasad conjecture [14], visible actions [39], etc.).

A.2. Spectrum of the restriction $\pi|_{G'}$:

- (discretely decomposable case) branching problems may be purely algebraic and combinatorial ([12, 13, 15, 26, 28, 29, 32, 49, 50, 59]);
- (continuous spectrum) branching problems may have analytic features [8, 52, 57, 63]. (For example, some special cases of branching laws of unitary representations are equivalent to a Plancherel-type theorem for homogeneous spaces.)

The goal of Stage A in branching problems is to analyze the aspects A.1 and A.2 in complete generality. A theorem in Stage A would be interesting on its own, but might also serve as a foundation for further detailed study of the restriction $\pi|_{G'}$ (Stages B and C). An answer in Stage A may also suggest an approach depending on specific features of the restrictions. For instance, if we know *a priori* that the restriction $\pi|_{G'}$ is discretely decomposable in Stage A, then one might use algebraic methods (e.g., combinatorics, $\mathcal{D}$-modules, etc.) to attack Stage B. If the restriction $\pi|_{G'}$ is known *a priori* to be multiplicity-free in Stage A, one might expect to find not only explicit irreducible decompositions (Stage B) but also quantitative estimates such as $L^p - L^q$ estimates, and Parseval–Plancherel type theorems for branching laws (Stage C).

In this article, we give some perspectives of the subject based on a general theory on A.1 and A.2, and recent progress in some classification theory:

- the multiplicities to be finite [bounded, one, $\cdots$],
- the spectrum to be discrete / continuous.

We also discuss a new phenomenon (*localness theorem*, Theorem 7.18) and open questions.

Stage B concerns the irreducible decomposition of the restriction. For a finite-dimensional representation such that the restriction $\pi|_{G'}$ is completely reducible, there is no ambiguity on a meaning of the irreducible decomposition. For a unitary representation π, we can consider Stage B by using the direct integral of Hilbert spaces (Fact 3.1). However, we would like to treat a more general setting where π is not necessarily a unitary representation. In this case, we may consider Stage B as the study of

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \quad \text{or} \quad \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \tag{1.1}$$

for irreducible representations π and τ of G and G', respectively.

Stage C is more involved than Stage B as it asks for concrete intertwining operators (e.g., the projection operator to an irreducible summand) rather than an abstract decomposition; it asks for the decomposition of vectors in addition to that of representations. Since Stage C depends on the realizations of the representations; it often interacts with geometric and analytic problems.

We organize this article not in the natural order, Stage A $\Rightarrow$ Stage B $\Rightarrow$ Stage C, but in an opposite order, Stage C $\Rightarrow$ Stages A and B. This is because it is only recently that a complete construction of all symmetry breaking operators has been carried out in some special settings, and because such examples and new methods might yield yet another interesting direction of branching problems in Stages A to C. The two spaces in (1.1) are discussed in Sections 4–6 from different perspectives (Stage A). The last section returns to Stage C together with comments on the general theory (Stages A and B).

2 Two concrete examples of Stage C

In this section, we illustrate Stage C in the branching program with two recent examples, namely, an explicit construction and a complete classification of *differential* symmetry breaking operators (Section 2.1) and *continuous* symmetry breaking operators (Section 2.2). They have been carried out only in quite special situations until now. In this section we examine these new examples by making some observations that may contain some interesting hints for future study. In later sections, we discuss to what extent the new results and methods apply to other situations and what the limitations of the general theory for Stage A would be.

2.1 Rankin–Cohen bidifferential operators for the tensor products of SL_2-modules

Taking the SL_2-case as a prototype, we explain what we have in mind for Stage C by comparing it with Stages A and B. We focus on *differential* symmetry breaking operators in this subsection, and point out that there are some missing operators even in the classical SL_2-case ([9, 62], see also van Dijk–Pevzner [11], Zagier [76]).

First, we begin with finite-dimensional representations. For every $m \in \mathbb{N}$, there exists the unique $(m+1)$-dimensional irreducible holomorphic representation of $\mathrm{SL}(2,\mathbb{C})$. These representations can be realized on the space $\mathrm{Pol}_m[z]$ of polynomials in z of degree at most m, by the following action of $\mathrm{SL}(2,\mathbb{C})$ with $\lambda = -m$:

$$(\pi_\lambda(g)f)(z) = (cz+d)^{-\lambda} f(\frac{az+b}{cz+d}) \quad \text{for } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.1}$$

The tensor product of two such representations decomposes into irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ subject to the classical Clebsch–Gordan formula:

$$\mathrm{Pol}_m[z] \otimes \mathrm{Pol}_n[z] \simeq \mathrm{Pol}_{m+n}[z] \oplus \mathrm{Pol}_{m+n-2}[z] \oplus \cdots \oplus \mathrm{Pol}_{|m-n|}[z]. \tag{2.2}$$

Secondly, we recall an analogous result for infinite-dimensional representations of $\mathrm{SL}(2,\mathbb{R})$. For this, let H_+ be the Poincaré upper half plane $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. Then $\mathrm{SL}(2,\mathbb{R})$ acts on the space $\mathcal{O}(H_+)$ of holomorphic functions on H_+ via π_λ ($\lambda \in \mathbb{Z}$). Further, we obtain an irreducible unitary representation of $\mathrm{SL}(2,\mathbb{R})$ on the following Hilbert space V_λ (the *weighted Bergman space*) via π_λ for $\lambda > 1$:

$$V_\lambda := \{f \in \mathcal{O}(H_+) : \int_{H_+} |f(x+\sqrt{-1}y)|^2 y^{\lambda-2} dxdy < \infty\},$$

where the inner product is given by

$$\int_{H_+} f(x+\sqrt{-1}y)\overline{g(x+\sqrt{-1}y)} y^{\lambda-2} dxdy \quad \text{for } f, g \in V_\lambda.$$

Repka [63] and Molchanov [57] obtained the irreducible decomposition of the tensor product of two such unitary representations, namely, there is a unitary equivalence between unitary representations of $\mathrm{SL}(2,\mathbb{R})$:

$$V_{\lambda_1} \widehat{\otimes} V_{\lambda_2} \simeq \sum_{a=0}^{\infty}{}^{\oplus} V_{\lambda_1+\lambda_2+2a}, \tag{2.3}$$

where the symbols $\widehat{\otimes}$ and $\sum^{\oplus}$ denote the Hilbert completion of the tensor product $\otimes$ and the algebraic direct sum $\oplus$, respectively. We then have:

Observation 2.1. (1) (multiplicity) *Both of the irreducible decompositions* (2.2) *and* (2.3) *are multiplicity-free.*
(2) (spectrum) *There is no continuous spectrum in either of the decompositions* (2.2) *or* (2.3).

These abstract features (Stage A) are immediate consequences of the decomposition formulæ (2.2) and (2.3) (Stage B), however, one could tell these properties without explicit formulæ from the general theory of visible actions on complex manifolds [34, 39] and a general theory of discrete decomposability [26, 28]. For instance, the following holds:

Fact 2.2. *Let π be an irreducible unitary highest weight representation of a real reductive Lie group G, and G' a reductive subgroup of G.*

(1) (multiplicity-free decomposition) *The restriction $\pi|_{G'}$ is multiplicity-free if π has a scalar minimal K-type and (G, G') is a symmetric pair.*
(2) (spectrum) *The restriction $\pi|_{G'}$ is discretely decomposable if the associated Riemannian symmetric spaces G/K and G'/K' carry Hermitian symmetric structures such that the embedding $G'/K' \hookrightarrow G/K$ is holomorphic.*

Stage C asks for a construction of the following explicit SL_2-intertwining operators (*symmetry breaking operators*):

$$\mathrm{Pol}_m[z] \otimes \mathrm{Pol}_n[z] \to \mathrm{Pol}_{m+n-2a}[z] \qquad \text{for } 0 \le a \le \min(m,n),$$
$$V_{\lambda_1} \widehat{\otimes} V_{\lambda_2} \to V_{\lambda_1+\lambda_2+2a} \qquad \text{for } a \in \mathbb{N},$$

for finite-dimensional and infinite-dimensional representations, respectively. We know a priori from Stages A and B that such intertwining operators exist uniquely (up to scalar multiplications) by Schur's lemma in this setting. A (partial) answer to this question is given by the classical Rankin–Cohen bidifferential operator, which is defined by

$$\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_1+\lambda_2+2a}(f_1, f_2)(z) := \sum_{l=0}^{a} \frac{(-1)^l}{l!(a-l)!} \frac{(\lambda_1+a-1)!(\lambda_2+a-1)!}{(\lambda_1+a-l-1)!(\lambda_2+l-1)!} \frac{\partial^{a-l} f_1}{\partial z^{a-l}}(z) \frac{\partial^l f_2}{\partial z^l}(z)$$

for $a \in \mathbb{N}$, $\lambda_1, \lambda_2 \in \{2, 3, 4, \dots\}$, and f_1, $f_2 \in \mathcal{O}(H_+)$. Then $\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_1+\lambda_2+2a}$ is an operator which intertwines $\pi_{\lambda_1} \widehat{\otimes} \pi_{\lambda_2}$ and $\pi_{\lambda_1+\lambda_2+2a}$.

More generally, we treat *non-unitary* representations π_λ on $\mathcal{O}(H_+)$ of the universal covering group $\mathrm{SL}(2,\mathbb{R})^\sim$ of $\mathrm{SL}(2,\mathbb{R})$ by the same formula (2.1) for $\lambda \in \mathbb{C}$, and consider a continuous linear map

$$T : \mathcal{O}(H_+ \times H_+) \to \mathcal{O}(H_+) \tag{2.4}$$

that intertwines $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ and π_{λ_3}, where $\mathrm{SL}(2,\mathbb{R})^\sim$ acts on $\mathcal{O}(H_+ \times H_+)$ via $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ under the diagonal action. We denote by $H(\lambda_1, \lambda_2, \lambda_3)$ the vector space of symmetry breaking operators T as in (2.4).

Question 2.3. (1) (Stage B) Find the dimension of $H(\lambda_1,\lambda_2,\lambda_3)$ for $(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{C}^3$.
(2) (Stage C) Explicitly construct a basis of $H(\lambda_1,\lambda_2,\lambda_3)$ when it is nonzero.

Even in the SL_2-setting, we could not find a complete answer to Question 2.3 in the literature, and thus we explain our solution below.

Replacing $\mu!$ by $\Gamma(\mu+1)$, we can define

$$\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_3}(f_1,f_2)(z) := \sum_{l=0}^{a} \frac{(-1)^l}{l!(a-l)!} \frac{\Gamma(\lambda_1+a)\Gamma(\lambda_2+a)}{\Gamma(\lambda_1+a-l)\Gamma(\lambda_2+l)} \frac{\partial^{a-l} f_1}{\partial z^{a-l}}(z) \frac{\partial^l f_2}{\partial z^l}(z), \tag{2.5}$$

where $a := \frac{1}{2}(\lambda_3-\lambda_1-\lambda_2)$ as long as $(\lambda_1,\lambda_2,\lambda_3)$ belongs to

$$\Omega := \{(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{C}^3 : \lambda_3-\lambda_1-\lambda_2 = 0,2,4,\dots\}.$$

We define a subset Ω_{sing} of Ω by

$$\Omega_{\mathrm{sing}} := \{(\lambda_1,\lambda_2,\lambda_3) \in \Omega : \lambda_1,\lambda_2,\lambda_3 \in \mathbb{Z}, \quad \lambda_3-|\lambda_1-\lambda_2| \geq 2 \geq \lambda_1+\lambda_2+\lambda_3\}.$$

Then we have the following classification of symmetry breaking operators by using the "F-method" ([51, Part II]). Surprisingly, it turns out that any symmetry breaking operator (2.4) is given by a differential operator.

Theorem 2.4. (1) $H(\lambda_1,\lambda_2,\lambda_3) \neq \{0\}$ *if and only if* $(\lambda_1,\lambda_2,\lambda_3) \in \Omega$.

From now on, we assume $(\lambda_1,\lambda_2,\lambda_3) \in \Omega$.
(2) $\dim_{\mathbb{C}} H(\lambda_1,\lambda_2,\lambda_3) = 1$ *if and only if* $\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_3} \neq 0$, *or equivalently,* $(\lambda_1,\lambda_2,\lambda_3) \notin \Omega_{\mathrm{sing}}$. *In this case,* $H(\lambda_1,\lambda_2,\lambda_3) = \mathbb{C}\,\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_3}$.
(3) *The following three conditions on* $(\lambda_1,\lambda_2,\lambda_3) \in \Omega$ *are equivalent:*

(i) $\dim_{\mathbb{C}} H(\lambda_1,\lambda_2,\lambda_3) = 2$.
(ii) $\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_3} = 0$.
(iii) $(\lambda_1,\lambda_2,\lambda_3) \in \Omega_{\mathrm{sing}}$.

In this case, the two-dimensional vector space $H(\lambda_1,\lambda_2,\lambda_3)$ *is spanned by*

$$\mathcal{RC}_{2-\lambda_1,\lambda_2}^{\lambda_3} \circ \left(\left(\frac{\partial}{\partial z_1}\right)^{1-\lambda_1} \otimes \mathrm{id}\right) \quad \textit{and} \quad \mathcal{RC}_{\lambda_1,2-\lambda_2}^{\lambda_3} \circ \left(\mathrm{id} \otimes \left(\frac{\partial}{\partial z_2}\right)^{1-\lambda_2}\right).$$

Theorem 2.4 answers Question 2.3 (1) and (2). Here are some observations.

Observation 2.5. (1) (localness property) *Any symmetry breaking operator from* $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ *to* π_{λ_3} *is given by a differential operator in the holomorphic realization of* π_{λ_j} *(*$j=1,2,3$*).*
(2) (multiplicity-two phenomenon) *The dimension of the space of symmetry breaking operators jumps up exactly when the holomorphic continuation of the Rankin–Cohen bidifferential operator vanishes.*

The localness property in Observation 2.5 (1) was recently proved in a more general setting (see Theorem 7.18 and Conjecture 7.23).

Remark 2.6 (higher multiplicities at Ω_{sing}).

(1) From the viewpoint of analysis (or the "F-method" [40, 47, 51]), the multiplicity-two phenomenon in Observation 2.5 (2) can be derived from the fact that Ω_{sing} is of codimension two in Ω and from the fact that $\{\frac{\partial}{\partial\lambda_1}\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2}, \frac{\partial}{\partial\lambda_2}\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2}\}$ forms a basis in $H(\lambda_1,\lambda_2,\lambda_3)$ when $\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2}=0$, namely, when $(\lambda_1,\lambda_2,\lambda_3)\in\Omega_{\mathrm{sing}}$.
(2) The basis given in Theorem 2.4 (3) is different from the basis in Remark 2.6 (1), and clarifies the representation-theoretic reason for the multiplicity-two phenomenon as it is expressed as the composition of two intertwining operators.
(3) Theorem 2.4 (3) implies a multiplicity-two phenomenon for Verma modules $M(\mu)=U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}_\mu$ for $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$:

$$\dim_{\mathbb{C}}\mathrm{Hom}_{\mathfrak{g}}(M(-\lambda_3),M(-\lambda_1)\otimes M(-\lambda_2))=2\quad\text{for }(\lambda_1,\lambda_2,\lambda_3)\in\Omega_{\mathrm{sing}}.$$

Again, the tensor product $M(-\lambda_1)\otimes M(-\lambda_2)$ of Verma modules decomposes into a multiplicity-free direct sum of irreducible $\mathfrak{g}$-modules for generic $\lambda_1,\lambda_2\in\mathbb{C}$, but not for singular parameters. See [51, Part II] for details.
(4) In turn, we shall get a two-dimensional space of differential symmetry breaking operators at Ω_{sing} for principal series representations with respect to $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})\downarrow\mathrm{diag}(\mathrm{SL}(2,\mathbb{R}))$, see Remark 7.15 in Section 7.

2.2 Symmetry breaking in conformal geometry

In contrast to the localness property for symmetry breaking operators in the holomorphic setting (Observation 2.5 (1)), there exist non-local symmetry breaking operators in a more general setting. We illustrate Stage C in the branching problem by an explicit construction and a complete classification of all local and non-local symmetry breaking operators that arise from conformal geometry. In later sections, we explain a key idea of the proof (Section 7) and present potential settings where we could expect that this example might serve as the prototype of analogous questions (Section 6). For full details of this subsection, see the monograph [52] joint with Speh.

For $\lambda\in\mathbb{C}$ we denote by $I(\lambda)^\infty$ the smooth (unnormalized) spherical principal series representation of $G=\mathrm{O}(n+1,1)$. In our parametrization, $\lambda\in\frac{n}{2}+\sqrt{-1}\mathbb{R}$ is the unitary axis, $\lambda\in(0,n)$ gives the complementary series representations, and $I(\lambda)^\infty$ contains irreducible finite-dimensional representations as submodules for $\lambda\in\{0,-1,-2,\dots\}$ and as quotients for $\lambda\in\{n,n+1,n+2,\dots\}$.

We consider the restriction of the representation $I(\lambda)^\infty$ and its subquotients to the subgroup $G':=\mathrm{O}(n,1)$. As we did for $I(\lambda)^\infty$, we denote by $J(\nu)^\infty$ for $\nu\in\mathbb{C}$, the (unnormalized) spherical principal series representations of $G'=\mathrm{O}(n,1)$. For

$(\lambda, \nu) \in \mathbb{C}^2$, we set

$$H(\lambda, \nu) := \mathrm{Hom}_{G'}(I(\lambda)^\infty, J(\nu)^\infty),$$

the space of (continuous) symmetry breaking operators. Similarly to Question 2.3, we ask:

Question 2.7. (1) (Stage B) Find the dimension of $H(\lambda, \nu)$ for $(\lambda, \nu) \in \mathbb{C}^2$.
(2) (Stage C) Explicitly construct a basis for $H(\lambda, \nu)$.
(3) (Stage C) Determine when $H(\lambda, \nu)$ contains a differential operator.

The following is a complete answer to Question 2.7 (1).

Theorem 2.8. (1) *For all* $\lambda, \nu \in \mathbb{C}$*, we have* $H(\lambda, \nu) \neq \{0\}$.
(2) $\dim_{\mathbb{C}} H(\lambda, \nu) = \begin{cases} 1 & \text{if} \quad (\lambda, \nu) \in \mathbb{C}^2 \setminus L_{\mathrm{even}}, \\ 2 & \text{if} \quad (\lambda, \nu) \in L_{\mathrm{even}}, \end{cases}$
where the "exceptional set" L_{even} *is the discrete subset of* $\mathbb{C}^2$ *defined by*

$$L_{\mathrm{even}} := \{(\lambda, \nu) \in \mathbb{Z}^2 : \lambda \leq \nu \leq 0, \quad \lambda \equiv \nu \mod 2\}.$$

The role of L_{even} in Theorem 2.8 is similar to that of Ω_{sing} in Section 2.1. For Stage C, we use the "N-picture" of the principal series representations, namely, realize $I(\lambda)^\infty$ and $J(\nu)^\infty$ in $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^{n-1})$, respectively. For $x \in \mathbb{R}^{n-1}$, we set $|x| = (x_1^2 + \cdots + x_{n-1}^2)^{\frac{1}{2}}$. For $(\lambda, \nu) \in \mathbb{C}^2$ satisfying $\mathrm{Re}(\nu - \lambda) \gg 0$ and $\mathrm{Re}(\nu + \lambda) \gg 0$, we construct explicitly a symmetry breaking operator (i.e., continuous G'-homomorphism) from $I(\lambda)^\infty$ to $J(\nu)^\infty$ as an integral operator given by

$$\begin{aligned}(\mathbb{A}_{\lambda,\nu} f)(y) &:= \int_{\mathbb{R}^n} |x_n|^{\lambda+\nu-n} (|x-y|^2 + x_n^2)^{-\nu} f(x, x_n) dx dx_n \\ &= \mathrm{rest}_{x_n=0} \circ (|x_n|^{\lambda+\nu-n} (|x|^2 + x_n^2)^{-\nu} *_{\mathbb{R}^n} f).\end{aligned} \tag{2.6}$$

One might regard $\mathbb{A}_{\lambda,\nu}$ as a generalization of the Knapp–Stein intertwining operator ($G = G'$ case), and also as the adjoint operator of a generalization of the Poisson transform.

The symmetry breaking operator $\mathbb{A}_{\lambda,\nu}$ extends meromorphically with respect to the parameter (λ, ν), and if we normalize $\mathbb{A}_{\lambda,\nu}$ as

$$\widetilde{\mathbb{A}}_{\lambda,\nu} := \frac{1}{\Gamma(\frac{\lambda+\nu-n+1}{2}) \Gamma(\frac{\lambda-\nu}{2})} \mathbb{A}_{\lambda,\nu},$$

then $\widetilde{\mathbb{A}}_{\lambda,\nu} : I(\lambda)^\infty \to J(\nu)^\infty$ is a continuous symmetry breaking operator that depends holomorphically on (λ, ν) in the entire complex plane $\mathbb{C}^2$, and $\widetilde{\mathbb{A}}_{\lambda,\nu} \neq 0$ if and only if $(\lambda, \nu) \notin L_{\mathrm{even}}$ ([52, Theorem 1.5]).

The singular set L_{even} is most interesting. To construct a symmetry breaking operator at L_{even}, we renormalize $\widetilde{\mathbb{A}}_{\lambda,\nu}$ for $\nu \in -\mathbb{N}$, by

$$\widetilde{\widetilde{\mathbb{A}}}_{\lambda,\nu} := \Gamma(\frac{\lambda-\nu}{2})\widetilde{\mathbb{A}}_{\lambda,\nu} = \frac{1}{\Gamma(\frac{\lambda+\nu-n+1}{2})}\mathbb{A}_{\lambda,\nu}.$$

In order to construct *differential* symmetry breaking operators, we recall that the Gegenbauer polynomial $C_l^\alpha(t)$ for $l \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ is given by

$$C_l^\alpha(t) := \sum_{k=0}^{[\frac{l}{2}]}(-1)^k \frac{\Gamma(l-k+\alpha)}{\Gamma(\alpha)\Gamma(l-2k+1)k!}(2t)^{l-2k}.$$

We note that $C_l^\alpha(t) \equiv 0$ if $l \geq 1$ and $\alpha = 0, -1, -2, \ldots, -[\frac{l-1}{2}]$. We renormalize $C_l^\alpha(t)$ by setting $\widetilde{C}_l^\alpha(t) := \frac{\Gamma(\alpha)}{\Gamma(\alpha+[\frac{l+1}{2}])}C_l^\alpha(t)$, so that $\widetilde{C}_l^\alpha(t)$ is a nonzero polynomial in t of degree l for all $\alpha \in \mathbb{C}$ and $l \in \mathbb{N}$. We inflate it to a polynomial of two variables u and v by

$$\widetilde{C}_k^\alpha(u,v) := u^{\frac{k}{2}}\widetilde{C}_k^\alpha(\frac{v}{\sqrt{u}}).$$

For instance, $\widetilde{C}_0^\alpha(u,v) = 1$, $\widetilde{C}_1^\alpha(u,v) = 2v$, $\widetilde{C}_2^\alpha(u,v) = 2(\alpha+1)v^2 - u$, etc. Substituting $u = -\Delta_{\mathbb{R}^{n-1}} = -\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}$ and $v = \frac{\partial}{\partial x_n}$, we get a differential operator of order $2l$:

$$\widetilde{\mathbb{C}}_{\lambda,\nu} := \mathrm{rest}_{x_n=0} \circ \widetilde{C}_{2l}^{\lambda-\frac{n-1}{2}}(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}).$$

This closed formula of the differential operator $\widetilde{\mathbb{C}}_{\lambda,\nu}$ was obtained by Juhl [21] (see also [47] for a short proof by the F-method, and [40] for yet another proof by using the residue formula), and the closed formula (2.6) of the symmetry breaking operator $\widetilde{\mathbb{A}}_{\lambda,\nu}$ was obtained by Kobayashi and Speh [52].

The following results answer Question 2.7 (2) and (3); see [52, Theorems 1.8 and 1.9]:

Theorem 2.9. (1) *With notation as above, we have*

$$H(\lambda,\nu) = \begin{cases} \mathbb{C}\widetilde{\mathbb{A}}_{\lambda,\nu} & \text{if } (\lambda,\nu) \in \mathbb{C}^2 \setminus L_{\mathrm{even}} \\ \mathbb{C}\widetilde{\widetilde{\mathbb{A}}}_{\lambda,\nu} \oplus \mathbb{C}\widetilde{\mathbb{C}}_{\lambda,\nu} & \text{if } (\lambda,\nu) \in L_{\mathrm{even}}. \end{cases}$$

(2) $H(\lambda,\nu)$ *contains a nontrivial differential operator if and only if* $\nu - \lambda = 0, 2, 4, 6, \ldots$. *In this case* $\widetilde{\mathbb{A}}_{\lambda,\nu}$ *is proportional to* $\widetilde{\mathbb{C}}_{\lambda,\nu}$, *and the proportionality constant vanishes if and only if* $(\lambda,\nu) \in L_{\mathrm{even}}$.

From Theorem 2.9 (2) and Theorem 2.8 (1), we have the following:

Observation 2.10. (1) *Unlike the holomorphic setting in Section 2.1, the localness property fails.*

(2) *Even if an irreducible smooth representation* $\pi^\infty = I(\lambda)^\infty$ *is unitarizable as a representation of* G, *the condition* $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \neq \{0\}$ *does not imply*

that the irreducible smooth representation $\tau^\infty = J(\nu)^\infty$ is unitarizable as a representation of G' (see Section 3.2 for the terminology).

For $\lambda \in \{n, n+1, n+2, \dots\}$, $I(\lambda)^\infty$ contains a unique proper infinite-dimensional closed G-submodule. We denote it by $A_\mathfrak{q}(\lambda - n)^\infty$, which is the Casselman–Wallach globalization of Zuckerman's derived functor module $A_\mathfrak{q}(\lambda - n)$ (see [69, 71]) for some θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. It is unitarizable ([70, 74]) and has nonzero $(\mathfrak{g}, K)$-cohomologies (Vogan–Zuckerman [73]).

By using the explicit formulæ of symmetry breaking operators and certain identities involving these operators, we can identify precisely the images of every subquotient of $I(\lambda)^\infty$ under these operators. In particular, we obtain the following corollary for the branching problem of $A_\mathfrak{q}(\lambda)$ modules. We note that in this setting, the restriction $A_\mathfrak{q}(\lambda)|_{\mathfrak{g}'}$ is not discretely decomposable as a $(\mathfrak{g}', K')$-module (Definition 4.3).

Corollary 2.11 ([52, Theorem 1.2]). *With notation as above, we have*

$$\dim_\mathbb{C} \mathrm{Hom}_{G'}(A_\mathfrak{q}(i)^\infty|_{G'}, A_{\mathfrak{q}'}(j)^\infty) = \begin{cases} 1 & \textit{if} \quad i \geq j \quad \textit{and} \quad i \equiv j \mod 2, \\ 0 & \textit{if} \quad i < j \quad \textit{and} \quad i \not\equiv j \mod 2. \end{cases}$$

There are some further applications of the explicit formulæ (2.6) (Stage C in the branching problems). For instance, J. Möllers and B. Ørsted recently found an interesting application of the explicit formulæ (2.6) to $L^p - L^q$ estimates of certain boundary-value problems, and to some questions in automorphic forms [58].

3 Preliminary results and basic notation

We review quickly some basic results on (infinite-dimensional) continuous representations of real reductive Lie groups and fix notation. There are no new results in this section.

By a continuous representation π of a Lie group G on a topological vector space V we shall mean that $\pi : G \to \mathrm{GL}_\mathbb{C}(V)$ is a group homomorphism from G into the group of invertible endomorphisms of V such that the induced map $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous. We say π is a (continuous) Hilbert [Banach, Fréchet, $\cdots$] representation if V is a Hilbert [Banach, Fréchet, $\cdots$] space. A continuous Hilbert representation π of G is said to be a unitary representation when all the operators $\pi(g)$ ($g \in G$) are unitary.

3.1 Decomposition of unitary representations

One of the most distinguished features of *unitary* representations is that they can be built up from the smallest objects, namely, irreducible unitary representations. To be precise, let G be a locally compact group. We denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G (the *unitary dual*), endowed with the Fell topology.

Fact 3.1 (Mautner–Teleman). *For every unitary representation π of a locally compact group G, there exist a Borel measure $d\mu$ on $\widehat{G}$ and a measurable function $n_\pi : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ such that π is unitarily equivalent to the direct integral of irreducible unitary representations:*

$$\pi \simeq \int_{\widehat{G}}^{\oplus} n_\pi(\sigma)\sigma \, d\mu(\sigma), \tag{3.1}$$

where $n_\pi(\sigma)\sigma$ stands for the multiple of an irreducible unitary representation σ with multiplicity $n_\pi(\sigma)$.

The decomposition (3.1) is unique if G is of type I in the sense of von Neumann algebras, in particular, if G (or G' in later notation) is a real reductive Lie group or a nilpotent Lie group. Then the *multiplicity function* n_π is well-defined up to a measure zero set with respect to $d\mu$. We say that π is *multiplicity-free* if $n_\pi(\sigma) \leq 1$ almost everywhere, or equivalently, if the ring of continuous G-endomorphisms of π is commutative.

The decomposition (3.1) splits into a direct sum of the discrete and continuous parts:

$$\pi \simeq (\pi)_{\mathrm{disc}} \oplus (\pi)_{\mathrm{cont}}, \tag{3.2}$$

where $(\pi)_{\mathrm{disc}}$ is a unitary representation defined on the maximal closed G-invariant subspace that is isomorphic to a discrete Hilbert sum of irreducible unitary representations and $(\pi)_{\mathrm{cont}}$ is its orthogonal complement.

Definition 3.2. We say a unitary representation π is *discretely decomposable* if $\pi = (\pi)_{\mathrm{disc}}$.

3.2 Continuous representations and smooth representations

We would like to treat non-unitary representations as well for branching problems. For this we recall some standard concepts of continuous representations of Lie groups.

Suppose π is a continuous representation of G on a Banach space V. A vector $v \in V$ is said to be *smooth* if the map $G \to V$, $g \mapsto \pi(g)v$ is of C^∞-class. Let V^∞ denote the space of smooth vectors of the representation (π, V). Then V^∞ carries a Fréchet topology with a family of semi-norms $\|v\|_{i_1 \cdots i_k} :=$

$\|d\pi(X_{i_1})\cdots d\pi(X_{i_k})v\|$, where $\{X_1,\dots,X_n\}$ is a basis of the Lie algebra $\mathfrak{g}_0$ of G. Then V^∞ is a G-invariant dense subspace of V, and we obtain a continuous Fréchet representation (π^∞, V^∞) of G. Similarly we can define a representation π^ω on the space V^ω of analytic vectors.

Suppose now that G is a real reductive linear Lie group, K a maximal compact subgroup of G, and $\mathfrak{g}$ the complexification of the Lie algebra $\mathfrak{g}_0$ of G. Let $\mathcal{HC}$ denote the category of Harish-Chandra modules whose objects and morphisms are $(\mathfrak{g},K)$-modules of finite length and $(\mathfrak{g},K)$-homomorphisms, respectively.

Let π be a continuous representation of G on a Fréchet space V. Suppose that π is of finite length, namely, there are at most finitely many closed G-invariant subspaces in V. We say π is *admissible* if

$$\dim \mathrm{Hom}_K(\tau, \pi|_K) < \infty$$

for any irreducible finite-dimensional representation τ of K. We denote by V_K the space of K-finite vectors. Then $V_K \subset V^\omega \subset V^\infty$ and the Lie algebra $\mathfrak{g}$ leaves V_K invariant. The resulting $(\mathfrak{g},K)$-module on V_K is called the underlying $(\mathfrak{g},K)$-module of π, and will be denoted by π_K.

For any admissible representation π on a Banach space V, the smooth representation (π^∞, V^∞) depends only on the underlying $(\mathfrak{g},K)$-module. We say (π^∞, V^∞) is an *admissible smooth representation*. By the Casselman–Wallach globalization theory, (π^∞, V^∞) has moderate growth, and there is a canonical equivalence of categories between the category $\mathcal{HC}$ of $(\mathfrak{g},K)$-modules of finite length and the category of admissible smooth representations of G ([74, Chapter 11]). In particular, the Fréchet representation π^∞ is uniquely determined by its underlying $(\mathfrak{g},K)$-module. We say π^∞ is the *smooth globalization* of $\pi_K \in \mathcal{HC}$.

For simplicity, by an *irreducible smooth representation*, we shall mean an irreducible admissible smooth representation of G. We denote by $\widehat{G}_{\mathrm{smooth}}$ the set of equivalence classes of irreducible smooth representations of G. Using the category $\mathcal{HC}$ of $(\mathfrak{g},K)$-modules, we may regard the unitary dual $\widehat{G}$ as a subset of $\widehat{G}_{\mathrm{smooth}}$.

4 Two spaces: $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ and $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$

Given irreducible continuous representations π of G and τ of a subgroup G', we may consider two settings for branching problems:

Case I. (embedding) continuous G'-homomorphisms from τ to $\pi|_{G'}$;

Case II. (symmetry breaking) continuous G'-homomorphisms from $\pi|_{G'}$ to τ.

We write $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ and $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ for the vector spaces of such continuous G'-homomorphisms, respectively. Needless to say, the existence of such G'-intertwining operators depends on the topology of the representation spaces of π and τ.

Cases I and II are related to each other by taking contragredient representations:

$$\mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \subset \mathrm{Hom}_{G'}(\pi^\vee|_{G'}, \tau^\vee),$$
$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \subset \mathrm{Hom}_{G'}(\tau^\vee, \pi^\vee|_{G'}).$$

Thus they are equivalent in the category of unitary representations (see Theorem 4.1 (3)). Furthermore, we shall use a variant of the above duality in analyzing differential symmetry breaking operators (Case II) by means of "discretely decomposable restrictions" of Verma modules (Case I); see the duality (7.3) for the proof of Theorem 7.13 below.

On the other hand, it turns out that Cases I and II are significantly different if we confine ourselves to irreducible smooth representations (see Section 3.2). Such a difference also arises in an analogous problem in the category $\mathcal{HC}$ of Harish-Chandra modules where no topology is specified.

Accordingly, we shall discuss some details for Cases I and II separately, in Sections 5 and 6, respectively.

4.1 K-finite vectors and K'-finite vectors

Let G be a real reductive linear Lie group, and G' a reductive subgroup. We take maximal compact subgroups K and K' of G and G', respectively, such that $K' = K \cap G'$.

We recall that for an admissible representation π of G on a Banach space V, any K-finite vector is contained in V^∞, and the underlying $(\mathfrak{g}, K)$-module π_K is defined on

$$V_K := V_{K\text{-finite}} \qquad (\subset V^\infty).$$

When we regard (π, V) as a representation of the subgroup G' by restriction, we denote by $(V|_{G'})^\infty$ the space of smooth vectors with respect to the G'-action, and write $(\pi|_{G'})^\infty$ for the continuous representation of G' on $(V|_{G'})^\infty$. In contrast to the case $G = G'$, we remark that K'-finite vectors are not necessarily contained in $(V|_{G'})^\infty$ if $G' \subsetneqq G$, because the G'-module $(\pi|_{G'}, V|_{G'})$ is usually not of finite length. Instead, we can define a $(\mathfrak{g}', K')$-module on

$$V_{K'} := V_{K'\text{-finite}} \cap (V|_{G'})^\infty,$$

which we denote simply by $\pi_{K'}$. Obviously we have the following inclusion relations:

$$\begin{array}{ccc} V_K & \subset & V_{K'} \\ \cap & & \cap \\ V^\infty & \subset & (V|_{G'})^\infty \subset V \end{array} \tag{4.1}$$

None of them coincides in general (e.g., $V_K = V_{K'}$ if and only if π_K is discretely decomposable as $(\mathfrak{g}', K')$-module, as we shall see in Theorem 4.5 below.

We set

$$H_K(\tau,\pi) := \mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}),$$
$$H_{K'}(\tau,\pi) := \mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_{K'}|_{\mathfrak{g}'}).$$

According to the inclusion relation (4.1), for irreducible representations τ of G' we have:

$$H_K(\tau,\pi) \subset H_{K'}(\tau,\pi).$$

In the case where π is a unitary representation of G, the latter captures discrete summands in the branching law of the restriction $\pi|_{G'}$ (see, Theorem 4.1 (3)), whereas the former vanishes even if the latter is nonzero when the continuous part $(\pi|_{G'})_{\mathrm{cont}}$ is not empty (see Theorem 4.5). The spaces of continuous G'-homomorphisms such as $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ or $\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$ are in between.

We begin with a general result:

Theorem 4.1. *Suppose that π and τ are admissible irreducible Banach representations of G and G'.*

(1) *We have natural inclusions and an isomorphism:*

$$\begin{aligned} H_K(\tau,\pi) &\hookrightarrow \mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'}) \\ &\hookrightarrow \mathrm{Hom}_{G'}(\tau^\infty, (\pi|_{G'})^\infty) \xrightarrow{\sim} H_{K'}(\tau,\pi). \end{aligned} \tag{4.2}$$

(2) *There are canonical injective homomorphisms:*

$$\begin{aligned} \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) &\hookrightarrow \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \\ &\hookrightarrow \mathrm{Hom}_{G'}(\pi^\omega|_{G'}, \tau^\omega) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}',K'}(\pi_K, \tau_{K'}). \end{aligned} \tag{4.3}$$

(3) (unitary case) *If τ and π are irreducible unitary representations of G' and G, respectively, then we have natural isomorphisms (where the last isomorphism is conjugate linear):*

$$\begin{aligned} H_{K'}(\tau,\pi) &\xleftarrow{\sim} \mathrm{Hom}_{G'}(\tau^\infty, (\pi|_{G'})^\infty) \\ &\xleftarrow{\sim} \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \simeq \mathrm{Hom}_{G'}(\pi|_{G'}, \tau). \end{aligned} \tag{4.4}$$

We write $m_\pi(\tau)$ for the dimension of one of (therefore, any of) the terms in (4.4). *Then the discrete part of the restriction $\pi|_{G'}$ (see Definition 3.2) decomposes discretely as*

$$(\pi|_{G'})_{\mathrm{disc}} \simeq \sum_{\tau\in\widehat{G'}}{}^{\oplus} m_\pi(\tau)\tau.$$

Remark 4.2. Even if π and τ are irreducible unitary representations of G and G', respectively, the canonical injective homomorphism

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \hookrightarrow \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \tag{4.5}$$

is not surjective in general.

In fact, we can give an example where the canonical homomorphism (4.5) is not surjective by using the classification of $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$ for the pair $(G, G') = (O(n+1,1), O(n,1))$ in Section 2.2 as follows. Recall $\mathrm{Hom}_{G'}(I(\lambda)^\infty|_{G'}, J(\nu)^\infty) \neq \{0\}$ for all $(\lambda, \nu) \in \mathbb{C}^2$ with the notation therein. However, for a fixed $\pi \in \widehat{G}$, there exist at most countably many $\tau \in \widehat{G'}$ that occur in the discrete part of the restriction $\pi|_{G'}$, and therefore $\{\tau \in \widehat{G'} : \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \neq \{0\}\}$ is an infinite set because we have the following bijection:

$$\{\tau \in \widehat{G'} : \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \neq \{0\}\} \simeq \{\tau \in \widehat{G'} : \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \neq \{0\}\}.$$

Hence, by taking $\pi^\infty = I(\lambda)^\infty$ for a fixed $\lambda \in \frac{n}{2} + \sqrt{-1}\mathbb{R}$ (unitary axis) or $\lambda \in (0, n)$ (complementary series), we see that the canonical homomorphism (4.5) must be zero when we take τ^∞ to be a representation $I(\nu)^\infty$ for $\nu \in \mathbb{C}$ such that $\nu \notin \frac{n-1}{2} + \sqrt{-1}\mathbb{R}$ and $\nu \notin \mathbb{R}$.

Let us give a proof of Theorem 4.1.

Proof. (1) To see the first inclusion, we prove that any $(\mathfrak{g}', K')$-homomorphism $\iota : \tau_{K'} \to \pi_K|_{\mathfrak{g}'}$ extends to a continuous map $\tau^\infty \to \pi^\infty|_{G'}$. We may assume that ι is nonzero, and therefore, is injective. Since $\iota(\tau_{K'}) \subset \pi_K \subset \pi^\infty$, we can define a Fréchet space W to be the closure of $\iota(\tau_{K'})$ in π^∞, on which G' acts continuously. Its underlying $(\mathfrak{g}', K')$-module is isomorphic to $\iota(\tau_{K'}) \simeq \tau_{K'}$.

Since the continuous representation π^∞ of G is of moderate growth, the Fréchet representation W of the subgroup G' is also of moderate growth. By the Casselman–Wallach globalization theory, there is a G'-homomorphism $\tau^\infty \overset{\sim}{\to} \overline{\iota(\tau_{K'})}\ (= W)$ extending the $(\mathfrak{g}', K')$-isomorphism $\iota : \tau_{K'} \overset{\sim}{\to} \iota(\tau_{K'})$. Hence we have obtained a natural map $\mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) \to \mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$, which is clearly injective because $\tau_{K'}$ is dense in τ^∞.

The second inclusion is obvious.

To see the third inclusion, it suffices to show that any $\iota \in \mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_{K'}|_{\mathfrak{g}'})$ extends to a continuous G'-homomorphism from τ^∞ to $(\pi|_{G'})^\infty$. Since $\tau_{K'}$ is an irreducible $(\mathfrak{g}', K')$-module, ι is injective unless ι is zero and $\iota(\tau_{K'})$ is isomorphic to $\tau_{K'}$ as $(\mathfrak{g}', K')$-modules.

Let V be the Banach space on which G acts via π, and W_1 and W_2 the closures of $\iota(\tau_{K'})$ in the Banach space V and the Fréchet space $(V|_{G'})^\infty$, respectively. Then the underlying $(\mathfrak{g}', K')$-modules of W_1 and W_2 are both isomorphic to $\tau_{K'}$. Moreover, $W_2 \subset W_1 \cap (V|_{G'})^\infty$ by definition, and W_2 is closed in $W_1 \cap (V|_{G'})^\infty$ with respect to the Fréchet topology. Since the subspace $\iota(\tau_{K'})$ of W_2 is dense in $W_1 \cap (V|_{G'})^\infty$, we conclude that W_2 coincides with $W_1 \cap (V|_{G'})^\infty$, which is the Casselman–Wallach globalization of the $(\mathfrak{g}', K')$-module $\iota(\tau_{K'}) \simeq \tau_{K'}$. By the uniqueness of the Casselman–Wallach globalization [74, Chapter 11], the $(\mathfrak{g}', K')$-isomorphism $\tau_{K'} \overset{\sim}{\to} \iota(\tau_{K'})$ extends to an isomorphism between Fréchet G'-modules $\tau^\infty \overset{\sim}{\to} W_2 (= W_1 \cap (V|_{G'})^\infty)$.

(2) If $\iota : \pi|_{G'} \to \tau$ is a continuous G'-homomorphism, then

$$\iota(\pi^\infty|_{G'}) \subset \iota((\pi|_{G'})^\infty) \subset \tau^\infty,$$

and thus we have obtained a continuous G'-homomorphism $\iota^\infty : \pi^\infty|_{G'} \to \tau^\infty$ between Fréchet representations. Furthermore $\iota \mapsto \iota^\infty$ is injective because V^∞ is dense in V. This shows the first inclusive relation of the statement (2). The proof for other inclusions are similar.
(3) The last isomorphism in (4.4) is given by taking the adjoint operator. The other isomorphisms are easy to see. The last statement follows from the fact that if $\varphi \in \mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ then φ is a scalar multiple of an *isometric* G'-homomorphism. □

The terms in (4.2) do not coincide in general. In order to clarify when they coincide, we recall from [29] the notion of discrete decomposability of $\mathfrak{g}$-modules.

Definition 4.3. A $(\mathfrak{g}, K)$-module X is said to be *discretely decomposable as a* $(\mathfrak{g}', K')$*-module* if there is a filtration $\{X_i\}_{i\in\mathbb{N}}$ of $(\mathfrak{g}', K')$-modules such that

- $\bigcup_{i\in\mathbb{N}} X_i = X$ and
- X_i is of finite length as a $(\mathfrak{g}', K')$-module for any $i \in \mathbb{N}$.

The idea was to exclude "hidden continuous spectrum" in an algebraic setting, and discrete decomposability here does not imply complete reducibility. Discrete decomposability is preserved by taking submodules, quotients, and the tensor product with finite-dimensional representations.

Remark 4.4 (see [29, Lemma 1.3]). Suppose that X is a unitarizable $(\mathfrak{g}, K)$-module. Then X is discretely decomposable as a $(\mathfrak{g}', K')$-module if and only if X is isomorphic to an algebraic direct sum of irreducible $(\mathfrak{g}', K')$-modules.

We get much stronger results than Theorem 4.1 in this setting:

Theorem 4.5 (discretely decomposable case). *Assume π is an irreducible admissible representation of G on a Banach space V. Let π_K be the underlying $(\mathfrak{g}, K)$-module. Then the following five conditions on the triple (G, G', π) are equivalent:*

(i) *There exists at least one irreducible $(\mathfrak{g}', K')$-module $\tau_{K'}$ such that* $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K) \neq \{0\}$.
(ii) *π_K is discretely decomposable as a $(\mathfrak{g}', K')$-module (see Definition 4.3).*
(iii) *All the terms in* (4.2) *are the same for any irreducible admissible Banach representation τ of G'.*
(iv) *All the terms in* (4.2) *are the same for some irreducible admissible Banach representation τ of G'.*
(v) $V_K = V_{K'}$.

Moreover, if (π, V) is a unitary representation, then one of (therefore, any of) the equivalent conditions (i) – (v) implies that the continuous part $(\pi|_{G'})_{\mathrm{cont}}$ of the restriction $\pi|_{G'}$ is empty.

Proof. See [29] for the first statement, and [32] for the second statement. □

4.2 Some observations on $\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$ and $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$

For a unitary representation (π, V) of G, Fact 3.1 gives an irreducible decomposition of the restriction $\pi|_{G'}$ into irreducible *unitary* representations of G'. However, symmetry breaking operators may exist between unitary and non-unitary representations:

Observation 4.6. *Suppose π is a unitary representation of G, and (τ, W) an irreducible admissible representation of a reductive subgroup G'.*

(1) *If $\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'}) \neq \{0\}$, then τ^∞ is unitarizable. Actually, τ occurs as a discrete part of $(\pi|_{G'})_{\mathrm{disc}}$ (see (3.2)).*
(2) *It may well happen that $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \neq \{0\}$ even when τ^∞ is not unitarizable.*

In fact, the first assertion is obtained by taking the completion of $\varphi(W^\infty)$ in the Hilbert space V for $\varphi \in \mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$ as in the proof of Theorem 4.1 (3), where we considered the case $(\pi|_{G'})^\infty$ instead of $\pi^\infty|_{G'}$. Theorem 2.9 gives an example of Observation 4.6 (2).

Here is another example that indicates a large difference between the two spaces, $\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$ and $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$.

Example 4.7. Suppose G is a real simple connected Lie group, and G' is a noncompact closed subgroup of G. Let π be any irreducible unitary representation such that $\dim \pi = \infty$ and $\mathrm{Hom}_G(\pi^\infty, C^\infty(G/G')) \neq \{0\}$. Then by Howe–Moore [20] we have

$$\mathrm{Hom}_{G'}(\mathbf{1}, \pi^\infty|_{G'}) = \{0\} \neq \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \mathbf{1}).$$

5 Features of the restriction, I : $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ (embedding)

In this section, we discuss Case I in Section 4, namely G'-homomorphisms from irreducible G'-modules τ into irreducible G-modules π. We put emphasis on its algebraic analogue in the category $\mathcal{HC}$ of Harish-Chandra modules.

The goals of this section are

(1) (criterion) to find a criterion for the triple (G, G', π) such that

$$\mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) \neq \{0\} \quad \text{for some } \tau; \tag{5.1}$$

(2) (classification theory) to classify the pairs (G, G') of reductive groups for which (5.1) occurs for at least one infinite-dimensional $\pi \in \widehat{G}$.

We also discuss recent progress in this direction as a refinement of (2):

(2)′ (classification theory) Classify the triples (G, G', π) for which (5.1) occurs in typical cases (e.g., π_K is Zuckerman's $A_{\mathfrak{q}}(\lambda)$ module, or a minimal representation).

In Section 7 we shall explain two new applications of discretely decomposable restrictions: one is a dimension estimate of differential symmetry breaking operators (Theorem 7.13), and the other is a proof of the "localness property" of symmetry breaking operators (Theorem 7.18); see Observation 2.5 (1).

5.1 Criteria for discrete decomposability of restriction

We review a necessary and sufficient condition for the restriction of Harish-Chandra modules to be discretely decomposable (Definition 4.3), which was established in [28] and [29].

An associated variety $\mathcal{V}_{\mathfrak{g}}(X)$ is a coarse approximation of the $\mathfrak{g}$-modules X, which we recall now from Vogan [72]. We shall use the associated variety for the study of the restrictions of Harish-Chandra modules.

Let $\{U_j(\mathfrak{g})\}_{j\in\mathbb{N}}$ be the standard increasing filtration of the universal enveloping algebra $U(\mathfrak{g})$. Suppose X is a finitely generated $\mathfrak{g}$-module. A filtration $\bigcup_{i\in\mathbb{N}} X_i = X$ is called a *good filtration* if it satisfies the following conditions:

- X_i is finite-dimensional for any $i \in \mathbb{N}$;
- $U_j(\mathfrak{g})X_i \subset X_{i+j}$ for any $i, j \in \mathbb{N}$;
- There exists n such that $U_j(\mathfrak{g})X_i = X_{i+j}$ for any $i \geq n$ and $j \in \mathbb{N}$.

The graded algebra $\operatorname{gr} U(\mathfrak{g}) := \bigoplus_{j\in\mathbb{N}} U_j(\mathfrak{g})/U_{j-1}(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$ by the Poincaré–Birkhoff–Witt theorem and we regard the graded module $\operatorname{gr} X := \bigoplus_{i\in\mathbb{N}} X_i/X_{i-1}$ as an $S(\mathfrak{g})$-module. Define

$$\operatorname{Ann}_{S(\mathfrak{g})}(\operatorname{gr} X) := \{f \in S(\mathfrak{g}) : fv = 0 \text{ for any } v \in \operatorname{gr} X\},$$
$$\mathcal{V}_{\mathfrak{g}}(X) := \{x \in \mathfrak{g}^* : f(x) = 0 \text{ for any } f \in \operatorname{Ann}_{S(\mathfrak{g})}(\operatorname{gr} X)\}.$$

Then $\mathcal{V}_{\mathfrak{g}}(X)$ does not depend on the choice of good filtration and is called the *associated variety* of X. We denote by $\mathcal{N}(\mathfrak{g}^*)$ the nilpotent variety of the dual space $\mathfrak{g}^*$. We have then the following basic properties of the associated variety [72].

Lemma 5.1. *Let X be a finitely generated $\mathfrak{g}$-module.*

(1) *If X is of finite length, then $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathcal{N}(\mathfrak{g}^*)$.*
(2) *$\mathcal{V}_{\mathfrak{g}}(X) = \{0\}$ if and only if X is finite-dimensional.*
(3) *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathfrak{h}^\perp$ if $\mathfrak{h}$ acts locally finitely on X, where $\mathfrak{h}^\perp := \{x \in \mathfrak{g}^* : x|_{\mathfrak{h}} = 0\}$.*

(1) and (3) imply that if X is a $(\mathfrak{g}, K)$-module of finite length, then $\mathcal{V}_{\mathfrak{g}}(X)$ is a $K_{\mathbb{C}}$-stable closed subvariety of $\mathcal{N}(\mathfrak{p}^*)$ because $\mathfrak{k}^\perp = \mathfrak{p}^*$.

Dual to the inclusion $\mathfrak{g}' \subset \mathfrak{g}$ of the Lie algebras, we write

$$\mathrm{pr} : \mathfrak{g}^* \to (\mathfrak{g}')^*$$

for the restriction map.

One might guess that irreducible summands of the restriction $\pi|_{G'}$ would be "large" if the irreducible representation π of G is "large". The following theorem shows that such a statement holds if the restriction of the Harish-Chandra module is discretely decomposable (Definition 4.3), however, it is false in general (see Counterexample 5.4 below).

Fact 5.2. *Let X be an irreducible $(\mathfrak{g}, K)$-module.*

(1) *If Y is an irreducible $(\mathfrak{g}', K')$-module such that $\mathrm{Hom}_{\mathfrak{g}',K'}(Y, X|_{\mathfrak{g}'}) \neq \{0\}$, then*

$$\mathrm{pr}(\mathcal{V}_{\mathfrak{g}}(X)) \subset \mathcal{V}_{\mathfrak{g}'}(Y).$$

(2) *If $Y^{(j)}$ are irreducible $(\mathfrak{g}', K')$-modules such that $\mathrm{Hom}_{\mathfrak{g}',K'}(Y^{(j)}, X|_{\mathfrak{g}'}) \neq \{0\}$ $(j = 1, 2)$, then*

$$\mathcal{V}_{\mathfrak{g}'}(Y_1) = \mathcal{V}_{\mathfrak{g}'}(Y_2).$$

In particular, the Gelfand–Kirillov dimension $\mathrm{GK\text{-}dim}(Y)$ of all irreducible $(\mathfrak{g}', K')$-submodules Y of $X|_{\mathfrak{g}'}$ are the same.

(3) (necessary condition [29, Corollary 3.5]) *If X is discretely decomposable as a $(\mathfrak{g}', K')$-module, then $\mathrm{pr}(\mathcal{V}_{\mathfrak{g}}(X)) \subset \mathcal{N}((\mathfrak{g}')^*)$, where $\mathcal{N}((\mathfrak{g}')^*)$ is the nilpotent variety of $(\mathfrak{g}')^*$.*

An analogous statement fails if we replace $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'})$ by the space $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ of continuous G'-intertwining operators:

False Statement 5.3. *Let π be an irreducible unitary representation of a real reductive Lie group G.*

(1) *If $\tau \in \widehat{G'}$ satisfies $\mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \neq \{0\}$, then $\mathrm{pr}(\mathcal{V}_{\mathfrak{g}}(\pi_K)) \subset \mathcal{V}_{\mathfrak{g}'}(\tau_{K'})$.*

(2) *If $\tau^{(j)} \in \widehat{G'}$ satisfy $\mathrm{Hom}_{G'}(\tau^{(j)}, \pi|_{G'}) \neq \{0\}$ $(j = 1, 2)$, then $\mathcal{V}_{\mathfrak{g}'}(\tau^{(1)}_{K'}) = \mathcal{V}_{\mathfrak{g}'}(\tau^{(2)}_{K'})$.*

Here are counterexamples to the "False Statement 5.3":

Counterexample 5.4. (1) There are many triples (G, G', π) such that $\pi \in \widehat{G}$ satisfies $(\pi|_{G'})_{\mathrm{cont}} \neq 0$; see [26, Introduction], [33, Section 3.3], and Theorem 5.14, for instance. In this case, $\mathrm{pr}(\mathcal{V}_{\mathfrak{g}}(\pi_K)) \not\subset \mathcal{V}_{\mathfrak{g}'}(\tau_{K'})$ for any $\tau \in \widehat{G'}$ by Fact 5.2 (3).

(2) Let $(G, G') = (G_1 \times G_1, \mathrm{diag}(G_1))$ with $G_1 = \mathrm{Sp}(n, \mathbb{R})$ $(n \geq 2)$. Take an irreducible unitary spherical principal series representation π_1 induced from the Siegel parabolic subgroup of G_1, and set $\pi = \pi_1 \boxtimes \pi_1$. Then there exist discrete series representations $\tau^{(1)}$ and $\tau^{(2)}$ of G' $(\simeq \mathrm{Sp}(n, \mathbb{R}))$, where $\tau^{(1)}$ is a holomorphic discrete series representation and $\tau^{(2)}$ is a non-holomorphic discrete series representation, such that

$$\mathrm{Hom}_{G'}(\tau^{(j)}, \pi) \neq \{0\} \quad (j = 1, 2) \quad \text{and} \quad \mathrm{GK\text{-}dim}(\tau^{(1)}) < \mathrm{GK\text{-}dim}(\tau^{(2)}).$$

In fact, it follows from Theorem 5.14 below that $\mathrm{Hom}_{G'}(\tau,\pi) \neq \{0\}$ if and only if τ is a discrete series representation for the reductive symmetric space $\mathrm{Sp}(n,\mathbb{R})/\mathrm{GL}(n,\mathbb{R})$. Then using the description of discrete series representations [55, 71], we get Counterexample 5.4 (2).

We now turn to an analytic approach to the question of discrete decomposability of the restriction. For simplicity, assume K is connected. We take a maximal torus T of K, and write $\mathfrak{t}_0$ for its Lie algebra. Fix a positive system $\Delta^+(\mathfrak{k},\mathfrak{t})$ and denote by C_+ ($\subset \sqrt{-1}\mathfrak{t}_0^*$) the dominant Weyl chamber. We regard $\widehat{T}$ as a subset of $\sqrt{-1}\mathfrak{t}_0^*$, and set $\Lambda_+ := C_+ \cap \widehat{T}$. Then Cartan–Weyl highest weight theory gives a bijection

$$\Lambda_+ \simeq \widehat{K}, \qquad \lambda \mapsto \tau_\lambda.$$

We recall for a subset S of $\mathbb{R}^N$, the asymptotic cone $S\infty$ is the closed cone defined by

$$S\infty := \{y \in \mathbb{R}^N : \text{ there exists a sequence } (y_n, \varepsilon_n) \in S \times \mathbb{R}_{>0} \text{ such that } \lim_{n\to\infty} \varepsilon_n y_n = y \text{ and } \lim_{n\to\infty} \varepsilon_n = 0\}.$$

The asymptotic K-support $\mathrm{AS}_K(X)$ of a K-module X is defined by Kashiwara and Vergne [22] as the asymptotic cone of the highest weights of irreducible K-modules occurring in X:

$$\mathrm{AS}_K(X) := \mathrm{Supp}_K(X)\infty,$$

where $\mathrm{Supp}_K(X)$ is the K-support of X given by

$$\mathrm{Supp}_K(X) := \{\lambda \in \Lambda_+ : \mathrm{Hom}_K(\tau_\lambda, X) \neq \{0\}\}.$$

For a closed subgroup K' of K, we write $\mathfrak{k}_0'$ for its Lie algebra, and regard $(\mathfrak{k}_0')^\perp = \mathrm{Ker}(\mathrm{pr} : \mathfrak{k}_0^* \to (\mathfrak{k}_0')^*)$ as a subspace of $\mathfrak{k}_0$ via a K-invariant inner product on $\mathfrak{k}_0$. We set

$$C_K(K') := C_+ \cap \sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}_0')^\perp.$$

An estimate of the singularity spectrum of the hyperfunction K-character of X yields a criterion of "K'-admissibility" of X for a subgroup K' of K ([28, Theorem 2.8] and [33]):

Fact 5.5. *Let $G \supset G'$ be a pair of real reductive linear Lie groups with compatible maximal compact subgroups $K \supset K'$, and X an irreducible $(\mathfrak{g}, K)$-module.*

(1) *The following two conditions on the triple (G, G', X) are equivalent:*

(i) *X is K'-admissible, i.e.,* $\dim \mathrm{Hom}_{K'}(\tau, X|_{K'}) < \infty$ *for all $\tau \in \widehat{K'}$.*
(ii) $C_K(K') \cap \mathrm{AS}_K(X) = \{0\}$.

(2) *If one of (therefore either of) (i) and (ii) is satisfied, then X is discretely decomposable as a $(\mathfrak{g}', K')$-module.*

5.2 Classification theory of discretely decomposable pairs

We begin with two observations.

First, for a Riemannian symmetric pair, that is, $(G, G') = (G, K)$ where $G' = K' = K$, the restriction $X|_{\mathfrak{g}'}$ is obviously discretely decomposable as a $(\mathfrak{g}', K')$-module for any irreducible $(\mathfrak{g}, K)$-module X, whereas the reductive pair $(G, G') = (\mathrm{SL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{R}))$ is an opposite extremal case as the restriction $X|_{\mathfrak{g}'}$ is never discretely decomposable as a $(\mathfrak{g}', K')$-module for any infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X ([29]). There are also intermediate cases such as $(G, G') = (\mathrm{SL}(n, \mathbb{R}), \mathrm{SO}(p, n-p))$ for which the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable for some infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X and is not for some other X.

Secondly, Harish-Chandra's admissibility theorem [16] asserts that

$$\dim_{\mathbb{C}} \mathrm{Hom}_K(\tau, \pi|_K) < \infty$$

for any $\pi \in \widehat{G}$ and $\tau \in \widehat{K}$.

This may be regarded as a statement for a Riemannian symmetric pair $(G, G') = (G, K)$. Unfortunately, there is a counterexample to an analogous statement for the reductive symmetric pair $(G, G') = (\mathrm{SO}(5, \mathbb{C}), \mathrm{SO}(3, 2))$, namely, we proved in [32] that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) = \infty \quad \text{for some } \pi \in \widehat{G} \text{ and } \tau \in \widehat{G'}.$$

However, it is plausible [32, Conjecture A] to have a generalization of Harish-Chandra's admissibility in the category $\mathcal{HC}$ of Harish-Chandra modules in the following sense:

$$\dim \mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) < \infty$$

for any irreducible $(\mathfrak{g}, K)$-module π_K and irreducible $(\mathfrak{g}', K')$-module $\tau_{K'}$.

In view of these two observations, we consider the following conditions (a) – (d) for a pair of real reductive Lie groups (G, G'), and raise a problem:

Problem 5.6. Classify the pairs (G, G') of real reductive Lie groups satisfying the condition (a) below (and also (b), (c) or (d)).

(a) there exist an infinite-dimensional irreducible unitary representation π of G and an irreducible unitary representation τ of G' such that

$$0 < \dim \mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) < \infty;$$

(b) there exist an infinite-dimensional irreducible unitary representation π of G and an irreducible unitary representation τ of G' such that

$$0 < \dim \mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'});$$

(c) there exist an infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X and an irreducible $(\mathfrak{g}', K')$-module Y such that

$$0 < \dim \mathrm{Hom}_{\mathfrak{g}',K'}(Y, X|_{\mathfrak{g}'}) < \infty;$$

(d) there exist an infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X and an irreducible $(\mathfrak{g}', K')$-module Y such that

$$0 < \dim \mathrm{Hom}_{\mathfrak{g}',K'}(Y, X|_{\mathfrak{g}'}).$$

Obviously we have the following implications:

$$\begin{array}{ccc} \text{(a)} & \Rightarrow & \text{(b)} \\ \Downarrow & & \Downarrow \\ \text{(c)} & \Rightarrow & \text{(d)} \end{array}$$

The vertical (inverse) implications (c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (b) will mean finite-multiplicity results like Harish-Chandra's admissibility theorem.

For symmetric pairs, Problem 5.6 has been solved in [50, Theorem 5.2]:

Theorem 5.7. *Let (G, G') be a reductive symmetric pair defined by an involutive automorphism σ of a simple Lie group G. Then the following five conditions (a), (b), (c), (d), and*

$$\sigma\beta \neq -\beta \tag{5.2}$$

are equivalent. Here β is the highest noncompact root with respect to a "$(-\sigma)$-compatible" positive system. (See [50] for a precise definition.)

Example 5.8. (1) $\sigma = \theta$ (Cartan involution). Then (5.2) is obviously satisfied because $\theta\beta = \beta$. Needless to say, the conditions (a)–(d) hold when $G' = K$.

(2) The reductive symmetric pairs $(G, G') = (\mathrm{SO}(p_1+p_2, q), \mathrm{SO}(p_1)\times\mathrm{SO}(p_2, q))$, $(\mathrm{SL}(2n, \mathbb{R}), \mathrm{Sp}(n, \mathbb{C}))$, $(\mathrm{SL}(2n, \mathbb{R}), \mathbb{T} \cdot \mathrm{SL}(n, \mathbb{C}))$ satisfy (5.2), and therefore (a)–(d).

The classification of irreducible symmetric pairs (G, G') satisfying one of (therefore all of) (a)–(d) was given in [50]. It turns out that there are fairly many reductive symmetric pairs (G, G') satisfying the five equivalent conditions in Theorem 5.7 when G does not carry a complex Lie group structure, whereas there are a few such pairs (G, G') when G is a complex Lie group. As a flavor of the classification, we present a list in this particular case. For this, we use the following notation, which is slightly different from that used in the other parts of this article. Let $G_{\mathbb{C}}$ be a complex simple Lie group, and $G_{\mathbb{R}}$ a real form. Take a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$, and let $K_{\mathbb{C}}$ be the complexification of $K_{\mathbb{R}}$ in $G_{\mathbb{C}}$. We denote by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{g}_{\mathbb{R}}$ the Lie algebras of $G_{\mathbb{C}}$, $K_{\mathbb{C}}$, and $G_{\mathbb{R}}$, respectively, and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the complexified Cartan decomposition.

Example 5.9 ([50, Corollary 5.9]). The following five conditions on the pairs $(G_{\mathbb{C}}, G_{\mathbb{R}})$ are equivalent:

(i) $(G_{\mathbb{C}}, K_{\mathbb{C}})$ satisfies (a) (or equivalently, (b), (c), or (d)).

(ii) $(G_{\mathbb{C}}, G_{\mathbb{R}})$ satisfies (a) (or equivalently, (b), (c), or (d)).
(iii) The minimal nilpotent orbit of $\mathfrak{g}$ does not intersect $\mathfrak{g}_{\mathbb{R}}$.
(iv) The minimal nilpotent orbit of $\mathfrak{g}$ does not intersect $\mathfrak{p}$.
(v) The Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{g}_{\mathbb{R}}$ are given in the following table:

$\mathfrak{g}$	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{so}(m,\mathbb{C})$	$\mathfrak{sp}(p+q,\mathbb{C})$	$\mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{e}_6^{\mathbb{C}}$
$\mathfrak{k}$	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{so}(m-1,\mathbb{C})$	$\mathfrak{sp}(p,\mathbb{C})+\mathfrak{sp}(q,\mathbb{C})$	$\mathfrak{so}(9,\mathbb{C})$	$\mathfrak{f}_4^{\mathbb{C}}$
$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{su}^*(2n)$	$\mathfrak{so}(m-1,1)$	$\mathfrak{sp}(p,q)$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-26)}$

where $m \geq 5$ and $n,p,q \geq 1$.

Remark 5.10. The equivalence (iv) and (v) was announced by Brylinski–Kostant in the context that there is no minimal representation of a Lie group $G_{\mathbb{R}}$ with the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ in the above table (see [7]). The new ingredient here is that this condition on the Lie algebras corresponds to a question of discretely decomposable restrictions of Harish-Chandra modules.

For nonsymmetric pairs, there are a few nontrivial cases where (a) (and therefore (b), (c), and (d)) holds, as follows.

Example 5.11 ([26]). The nonsymmetric pairs $(G,G') = (\mathrm{SO}(4,3), \mathrm{G}_{2(2)})$ and $(\mathrm{SO}(7,\mathbb{C}), \mathrm{G}_2^{\mathbb{C}})$ satisfy (a) (and also (b), (c), and (d)).

Once we classify the pairs (G,G') such that there exists at least one irreducible infinite-dimensional $(\mathfrak{g},K)$-module X which is discretely decomposable as a $(\mathfrak{g}',K')$-module, then we would like to find all such Xs.

In [49] we carried out this project for $X = A_{\mathfrak{q}}(\lambda)$ by applying the general criterion (Facts 5.2 and 5.5) to reductive symmetric pairs (G,G'). This is a result in Stage A of the branching problem, and we think it will serve as a foundational result for Stage B (explicit branching laws). Here is another example of the classification of the triples (G,G',X) when $G \simeq G' \times G'$, see [50, Theorem 6.1]:

Example 5.12 (tensor product). Let G be a noncompact connected simple Lie group, and let X_j $(j=1,2)$ be infinite-dimensional irreducible $(\mathfrak{g},K)$-modules.

(1) Suppose G is not of Hermitian type. Then the tensor product representation $X_1 \otimes X_2$ is never discretely decomposable as a $(\mathfrak{g},K)$-module.
(2) Suppose G is of Hermitian type. Then the tensor product representation $X_1 \otimes X_2$ is discretely decomposable as a $(\mathfrak{g},K)$-module if and only if both X_1 and X_2 are simultaneously highest weight $(\mathfrak{g},K)$-modules or simultaneously lowest weight $(\mathfrak{g},K)$-modules.

5.3 *Two spaces* $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ *and* $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'})$

There is a canonical injective homomorphism

$$\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) \hookrightarrow \mathrm{Hom}_{G'}(\tau, \pi|_{G'}),$$

however, it is not bijective for $\tau \in \widehat{G'}$ and $\pi \in \widehat{G}$. In fact, we have:

Proposition 5.13. *Suppose that π is an irreducible unitary representation of G. If the restriction $\pi|_{G'}$ contains a continuous spectrum and if an irreducible unitary representation τ of G' appears as an irreducible summand of the restriction $\pi|_{G'}$, then we have*

$$\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) = \{0\} \neq \mathrm{Hom}_{G'}(\tau, \pi|_{G'}).$$

Proof. If $\mathrm{Hom}_{\mathfrak{g}'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) = \mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'})$ were nonzero, then the $(\mathfrak{g}, K)$-module π_K would be discretely decomposable as a $(\mathfrak{g}', K')$-module by Theorem 4.5. In turn, the restriction $\pi|_{G'}$ of the unitary representation π would decompose discretely into a Hilbert direct sum of irreducible unitary representations of G' by [32, Theorem 2.7], contradicting the assumption. Hence we conclude $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) = \{0\}$. $\square$

An example of Proposition 5.13 may be found in [45, Part II] where π is the minimal representation of $G = \mathrm{O}(p, q)$ and τ is the unitarization of a Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda)$ for $G' = \mathrm{O}(p', q') \times \mathrm{O}(p'', q'')$ with $p = p' + p''$ and $q = q' + q''$ ($p', q', p'', q'' > 1$ and $p + q$ even).

Here is another example of Proposition 5.13:

Theorem 5.14. *Let G be a real reductive linear Lie group, and let $\pi = \mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ be a spherical unitary degenerate principal series representation of G induced from a unitary character $\mathbb{C}_\lambda$ of a parabolic subgroup $P = LN$ of G.*

(1) *For any irreducible $(\mathfrak{g}, K)$-module τ_K, we have*

$$\mathrm{Hom}_{\mathfrak{g},K}(\tau_K, \pi_K \otimes \pi_K) = \{0\}.$$

(2) *Suppose now G is a classical group. If N is abelian and P is conjugate to the opposite parabolic subgroup $\overline{P} = L\overline{N}$, then we have a unitary equivalence of the discrete part:*

$$L^2(G/L)_{\mathrm{disc}} \simeq \sum_{\tau \in \widehat{G}}^{\oplus} \dim_{\mathbb{C}} \mathrm{Hom}_G(\tau, \pi \widehat{\otimes} \pi)\, \tau. \tag{5.3}$$

In particular, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\tau, \pi \widehat{\otimes} \pi) \leq 1$$

for any irreducible unitary representation τ of G. Moreover there exist countably many irreducible unitary representations τ of G such that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\tau, \pi \widehat{\otimes} \pi) = 1.$$

A typical example of the setting in Theorem 5.14 (2) is the Siegel parabolic subgroup $P = LN = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathrm{Sym}(n, \mathbb{R})$ in $G = \mathrm{Sp}(n, \mathbb{R})$.

Proof. (1) This is a direct consequence of Example 5.12.

(2) Take $w_0 \in G$ such that $w_0 L w_0^{-1} = L$ and $w_0 N w_0^{-1} = \overline{N}$. Then the G-orbit through $(w_0 P, eP)$ in $G/P \times G/P$ under the diagonal action is open dense, and therefore Mackey theory gives a unitary equivalence

$$L^2(G/L) \simeq \pi_\lambda \widehat{\otimes} \pi_\lambda \tag{5.4}$$

because $\mathrm{Ad}^*(w_0)\lambda = -\lambda$, see [30] for instance. Since N is abelian, (G, L) forms a symmetric pair (see [64]). Therefore the branching law of the tensor product representation $\pi \widehat{\otimes} \pi$ reduces to the Plancherel formula for the regular representation on the reductive symmetric space G/L, which is known; see [10]. In particular, we have the unitary equivalence (5.3), and the left-hand side of (5.3) is nonzero if and only if $\operatorname{rank} G/L = \operatorname{rank} K/L \cap K$ due to Flensted-Jensen and Matsuki–Oshima [55]. By the description of discrete series representation for G/L by Matsuki–Oshima [55] and Vogan [71], we have the conclusion.

□

5.4 Analytic vectors and discrete decomposability

Suppose π is an irreducible unitary representation of G on a Hilbert space V, and G' is a reductive subgroup of G as before. Any G'-invariant closed subspace W in V contains G'-analytic vectors (hence, also G'-smooth vectors) as a dense subspace. However, W may not contain nonzero G-smooth vectors (hence, also G-analytic vectors). In view of Theorem 4.5 in the category $\mathcal{HC}$ of Harish-Chandra modules, we think that this is related to the existence of a continuous spectrum in the branching law of the restriction $\pi|_{G'}$. We formulate a problem related to this delicate point below. As before, π^∞ and τ^∞ denote the space of G-smooth vectors and G'-smooth vectors for representations π and τ of G and G', respectively. An analogous notation is applied to π^ω and τ^ω.

Problem 5.15. Let (π, V) be an irreducible unitary representation of G, and G' a reductive subgroup of G. Are the following four conditions on the triple (G, G', π) equivalent?

(i) There exists an irreducible $(\mathfrak{g}', K')$-module $\tau_{K'}$ such that

$$\mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) \neq \{0\}.$$

(ii) There exists an irreducible unitary representation τ of G' such that

$$\mathrm{Hom}_{G'}(\tau^\omega, \pi^\omega|_{G'}) \neq \{0\}.$$

(iii) There exists an irreducible unitary representation τ of G' such that

$$\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'}) \neq \{0\}.$$

(iv) The restriction $\pi|_{G'}$ decomposes discretely into a Hilbert direct sum of irreducible unitary representations of G'.

Here are some remarks on Problem 5.15.

Remark 5.16. (1) In general, the implication (i) $\Rightarrow$ (iv) holds ([32, Theorem 2.7]).
(2) If the restriction $\pi|_{K'}$ is K'-admissible, then (i) holds by [29, Proposition 1.6] and (iv) holds by [26, Theorem 1.2].
(3) The implication (iv) $\Rightarrow$ (i) was raised in [32, Conjecture D], and some affirmative results has been announced by Duflo and Vargas in a special setting where π is Harish-Chandra's discrete series representation (cf. [12]). A related result is given in [77].
(4) Even when the unitary representation $\pi|_{G'}$ decomposes discretely (i.e., (iv) in Problem 5.15 holds), it may happen that $V^\infty \subsetneqq (V|_{G'})^\infty$. The simplest example for this is as follows. Let (π', V') and (π'', V'') be infinite-dimensional unitary representations of noncompact Lie groups G' and G'', respectively. Set $G = G' \times G''$, with G' realized as a subgroup of G as $G' \times \{e\}$, and set $\pi = \pi' \boxtimes \pi''$. Then $V^\infty \subsetneqq (V|_{G'})^\infty$ because $(V'')^\infty \subsetneqq V''$.

6 Features of the restriction, II : $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ (symmetry breaking operators)

In the previous section, we discussed embeddings of irreducible G'-modules τ into irreducible G-modules π (or the analogous problem in the category $\mathcal{HC}$ of Harish-Chandra modules); see Case I in Section 4. In contrast, we consider the opposite order in this section, namely, continuous G'-homomorphisms from irreducible G-modules π to irreducible G'-modules τ, see Case II in Section 4. We highlight the case where π and τ are admissible smooth representations (Casselman–Wallach globalization of modules in the category $\mathcal{HC}$). Then it turns out that the spaces $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$ or $\mathrm{Hom}_{\mathfrak{g}',K'}(\pi_K|_{\mathfrak{g}'}, \tau_{K'})$ are much larger in general than the spaces $\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$ or $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'})$ considered in Section 5. Thus the primary concern here will be with obtaining an upper estimate for the dimensions of those spaces.

It would make reasonable sense to find branching laws (Stage B) or to construct symmetry breaking operators (Stage C) if we know *a priori* the nature of the multiplicities in branching laws. The task of Stage A of the branching problem is to establish a criterion and to give a classification of desirable settings. In this section, we consider:

Problem 6.1. (1) (finite multiplicities) Find a criterion for when a pair (G, G') of real reductive Lie groups satisfies

$$\dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty \quad \text{for any } \pi^\infty \in \widehat{G}_{\mathrm{smooth}} \text{ and } \tau^\infty \in \widehat{G'}_{\mathrm{smooth}}.$$

Classify all such pairs (G, G').

(2) (uniformly bounded multiplicities) Find a criterion for when a pair (G, G') of real reductive Lie groups satisfies

$$\sup_{\pi^\infty \in \widehat{G}_{\mathrm{smooth}}} \sup_{\tau^\infty \in \widehat{G'}_{\mathrm{smooth}}} \dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty.$$

Classify all such pairs (G, G').

One may also think of variants of Problem 6.1. For instance, we may refine Problem 6.1 by considering it as a condition on the triple (G, G', π) instead of a condition on the pair (G, G'):

Problem 6.2. (1) Classify the triples (G, G', π^∞) with $G \supset G'$ and $\pi^\infty \in \widehat{G}_{\mathrm{smooth}}$ such that

$$\dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty \quad \text{for any } \tau^\infty \in \widehat{G'}_{\mathrm{smooth}}. \tag{6.1}$$

(2) Classify the triples (G, G', π^∞) such that

$$\sup_{\tau^\infty \in \widehat{G'}_{\mathrm{smooth}}} \dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty. \tag{6.2}$$

Problem 6.1 has been solved recently for all reductive symmetric pairs (G, G'); see Sections 6.3 and 6.4. On the other hand, Problem 6.2 has no complete solution even when (G, G') is a reductive symmetric pair. Here are some partial answers to Problem 6.2 (1):

Example 6.3. (1) If (G, G') satisfies (PP) (see the list in Theorem 6.14), then the triple (G, G', π) satisfies (6.1) whenever $\pi^\infty \in \widehat{G}_{\mathrm{smooth}}$.
(2) If π is K'-admissible, then (6.1) is satisfied. A necessary and sufficient condition for the K'-admissibility of $\pi|_{K'}$, Fact 5.5, is easy to check in many cases. In particular, a complete classification of the triples (G, G', π) such that $\pi|_{K'}$ is K'-admissible was recently accomplished in [49] in the setting where $\pi_K = A_{\mathfrak{q}}(\lambda)$ and where (G, G') is a reductive symmetric pair.

We give a conjectural statement concerning Problem 6.2 (2).

Conjecture 6.4. *Let (G, G') be a reductive symmetric pair. If π is an irreducible highest weight representation of G or if π is a minimal representation of G, then the uniform boundedness property* (6.2) *would hold for the triple (G, G', π^∞).*

Some evidence was given in [35, Theorems B and D] and in [45, 46].

6.1 Real spherical homogeneous spaces

A complex manifold $X_{\mathbb{C}}$ with an action of a complex reductive group $G_{\mathbb{C}}$ is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. Spherical varieties have been studied extensively in the context of algebraic geometry and finite-dimensional representation theory. In the real setting, in search of a broader framework for global analysis on homogeneous spaces than the usual (e.g., reductive symmetric spaces), we propose the following:

Definition 6.5 ([27]). Let G be a real reductive Lie group. We say a connected smooth manifold X with G-action is *real spherical* if a minimal parabolic subgroup P of G has an open orbit in X, or equivalently $\#(P\backslash X) < \infty$.

The equivalence in Definition 6.5 was proved in [5] by using Kimelfeld [23] and Matsuki [54]; see [48, Remark] and references therein for related earlier results.

Here are some partial results on the classification of real spherical homogeneous spaces.

Example 6.6. (1) If G is compact then all G-homogeneous spaces are real spherical.
(2) Any semisimple symmetric space G/H is real spherical. The (infinitesimal) classification of semisimple symmetric spaces was accomplished by Berger [3].
(3) G/N is real spherical where N is a maximal unipotent subgroup of G.
(4) For G of real rank one, real spherical homogeneous spaces of G are classified by Kimelfeld [23].
(5) Any real form G/H of a spherical homogeneous space $G_{\mathbb{C}}/H_{\mathbb{C}}$ is real spherical [48, Lemma 4.2]. The latter were classified by Krämer [53], Brion, [6], and Mikityuk [56]. In particular, if G is quasi-split, then the classification problem of real spherical homogeneous spaces G/H reduces to that of the known classification of spherical homogeneous spaces.
(6) The triple product space $(G \times G \times G)/\operatorname{diag} G$ is real spherical if and only if G is locally isomorphic to the direct product of compact Lie groups and some copies of $\mathrm{O}(n, 1)$ (Kobayashi [27]).
(7) Real spherical homogeneous spaces of the form $(G \times G')/\operatorname{diag} G'$ for symmetric pairs (G, G') were recently classified. We review this in Theorem 6.14 below.

The second and third examples form the basic geometric settings for analysis on reductive symmetric spaces and Whittaker models. The last two examples play a role in Stage A of the branching problem, as we see in the next subsection.

The significance of this geometric property is that the group G controls the space of functions on X in the sense that the finite-multiplicity property holds for the regular representation of G on $C^\infty(X)$:

Fact 6.7 ([48, Theorems A and C]). *Suppose G is a real reductive linear Lie group, and H is an algebraic reductive subgroup.*

(1) *The homogeneous space G/H is real spherical if and only if*

$$\operatorname{Hom}_G(\pi^\infty, C^\infty(G/H)) \text{ is finite-dimensional for all } \pi^\infty \in \widehat{G}_{\mathrm{smooth}}.$$

(2) *The complexification $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if and only if*

$$\sup_{\pi^\infty \in \widehat{G}_{\mathrm{smooth}}} \dim_{\mathbb{C}} \mathrm{Hom}_G(\pi^\infty, C^\infty(G/H)) < \infty.$$

See [48] for upper and lower estimates of the dimension, and also for the non-reductive case. The proof uses the theory of regular singularities of a system of partial differential equations by taking an appropriate compactification with normal crossing boundaries.

6.2 A geometric estimate of multiplicities : (PP) and (BB)

Suppose that G' is an algebraic reductive subgroup of G. For Stage A in the branching problem for the restriction $G \downarrow G'$, we apply the general theory of Section 6.1 to the homogeneous space $(G \times G')/\operatorname{diag} G'$.

Let P be a minimal parabolic subgroup of G, and P' a minimal parabolic subgroup of G'.

Definition-Lemma 6.8 ([48]). *We say the pair (G, G') satisfies the property (PP) if one of the following five equivalent conditions is satisfied:*
(PP1) $(G \times G')/\operatorname{diag} G'$ *is real spherical as a* $(G \times G')$*-space.*
(PP2) G/P' *is real spherical as a* G*-space.*
(PP3) G/P *is real spherical as a* G'*-space.*
(PP4) G *has an open orbit in* $G/P \times G/P'$ *via the diagonal action.*
(PP5) $\#(P' \backslash G/P) < \infty$.

Since the above five equivalent conditions are determined by the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, we also say that the pair $(\mathfrak{g}, \mathfrak{g}')$ of reductive Lie algebras satisfies (PP), where $\mathfrak{g}$ and $\mathfrak{g}'$ are the Lie algebras of the Lie groups G and G', respectively.

Remark 6.9. If the pair $(\mathfrak{g}, \mathfrak{g}')$ satisfies (PP), in particular, (PP5), then there are only finitely many possibilities for $\operatorname{Supp} T$ for symmetry breaking operators $T : C^\infty(G/P, \mathcal{V}) \to C^\infty(G'/P', \mathcal{W})$ (see Definition 7.9 below). This observation has become a guiding principle to formalise a strategy in classifying all symmetry breaking operators used in [52], as we shall discuss in Section 7.2

Next we consider another property, to be denoted (BB), which is stronger than (PP). Let $G_{\mathbb{C}}$ be a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and $G'_{\mathbb{C}}$ a subgroup of $G_{\mathbb{C}}$ with complexified Lie algebra $\mathfrak{g}'_{\mathbb{C}} = \mathfrak{g}' \otimes_{\mathbb{R}} \mathbb{C}$. We do not assume either $G \subset G_{\mathbb{C}}$ or $G' \subset G'_{\mathbb{C}}$. Let $B_{\mathbb{C}}$ and $B'_{\mathbb{C}}$ be Borel subgroups of $G_{\mathbb{C}}$ and $G'_{\mathbb{C}}$, respectively.

Definition-Lemma 6.10. *We say the pair (G, G') (or the pair $(\mathfrak{g}, \mathfrak{g}')$) satisfies the property* (BB) *if one of the following five equivalent conditions is satisfied:*
(BB1) $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag} G'_{\mathbb{C}}$ *is spherical as a* $(G_{\mathbb{C}} \times G'_{\mathbb{C}})$*-space.*
(BB2) $G_{\mathbb{C}}/B'_{\mathbb{C}}$ *is spherical as a* $G_{\mathbb{C}}$*-space.*

(BB3) *$G_{\mathbb{C}}/B_{\mathbb{C}}$ is spherical as a $G'_{\mathbb{C}}$-space.*
(BB4) *$G_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/B_{\mathbb{C}} \times G_{\mathbb{C}}/B'_{\mathbb{C}}$ via the diagonal action.*
(BB5) *$\#(B'_{\mathbb{C}}\backslash G_{\mathbb{C}}/B_{\mathbb{C}}) < \infty$.*

The above five equivalent conditions (BB1) – (BB5) are determined only by the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$.

Remark 6.11. (1) (BB) implies (PP).
(2) If both G and G' are quasi-split, then (BB) $\Leftrightarrow$ (PP).
In fact, the first statement follows immediately from [48, Lemmas 4.2 and 5.3], and the second statement is clear.

6.3 Criteria for finiteness/boundedness of multiplicities

In this and the next subsections, we give an answer to Problem 6.1. The following criteria are direct consequences of Fact 6.7 and a careful consideration of the topology of representation spaces, and are proved in [48].

Theorem 6.12. *The following three conditions on a pair of real reductive algebraic groups $G \supset G'$ are equivalent:*

(i) (Symmetry breaking) $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$ *is finite-dimensional for any pair $(\pi^\infty, \tau^\infty)$ of irreducible smooth representations of G and G'.*
(ii) (Invariant bilinear form) *There exist at most finitely many linearly independent G'-invariant bilinear forms on $\pi^\infty|_{G'} \widehat{\otimes} \tau^\infty$, for any $\pi^\infty \in \widehat{G}_{\mathrm{smooth}}$ and $\tau^\infty \in \widehat{G'}_{\mathrm{smooth}}$.*
(iii) (Geometry) *The pair (G, G') satisfies the condition (PP) (Definition-Lemma 6.8).*

Theorem 6.13. *The following three conditions on a pair of real reductive algebraic groups $G \supset G'$ are equivalent:*

(i) (Symmetry breaking) *There exists a constant C such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \leq C$$

for any $\pi^\infty \in \widehat{G}_{\mathrm{smooth}}$ and $\tau^\infty \in \widehat{G'}_{\mathrm{smooth}}$.
(ii) (Invariant bilinear form) *There exists a constant C such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty|_{G'} \widehat{\otimes} \tau^\infty, \mathbb{C}) \leq C$$

for any $\pi^\infty \in \widehat{G}_{\mathrm{smooth}}$ and $\tau^\infty \in \widehat{G'}_{\mathrm{smooth}}$.
(iii) (Geometry) *The pair (G, G') satisfies the condition (BB) (Definition-Lemma 6.10).*

6.4 Classification theory of finite-multiplicity branching laws

This section gives a complete list of the reductive symmetric pairs (G, G') such that $\dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$ is finite or bounded for all $\pi^\infty \in \widehat{G}_{\mathrm{smooth}}$ and $\tau^\infty \in \widehat{G'}_{\mathrm{smooth}}$. Owing to the criteria in Theorems 6.12 and 6.13, the classification is reduced to that of (real) spherical homogeneous spaces of the form $(G \times G')/\operatorname{diag} G'$, which was accomplished in [44] by using an idea of "linearization" :

Theorem 6.14. *Suppose (G, G') is a reductive symmetric pair. Then the following two conditions are equivalent:*

(i) $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$ *is finite-dimensional for any pair $(\pi^\infty, \tau^\infty)$ of admissible smooth representations of G and G'.*
(ii) *The pair $(\mathfrak{g}, \mathfrak{g}')$ of their Lie algebras is isomorphic (up to outer automorphisms) to a direct sum of the following pairs:*

A) Trivial case: $\mathfrak{g} = \mathfrak{g}'$.
B) Abelian case: $\mathfrak{g} = \mathbb{R}$, $\mathfrak{g}' = \{0\}$.
C) Compact case: $\mathfrak{g}$ *is the Lie algebra of a compact simple Lie group.*
D) Riemannian symmetric pair: $\mathfrak{g}'$ *is the Lie algebra of a maximal compact subgroup K of a noncompact simple Lie group G.*
E) Split rank one case ($\operatorname{rank}_{\mathbb{R}} G = 1$):
E1) $(\mathfrak{o}(p+q,1), \mathfrak{o}(p) + \mathfrak{o}(q,1))$ $(p+q \geq 2)$,
E2) $(\mathfrak{su}(p+q,1), \mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(q,1)))$ $(p+q \geq 1)$,
E3) $(\mathfrak{sp}(p+q,1), \mathfrak{sp}(p) + \mathfrak{sp}(q,1))$ $(p+q \geq 1)$,
E4) $(\mathfrak{f}_{4(-20)}, \mathfrak{o}(8,1))$.
F) Strong Gelfand pairs and their real forms:
F1) $(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}))$ $(n \geq 2)$,
F2) $(\mathfrak{o}(n+1,\mathbb{C}), \mathfrak{o}(n,\mathbb{C}))$ $(n \geq 2)$,
F3) $(\mathfrak{sl}(n+1,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}))$ $(n \geq 1)$,
F4) $(\mathfrak{su}(p+1,q), \mathfrak{u}(p,q))$ $(p+q \geq 1)$,
F5) $(\mathfrak{o}(p+1,q), \mathfrak{o}(p,q))$ $(p+q \geq 2)$.
G) Group case: $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{g}_1 + \mathfrak{g}_1, \operatorname{diag} \mathfrak{g}_1)$ *where*
G1) $\mathfrak{g}_1$ *is the Lie algebra of a compact simple Lie group,*
G2) $(\mathfrak{o}(n,1) + \mathfrak{o}(n,1), \operatorname{diag} \mathfrak{o}(n,1))$ $(n \geq 2)$.
H) Other cases:
H1) $(\mathfrak{o}(2n,2), \mathfrak{u}(n,1))$ $(n \geq 1)$.
H2) $(\mathfrak{su}^*(2n+2), \mathfrak{su}(2) + \mathfrak{su}^*(2n) + \mathbb{R})$ $(n \geq 1)$.
H3) $(\mathfrak{o}^*(2n+2), \mathfrak{o}(2) + \mathfrak{o}^*(2n))$ $(n \geq 1)$.
H4) $(\mathfrak{sp}(p+1,q), \mathfrak{sp}(p,q) + \mathfrak{sp}(1))$.
H5) $(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1) + \mathbb{R})$.

Among the pairs $(\mathfrak{g}, \mathfrak{g}')$ in the list (A)–(H) in Theorem 6.14 describing finite multiplicities, those pairs having uniform bounded multiplicities are classified as follows.

Theorem 6.15. *Suppose* (G, G') *is a reductive symmetric pair. Then the following two conditions are equivalent:*

(i) *There exists a constant* C *such that*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \leq C$$

for any $\pi^\infty \in \widehat{G}_{\mathrm{smooth}}$ *and* $\tau^\infty \in \widehat{G'}_{\mathrm{smooth}}$.

(ii) *The pair of their Lie algebras* $(\mathfrak{g}, \mathfrak{g}')$ *is isomorphic* (*up to outer automorphisms*) *to a direct sum of the pairs in* (A), (B) *and* (F1) – (F5).

Proof. Theorem 6.14 follows directly from Theorem 6.12 and [44, Theorem 1.3]. Theorem 6.15 follows directly from Theorem 6.13 and [44, Proposition 1.6]. □

Example 6.16. In connection with branching problems, some of the pairs appeared earlier in the literature. For instance,

(F1), (F2) $\cdots$ finite-dimensional representations (strong Gelfand pairs) [53];
(F2), (F5) $\cdots$ tempered unitary representations (Gross–Prasad conjecture) [14];
(G2) $\cdots$ tensor product, trilinear forms [8, 27];
(F1)–(F5) $\cdots$ multiplicity-free restrictions [2, 68].

7 Construction of symmetry breaking operators

Stage C in the branching problem asks for an explicit construction of intertwining operators. This problem depends on the geometric models of representations of a group G and its subgroup G'. In this section we discuss symmetry breaking operators in two models, i.e., in the setting of real flag manifolds (Sections 7.1–7.3) and in the holomorphic setting (Sections 7.4–7.5).

7.1 Differential operators on different base spaces

We extend the usual notion of differential operators between two vector bundles on the *same* base space to those on *different* base spaces X and Y with a morphism $p : Y \to X$ as follows.

Definition 7.1. Let $\mathcal{V} \to X$ and $\mathcal{W} \to Y$ be two vector bundles, and $p : Y \to X$ a smooth map between the base manifolds. A continuous linear map $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ is said to be a *differential operator* if

$$p(\operatorname{Supp}(Tf)) \subset \operatorname{Supp} f \quad \text{for all } f \in C^\infty(X, \mathcal{V}), \tag{7.1}$$

where Supp stands for the support of a section.

The condition (7.1) shows that T is a local operator in the sense that for any open subset U of X, the restriction $(Tf)|_{p^{-1}(U)}$ is determined by the restriction $f|_U$.

Example 7.2. (1) If $X = Y$ and p is the identity map, then the condition (7.1) is equivalent to the condition that T is a differential operator in the usual sense, due to Peetre's theorem [61].

(2) If $p : Y \to X$ is an immersion, then any operator T satisfying (7.1) is locally of the form

$$\sum_{(\alpha,\beta)\in\mathbb{N}^{m+n}} g_{\alpha\beta}(y) \frac{\partial^{|\alpha|+|\beta|}}{\partial y^\alpha \partial z^\beta}\Bigg|_{z_1=\cdots=z_n=0} \qquad \text{(finite sum)},$$

where $\{(y_1, \ldots, y_m, z_1, \ldots, z_n)\}$ are local coordinates of X such that Y is given locally by the equation $z_1 = \cdots = z_n = 0$, and $g_{\alpha\beta}(y)$ are matrix-valued functions on Y.

7.2 Distribution kernels for symmetry breaking operators

In this section, we discuss symmetry breaking operators in a geometric setting, where representations are realized in the space of smooth sections for homogeneous vector bundles.

Let G be a Lie group, and $\mathcal{V} \to X$ a homogeneous vector bundle, namely, a G-equivariant vector bundle such that the G-action on the base manifold X is transitive. Likewise, let $\mathcal{W} \to Y$ be a homogeneous vector bundle for a subgroup G'. The main assumption of our setting is that there is a G'-equivariant map $p : Y \to X$. For simplicity, we also assume that p is injective, and do not assume any relationship between $p^*\mathcal{V}$ and $\mathcal{W}$. Then we have continuous representations of G on the Fréchet space $C^\infty(X, \mathcal{V})$ and of the subgroup G' on $C^\infty(Y, \mathcal{W})$, but it is not obvious if there exists a nonzero continuous G'-homomorphism (symmetry breaking operator)

$$T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W}).$$

In this setting, a basic problem is:

Problem 7.3. (1) (Stage A) Find an upper and lower estimate of the dimension of the space $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$ of symmetry breaking operators.

(2) (Stage A) When is $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$ finite-dimensional for any G-equivariant vector bundle $\mathcal{V} \to X$ and any G'-equivariant vector bundle $\mathcal{W} \to Y$?

(3) (Stage B) Given equivariant vector bundles $\mathcal{V} \to X$ and $\mathcal{W} \to Y$, determine the dimension of $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$.

(4) (Stage C) Construct explicit elements in $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$.

Here are some special cases:

Example 7.4. Suppose $G = G'$, X is a (full) real flag manifold G/P where P is a minimal parabolic subgroup of G, and Y is algebraic.

(1) In this setting, Problem 7.3 (1) and (2) were solved in [48]. In particular, a necessary and sufficient condition for Problem 7.3 (2) is that Y is real spherical, by Fact 6.7 (1) (or directly from the original proof of [48, Theorem A]).
(2) Not much is known about precise results for Problem 7.3 (3), even when $G = G'$. On the other hand, Knapp–Stein intertwining operators or Poisson transforms are examples of explicit intertwining operators when Y is a real flag manifold or a symmetric space, respectively, giving a partial solution to Problem 7.3 (4).

Example 7.5. Let G be the conformal group of the standard sphere $X = S^n$, let G' be the subgroup that leaves the totally geodesic submanifold $Y = S^{n-1}$ invariant, and let $\mathcal{V} \to X$, $\mathcal{W} \to Y$ be G-, G'-equivariant line bundles, respectively. Then $\mathcal{V}$ and $\mathcal{W}$ are parametrized by complex numbers λ and ν, respectively, up to signatures. In this setting Problem 7.3 (3) and (4) were solved in [52]. This is essentially the geometric setup for the classification of $\mathrm{Hom}_{\mathrm{O}(n,1)}(I(\lambda)^\infty, J(\nu)^\infty)$ which was discussed in Section 2.2.

We return to the general setting. Let H be an algebraic subgroup of G, (λ, V) a finite-dimensional representation of H, and $\mathcal{V} := G \times_H V \to X := G/H$ the associated G-homogeneous bundle. Likewise, let (ν, W) be a finite-dimensional representation of $H' := H \cap G'$, and $\mathcal{W} := G' \times_{H'} W \to Y := G'/H'$ the associated G'-equivariant bundle. Denote by $\mathbb{C}_{2\rho}$ the one-dimensional representation of H defined by $h \mapsto |\det(\mathrm{Ad}(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h})|^{-1}$. Then the volume density bundle $\Omega_{G/H}$ of G/H is given as a homogeneous bundle $G \times_H \mathbb{C}_{2\rho}$. Let $(\lambda^\vee, V^\vee)$ be the contragredient representation of the finite-dimensional representation (λ, V) of H. Then the dualizing bundle $\mathcal{V}^* := \mathcal{V}^\vee \otimes \Omega_{G/H}$ is given by $\mathcal{V}^* \simeq G \times_H (V^\vee \otimes \mathbb{C}_{2\rho})$ as a homogeneous vector bundle.

By the Schwartz kernel theorem, any continuous operator $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ is given by a distribution kernel $k_T \in \mathcal{D}'(X \times Y, \mathcal{V}^* \boxtimes \mathcal{W})$. We write

$$m : G \times G' \to G, \quad (g, g') \mapsto (g')^{-1} g,$$

for the multiplication map. If T intertwines G'-actions, then k_T is G'-invariant under the diagonal action, and therefore k_T is of the form $m^* K_T$ for some $K_T \in \mathcal{D}'(X, \mathcal{V}^*) \otimes W$. We have shown in [52, Proposition 3.1] the following proposition:

Proposition 7.6. *Suppose X is compact. Then the correspondence $T \mapsto K_T$ induces a bijection:*

$$\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) \xrightarrow{\sim} (\mathcal{D}'(X, \mathcal{V}^*) \otimes W)^{\Delta(H')}.$$

Using Proposition 7.6, we can give a solution to Problem 7.3 (2) when X is a real flag manifold:

Theorem 7.7. *Suppose P is a minimal parabolic subgroup of G, $X = G/P$, and $Y = G'/(G' \cap P)$. Then $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$ is finite-dimensional for any G-equivariant vector bundle $\mathcal{V} \to X$ and any G'-equivariant vector bundle $\mathcal{W} \to Y$ if and only if $G/(G' \cap P)$ is real spherical.*

Proof. We set $\widetilde{Y} := G/(G' \cap P)$ and $\widetilde{\mathcal{W}} := G \times_{(G' \cap P)} W$. Then Proposition 7.6 implies that there is a canonical bijection:

$$\mathrm{Hom}_G(C^\infty(X, \mathcal{V}), C^\infty(\widetilde{Y}, \widetilde{\mathcal{W}})) \stackrel{\sim}{\to} \mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})).$$

We apply [48, Theorem A] to the left-hand side, and get the desired conclusion for the right-hand side. □

The smaller X is, the more likely it will be that there exists Y satisfying the finiteness condition posed in Problem 7.3 (2). Thus one might be interested in replacing the *full* real flag manifold by a *partial* real flag manifold in Theorem 7.7. By applying the same argument as above to a generalization of [48] to a partial flag manifold in [41, Corollary 6.8], we get

Proposition 7.8. *Suppose P is a (not necessarily minimal) parabolic subgroup of G and $X = G/P$. Then the finiteness condition for symmetry breaking operators in Problem 7.3 (2) holds only if the subgroup $G' \cap P$ has an open orbit in G/P.*

Back to the general setting, we endow the double coset space $H'\backslash G/H$ with the quotient topology via the canonical quotient $G \to H'\backslash G/H$. Owing to Proposition 7.6, we associate a closed subset of $H'\backslash G/H$ to each symmetry breaking operator:

Definition 7.9. Given a continuous symmetry breaking operator $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$, we define a closed subset $\mathrm{Supp}\, T$ in the double coset space $H'\backslash G/H$ as the support of $K_T \in \mathcal{D}'(X, \mathcal{V}^*) \otimes W$.

Example 7.10. If $H = P$, a minimal parabolic subgroup of G, and if H' has an open orbit in G/P, then $\#(H'\backslash G/P) < \infty$. In particular, there are only finitely many possibilities for $\mathrm{Supp}\, T$.

Definition 7.11. Let $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ be a continuous symmetry breaking operator.

1) We say T is a *regular* symmetry breaking operator if $\mathrm{Supp}\, T$ contains an interior point of $H'\backslash G/H$. We say T is *singular* if T is not regular.
2) We say T is a *differential* symmetry breaking operator if $\mathrm{Supp}\, T$ is a singleton in $H'\backslash G/H$.

Remark 7.12. The terminology “differential symmetry breaking operator” in Definition 7.11 makes reasonable sense. In fact, T is a differential operator in the sense of Definition 7.1 if and only if $\mathrm{Supp}\, T$ is a singleton in $H'\backslash G/H$ (see [51, Part I, Lemma 2.3]).

The strategy of [52] for the classification of *all* symmetry breaking operators for (G, G') satisfying (PP) is to use the stratification of H'-orbits in G/H by the closure relation. To be more precise, the strategy is:

- to obtain all differential symmetry breaking operators, which corresponds to the singleton in $H'\backslash G/H$, or equivalently, to solve certain branching problems for generalized Verma modules (see Section 7.3 below) via the duality (7.3),
- to construct and classify $\{T \in H(\lambda, \nu) : \operatorname{Supp} T \subset \overline{S}\}$ modulo $\{T \in H(\lambda, \nu) : \operatorname{Supp} T \subset \partial S\}$ for $S \in G'\backslash G/H$ inductively.

The "F-method" [38, 40, 47, 51] gives a conceptual and a practical tool to construct differential symmetry breaking operators in Step 1. The second step may involve analytic questions such as the possibility of an extension of an H'-invariant distribution on an H'-invariant subset of G/H satisfying a differential equation to an H'-invariant distribution solution on the whole of G/H (e.g., [52, Chapter 11, Sect. 4]), and an analytic continuation and residue calculus with respect to some natural parameter (e.g., [52, Chapters 8 and 12]).

We expect that the methods developed in [52] for the classification of symmetry breaking operators for the pair $(G, G') = (\mathrm{O}(n+1, 1), \mathrm{O}(n, 1))$ would work for some other pairs (G, G') such as those satisfying (PP) (see Theorem 6.14 for the list), or more strongly those satisfying (BB) (see Theorem 6.15 for the list).

7.3 Finiteness criterion for differential symmetry breaking operators

As we have seen in Theorem 7.7 and Proposition 7.8, it is a considerably strong restriction on the G'-manifold Y for the space $\operatorname{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$ of symmetry breaking operators to be finite-dimensional, which would be a substantial condition for further study in Stages B and C of the branching problem. On the other hand, if we consider only *differential* symmetry breaking operators, then it turns out that there are much broader settings for which the finite-multiplicity property (or even the multiplicity-free property) holds. The aim of this subsection is to formulate this property.

In order to be precise, we write $\operatorname{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$ for the space of continuous symmetry breaking operators, and $\operatorname{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$ for that of differential symmetry breaking operators. Clearly we have

$$\operatorname{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) \subset \operatorname{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})). \tag{7.2}$$

We now consider the problem analogous to Problem 7.3 by replacing the right-hand side of (7.2) with the left-hand side.

For simplicity, we consider the case where $\mathcal{V} \to X$ is a G-equivariant line bundle over a real flag manifold G/P, and write $\mathcal{L}_\lambda \to X$ for the line bundle associated to a one-dimensional representation λ of P. We use the same letter λ to denote the corresponding infinitesimal representation of the Lie algebra $\mathfrak{p}$, and write $\lambda \gg 0$ if $\langle \lambda|_{\mathfrak{j}}, \alpha \rangle \gg 0$ for all $\alpha \in \Delta(\mathfrak{n}^+, \mathfrak{j})$ where $\mathfrak{j}$ is a Cartan subalgebra contained in the Levi part $\mathfrak{l}$ of the parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$.

We say a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $\mathfrak{g}'$-*compatible* if $\mathfrak{p}$ is defined as the sum of eigenspaces with nonnegative eigenvalues for some hyperbolic element in $\mathfrak{g}'$. Then $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{g}'$ is a parabolic subalgebra of $\mathfrak{g}'$ and we have compatible Levi decompositions $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ and $\mathfrak{p}' = (\mathfrak{l} \cap \mathfrak{g}') + (\mathfrak{n}^+ \cap \mathfrak{g}')$. We are ready to state an answer to a question analogous to Problem 7.3 (1) and (2) for *differential* symmetry breaking operators (cf. [40]).

Theorem 7.13 (local operators). *Let G' be a reductive subgroup of a real reductive linear Lie group G, $X = G/P$ and $Y = G'/P'$ where P is a parabolic subgroup of G and $P' = P \cap G'$ such that the parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ of $\mathfrak{g}$ is $\mathfrak{g}'$-compatible.*
(1) (finite multiplicity) *For any finite-dimensional representations V and W of the parabolic subgroups P and P', respectively, we have*

$$\dim_{\mathbb{C}} \mathrm{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) < \infty,$$

where $\mathcal{V} = G \times_P V$ and $\mathcal{W} = G' \times_{P'} W$ are equivariant vector bundles over X and Y, respectively.
(2) (uniformly bounded multiplicity) *If $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair and $\mathfrak{n}^+$ is abelian, then for any finite-dimensional representation V of P,*

$$C_V := \sup_W \dim_{\mathbb{C}} \mathrm{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) < \infty.$$

Here W runs over all finite-dimensional irreducible representations of P'. Furthermore, $C_V = 1$ if V is a one-dimensional representation λ of P with $\lambda \gg 0$.

Proof. The classical duality between Verma modules and principal series representations in the case $G = G'$ (e.g., [17]) can be extended to the context of the restriction of reductive groups $G \downarrow G'$, and the following bijection holds (see [51, Part I, Corollary 2.9]):

$$\mathrm{Hom}_{(\mathfrak{g}', P')}(U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} W^\vee, U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee) \simeq \mathrm{Diff}_{G'}(C^\infty(G/P, \mathcal{V}), C^\infty(G'/P', \mathcal{W})). \quad (7.3)$$

Here $(\lambda^\vee, V^\vee)$ denotes the contragredient representation of (λ, V). The right-hand side of (7.3) concerns Case II (symmetry breaking) in Section 4, whereas the left-hand side of (7.3) concerns Case I (embedding) in the BGG category $\mathcal{O}$. An analogous theory of discretely decomposable restriction in the Harish-Chandra category $\mathcal{HC}$ (see Sections 4 and 5) can be developed more easily and explicitly in the BGG category $\mathcal{O}$, which was done in [37]. In particular, the $\mathfrak{g}'$-compatibility is a sufficient condition for the "discrete decomposability" of generalized Verma modules

$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F$ when restricted to the reductive subalgebra $\mathfrak{g}'$. Thus the proof of Theorem 7.13 is reduced to the next proposition.

Proposition 7.14. *Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$. Suppose that a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ is $\mathfrak{g}'$-compatible.*
(1) *For any finite-dimensional $\mathfrak{p}$-module F and $\mathfrak{p}'$-module F',*

$$\dim \operatorname{Hom}_{\mathfrak{g}'}(U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} F', U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F) < \infty.$$

(2) *If $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair and $\mathfrak{n}^+$ is abelian, then*

$$\sup_{F'} \dim \operatorname{Hom}_{\mathfrak{g}'}(U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} F', U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda) = 1$$

for any one-dimensional representation λ of $\mathfrak{p}$ with $\lambda \ll 0$. Here the supremum is taken over all finite-dimensional simple $\mathfrak{p}'$-modules F'.

Proof. (1) The proof is parallel to [37, Theorem 3.10] which treated the case where F and F' are simple modules of P and P', respectively.
(2) See [37, Theorem 5.1]. □

Hence Theorem 7.13 is proved. □

Remark 7.15. If we drop the assumption $\lambda \gg 0$ in Theorem 7.13 (2) or $\lambda \ll 0$ in Proposition 7.14 (2), then the multiplicity-free statement may fail. In fact, the computation in Section 2.1 gives a counterexample where $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{sl}(2, \mathbb{C}) + \mathfrak{sl}(2, \mathbb{C}), \operatorname{diag}(\mathfrak{sl}(2, \mathbb{C})))$; see Remark 2.6 (3).

Remark 7.16. (1) (Stage B) In the setting of Proposition 7.14 (2), Stage B in the branching problem (finding explicit branching laws) have been studied in [35, 37] in the BGG category $\mathcal{O}$ generalizing earlier results by Kostant and Schmid [65].
(2) (Stage C) In the setting of Theorem 7.13 (2), one may wish to find an explicit formula for the unique *differential* symmetry breaking operators. So far, this has been done only in some special cases; see [9, 11] for the Rankin–Cohen bidifferential operator, Juhl [21] in connection with conformal geometry, and [47, 51] using the Fourier transform ("F-method" in [38]).

We end this subsection by applying Theorem 7.13 and Theorem 6.12 to the reductive symmetric pair $(G, G') = (\mathrm{GL}(n_1 + n_2, \mathbb{R}), \mathrm{GL}(n_1, \mathbb{R}) \times \mathrm{GL}(n_2, \mathbb{R}))$, and observe a sharp contrast between differential and continuous symmetry breaking operators, i.e., the left-hand and right-hand sides of (7.2), respectively.

Example 7.17. Let $n = n_1 + n_2$ with $n_1, n_2 \geq 2$. Let P, P' be minimal parabolic subgroups of

$$(G, G') = (\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n_1, \mathbb{R}) \times \mathrm{GL}(n_2, \mathbb{R})),$$

respectively, and set $X = G/P$ and $Y = G'/P'$. Then:

(1) For all finite-dimensional representations V of P and W of P',

$$\dim_{\mathbb{C}} \mathrm{Diff}_{G'}(\mathrm{Ind}_P^G(V)^\infty, \mathrm{Ind}_{P'}^{G'}(W)^\infty) < \infty.$$

Furthermore if V is a one-dimensional representation $\mathbb{C}_\lambda$ with $\lambda \gg 0$ in the notation of Theorem 7.13, then the above dimension is 0 or 1.

(2) For some finite-dimensional representations V of P and W of P',

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(V)^\infty, \mathrm{Ind}_{P'}^{G'}(W)^\infty) = \infty.$$

7.4 Localness theorem in the holomorphic setting

In the last example (Example 7.17) and also Theorem 2.9 in Section 2.2, we have seen in the real setting that differential symmetry breaking operators are "very special" among continuous symmetry breaking operators. In this subsection we explain the remarkable phenomenon in the holomorphic framework that any continuous symmetry breaking operator between two representations under certain special geometric settings is given by a differential operator; see Observation 2.5 (1) for the $SL(2,\mathbb{R})$ case. A general case is formulated in Theorem 7.18 below. The key idea of the proof is to use the theory of discretely decomposable restrictions [26, 28, 29], briefly explained in Section 5. A conjectural statement is given in the next subsection.

Let $G \supset G'$ be real reductive linear Lie groups, $K \supset K'$ their maximal compact subgroups, and $G_{\mathbb{C}} \supset G'_{\mathbb{C}}$ connected complex reductive Lie groups containing $G \supset G'$ as real forms, respectively. The main assumption of this subsection is that $X := G/K$ and $Y := G'/K'$ are Hermitian symmetric spaces. To be more precise, let $Q_{\mathbb{C}}$ and $Q'_{\mathbb{C}}$ be parabolic subgroups of $G_{\mathbb{C}}$ and $G'_{\mathbb{C}}$ with Levi subgroups $K_{\mathbb{C}}$ and $K'_{\mathbb{C}}$, respectively, such that the following commutative diagram consists of holomorphic maps:

$$\begin{array}{ccccc} Y = & G'/K' & \subset X = & G/K & \\ \text{Borel embedding} & \cap & & \cap & \text{Borel embedding} \\ & G'_{\mathbb{C}}/Q'_{\mathbb{C}} & \subset & G_{\mathbb{C}}/Q_{\mathbb{C}}. & \end{array} \tag{7.4}$$

Theorem 7.18 ([51, Part I]). *Let $\mathcal{V} \to X$, $\mathcal{W} \to Y$ be G-equivariant, G'-equivariant holomorphic vector bundles, respectively.*

(1) (localness theorem) *Any G'-homomorphism from $\mathcal{O}(X,\mathcal{V})$ to $\mathcal{O}(Y,\mathcal{W})$ is given by a holomorphic differential operator, in the sense of Definition 7.1, with respect to a holomorphic embedding $Y \hookrightarrow X$.*

We extend $\mathcal{V}$ and $\mathcal{W}$ to holomorphic vector bundles over $G_{\mathbb{C}}/Q_{\mathbb{C}}$ and $G'_{\mathbb{C}}/Q'_{\mathbb{C}}$, respectively.

(2) (extension theorem) *Any differential symmetry breaking operator in (1) defined on Hermitian symmetric spaces extends to a $G'_{\mathbb{C}}$-equivariant holomorphic differential operator $\mathcal{O}(G_{\mathbb{C}}/Q_{\mathbb{C}}, \mathcal{V}) \to \mathcal{O}(G'_{\mathbb{C}}/Q'_{\mathbb{C}}, \mathcal{W})$ with respect to a holomorphic map between the flag varieties $G'_{\mathbb{C}}/Q'_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}/Q_{\mathbb{C}}$.*

Remark 7.19. The representation π on the Fréchet space $\mathcal{O}(G/K, \mathcal{V})$ is a maximal globalization of the underlying $(\mathfrak{g}, K)$-module π_K in the sense of Schmid [66], and contains some other globalizations having the same underlying $(\mathfrak{g}, K)$-module π_K (e.g., the Casselman–Wallach globalization π^∞). One may ask whether an analogous statement holds if we replace $(\pi, \mathcal{O}(G/K, \mathcal{V}))$ and $(\tau, \mathcal{O}(G'/K', \mathcal{W}))$ by other globalizations such as π^∞ and τ^∞. This question was raised by D. Vogan during the conference at MIT in May 2014. We gave an affirmative answer in [51, Part I] by proving that the natural inclusions

$$\mathrm{Hom}_{G'}(\pi, \tau) \subset \mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty) \subset \mathrm{Hom}_{\mathfrak{g}', K'}(\pi_K, \tau_{K'})$$

are actually bijective in our setting.

7.5 Localness conjecture for symmetry breaking operators on cohomologies

It might be natural to ask a generalization of Theorem 7.18 to some other holomorphic settings, from holomorphic sections to Dolbeault cohomologies, and from highest weight modules to $A_{\mathfrak{q}}(\lambda)$ modules.

Problem 7.20. To what extent does the localness and extension theorem hold for symmetry breaking operators between Dolbeault cohomologies?

In order to formulate the problem more precisely, we introduce the following assumption on the pair (G, G') of real reductive groups:

$$K \text{ has a normal subgroup of positive dimension which is contained in } K'. \quad (7.5)$$

Here, K and $K' = K \cap G'$ are maximal compact subgroups of G and G', respectively, as usual. We write $K^{(2)}$ for the normal subgroup in (7.5), $\mathfrak{k}_0^{(2)}$ for the corresponding Lie algebra, and $\mathfrak{k}^{(2)}$ for its complexification. Then the assumption (7.5) means that we have direct sum decompositions

$$\mathfrak{k} = \mathfrak{k}^{(1)} \oplus \mathfrak{k}^{(2)}, \qquad \mathfrak{k}' = \mathfrak{k}'^{(1)} \oplus \mathfrak{k}^{(2)}$$

for some ideals $\mathfrak{k}^{(1)}$ of $\mathfrak{k}$ and $\mathfrak{k}'^{(1)}$ of $\mathfrak{k}'$, respectively. The point here is that $\mathfrak{k}^{(2)}$ is common to both $\mathfrak{k}$ and $\mathfrak{k}'$.

We take $H \in \sqrt{-1}\mathfrak{k}_0^{(2)}$, define a θ-stable parabolic subalgebra of $\mathfrak{g}$ by

$$\mathfrak{q} \equiv \mathfrak{q}(H) = \mathfrak{l} + \mathfrak{u}$$

as the sum of eigenspaces of $\mathrm{ad}(H)$ with nonnegative eigenvalues, and set $L := G \cap Q_{\mathbb{C}}$ where $Q_{\mathbb{C}} = N_{G_{\mathbb{C}}}(\mathfrak{q})$ is the parabolic subgroup of $G_{\mathbb{C}}$. Then L is a reductive subgroup of G with complexified Lie algebra $\mathfrak{l}$, and we have an open embedding $X := G/L \subset G_{\mathbb{C}}/Q_{\mathbb{C}}$ through which G/L carries a complex structure. The same element H defines complex manifolds $Y := G'/L' \subset G'_{\mathbb{C}}/Q'_{\mathbb{C}}$ with the obvious notation.

In summary, we have the following geometry that generalizes (7.4):

$$\begin{array}{ccccc} Y = & G'/L' & \subset X = & G/L & \\ \text{open} & \cap & & \cap & \text{open} \\ & G'_{\mathbb{C}}/Q'_{\mathbb{C}} & \subset & G_{\mathbb{C}}/Q_{\mathbb{C}}. & \end{array}$$

It follows from the assumption (7.5) that the compact manifold $K/L \cap K$ coincides with $K'/L' \cap K'$. Let S denote the complex dimension of the complex compact manifolds $K/L \cap K \simeq K'/L' \cap K'$.

Example 7.21. (1) (Hermitian symmetric spaces) Suppose that $K^{(2)}$ is abelian. Then $Y \subset X$ are Hermitian symmetric spaces, $S = 0$, and we obtain the geometric setting of Theorem 7.18.

(2) $(G, G') = (\mathrm{U}(p, q; \mathbb{F}), \mathrm{U}(p'; \mathbb{F}) \times \mathrm{U}(p'', q; \mathbb{F}))$ with $p = p' + p''$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and $K^{(2)} = \mathrm{U}(q; \mathbb{F})$. Then neither G/L nor G'/L' is a Hermitian symmetric space but the assumption (7.5) is satisfied. Thus the conjecture below applies.

For a finite-dimensional holomorphic representation V of $Q_{\mathbb{C}}$, we define a holomorphic vector bundle $G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} V$ over the generalized flag variety $G_{\mathbb{C}}/Q_{\mathbb{C}}$, and write $\mathcal{V} := G \times_L V$ for the G-equivariant holomorphic vector bundle over $X = G/L$ as the restriction $(G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} V)|_{G/L}$. Then the Dolbeault cohomology $H^j_{\bar{\partial}}(X, \mathcal{V})$ naturally carries a Fréchet topology by the closed range theorem of the $\bar{\partial}$-operator, and gives the maximal globalization of the underlying $(\mathfrak{g}, K)$-modules, which are isomorphic to Zuckerman's derived functor modules $\mathcal{R}^j_{\mathfrak{q}}(V \otimes \mathbb{C}_{-\rho})$ [69, 75]. Similarly for G', given a finite-dimensional holomorphic representation W of $Q'_{\mathbb{C}}$, we form a G'-equivariant holomorphic vector bundle $\mathcal{W} := G' \times_{L'} W$ over $Y = G'/L'$ and define a continuous representation of G' on the Dolbeault cohomologies $H^j_{\bar{\partial}}(Y, \mathcal{W})$. In this setting we have the discrete decomposability of the restriction by the general criterion (see Fact 5.5).

Proposition 7.22. *The underlying $(\mathfrak{g}, K)$-modules $H^j_{\bar{\partial}}(X, \mathcal{V})_K$ are K'-admissible. In particular, they are discretely decomposable as $(\mathfrak{g}', K')$-modules.*

Explicit branching laws in some special cases (in particular, when $\dim V = 1$) of Example 7.21 (1) and (2) may be found in [35] and [15, 25], respectively.

We are now ready to formulate a possible extension of the localness and extension theorem for holomorphic functions (Theorem 7.18) to Dolbeault cohomologies that gives geometric realizations of Zuckerman's derived functor modules.

Conjecture 7.23. *Suppose we are in the above setting, and let V and W be finite-dimensional representations of $Q_{\mathbb{C}}$ and $Q'_{\mathbb{C}}$, respectively.*

(1) (localness theorem) *Any continuous G'-homomorphism*

$$H^S_{\bar{\partial}}(X,\mathcal{V}) \to H^S_{\bar{\partial}}(Y,\mathcal{W})$$

is given by a holomorphic differential operator with respect to a holomorphic embedding $Y \hookrightarrow X$.

(2) (extension theorem) *Any such operator in (1) defined on the open subsets $Y \subset X$ of $G'_{\mathbb{C}}/Q'_{\mathbb{C}} \subset G_{\mathbb{C}}/Q_{\mathbb{C}}$, respectively, extends to a $G'_{\mathbb{C}}$-equivariant holomorhic differential operator with respect to a holomorphic map between the flag varieties $G'_{\mathbb{C}}/Q'_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}/Q_{\mathbb{C}}$.*

The key ingredient of the proof of Theorem 7.18 for Hermitian symmetric spaces was the discrete decomposability of the restriction of the representation (Fact 2.2 (2)). Proposition 7.22 is a part of the evidence for Conjecture 7.23 in the general setting.

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References

1. J. Adams, *Unitary highest weight modules*, Adv. in Math. **63** (1987), 113–137.
2. A. Aizenbud and D. Gourevitch, *Multiplicity one theorem for* $(GL_{n+1}(\mathbb{R}), GL_n(\mathbb{R}))$, Selecta Math. **15** (2009), 271–294.
3. M. Berger, *Les espaces symétriques non compacts*, Ann. Sci. École Norm. Sup. **74** (1957), 85–177.
4. I. N. Bernstein and S. I. Gelfand, *Meromorphic property of the functions* P^λ, Funktsional Anal, i Prilozhen., **3** (1969), 84–85.
5. F. Bien, *Orbit, multiplicities, and differential operators*, Contemp. Math. **145** Amer. Math. Soc., 1993, 199–227.
6. M. Brion, *Classification des espaces homogènes sphériques*, Compos. Math. **63** (1986), 189–208.
7. R. Brylinski, *Geometric quantization of real minimal nilpotent orbits*, Differential Geom. Appl. **9** (1998), 5–58.
8. J.-L. Clerc, T. Kobayashi, B. Ørsted, and M. Pevzner, *Generalized Bernstein–Reznikov integrals*, Math. Ann., **349** (2011), 395–431.
9. H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*, Math. Ann., **217** (1975), 271–285.
10. P. Delorme, *Formule de Plancherel pour les espaces symétriques réductifs*, Ann. of Math. (2), **147** (1998), 417–452.

11. G. van Dijk and M. Pevzner, *Ring structures for holomorphic discrete series and Rankin-Cohen brackets*, J. Lie Theory, **17**, (2007), 283–305.
12. M. Duflo and J. A. Vargas, *Branching laws for square integrable representations*, Proc. Japan Acad. Ser. A, Math. Sci. **86** (2010), 49–54.
13. T. Enright and J. Willenbring, *Hilbert series, Howe duality and branching for classical groups*, Ann. of Math. (2) **159** (2004), 337–375.
14. B. Gross and D. Prasad, *On the decomposition of a representations of SO_n when restricted to SO_{n-1}*, Canad. J. Math. **44** (1992), 974–1002.
15. B. Gross and N. Wallach, *Restriction of small discrete series representations to symmetric subgroups*, Proc. Sympos. Pure Math., **68** (2000), Amer. Math. Soc., 255–272.
16. Harish-Chandra, *Representations of semisimple Lie groups on a Banach space*, Proc. Nat. Acad. Sci. U. S. A. **37** (1951), 170–173.
17. M. Harris and H. P. Jakobsen, *Singular holomorphic representations and singular modular forms*, Math. Ann., **259** (1982), 227–244.
18. R. Howe, *θ-series and invariant theory*, Proc. Symp. Pure Math. **33** (1979), Amer. Math. Soc., 275–285.
19. R. Howe, *Reciprocity laws in the theory of dual pairs*, Progr. Math. Birkhäuser, **40**, (1983), 159–175.
20. R. Howe and C. Moore, *Asymptotic properties of unitary representations*. J. Funct. Anal. **32**, (1979), 72–96.
21. A. Juhl, *Families of conformally covariant differential operators, Q-curvature and holography.* Progr. Math., **275**. Birkhäuser, 2009.
22. M. Kashiwara and M. Vergne, *K-types and singular spectrum*, In: Lect. Notes in Math. **728**, 1979, Springer-Verlag, 177–200.
23. B. Kimelfeld, *Homogeneous domains in flag manifolds of rank 1*, J. Math. Anal. & Appl. **121**, (1987), 506–588.
24. A. W. Knapp and D. Vogan, Jr., *Cohomological Induction and Unitary Representations*, Princeton U.P., 1995.
25. T. Kobayashi, *The restriction of $A_{\mathfrak{q}}(\lambda)$ to reductive subgroups*, Proc. Japan Acad., **69** (1993), 262–267.
26. T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications*, Invent. Math., **117**, (1994), 181–205.
27. T. Kobayashi, *Introduction to harmonic analysis on real spherical homogeneous spaces*, Proceedings of the 3rd Summer School on Number Theory "Homogeneous Spaces and Automorphic Forms" in Nagano (F. Sato, ed.), 1995, 22–41 (in Japanese).
28. T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups* II – *micro-local analysis and asymptotic K-support*, Ann. of Math., **147**, (1998), 709–729.
29. T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups* III – *restriction of Harish-Chandra modules and associated varieties*, Invent. Math., **131**, (1998), 229–256.
30. T. Kobayashi, *Harmonic analysis on homogeneous manifolds of reductive type and unitary representation theory, Translations, Series II*, Selected Papers on Harmonic Analysis, Groups, and Invariants (K. Nomizu, ed.), **183**, (1998), Amer. Math. Soc., 1–31.
31. T. Kobayashi, *Discrete series representations for the orbit spaces arising from two involutions of real reductive Lie groups*, J. Funct. Anal., **152**, (1998), 100–135.
32. T. Kobayashi, *Discretely decomposable restrictions of unitary representations of reductive Lie groups —examples and conjectures*, Advanced Study in Pure Math., **26**, (2000), 98–126.
33. T. Kobayashi, *Restrictions of unitary representations of real reductive groups*, Progr. Math. **229**, pages 139–207, Birkhäuser, 2005.
34. T. Kobayashi, *Multiplicity-free representations and visible actions on complex manifolds*, Publ. Res. Inst. Math. Sci. **41**, (2005), 497–549.
35. T. Kobayashi, *Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs*, 45–109, Progr. Math., **255**, Birkhäuser, Boston, 2008.

36. T. Kobayashi, *Branching problems of Zuckerman derived functor modules*, In: *Representation Theory and Mathematical Physics (in honor of Gregg Zuckerman)*, Contemporary Mathematics, **557**, 23–40. Amer. Math. Soc., Providence, RI, 2011.
37. T. Kobayashi, *Restrictions of generalized Verma modules to symmetric pairs*, Transform. Group, **17**, (2012), 523–546.
38. T. Kobayashi, *F-method for constructing equivariant differential operators*, Geometric Analysis and Integral Geometry Contemporary Mathematics (in honor of S. Helgason), **598**, Amer. Math. Soc., 2013, 141–148.
39. T. Kobayashi, *Propagation of multiplicity-freeness property for holomorphic vector bundles*, Progr. Math. **306**, Birkhäuser, 2013, 113–140.
40. T. Kobayashi *F-method for symmetry breaking operators*, Differential Geom. Appl. **33** (2014), 272–289, Special issue in honor of M. Eastwood.
41. T. Kobayashi, *Shintani functions, real spherical manifolds, and symmetry breaking operators*, Developments in Mathematics, **37** (2014), 127–159.
42. T. Kobayashi, T. Kubo, and M. Pevzner, *Vector-valued covariant differential operators for the Möbius transformation*, Springer Proceedings in Mathematics $\&$ Statistics, **111**, 67–86, 2015.
43. T. Kobayashi and G. Mano, *The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p,q)$*, Mem. Amer. Math. Soc. (2011), **212**, no. 1000 , vi+132 pages.
44. T. Kobayashi and T. Matsuki, *Classification of finite-multiplicity symmetric pairs*, Transform. Groups **19** (2014), 457–493, Special issue in honor of Dynkin for his 90th birthday.
45. T. Kobayashi and B. Ørsted, *Analysis on the minimal representation of* $\mathrm{O}(p,q)$. Part I, Adv. Math., **180**, (2003) 486–512; Part II, *ibid*, 513–550; Part III, *ibid*, 551–595.
46. T. Kobayashi, B. Ørsted, and M. Pevzner, *Geometric analysis on small unitary representations of $GL(n,\mathbb{R})$*, J. Funct. Anal., **260**, (2011) 1682–1720.
47. T. Kobayashi, B. Ørsted, P. Somberg, and V. Souček, *Branching laws for Verma modules and applications in parabolic geometry*, Part I, Adv. Math., **285**, (2015), 1796-1852.
48. T. Kobayashi and T. Oshima, *Finite multiplicity theorems for induction and restriction*, Adv. Math., **248**, (2013), 921–944.
49. T. Kobayashi and Y. Oshima, *Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs*, Adv. Math., **231** (2012), 2013–2047.
50. T. Kobayashi and Y. Oshima, *Classification of symmetric pairs with discretely decomposable restrictions of $(\mathfrak{g},K)$-modules*, Journal für die reine und angewandte Mathematik, **2015**, (2015), no.703, 201–223.
51. T. Kobayashi and M. Pevzner, *Differential symmetry breaking operators.* I. *General theory and F-method*, to appear in Selecta Math. 44 pages; II. *Rankin–Cohen operators for symmetric pairs*, to appear in Selecta Math. 65 pages, arXiv:1301.2111.
52. T. Kobayashi and B. Speh, *Symmetry breaking for representations of rank one orthogonal groups*, (2015), Memoirs of Amer. Math. Soc. **238**, no.1126, 118 pages.
53. M. Krämer, *Multiplicity free subgroups of compact connected Lie groups*, Arch. Math. (Basel) **27**, (1976), 28–36.
54. T. Matsuki, *Orbits on flag manifolds*, Proceedings of the International Congress of Mathematicians, Kyoto 1990, Vol. II (1991), Springer-Verlag, 807–813.
55. T. Matsuki and T. Oshima *A description of discrete series for semisimple symmetric spaces*, Adv. Stud. Pure Math. **4**, (1984), 331–390.
56. I. V. Mikityuk, *Integrability of invariant Hamiltonian systems with homogeneous configuration spaces*, Math. USSR-Sbornik **57**, (1987), 527–546.
57. V. F. Molchanov, *Tensor products of unitary representations of the three-dimensional Lorentz group*, Math. USSR, Izv. **15**, (1980), 113–143.
58. J. Möllers and B. Ørsted, *Estimates for the restriction of automorphic forms on hyperbolic manifolds to compact geodesic cycles*, to appear in Int. Math. Res. Not. IMRN.
59. B. Ørsted and B. Speh, *Branching laws for some unitary representations of $SL(4,\mathbb{R})$*, SIGMA **4**, (2008), doi:10.3842/SIGMA.2008.017.
60. Y. Oshima, *Discrete branching laws of Zuckerman's derived functor modules*, Ph.D. thesis, the University of Tokyo, 2013.

61. J. Peetre, *Une caractérisation abstraite des opérateurs différentiels*, Math. Scand., **7**, (1959), 211–218.
62. R. A. Rankin, *The construction of automorphic forms from the derivatives of a given form*, J. Indian Math. Soc., **20**, (1956), 103–116.
63. J. Repka, *Tensor products of holomorphic discrete series representations*, Can. J. Math., **31**, (1979), 836–844.
64. R. Richardson, G. Röhrle, and R. Steinberg, *Parabolic subgroup with abelian unipotent radical*, Invent. Math., **110** (1992), 649–671.
65. W. Schmid, *Die Randwerte holomorphe Funktionen auf hermetisch symmetrischen Raumen*, Invent. Math., **9**, (1969–70), 61–80.
66. W. Schmid, *Boundary value problems for group invariant differential equations*, Astérisque (1985), 311–321.
67. J. Sekiguchi, *Remarks on real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39**, (1987), 127–138.
68. B. Sun and C.-B. Zhu, *Multiplicity one theorems: the Archimedean case*, Ann. of Math. **175**, (2012), 23–44.
69. D. A. Vogan, Jr., *Representations of Real Reductive Lie Groups*, Progr. Math., **15**, Birkhäuser, 1981.
70. D. A. Vogan, Jr., *Unitarizability of certain series of representations*, Ann. of Math. **120**, (1984), 141–187.
71. D. A. Vogan, Jr., *Irreducibility of discrete series representations for semisimple symmetric spaces*, Adv. Stud. Pure Math. **14**, (1988), 191–221.
72. D. A. Vogan, Jr., *Associated varieties and unipotent representations*, Progr. Math. **101**, (1991), Birkhäuser, 315–388.
73. D. A. Vogan, Jr. and G. J. Zuckerman, *Unitary representations with nonzero cohomology*, Compositio Math. **53** (1984), 51–90.
74. N. R. Wallach, *Real reductive groups*. I, II, Pure and Applied Mathematics, **132**, Academic Press, Inc., Boston, MA, 1988.
75. H. Wong, *Dolbeault cohomological realization of Zuckerman modules associated with finite rank representations*, J. Funct. Anal. **129**, (1995), 428–454.
76. D. Zagier, *Modular forms and differential operators*, Proc. Indian Acad. Sci. (Math. Sci.) **104**, (1994), 57–75.
77. F. Zhu and K. Liang, *On a branching law of unitary representations and a conjecture of Kobayashi*, C. R. Acad. Sci. Paris, Ser. I, **348**, (2010), 959–962.