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Representation Theory and Analysis of Reductive Groups: Spherical Spaces and Hecke Algebras

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Analysis on real spherical manifolds and their applications to Shintani functions and symmetry breaking operators

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A complex manifold $X_{\mathbb{C}}$ with action of a complex reductive group $G_{\mathbb{C}}$ is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. In the real setting, in search of a good framework for global analysis on homogeneous spaces which are broader than the usual (*e.g.* symmetric spaces), we proposed to call:

Definition 1 ([K95, §2])**.** Let G be a real reductive Lie group. We say a smooth manifold X with G-action is *real spherical* if a minimal parabolic subgroup P of G has an open orbit in X.

The significance of this geometric property is the finite-multiplicity property in the regular representation of G on $C^\infty(X)$, which we discovered and proved by using the theory of hyperfunctions and regular singularities of a system of partial differential equations:

Fact 2 ([KO13, Theorems A and B])**.** *Let G be a real reductive linear Lie group, and H an algebraic subgroup. For an algebraic representation W of H, we form a G-equivariant vector bundle $\mathcal{W} := G \times_H W$ on G/H.*

1) *The following two conditions on the pair (G, H) are equivalent:*
   (i) *The homogeneous space G/H is real spherical.*
   (ii) $\dim \operatorname{Hom}_G(\pi^\infty, C^\infty(G/H, \mathcal{W}))\,(= \dim(\pi^{-\infty} \otimes W)^H) < \infty$ *for any smooth admissible representation π^∞ of G and for any algebraic representation W of H.*
2) *The following two conditions on the pair (G, H) are equivalent:*
   (i) *The complexification $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical.*
   (ii) *There exists a constant $C > 0$ such that*

$$\dim \operatorname{Hom}_G(\pi^\infty, C^\infty(G/H, \mathcal{W}))\,(= \dim(\pi^{-\infty} \otimes W)^H) \leq C \dim W$$

   for any smooth irreducible admissible representation π^∞ of G and any algebraic representation W of H.

More precisely, a quantitative estimate for upper and lower bounds of the dimension was also given in [KO13].

Instead of smooth sections, we may also consider other function spaces such as the Hilbert space of square integrable functions. An earlier work for the construction of discrete series representations for some non-symmetric spherical homogeneous spaces can be found in [Ko94].

The primary purpose of the talk was to explain some application of Fact 2 to the relationship among Shintani functions, branching problems, and real spherical varieties. For this, we fix some terminologies. Denote by $\mathfrak{g}$ the Lie algebra of G, and by $U(\mathfrak{g}_{\mathbb{C}})$ the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. For $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, we set

$$(L_X f)(g) := \frac{d}{dt}|_{t=0} f(\exp(-tX)g), \quad (R_X f)(g) := \frac{d}{dt}|_{t=0} f(g \exp(tX)),$$

and extend these actions to those of $U(\mathfrak{g}_{\mathbb{C}})$. We denote by $\mathfrak{Z}_G$ the $\mathbb{C}$-algebra of G-invariant elements in $U(\mathfrak{g}_{\mathbb{C}})$. Let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}$. Then any $\lambda \in \mathfrak{j}_{\mathbb{C}}^\vee$ gives rise to a $\mathbb{C}$-algebra homomorphism $\chi_\lambda : \mathfrak{Z}_G \to \mathbb{C}$ via the Harish-Chandra isomorphism $\mathfrak{Z}_G \xrightarrow{\sim} S(\mathfrak{j}_{\mathbb{C}})^{W_G}$. The finite group W_G is the Weyl group of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ if G is connected.

Suppose that G' is an algebraic reductive subgroup. Analogous notation will be applied to G', *e.g.*, $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_{G'}, \mathbb{C}) \simeq (\mathfrak{j}'_{\mathbb{C}})^{\vee}/W_{G'}$, $\chi_\nu \leftrightarrow \nu$, where $\mathfrak{j}'$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}'$ of G'.

We take a maximal compact subgroup K of G such that $K' := K \cap G'$ is a maximal compact subgroup. Following Murase–Sugano [MS96], we call:

Definition 3 (Shintani function)**.** We say $f \in C^\infty(G)$ is a *Shintani function* of $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters (λ, ν) if f satisfies the following three properties:

(1) $f(k'gk) = f(g)$ for any $k' \in K'$, $k \in K$.
(2) $R_u f = \chi_\lambda(u) f$ for any $u \in \mathfrak{Z}_G$.
(3) $L_v f = \chi_\nu(v) f$ for any $v \in \mathfrak{Z}_{G'}$.

We denote by $\mathrm{Sh}(\lambda, \nu)$ the space of Shintani functions of type (λ, ν).

For $G = G'$ and $\lambda = -\nu \mod W_G$, Shintani functions are nothing but Harish-Chandra's zonal spherical functions.

The following two theorems (Theorems A and B) go back to [K95], and the proof was given in [K14] (and partly in [KO13]).

Theorem A. *The following four conditions on a pair of real reductive algebraic groups $G \supset G'$ are equivalent:*

(i) (Shintani function) $\mathrm{Sh}(\lambda, \nu)$ *is finite-dimensional for any pair* (λ, ν) *of* $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$*-infinitesimal characters.*
(ii) (Symmetry breaking) $\mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty)$ *is finite-dimensional for any pair* $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of G and G'.*
(iii) (Invariant bilinear form) $\mathrm{Hom}_{G'}(\pi^\infty \widehat{\otimes} \tau^\infty, \mathbb{C})$ *is finite-dimensional for any pair* $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of G and G'.*
(iv) (Geometric property (PP)) *The homogeneous space* $(G' \times G')/\Delta G'$ *is real spherical.*

The last geometric condition (PP) may be restated as follows. Let P' be a minimal parabolic subgroup of G'.

Definition 4. We say the pair (G, G') satisfies (PP) if one of the following five equivalent conditions are satisfied.

(PP1) $(G \times G')/\operatorname{diag} G'$ is real spherical as a $(G \times G')$-space.
(PP2) G/P' is real spherical as a G-space.
(PP3) G/P is real spherical as a G'-space.
(PP4) G has an open orbit in $G/P \times G/P'$ via the diagonal action.
(PP5) There are finitely many G-orbits in $G/P \times G/P'$ via the diagonal action.

The dimension of the Shintani space $\mathrm{Sh}(\lambda, \nu)$ depends on λ and ν in general. We give a characterization of the uniform boundedness property:

Theorem B. *The following four conditions on a pair of real reductive algebraic groups $G \supset G'$ are equivalent:*

(i) (Shintani function) *There exists a constant C such that* $\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu) \le C$ *for any pair* (λ, ν) *of* $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$*-infinitesimal characters.*
(ii) (Symmetry breaking) *There exists a constant C such that* $\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty) \le C$ *for any pair* $(\pi^\infty, \tau^\infty)$ *of irreducible admissible smooth representations of G and G'.*

(iii) (Invariant bilinear form) *There exists a constant C such that* $\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty \widehat{\otimes} \tau^\infty, \mathbb{C}) \leq C$ *for any pair* $(\pi^\infty, \tau^\infty)$ *of irreducible admissible smooth representations of* G *and* G'.

(iv) (Geometric property (BB)) *The homogeneous space* $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\Delta G'_{\mathbb{C}}$ *is spherical.*

Theorems A and B hold for general pairs of real reductive groups (G, G'). Among others, typical examples are obtained in the case where (G, G') are symmetric pairs.

Example 5 ([K95, Example 2.8.6]). Let G be a simple Lie group. Then (i) and (ii) are equivalent:

(i) $(G \times G \times G)/\Delta G$ is real spherical ($\Leftrightarrow$ $(G \times G, \Delta G)$ satisfies (PP)).

(ii) G is compact or $\mathfrak{g} \simeq \mathfrak{o}(n, 1)$.

Thus we could expect detailed analysis on invariant trilinear forms for $G = O(n, 1)$. See [CKØP11] for an example of the research in this direction.

Example 6 ([K14]). Let $(G, G') = (GL(n+1, \mathbb{F}), GL(n, \mathbb{F}) \times GL(1, \mathbb{F}))$.

1) For $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, $\dim \mathrm{Sh}(\lambda, \nu) < \infty$ for all λ and ν.

2) $\sup_\lambda \sup_\nu \dim \mathrm{Sh}(\lambda, \nu) < \infty$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, whereas $\sup_\lambda \sup_\nu \dim \mathrm{Sh}(\lambda, \nu) = \infty$ for $\mathbb{F} = \mathbb{H}$.

Under an additional assumption that (G, G') is a symmetric pair, we gave a complete classification of the pairs (G, G') satisfying (PP), or equivalently, satisfying any of (i), (ii) or (iii) in Theorem A (see [KM13]). The idea of the classification extends the idea of 'linearization' which was the key in the proof of Example 5, together with some further ideas such as constructing invariants for quivers.

In [KS13], we constructed and classified all the symmetry breaking operators for spherical principal series representations for $(G, G') = (O(n+1, 1), O(n, 1))$. In this connection, we explained in the talk how symmetry breaking operators of the restriction of smooth admissible representations yield Shintani functions of moderate growth, and presented an explicit dimension formula (*cf.* [K14]).

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