Branching laws for Verma modules and applications in parabolic geometry. I

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Abstract

We initiate a new study of differential operators with symmetries and combine this with the study of branching laws for Verma modules of reductive Lie algebras. By the criterion for discretely decomposable and multiplicity-free restrictions of generalized Verma modules [T. Kobayashi, Transf. Groups (2012)], we are brought to natural settings of parabolic geometries for which there exist unique equivariant differential operators to submanifolds. Then we apply a new method (F-method) relying on the Fourier transform to find singular vectors in generalized Verma modules, which significantly simplifies and generalizes many preceding works. In certain cases, it also determines the Jordan–Hölder series of the restriction for singular parameters. The F-method yields an explicit formula of such unique operators, for example, giving an intrinsic and new proof of Juhl's conformally invariant differential operators [Juhl, Progr. Math. 2009] and its generalizations to spinor bundles. This article is the first in the series, and the next ones include their extension to curved cases together with more applications of the F-method to various settings in parabolic geometries.

Key words: F-method, branching law, conformal geometry, parabolic geometry, equivariant differential operator, Verma module, symmetric pairs .

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Contents

1 Introduction

Let $G' \subset G$ be a pair of real reductive Lie groups. The main objects of this paper are G'-equivariant differential operators between two homogeneous vector bundles over two real flag manifolds $N = G'/P'$ and $M = G/P$, where N is a submanifold of M corresponding to $G' \subset G$. We provide a new method ("F-method") for constructing explicitly such operators, and demonstrate its effectiveness in several concrete examples.

On the algebraic level — in the dual language of homomorphisms between (generalized) Verma modules, the whole construction is connected to natural questions in representation theory, namely branching laws; no attempts at a systematic approach to branching laws for Verma modules has been made until quite recently, and our results might be of independent interest from this point of view. Restricting Verma modules to reductive subalgebras appears manageable at a first glance, however, it involves sometimes wild problems such as the effect of "hidden continuous spectrum" as was revealed in [29]. Nevertheless there is a considerably rich family of examples with good behaviour such as discretely decomposable and multiplicity-free restrictions [28], which are at the same time particularly important for our geometric purposes.

Some of the operators we construct for example, (powers of) the wave operator and Dirac operator appeared previously in physics. Since a large amount of natural differential operators have been already found in parabolic geometries, it is worth pointing out, that the ones treated as a prototype here are exactly the ones that are the hardest to find by the previous methods (essentially coming from the BGG resolution). Further we extend this prototype in two folds to arbitrary signatures in pseudo-Riemannian manifolds (Theorem 4.3) and to Dirac operators (Theorem 5.7) by the new method. We work primarily in the model case situation where the manifold is a real flag manifold, but we see in the second part of the series [33] that (as seen for example in the case of conformal geometry) it is both possible and interesting to extend to the "curved case" of manifolds equipped with the corresponding parabolic geometry.

The results we are going to present are inspired by geometrical considerations. In particular, they correspond to differential invariants (of higher order in general) in the case of models for parabolic geometries. To be precise, let G be a real reductive Lie group, P a parabolic subgroup of G, and G' a reductive subgroup of G such that $P' := P \cap G'$ is a parabolic subgroup of G'. We consider G'-equivariant differential operators acting on sections of homogeneous vector bundles over homogeneous models over G'/P' and G/P. In effect, what happens is that initial sections on G/P are differentiated and then restricted to the submanifolds G'/P', and this combined operation commutes with the action of the group G'.

Explicit formulae for invariant differential operators constructed in the paper are described in the simplest possible coordinates, i.e., in the noncompact picture. In principle, there are methods (based on factorization identities) how to compute explicit form of the differential operators in compact picture but the work needed to do so is nontrivial. An example of such computation can be found in [23, Chapt. 5.2].

Our language chosen for presenting these results is algebraic, however, relies at a stage on certain analytic techniques. The first step is to translate geometrical problems into branching problems of generalized Verma modules for the Lie algebra of G induced from P when restricted to the Lie algebra of G'. Let us recall what is known and what is not known.

The *existence* of equivariant differential operators is assured by the discrete decomposability (Definition 3.1) of the restriction in the BGG category $\mathcal{O}$. A general theory of discretely decomposable restrictions in the BGG category $\mathcal{O}$ as well as in the category of Harish-Chandra modules was established in [25, 26, 27, 29]. Moreover, the *uniqueness* of such operators is guaranteed by the multiplicity-freeness of the restriction. An explicit formula of the branching law for reductive symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$ was proved in [28, 29], which includes the classical Hua–Kostant–Schmid formula and also the decomposition of the tensor product of two modules as special cases. A short summary is given in Section 3 in a way that we need. These general results help us to single out appropriate geometric settings for which we could expect to construct natural equivariant operators to submanifolds, however, we need another idea to answer the following delicate algebraic problems of branching laws, which are closely related with our geometric interest in finding explicit formulae equivariant operators in parabolic geometries. In what follows, $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \equiv M_{\mathfrak{p}}^{\mathfrak{g}}(F_\lambda)$ denotes the $\mathfrak{g}$-module induced from an irreducible finite-dimensional $\mathfrak{p}$-module F_λ with highest weight λ.

Problem A. Find precisely where irreducible $\mathfrak{g}'$-submodules are located in a generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ of $\mathfrak{g}$.

Problem B. Find the Jordan–Hölder series of a generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ of $\mathfrak{g}$ regarded as a module of a reductive subalgebra $\mathfrak{g}'$ by the restriction.

Problem A is to ask for an explicit description of $\mathfrak{g}'$-*singular vectors* (i.e., vectors annihilated by the action of the nilpotent radical of $\mathfrak{p}' = \mathfrak{g}' \cap \mathfrak{p}$, see Section 3 for the definition) in the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$) of $\mathfrak{g}$, and in turn, is equivalent to our geometric question, namely, to construct equivariant differential operators explicitly from real flag varieties to subvarieties (see Theorem 2.4). Problem B concerns with the case where $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is not completely reducible, in particular, for non-generic parameter λ. It should be noted that even in the case $\mathfrak{g} = \mathfrak{g}'$, Problem B is already difficult and unsolved in general. Furthermore, complete reducibility as a $\mathfrak{g}'$-module is another thing than complete reducibility as a $\mathfrak{g}$-module, and it seems that Problem B has never been studied before in the case where $\mathfrak{g}' \subsetneqq \mathfrak{g}$ (even for Lie algebras of small dimensions). The new ingredient of Problem B is to understand how non-trivial $\mathfrak{g}$-extensions occurring in $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ behave when restricted to the subalgebra $\mathfrak{g}'$.

We are interested in Problems A and B, in particular, when we know a priori the restriction $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}'}$ is isomorphic to a multiplicity-free direct sum of irreducible $\mathfrak{g}'$-modules for generic parameter λ.

In the category $\mathcal{O}$, every irreducible $\mathfrak{g}'$-submodule contains a singular vector, and conversely, every singular vector generates a $\mathfrak{g}'$-submodule of finite length. Thus the structure of the set of all singular vectors is a key to the above mentioned problems. In the case of conformal densities, singular vectors were found by using the recurrence relations in certain generalized Verma modules by A. Juhl [23]. However, it seems hard to apply such a combinatorial method in a more general setting due to its computational complexity.

Our method to attack Problems A and B is based on the "Fourier transform" of generalized Verma modules; we call it *the F-method*. The idea is to characterize the set of all singular vectors by means of a system of partial differential equations on the Fourier transform side. It was first suggested by T. Kobayashi, March 2010, with a number of new examples. In contrast to the existing combinatorial techniques to find singular vectors, the F-method is more conceptual.

For example, the coefficients of Juhl's families of equivariant differential operators for the conformal group (see (4.22)) coincide with those of the Gegenbauer polynomials. This was discovered by Juhl [23], but the combinatorial proof there based on recurrence relations did not explain the origin of the special functions in formulae. Our new method is completely different and explains their appearance in a natural way (see Section 4).

The F-method itself is briefly described in Section 2. The key idea of the F-method is to take the Fourier transform of Verma modules after realizing them in the space of distributions supported at the origin on the flag variety. Then we can transfer the algebraic branching problem for generalized Verma modules into an analytic problem, to find polynomial solutions to a system of partial differential equations. In the setting we consider, it leads to an ordinary differential equation (due to symmetry involved). The resulting second-order differential equations control all the family of equivariant differential operators (of arbitrarily high order). The polynomial solutions are the Fourier transform of singular vectors. Hence this new method offers a uniform and effective tool to find explicitly singular vectors in many different cases.

In Section 3, we discuss a class of branching problems for modules in the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ having a discrete decomposability property with respect to reductive subalgebras $\mathfrak{g}'$. Moreover, one of our guiding principles is to focus on multiplicity-free cases which were obtained systematically in [28, 29] by two methods — by visible actions on complex manifolds and by purely algebraic methods. Branching rules are given in terms of the Grothendieck group of the category $\mathcal{O}^{\mathfrak{p}}$, and they give geometric settings where we shall apply the F-method.

The rest of the paper contains applications of the F-method for descriptions of the space of all singular vectors in particular cases of conformal geometry. It contains a complete answer to Problems A and B for generalized Verma modules of scalar type in the case where $(G, G') = (SO_0(p, q), SO_0(p, q-1))$, see Theorems 4.2 and 4.10, respectively. The explicit construction of equivariant differential operators for the particular examples of pseudo-Riemannian manifolds of arbitrary signature (p, q) is given in Theorem 4.3, extending a theorem of Juhl. A further generalization to spinor-valued sections is discussed in Section 5, and explicit formulae of equivariant differential operators for the conformal group are given in Theorem 5.7 by using the Dirac operator and the coefficients of Gegenbauer polynomials. Again the main machinery is the F-method.

As we already emphasized, our original motivation for the study of branching rules for generalized Verma modules came from differential geometry. In fact, there is a substantial relation of the curved version of the Juhl family and a notion of Q-curvature and conformally invariant powers of the Laplace operator. In the second part of the series [33], we construct the curved version of the Juhl family and its generalization by using the result of this article and the ideas of semi-holonomic Verma modules.

To summarize, we have in this paper provided a new method, and some new results concerning the relation between several important topics in representation theory and parabolic geometry, namely branching laws for generalized Verma modules and the construction of equivariant differential operators to submanifolds. In the second part of the series, [33], we give further applications of the F-method to some other examples of parabolic geometries with commutative nilradical, e.g., the projective geometry, Grassmannian geometry and Rankin–Cohen brackets as an example of branching rules for the symmetric pair $(G \times G, \Delta(G))$, where $\Delta : G \to G \times G$ is a diagonal embedding, and discuss their "curved versions".

In the paper, we use the following notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}_+ = \{1, 2, \cdots\}$.

2 Problems and methods for their solutions

The first aim of this section is to explain in more details the connection between geometric and algebraic side of the problem and a method to find $\mathfrak{g}'$-singular vectors in Verma modules of $\mathfrak{g}$, where $\mathfrak{g}' \subset \mathfrak{g}$ are a pair of complex reductive Lie algebras. The second aim is to discuss the general idea of a new approach how to describe equations for singular vectors by using Fourier analysis. The main advantage of the method, which we call "F-method", is that a combinatorially complicated problem of finding singular vectors by the existing methods is converted to a more conceptual question to find polynomial solutions of a certain system of partial or ordinary differential equations, see (2.7). Explicit examples showing how the method works in various situations (related in problems in differential geometry) can be found in the latter half of the paper, in the second part [33] of the series, and [35].

2.1 Two dual faces of the problem

Let $G' \subset G$ be real reductive Lie groups, and $\mathfrak{g}' \subset \mathfrak{g}$ their complexified Lie algebras. In the paper, we are studying two closely related problems. On the side of geometry, we are going to construct intertwining differential operators between principal series representations of the two groups G and G'. On algebraic side, we are going to construct homomorphisms between generalized Verma modules of the two Lie algebras $\mathfrak{g}'$ and $\mathfrak{g}$. The relation between these geometric and algebraic sides is classically known when $G' = G$ (see [3], for instance). We generalize it to the case $G' \neq G$ in connection with branching problems in representation theory. The treatment here is based on recent publications [7, 8] by using jet bundles. A self-contained account (in a more general situation) by a different approach can be found in [34, Sect. 2].

Let G be a connected real reductive Lie group with Lie algebra $\mathfrak{g}(\mathbb{R})$. Let $x \in \mathfrak{g}(\mathbb{R})$ be a hyperbolic element. This means that $\mathrm{ad}(x)$ is diagonalizable and its eigenvalues are all real. Then we have the following Gelfand–Naimark decomposition

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{n}_-(\mathbb{R}) + \mathfrak{l}(\mathbb{R}) + \mathfrak{n}_+(\mathbb{R}),$$

according to the negative, zero, and positive eigenvalues of $\mathrm{ad}(x)$. The subalgebra $\mathfrak{p}(\mathbb{R}) := \mathfrak{l}(\mathbb{R}) + \mathfrak{n}_+(\mathbb{R})$ is a parabolic subalgebra of $\mathfrak{g}(\mathbb{R})$, and its normalizer P in G is a parabolic subgroup of G. Subgroups $N_\pm \subset G$ are defined by $N_\pm = \exp \mathfrak{n}_\pm(\mathbb{R})$.

Given a complex finite-dimensional P-module V, we consider the unnormalized induced representation π of G on the space $\mathrm{Ind}_P^G(V)$ of smooth sections for the homogeneous vector bundle $\mathcal{V} := G \times_P V \to G/P$. We can identify this space with

$$\mathcal{C}^\infty(G, V)^P := \{f \in \mathcal{C}^\infty(G, V) : f(g\,p) = p^{-1} \cdot f(g),\ g \in G, p \in P\}.$$

Moreover, we shall also need the space $J^k_e(G,V)^P$ of k-jets in $e \in G$ of P-equivariant maps and its projective limit

$$J^\infty_e(G,V)^P = \varinjlim_k J^k_e(G,V)^P.$$

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}$ of $\mathfrak{g}(\mathbb{R})$. Let $V^\vee$ denote the contragredient representation. Then $V^\vee$ extends to a representation of the whole enveloping algebra $U(\mathfrak{p})$. The generalized Verma module $M^{\mathfrak{g}}_{\mathfrak{p}}(V^\vee)$ is defined by

$$M^{\mathfrak{g}}_{\mathfrak{p}}(V^\vee) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee.$$

It is a well-known fact that there is a non-degenerate invariant pairing between $J^\infty_e(G,V)^P$ and $M^{\mathfrak{g}}_{\mathfrak{p}}(V^\vee)$. In what follows, we present a version better adapted for our needs. All details of the proof of the version of the claim presented below can be found in [8, App.1], which is an extended version of the paper [7].

Fact 2.1. *Let G be a connected semisimple Lie group with complexified Lie algebra $\mathfrak{g}$, and P a parabolic subgroup of G with complexified Lie algebra $\mathfrak{p}$. Suppose further that V is a finite-dimensional P-module and $V^\vee$ its dual. Then there is a $(\mathfrak{g}, P)$-invariant pairing between $J^\infty_e(G,V)^P$ and $M^{\mathfrak{g}}_{\mathfrak{p}}(V^\vee)$, which identifies $M^{\mathfrak{g}}_{\mathfrak{p}}(V^\vee)$ with the space of all linear maps from $J^\infty_e(G,V)^P$ to $\mathbb{C}$ that factor through $J^k_e(G,V)^P$ for some k.*

The statement above has the following classical corollary (see [3] for instance), which explains the relation between the geometrical problem of finding G-equivariant differential operators between induced representations and the algebraic problem of finding homomorphisms between generalized Verma modules.

Corollary 2.2. *Let V and V' be two finite-dimensional P-modules. Then the space of G-equivariant differential operators from $\mathrm{Ind}^G_P(V)$ to $\mathrm{Ind}^G_P(V')$ is isomorphic to the space of $(\mathfrak{g}, P)$-homomorphisms from $M^{\mathfrak{g}}_{\mathfrak{p}}((V')^\vee)$ to $M^{\mathfrak{g}}_{\mathfrak{p}}(V^\vee)$.*

We now discuss a generalization of Corollary 2.2 to the case of homogeneous vector bundles over *different* flag manifolds, which is formulated and proved below. Suppose that, in addition to the pair $P \subset G$ used above, we consider another pair $P' \subset G'$, such that G' is a reductive subgroup of G and $P' = P \cap G'$ is a parabolic subgroup of G'. We shall see that this happens if $\mathfrak{p}$ is $\mathfrak{g}'$-compatible in the sense of Definition 3.3. In this case, $\mathfrak{n}'_+ := \mathfrak{n}_+ \cap \mathfrak{g}'$ is the nilradical of $\mathfrak{p}'$, and we set $L' = L \cap G'$ for the corresponding Levi subgroup in G'.

For any smooth vector bundle $\mathcal{V} \to M$, there exists a unique (up to isomorphism) vector bundle $J^k(\mathcal{V})$ over M (called the k-th jet prolongantion of $\mathcal{V}$) together with the canonical differential operator

$$J^k : \mathcal{C}^\infty(M, \mathcal{V}) \to \mathcal{C}^\infty(M, J^k\mathcal{V})$$

of order k. Recall that a linear operator $D : \mathcal{C}^\infty(M, \mathcal{V}) \to \mathcal{C}^\infty(M, \mathcal{V}')$ between two smooth vector bundles over M is called a differential operator of order at most k, if there is a bundle morphism $Q : J^k\mathcal{V} \to \mathcal{V}'$ such that $D = Q \circ J^k$. We need a generalization of this definition to the case of a linear operator acting between vector fibre bundles over two *different* smooth manifolds.

Definition 2.3. Fix $k \in \mathbb{N}$. Let $p : N \to M$ be a smooth map between two smooth manifolds and let $\mathcal{V} \to M$ (respectively, $\mathcal{V}' \to N$) be two smooth vector bundles.

A linear map $D : \mathcal{C}^\infty(M, \mathcal{V}) \to \mathcal{C}^\infty(N, \mathcal{V}')$ is said to be a differential operator of order k, if there exists a bundle map $Q : \mathcal{C}^\infty(N, p^*(J^k\mathcal{V})) \to \mathcal{C}^\infty(N, \mathcal{V}')$ such that

$$D = Q \circ p^* \circ J^k.$$

An alternative definition of a differential operator can be based on suitable local properties. For operators between bundles over the same manifold, the relation between both definitions is contained in the classical theorem of Peetre ([40]). For a more general situation (including the case of an linear operator between bundles over different manifolds), an analogue of the Peetre theorem also holds, see [34, Sect.2] and [37, Chapt. 19].

Now we can formulate and prove a generalization of Corollary 2.2 in a more general situation as follows:

Theorem 2.4. *The set of all G'-equivariant differential operators from $\mathrm{Ind}_P^G(V)$ to $\mathrm{Ind}_{P'}^{G'}(V')$ is in one-to-one correspondence with the space of all $(\mathfrak{g}', P')$-homomorphisms from $M_{\mathfrak{p}'}^{\mathfrak{g}'}(V'^\vee)$ to $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$.*

Proof. The inclusion $i : G' \to G$ induces a smooth map $i : G'/P' \to G/P$. The fibers of $J^k(\mathcal{V})$ and $i^*(J^k(V))$ over $o \in G'/P'$ are both isomorphic to $J_e^k(G,V)^P$, hence G'-equivariant bundle maps from $i^*(J^k(\mathcal{V}))$ to V' over G'/P' are in bijective correspondence with elements in $\mathrm{Hom}_{P'}(J^k(G,V)^P, V')$. It implies that G'-equivariant differential operators from $\mathrm{Ind}_P^G(V)$ to $\mathrm{Ind}_{P'}^{G'}(V')$ are in one-to-one correspondence with P'-homomorphisms from $J_e^\infty(G,V)^P$ to V' that factor through $J_e^k(G,V)^P$ for some k. By using the pairing in Fact 2.1, such homomorphisms are in bijective correspondence with elements in $(M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \otimes V')^{P'} \simeq \mathrm{Hom}_{P'}((V')^\vee, M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$. Finally, the Frobenius reciprocity gives us the bijective correspondence between the spaces $\mathrm{Hom}_{P'}((V')^\vee, M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$ and $\mathrm{Hom}_{(\mathfrak{g}',P')}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(V'^\vee), M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$. $\square$

A detailed account of Theorem 2.4 (in a more general setting) with a different proof can be found in [34]. See also [36, Chap. 3] for the general perspectives of *continuous* symmetry breaking operators which include G'-equivariant *differential* operators as a special case.

2.2 F-method

Let us recall that we now consider a general setting with a given pair (P, G) together with another pair (P', G'), such that $G' \subset G$ is a reductive subgroup of G, $P = LN_+$ is a Levi decomposition of a (real) parabolic subgroup of G, and $P' = P \cap G'$ is a parabolic subgroup of G' with Levi decomposition $L'N'_+ \equiv (L \cap G')(N_+ \cap G')$. We write $\mathfrak{g}, \mathfrak{g}', \mathfrak{p}, \mathfrak{p}', \mathfrak{l}, \mathfrak{l}', \mathfrak{n}_+$, and $\mathfrak{n}'_+$ for the complexified Lie algebras of G, G', P, P', L, L', N_+, and N'_+, respectively. Then $\mathfrak{n}'_+ = \mathfrak{n}_+ \cap \mathfrak{g}'$ is the nilradical of $\mathfrak{p}'$. A usual classical setting is that $G = G'$ and $P = P'$.

As explained in Theorem 2.4, a study of intertwining differential operators between principal series of representations of the two groups G and G' can be translated to a study of homomorphisms between generalized Verma modules of the two Lie algebras $\mathfrak{g}'$ and $\mathfrak{g}$. By the universality of the tensor product, the latter homomorphisms are characterized by the image of the highest weight vectors with respect of the parabolic subalgebra $\mathfrak{p}'$, which are sometimes referred to as *singular vectors.* These vectors are annihilated by the nilradical $\mathfrak{n}'_+$.

The whole procedure of the F-method to find explicit singular vectors may be divided into the following three main steps.

Step 1. Computation of the infinitesimal action $d\pi(X)$ for $X \in \mathfrak{n}_+(\mathbb{R})$ on a chosen principal series representation of G (in the non-compact picture).

Step 2. Computation of the dual infinitesimal action $d\pi^\vee(X)$ for $X \in \mathfrak{n}_+(\mathbb{R})$ on the dual space $\mathcal{D}'_{[o]}(N_-, V^\vee)$ of distributions on N_- with values in $V^\vee$ with support in $[o]$. This space is isomorphic with $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ as $\mathfrak{g}$-modules.

Step 3. Suppose now that N_- is commutative and we identify N_- with the Lie algebra $\mathfrak{n}_-(\mathbb{R})$ via the exponential map. The Fourier transform defines an isomorphism $F \otimes \mathrm{Id}_{V^\vee}$: $\mathcal{D}'_{[o]}(N_-, V^\vee) \to \mathrm{Pol}[\mathfrak{n}_+] \otimes V^\vee$ and the representation $d\pi^\vee$ induces the action $d\tilde{\pi}$ of $\mathfrak{g}$ on $\mathrm{Pol}[\mathfrak{n}_+] \otimes V^\vee$.

2.3 Realization of F-method.

We shall describe now these three steps in more details.

Step 1. The fibration $p : G \to G/P$ is a principal fiber bundle with the structure group P over the compact manifold G/P. The manifold $p(N_-\, P)$ is an open dense subset of G/P, sometimes referred to as the big Schubert cell or the open Bruhat cell of G/P. It is naturally identified with N_-. Let $o := e \cdot P \in M \subset G/P$. The exponential map

$$\phi : \mathfrak{n}_-(\mathbb{R}) \to G/P, \quad \phi(X) := \exp(X) \cdot o \in G/P$$

gives the natural identification of the vector space $\mathfrak{n}_-(\mathbb{R})$ with the open Bruhat cell $N_- \simeq p(N_-\, P)$.

Let V be an irreducible complex finite-dimensional P-module, and let us consider the corresponding induced representation π of G on $\mathrm{Ind}_P^G(V) \simeq \mathcal{C}^\infty(G, V)^P$. The representation of G on $\mathrm{Ind}_P^G(V)$ will be denoted by π, and the infinitesimal representation $d\pi$ of its complexified Lie algebra $\mathfrak{g}$ will be considered in the non-compact picture as follows: We identify the space of equivariant smooth maps $\mathcal{C}^\infty(N_-\, P, V)^P$ with the space $\mathcal{C}^\infty(N_-, V)$ as follows: A function $f \in \mathcal{C}^\infty(N_-, V)$ corresponds to $\tilde{f} \in \mathcal{C}^\infty(N_-\, P, V)^P$ defined by $\tilde{f}(n_-\, p) = p^{-1} \cdot f(n_-)$, $n_- \in N_-$, $p \in P$. The induced representation $d\pi$ defines by restriction the representation of $\mathfrak{g}$ on the space $\mathcal{C}^\infty(N_-, V)$.

An actual computation of the representation $d\pi(Z), Z \in \mathfrak{n}_+(\mathbb{R})$ can be carried out by the usual scheme: For a given $Z \in \mathfrak{n}_+(\mathbb{R})$, we consider the one-parameter subgroup $n(t) = \exp(tZ) \in N_+$ and rewrite the product $n(t)^{-1}x$, for $x \in N_-$ and for small $t \in \mathbb{R}$ as

$$n(t)^{-1}x = \tilde{x}(t)p(t),\ \tilde{x}(t) \in N_-,\ p(t) \in P.$$

Then, for $f \in \mathcal{C}^\infty(N_-, V)$, we have

$$[d\pi(Z)f](x) = \frac{d}{dt}\Big|_{t=0}(p(t))^{-1} \cdot f(\tilde{x}(t)). \tag{2.1}$$

Step 2. A simple way how to describe the dual representation $d\pi^\vee$ on the corresponding generalized Verma module is to use a well-known identification of generalized Verma modules with the spaces of distributions supported at the origin of G/P or on the open Bruhat cell N_-. It goes as follows.

Let $\mathcal{D}'_{[o]}(N_-, V^\vee)$ denote the space of $V^\vee$-valued distributions on N_- with support in the point $\{o\}$. The Lie algebra $\mathfrak{g}$ acts on this space by the dual action $d\pi^\vee$:

$$d\pi^\vee(X)(T)(f) = -T(d\pi(X)(f)), X \in \mathfrak{g}, f \in \mathcal{C}^\infty(N_-, V).$$

The action can be extended to the action of $U(\mathfrak{g})$ by

$$d\pi^\vee(u)(T)(f) = -T(d\pi(X)(u^o)), X \in \mathfrak{g}, f \in \mathcal{C}^\infty(N_-, V),$$

where the map $u \to u^o$ is the antiautomorphism of $U(\mathfrak{g})$ acting as $X \mapsto -X$ on $\mathfrak{g}$.

The space $\mathcal{D}'_{[o]}(N_-, V^\vee)$ can be identified with a suitable generalized Verma module:

Fact 2.2 The linear map

$$\phi : U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee \to \mathcal{D}'_{[o]}(N_-, V^\vee)$$

determined by

$$\phi(u \otimes v^\vee) : f \mapsto \langle v^\vee, (d\pi(u^o)f)(o)\rangle$$

is a $U(\mathfrak{g})$-module isomorphism.

The proof of this claim can be found in [1, 34].

Step 3. Suppose now that the unipotent group N_- is commutative. We identify $\mathfrak{n}_-(\mathbb{R})$ with N_- via the exponential map. The Fourier transform gives an isomorphism

$$\mathcal{F} : \mathcal{D}'_{[0]}(\mathfrak{n}_-(\mathbb{R})) \xrightarrow{\sim} \mathrm{Pol}[\mathfrak{n}_+], \quad T \mapsto \mathcal{F}(T) \tag{2.2}$$

defined by

$$\mathcal{F}(T)(\xi) = T_x(e^{i\langle x,\xi\rangle}), \text{ for } x \in \mathfrak{n}_-(\mathbb{R}), \xi \in \mathfrak{n}_+,$$

where $\langle x, \xi\rangle$ is given by the Killing form on $\mathfrak{g}$.

If we consider the space $\mathcal{D}'_{[0]}(\mathfrak{n}_-(\mathbb{R}))$ as a convolution algebra with the delta function being the unit, then $\mathcal{F}$ is an algebra isomorphism mapping the delta function to the constant polynomial 1.

The isomorphism (2.2) can be extended to distributions with values in $V^\vee$ by

$$\mathcal{F} \otimes \mathrm{Id}_{V^\vee} : \mathcal{D}'_{[o]}(N_-, V^\vee) \to \mathrm{Pol}[\mathfrak{n}_+] \otimes V^\vee.$$

The Fourier transform $\mathcal{F} \otimes \mathrm{Id}_{V^\vee}$ can be then used to define the action $d\tilde{\pi}$ of $\mathfrak{g}$ on the space $\mathrm{Pol}[\mathfrak{n}_+] \otimes V^\vee$. Elements of $\mathfrak{n}_+$ act by differential operators of second order as $\mathfrak{n}_+$ is commutative. Let $\mathcal{F}^{-1}$ be the inverse Fourier transform, and we set $\varphi := \phi^{-1} \circ (\mathcal{F}^{-1} \otimes \mathrm{Id}_{V^\vee})$. Then φ gives a bijection

$$\varphi : \mathrm{Pol}[\mathfrak{n}_+] \otimes V^\vee \xrightarrow{\sim} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee. \tag{2.3}$$

2.4 Singular vectors in F-method

Definition 2.5. Let V be any irreducible finite-dimensional $\mathfrak{p}$-module. Let us define the L'-module

$$M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+} := \{v \in M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) : d\pi^\vee(Z)v = 0 \text{ for any } Z \in \mathfrak{n}'_+\}. \tag{2.4}$$

The space $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ of singular vectors is a principal object of our interest. For $G' = G$, the space $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ is of finite-dimension. Note that for $G' \subsetneq G$, the space $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ is infinite-dimensional but it is still completely reducible as an $\mathfrak{l}'$-module. Let us decompose $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ into irreducible $\mathfrak{l}'$-submodules and take $W^\vee$ to be one of its irreducible submodules. Then we get a $\mathfrak{g}'$-homomorphism from $M_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee)$ to $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$.

If V is a P-module, then $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ carries a $(\mathfrak{g}, P)$-module structure, and therefore, $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ becomes an L'-module. In this case, we consider irreducible submodules of L' in $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ for $W^\vee$, and regard $W^\vee$ as a P'-module by letting N'_+ act trivially. Then we get a $(\mathfrak{g}', P')$-homomorphism from $M_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee)$ to $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ via the canonical isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}',P'}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) &\simeq \mathrm{Hom}_{\mathfrak{p}',P'}(W^\vee, M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \\ &\simeq \mathrm{Hom}_{L'}(W^\vee, M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}). \end{aligned} \tag{2.5}$$

Dually, in the language of differential operators, we shall get an invariant differential operator from $\mathrm{Ind}_P^G(V)$ to $\mathrm{Ind}_{P'}^{G'}(W)$ by Theorem 2.4. So the knowledge of all irreducible summands $W^\vee$ of the L'-module $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ gives the knowledge of all possible targets $\mathrm{Ind}_{P'}^{G'}(W)$ for equivariant differential operators on $\mathrm{Ind}_P^G(V)$.

In the F-method, we then realize the space $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ in the space of polynomials on $\mathfrak{n}_+$ with values in $V^\vee$ with action $d\tilde{\pi}$. It can be done efficiently using the Fourier transform as follows:

Definition 2.6. We define

$$\begin{aligned} \mathrm{Sol} \equiv& \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; V^\vee) \\ :=& \{f \in \mathrm{Pol}[\mathfrak{n}_+] \otimes V^\vee : d\tilde{\pi}(Z)f = 0 \text{ for any } Z \in \mathfrak{n}'_+\}. \end{aligned} \tag{2.6}$$

The inverse Fourier transform gives an L'-isomorphism

$$\varphi : \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; V^\vee) \xrightarrow{\sim} M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}. \tag{2.7}$$

An explicit form of the action $d\tilde{\pi}(Z)$ leads to a (system of) differential equation for elements in Sol which makes it possible to describe its structure completely in some particular cases of interest. We shall see in Sections 4 and 5 in certain settings the full understanding of the structure of the set Sol as an L'-module gives complete classification of $\mathfrak{g}'$-homomorphisms from $M_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee)$ to $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ and answers Problems A and B.

The transition from $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ to $\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; V^\vee)$ is the key point of the F-method. It transforms the algebraic problem of finding singular vectors in generalized Verma modules (Problem A) into an analytic problem of solving certain differential equations. It turns out that the F-method is often more efficient than other existing algebraic methods in finding singular vectors. Furthermore, the F-method clarifies why the combinatorial formula appearing in the coefficients of intertwining differential operators in the example of Juhl [23, Chapter 5] are related to those of the Gegenbauer polynomials. It also reduces substantially the amount of computation needed and gives a complete description of the set of singular vectors (e.g. Theorem 4.2); finally it offers a systematic and effective tool for investigation of singular vectors in many cases (e.g. Theorem 4.10). It will be illustrated in Sections 4 and 5 of this article as well as in the second part of the series with a series of different examples.

3 Discretely decomposable branching laws

Suppose that $\mathfrak{g} \supset \mathfrak{g}'$ are a pair of complex reductive Lie algebras, and that X is an irreducible $\mathfrak{g}$-module. It often happens that the restriction $X|_{\mathfrak{g}'}$ does not contain any irreducible $\mathfrak{g}'$-module ([32, 34]). On the other hand, for $X = M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ belonging to the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ (see below), the restriction $X|_{\mathfrak{g}'}$ needs to contain some irreducible $\mathfrak{g}'$-module for the existence of nonzero G'-equivariant differential operators by Theorem 2.4. This algebraic property is said to be "discretely decomposable restrictions" in representation theory, which gives a certain constraint on the triple $(\mathfrak{g}, \mathfrak{g}', \mathfrak{p})$ (Proposition 3.2).

Further, the uniqueness (up to scaling) of equivariant differential operators to submanifolds is assured if (2.5) is one-dimensional, or if the branching law of the restriction $X|_{\mathfrak{g}'}$ is multiplicity free, by Theorem 2.4.

In this section we fix notation for the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$, and summarize the algebraic framework on discretely decomposable restrictions and multiplicity-free theorems in branching laws that were established in [25, 28, 29]. These algebraic results are a guiding principle in this current article and in the second part [33] of the series for finding appropriate settings in parabolic geometry, where one could expect to obtain explicit formulas of equivariant differential operators.

3.1 Category $\mathcal{O}$ and $\mathcal{O}^{\mathfrak{p}}$

We begin with a quick review of the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ (see [21] for an introduction to this area).

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{j}$ a Cartan subalgebra. We write $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{j})$ for the root system, $\mathfrak{g}_\alpha$ ($\alpha \in \Delta$) for the root space, and $\alpha^\vee$ for the coroot, and $W \equiv W(\mathfrak{g})$ for the Weyl group for the root system $\Delta(\mathfrak{g}, \mathfrak{j})$. We fix a positive system Δ^+, write $\rho \equiv \rho(\mathfrak{g})$ for half the sum of the positive roots, and define a Borel subalgebra $\mathfrak{b} = \mathfrak{j} + \mathfrak{n}$ with nilradical $\mathfrak{n} := \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. The Bernstein–Gelfand–Gelfand category $\mathcal{O}$ (BGG category for short) is defined to be the full subcategory of $\mathfrak{g}$-modules whose objects are finitely generated $\mathfrak{g}$-modules X such that X are $\mathfrak{j}$-semisimple and locally $\mathfrak{n}$-finite [2].

Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{b}$, and $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ its Levi decomposition with $\mathfrak{j} \subset \mathfrak{l}$. We set $\Delta^+(\mathfrak{l}) := \Delta^+ \cap \Delta(\mathfrak{l}, \mathfrak{j})$, and define

$$\mathfrak{n}_-(\mathfrak{l}) := \sum_{\alpha \in \Delta^+(\mathfrak{l})} \mathfrak{g}_{-\alpha}.$$

The parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of $\mathcal{O}$ whose objects X are locally $\mathfrak{n}_-(\mathfrak{l})$-finite. We note that $\mathcal{O}^{\mathfrak{b}} = \mathcal{O}$ by definition.

The set of λ for which $\lambda|_{\mathfrak{j} \cap [\mathfrak{l}, \mathfrak{l}]}$ is dominant integral is denoted by

$$\Lambda^+(\mathfrak{l}) := \{\lambda \in \mathfrak{j}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{N} \text{ for all } \alpha \in \Delta^+(\mathfrak{l})\}.$$

We write F_λ for the finite-dimensional simple $\mathfrak{l}$-module with highest weight λ, inflate F_λ to a $\mathfrak{p}$-module via the projection $\mathfrak{p} \to \mathfrak{p}/\mathfrak{n}_+ \simeq \mathfrak{l}$, and define the generalized Verma module by

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \equiv M_{\mathfrak{p}}^{\mathfrak{g}}(F_\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\lambda. \tag{3.1}$$

Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \in \mathcal{O}^{\mathfrak{p}}$, and any simple object in $\mathcal{O}^{\mathfrak{p}}$ is the quotient of some $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$. We say $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of *scalar type* if F_λ is one-dimensional, or equivalently, if $\langle \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta(\mathfrak{l})$.

If $\lambda \in \Lambda^+(\mathfrak{l})$ satisfies

$$\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N}_+ \text{ for all } \beta \in \Delta^+ \setminus \Delta(\mathfrak{l}), \tag{3.2}$$

then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple, see [9].

Let $\mathfrak{Z}(\mathfrak{g})$ be the center of the enveloping algebra $U(\mathfrak{g})$, and we parameterize maximal ideals of $\mathfrak{Z}(\mathfrak{g})$ by the Harish-Chandra isomorphism:

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}), \mathbb{C}) \simeq \mathfrak{j}^*/W, \quad \chi_\lambda \leftrightarrow \lambda.$$

In our normalization, the trivial one-dimensional representation has a $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character $\rho \in \mathfrak{j}^*/W$. Then the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ has a $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character $\lambda + \rho \in \mathfrak{j}^*/W$.

We denote by $\mathcal{O}_\lambda^{\mathfrak{p}}$ the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ whose objects have generalized $\mathfrak{Z}(\mathfrak{g})$-infinitesimal characters $\lambda \in \mathfrak{j}^*/W$, namely,

$$\mathcal{O}_\lambda^{\mathfrak{p}} = \bigcup_{n=1}^{\infty} \{X \in \mathcal{O}^{\mathfrak{p}} : (z - \chi_\lambda(z))^n v = 0 \quad \text{for any } v \in X \text{ and } z \in \mathfrak{Z}(\mathfrak{g})\}.$$

Any $\mathfrak{g}$-module in $\mathcal{O}^{\mathfrak{p}}$ is a finite direct sum of $\mathfrak{g}$-modules belonging to some $\mathcal{O}_\lambda^{\mathfrak{p}}$. Let $K(\mathcal{O}_\lambda^{\mathfrak{p}})$ be the Grothendieck group of $\mathcal{O}_\lambda^{\mathfrak{p}}$, and set

$$K(\mathcal{O}^{\mathfrak{p}}) := {\prod_{\lambda \in \mathfrak{j}^*/W}}' K(\mathcal{O}_\lambda^{\mathfrak{p}}),$$

where $\prod'$ denotes the direct product for which the components are zero except for countably many constituents. Then $K(\mathcal{O}^{\mathfrak{p}})$ is a free $\mathbb{Z}$-module with basis elements $\mathrm{Ch}(X)$ in one-to-one correspondence with simple modules $X \in \mathcal{O}^{\mathfrak{p}}$. We note that $K(\mathcal{O}^{\mathfrak{p}})$ allows a formal sum of countably many $\mathrm{Ch}(X)$, and contains the Grothendieck group of $\mathcal{O}^{\mathfrak{p}}_{\lambda}$ as a subgroup.

3.2 Discretely decomposable branching laws for $\mathcal{O}^{\mathfrak{p}}$

Retain the notation of Section 3.1. Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$. Our subject here is to understand the $\mathfrak{g}'$-module structure of a $\mathfrak{g}$-module $X \in \mathcal{O}^{\mathfrak{p}}$, to which we simply refer as the *restriction* $X|_{\mathfrak{g}'}$. We allow the case where $\mathfrak{g}'$ is not of maximal rank in $\mathfrak{g}$. This question might look easy in the category $\mathcal{O}$ at first glance, however, the restriction $X|_{\mathfrak{g}'}$ behaves surprisingly in a various (and sometimes "wild") manner even when $(\mathfrak{g}, \mathfrak{g}')$ is a reductive symmetric pair. In particular, it may well happen that the restriction $X|_{\mathfrak{g}'}$ does not contain any simple module of $\mathfrak{g}'$, which may be considered as the effect of "hidden continuous spectrum" (see [29]).

In order to exclude "hidden continuous spectrum", we introduced the following notion:

Definition 3.1 ([27, Definition 1.1.]). A $\mathfrak{g}'$-module X is *discretely decomposable* if there exists an increasing sequence of $\mathfrak{g}'$-modules X_j of finite length ($j \in \mathbb{N}$) such that $X = \cup_{j=0}^{\infty} X_j$.

For an irreducible $\mathfrak{g}$-module X, the restriction $X|_{\mathfrak{g}'}$ contains an irreducible $\mathfrak{g}'$-module if and only if the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable [27, Sect.1]. Applying this to $X = M^{\mathfrak{g}}_{\mathfrak{p}}(V^{\vee})$, we see from Theorem 2.4 that the concept of "discretely decomposable restrictions" exactly guarantees the existence of our main objects, namely equivariant differential operators between two real flag varieties.

We then ask for which triple $\mathfrak{g}' \subset \mathfrak{g} \supset \mathfrak{p}$ the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable as a $\mathfrak{g}'$-module for any $X \in \mathcal{O}^{\mathfrak{p}}$. A criterion for this was established in [29] by using an idea of $\mathcal{D}$-modules as follows: Let G be the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$, $P \subset G$ the parabolic subgroup of G with Lie algebra $\mathfrak{p}$, and $G' \subset G$ a reductive subgroup with Lie algebra $\mathfrak{g}' \subset \mathfrak{g}$.

Proposition 3.2. *If $G'P$ is closed in G, then the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable for any $X \in \mathcal{O}^{\mathfrak{p}}$. The converse statement also holds if (G, G') is a symmetric pair.*

Proof. See [29, Proposition 3.5 and Theorem 4.1]. □

Let us consider a simple sufficient condition for the closedness of $G'P$ in G, which will be fulfilled in all the examples discussed in this article. To that aim, let E be a hyperbolic element of $\mathfrak{g}$ defining a parabolic subalgebra $\mathfrak{p}(E) = \mathfrak{l}(E) + \mathfrak{n}(E)$, namely, $\mathfrak{l}(E)$ and $\mathfrak{n}(E)$ are the sum of eigenspaces of $\mathrm{ad}(E)$ with zero, positive eigenvalues.

Definition 3.3 ([29, Definition 3.7]). A parabolic subalgebra $\mathfrak{p}$ is $\mathfrak{g}'$-*compatible* if there exists a hyperbolic element $E' \in \mathfrak{g}'$ such that $\mathfrak{p} = \mathfrak{p}(E')$.

If $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ is $\mathfrak{g}'$-compatible, then $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{g}'$ becomes a parabolic subalgebra of $\mathfrak{g}'$ with the following Levi decomposition:

$$\mathfrak{p}' = \mathfrak{l}' + \mathfrak{n}'_+ := (\mathfrak{l} \cap \mathfrak{g}') + (\mathfrak{n}_+ \cap \mathfrak{g}'),$$

and $P' := P \cap G'$ becomes a parabolic subgroup of G'. Hence, G'/P' becomes automatically a closed submanifold of G/P, or equivalently, $G'P$ is closed in G. Here is a direct consequence of Proposition 3.2:

Proposition 3.4 ([29, Proposition 3.8]). *If $\mathfrak{p}$ is $\mathfrak{g}'$-compatible, then the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable for any $X \in \mathcal{O}^{\mathfrak{p}}$, and any X_j in Definition 3.1 belongs to the parabolic BGG category $\mathcal{O}^{\mathfrak{p}'}$ for $\mathfrak{g}'$-modules.*

Let $\mathfrak{p}$ be a $\mathfrak{g}'$-compatible parabolic subalgebra, and keep the above notation. We denote by F'_μ a finite-dimensional simple $\mathfrak{l}'$-module with highest weight $\mu \in \Lambda^+(\mathfrak{l}')$. The $\mathfrak{l}'$-module structure on the opposite nilradical $\mathfrak{n}_-$ descends to $\mathfrak{n}_-/(\mathfrak{n}_- \cap \mathfrak{g}')$, and consequently extends to the symmetric tensor algebra $S(\mathfrak{n}_-/(\mathfrak{n}_- \cap \mathfrak{g}'))$. We set

$$m(\lambda, \mu) := \dim \operatorname{Hom}_{\mathfrak{l}'}(F'_\mu, F_\lambda|_{\mathfrak{l}'} \otimes S(\mathfrak{n}_-/(\mathfrak{n}_- \cap \mathfrak{g}'))).$$

The following identity is a key step to find branching laws (in a generic case) for the restriction $X|_{\mathfrak{g}'}$ for $X \in \mathcal{O}^{\mathfrak{p}}$:

Theorem 3.5 ([29, Proposition 5.2]). *Suppose that $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ is a $\mathfrak{g}'$-compatible parabolic subalgebra of $\mathfrak{g}$, and $\lambda \in \Lambda^+(\mathfrak{l})$. Then*

1) $m(\lambda, \mu) < \infty$ *for all* $\mu \in \Lambda^+(\mathfrak{l}')$.

2) *We have the following identity in $K(\mathcal{O}^{\mathfrak{p}})$:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}'} \simeq \bigoplus_{\mu \in \Lambda^+(\mathfrak{l}')} m(\lambda, \mu) M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mu)$$

for any generalized Verma modules $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ and $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mu)$ defined respectively by $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\lambda$, $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mu) = U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} F'_\mu$.

Finally, we highlight the multiplicity-free case, namely, when $m(\lambda, \mu) \leq 1$ and give a closed formula of branching laws. Suppose now that $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ is a parabolic subalgebra such that the nilradical $\mathfrak{n}_+$ is abelian. We write $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{l} + \mathfrak{n}_+$ for the Gelfand–Naimark decomposition. Let θ be an endomorphism of $\mathfrak{g}$ such that $\theta|_{\mathfrak{l}} = \mathrm{id}$ and $\theta|_{\mathfrak{n}_+ + \mathfrak{n}_-} = -\mathrm{id}$. Then θ is an involutive automorphism of $\mathfrak{g}$ because $\mathfrak{n}_+$ is abelian.

Suppose τ is another involutive automorphism of the complex Lie algebra $\mathfrak{g}$ such that $\tau\mathfrak{l} = \mathfrak{l}$ and $\tau\mathfrak{n}_\pm = \mathfrak{n}_\pm$. Then $\tau\theta = \theta\tau$ and the parabolic subalgebra $\mathfrak{p}$ is $\mathfrak{g}^\tau$-compatible. We take a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{l}$ such that $\mathfrak{j}^\tau$ is a maximal abelian subspace of $\mathfrak{l}^\tau$. Here, for a subspace V in $\mathfrak{g}$, we write $V^{\pm\tau} := \{v \in V : \tau v = \pm v\}$ for the ± 1 eigenspaces of τ, respectively. Then

$$\mathfrak{g}^{\tau\theta} := \mathfrak{l}^\tau + \mathfrak{n}_-^{-\tau} + \mathfrak{n}_+^{-\tau}$$

is a reductive subalgebra of $\mathfrak{g}$. We write $\mathfrak{g}^{\tau\theta} = \bigoplus_i \mathfrak{g}_i^{\tau\theta}$ for the decomposition into simple or abelian ideals, and decompose $\mathfrak{n}_-^{-\tau} = \bigoplus_i \mathfrak{n}_{-,i}^{-\tau}$ correspondingly. Each $\mathfrak{n}_{-,i}^{-\tau}$ is a $\mathfrak{j}^\tau$-module, and we denote by $\Delta(\mathfrak{n}_{-,i}^{-\tau}, \mathfrak{j}^\tau)$ the set of weights of $\mathfrak{n}_{-,i}^{-\tau}$ with respect to $\mathfrak{j}^\tau$. (We note that $\mathfrak{n}_{-,i}^{-\tau} = \{0\}$ except for a single i in the case where we shall treat, and thus we may simply replace $\mathfrak{n}_{-,i}^{-\tau}$ by $\mathfrak{n}_-^{-\tau}$ below for actual computations below.)

The roots α and β are said to be *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. We take a maximal set of strongly orthogonal roots $\{\nu_1^{(i)}, \cdots, \nu_{k_i}^{(i)}\}$ in $\Delta(\mathfrak{n}_{-,i}^{-\tau}, \mathfrak{j}^\tau)$ inductively as follows:

1) $\nu_1^{(i)}$ is the highest root of $\Delta(\mathfrak{n}_{-,i}^{-\tau}, \mathfrak{j}^\tau)$.

2) $\nu_{j+1}^{(i)}$ is the highest root among the elements in $\Delta(\mathfrak{n}_{-,i}^{-\tau}, \mathfrak{j}^\tau)$ that are strongly orthogonal to $\nu_1^{(i)}, \cdots, \nu_j^{(i)}$ $(1 \leq j \leq k_i - 1)$.

Then we recall from [28, 29] the multiplicity-free branching law:

Theorem 3.6. *Suppose that $\mathfrak{p}$, τ, and λ are as above. Then the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ decomposes into a multiplicity-free direct sum of generalized Verma modules of $\mathfrak{g}^\tau$ in $K(\mathcal{O}^{\mathfrak{p}})$:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau} \simeq \bigoplus M_{\mathfrak{p}^\tau}^{\mathfrak{g}^\tau}(\lambda|_{\mathfrak{j}^\tau} + \sum_i \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)}). \tag{3.3}$$

Here the summation is taken over the following subset of $\mathbb{N}^k$ $(k = \sum k_i)$ defined by

$$\prod_i A_i, \qquad A_i := \{(a_j^{(i)})_{1 \leq j \leq k_i} \in \mathbb{N}^{k_i} : a_1^{(i)} \geq \cdots \geq a_{k_i}^{(i)} \geq 0\}.$$

The restriction $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau}$ is actually a direct sum in the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ if λ is sufficiently negative, or more generally, if the following two conditions are satisfied for $\{a_j^{(i)}\}$ in the above range:

$$\langle \lambda|_{\mathfrak{j}^\tau} + \rho(\mathfrak{g}^\tau) + \sum_i \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)}, \beta^\vee \rangle \notin \mathbb{N}_+ \quad \textit{for all } \beta \in \Delta(\mathfrak{n}_+^\tau, \mathfrak{j}^\tau), \tag{3.4}$$

$$\lambda|_{\mathfrak{j}^\tau} + \rho(\mathfrak{g}^\tau) + \sum_i \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)} \textit{ are all distinct in } (\mathfrak{j}^\tau)^*/W(\mathfrak{g}^\tau). \tag{3.5}$$

Proof. The formula (3.3) was proved in [28, Theorem 8.3] (in the framework of holomorphic discrete series representations) and in [29, Theorem 5.2] (in the framework of generalized Verma modules) under the assumption that λ is sufficiently negative and that $\mathfrak{g}^{\tau\theta}$ is simple. The latter proof shows in fact that the identity (3.3) holds in $K(\mathcal{O}^{\mathfrak{p}})$ for all λ. Since two modules with different infinitesimal characters do not have extension, the last statement follows. $\square$

Remark 3.7. In the case $\mathfrak{l} = \mathfrak{g}^\tau$, each summand of the right-hand side is finite-dimensional because the parabolic subalgebra $\mathfrak{p}^\tau$ coincides with $\mathfrak{g}^\tau$. In this very special case, Theorem 3.6 for sufficiently negative λ was proved earlier by B. Kostant and W. Schmid. We also note that Theorem 3.6 includes the decomposition of the tensor product of two representations for generic parameters because the pair $(\mathfrak{g} \oplus \mathfrak{g}, \mathrm{diag}(\mathfrak{g}))$ is regarded as an example of a symmetric pair.

4 Conformal geometry with arbitrary signature

In Sections 4 and 5, we illustrate the F-method by examples of pseudo-Riemannian manifolds with arbitrary signatures, whose symmetries are given by

$$(G, G') = (SO_o(p,q), SO_o(p,q-1)) \quad \text{or} \quad (Spin_o(p,q), Spin_o(p,q-1)).$$

In answer to the Problem A posed in Introduction, we see that the F-method brings us to the Gegenbauer differential equation (6.3) in this case, and prove that all singular vectors can be described by using the classical orthogonal polynomials. In turn, these orthogonal polynomials yield a generalization of Juhl's equivariant differential operators between sections of line bundles over two conformal manifolds (of different dimensions), of which the original form was constructed by completely different (combinatorial) techniques in [23].

Concerning Problem B in Introduction, by using the explicit singular vectors obtained by the F-method, we can determine the Jordan–Hölder series of the generalized Verma modules for $\mathfrak{g}$ with exceptional (discrete) values of parameters, when we restrict them to the reductive subalgebra $\mathfrak{g}'$, see Theorem 4.10.

4.1 Notation

Let $p \geq 1$ and $q \geq 2$. We set $n = p + q - 2$ and

$$\epsilon_i := \begin{cases} 1 & (1 \leq i \leq p-1), \\ -1 & (p \leq i \leq n). \end{cases}$$

We note $\epsilon_n = -1$ because $q \geq 2$. Let us consider the quadratic form

$$2x_0\, x_{n+1} + \sum_{i=1}^{n} \epsilon_i x_i^2 \quad \text{for} \quad x = (x_0, \cdots, x_{n+1}). \tag{4.1}$$

on $\mathbb{R}^{p+q} \simeq \mathbb{R}^{n+2}$, and set $G := SO_o(p,q)$, the identity component of the group preserving the quadratic form. Then the group G preserves the null cone

$$\mathcal{N} = \mathcal{N}_{p,q} := \{x = (x_0, \cdots, x_{n+1}) \in \mathbb{R}^{p,q} \setminus \{0\} : 2x_0\, x_{n+1} + \sum_{i=1}^{n} \epsilon_i x_i^2 = 0\}. \tag{4.2}$$

We define the parabolic subgroups P and P_- to be the isotropy subgroups of the line in the null cone $\mathcal{N}$ generated by $e_0 = {}^t(1, 0, \ldots, 0)$ and $e_{n+1} = {}^t(0, \ldots, 0, 1)$, respectively, and set $L := P \cap P_-$. The homogeneous space G/P is the projective null cone $\mathbb{P}\mathcal{N}$ with its conformal structure. We write $P = LN_+ = MAN_+$ for the Langlands decomposition. Let $\{E_j\}_{j=1,\cdots,n}$ be the standard basis of the Lie algebra $\mathfrak{n}_+(\mathbb{R})$ of N_+, which we identify with $\mathbb{R}^n$, and likewise $\mathfrak{n}_-(\mathbb{R})$ with $\mathbb{R}^n$ by using the standard basis:

$$\mathfrak{n}_+(\mathbb{R}) \simeq \{Z : Z = (z_1, \ldots, z_n)\}, \quad \mathfrak{n}_-(\mathbb{R}) \simeq \{X : {}^tX = (x_1, \ldots, x_n)\}. \tag{4.3}$$

The group M is isomorphic to $SO(p-1, q-1)$, and acts on $\mathfrak{n}_+(\mathbb{R}) \simeq \mathbb{R}^n (= \mathbb{R}^{p+q-2})$ as the natural representation, preserving the quadratic form $\sum_{i=1}^n \epsilon_i x_i^2$. Denote by $\mathbb{J}$ the $n \times n$ matrix of this quadratic form with elements ϵ_i on the diagonal.

Elements in G can be written as block matrices with respect to the direct sum decomposition

$$\mathbb{R}^{n+2} = \mathbb{R}e_0 \oplus \sum_{j=1}^{n} \mathbb{R}e_j \oplus \mathbb{R}e_{n+1}. \tag{4.4}$$

Then elements in the real parabolic subgroup P are given by block triangular matrices

$$p = \begin{pmatrix} \epsilon(m)a & \star & \star \\ 0 & m & \star \\ 0 & 0 & \epsilon(m)a^{-1} \end{pmatrix} \tag{4.5}$$

with $a \in \mathbb{R}_+$, $m \in SO(p-1, q-1)$. Here $\epsilon(m) = +1$ or -1 according to whether m belongs to the identity component $SO_o(p-1, q-1)$ or not. In the coordinates (4.3) we have

$$n = \exp Z = \begin{pmatrix} 1 & Z & -\frac{|Z|^2}{2} \\ 0 & \mathrm{Id} & -\mathbb{J}{}^tZ \\ 0 & 0 & 1 \end{pmatrix} \in N_+, \; x = \exp X = \begin{pmatrix} 1 & 0 & 0 \\ X & \mathrm{Id} & 0 \\ -\frac{|X|^2}{2} & -{}^tX\mathbb{J} & 1 \end{pmatrix} \in N_-, \tag{4.6}$$

where we set $|X|^2 := {}^tX\mathbb{J}X$ and $|Z|^2 := Z\mathbb{J}{}^tZ$.

4.2 The representations $d\pi_\lambda$ and $d\tilde{\pi}_\lambda$.

We are going now to apply the F-method explained in Section 2 to the conformal case of the general signature (p,q),

$$(G,G') = (SO_o(p,q), SO_o(p,q-1)).$$

The first goal is to describe the action of elements in $\mathfrak{n}_+$ in terms of differential operators acting on the "Fourier image" of the generalized Verma module. This can be deduced from the explicit form of the (easily described) action of the induced representation in the non-compact picture. We shall then find singular vectors in $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ by using the F-method. Later on, we use them to obtain equivariant differential operators from $\mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ to $\mathrm{Ind}_{P'}^{G'}(\mathbb{C}_{\lambda+K})$ for some $K\in\mathbb{N}$ after switching λ to $-\lambda$ corresponding to the dual representation.

For $\lambda\in\mathbb{C}$, we define a family of differential operators on $\mathfrak{n}_-(\mathbb{R})\simeq\mathbb{R}^n_x$ by

$$Q_j(\lambda) := -\frac{1}{2}\epsilon_j|X|^2\partial_{x_j} + x_j(\lambda+\sum_k x_k\partial_{x_k}),\ j=1,\dots,n,$$

and on its dual space $\mathbb{R}^n_\xi$ by

$$P_j(\lambda) := -i\left(\frac{1}{2}\epsilon_j\xi_j\Box + (\lambda-E)\partial_{\xi_j}\right),\ j=1,\dots,n, \tag{4.7}$$

where

$$\Box := \partial^2_{\xi_1}+\cdots+\partial^2_{\xi_{p-1}} - \partial^2_{\xi_p} - \cdots - \partial^2_{\xi_{p+q-2}}$$

is the Laplace–Beltrami operator of signature $(p-1,q-1)$ and $E=\sum_{k=1}^n \xi_k\partial_{\xi_k}$ is the Euler homogeneity operator. The mutually commuting operators $P_j(\lambda)$ $(1\le j\le n)$ were introduced in [32, Chapter 1], and include "fundamental differential operators" on the isotropic cone as a special case (i.e., $\lambda=1$).

Let $\mathbb{C}_\lambda$ be the one-dimensional representation of P given by $p\mapsto a^\lambda$, with the notation of (4.5). By a little abuse of notation we shall use $\mathbb{C}_\lambda$ to stand for the one-dimensional representation space $\mathbb{C}$. Let π_λ $(\equiv \pi_{\lambda,+})$ be the complex representation of G on the unnormalized induced representation

$$\mathrm{Ind}_P^G(\mathbb{C}_\lambda) := \{f\in\mathcal{C}^\infty(G) : f(gp)=a^{-\lambda}f(g) \quad \text{for any} \quad p\in P\}$$

with p in the notation (4.5). The infinitesimal representation $d\pi_\lambda$ of the Lie algebra $\mathfrak{g}$ acts on $\mathcal{C}^\infty(\mathfrak{n}_-(\mathbb{R}))\otimes\mathbb{C}_\lambda$. We write $\mathbb{C}_\lambda^\vee$ for the contragredient representation of $\mathbb{C}_\lambda$.

Lemma 4.1. *The elements $E_j\in\mathfrak{n}_+(\mathbb{R})$ act on $\mathcal{C}^\infty(\mathfrak{n}_-(\mathbb{R}))\otimes\mathbb{C}_\lambda$ by*

$$d\pi_\lambda(E_j)(g\otimes v) = Q_j(\lambda)(g)\otimes v \quad \text{for } g\in\mathcal{C}^\infty(\mathfrak{n}_-(\mathbb{R})), v\in\mathbb{C}_\lambda. \tag{4.8}$$

The action of $d\tilde{\pi}_\lambda$ on $\mathrm{Pol}[\xi_1,\dots,\xi_n]\otimes\mathbb{C}_\lambda^\vee$ is given by

$$d\tilde{\pi}_\lambda(E_j)(f\otimes v) = P_j(\lambda)(f)\otimes v \quad \text{for } f\in\mathrm{Pol}[\xi_1,\dots,\xi_n], v\in\mathbb{C}_\lambda^\vee. \tag{4.9}$$

Proof. For $n=\exp Z\in N_+$ and $x=\exp X\in N_-$ we have from (4.5)

$$n^{-1}x = \begin{pmatrix} a & -Z+\frac{1}{2}|Z|^2\,{}^tX\mathbb{J} & -\frac{1}{2}|Z|^2 \\ X-\frac{1}{2}|X|^2\mathbb{J}^tZ & \mathrm{Id}-\mathbb{J}^tZ\otimes{}^tX\mathbb{J} & \mathbb{J}^tZ \\ -\frac{1}{2}|X|^2 & -{}^tX\mathbb{J} & 1 \end{pmatrix},$$

where

$$a := 1 - Z \cdot X + \frac{|Z|^2|X|^2}{4}. \tag{4.10}$$

If Z and X are sufficiently small, then $a \neq 0$ and we can define

$$\tilde{x} := \begin{pmatrix} 1 & 0 & 0 \\ a^{-1}(X - \frac{1}{2}|X|^2 \mathbb{J}^t Z) & \mathrm{Id} & 0 \\ -\frac{1}{2}a^{-1}|X|^2 & a^{-1}({}^t X \mathbb{J} + \frac{1}{2}|X|^2 Z) & 1 \end{pmatrix} \in N_-.$$

Then $p := \tilde{x}^{-1} n^{-1} x \in P$, and in the expression (4.5), a is given by (4.10) and m satisfies $\epsilon(m) = 1$. If we let Z tend to be zero, then the elements $\tilde{x}, a$ and m behave up to the first order in $\|Z\| = (\sum_{i=1}^{n} |z_i|^2)^{\frac{1}{2}}$ as

$$a \sim 1 - Z \cdot X, \quad m \sim \mathrm{Id} - \mathbb{J}^t Z \otimes {}^t X \mathbb{J} + X \otimes Z; \tag{4.11}$$

$$\tilde{x} = \exp \tilde{X}, \ \tilde{X} \sim (1 + Z \cdot X) \left(X - \frac{|X|^2 \mathbb{J}^t Z}{2} \right). \tag{4.12}$$

Taking $Z = tE_j$, we have for $X = {}^t(x_1, \ldots, x_n) \in \mathfrak{n}_-(\mathbb{R}) \simeq \mathbb{R}^n$,

$$\begin{aligned} &a = 1 - t x_j + o(t), \\ &\widetilde{X} = X + t x_j {}^t(x_1, \ldots, x_n) - \frac{1}{2}\epsilon_j t |X|^2 \, {}^t E_j + o(t), \end{aligned}$$

where $o(t)$ denotes the Landau symbol. Therefore, for $F \in \mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ and $x = \exp(X) \in N_-$, we have

$$\begin{aligned} (\mathrm{d}\pi_\lambda(E_j)F)(x) &= \frac{d}{dt}|_{t=0} \ F(\exp(-tE_j)x) \\ &= \frac{d}{dt}|_{t=0} \ F(\tilde{x}p) \\ &= \frac{d}{dt}|_{t=0} \ a^{-\lambda} F(\exp(\tilde{X})) \\ &= \big(Q_j(\lambda)(F \circ \exp)\big)(X). \end{aligned}$$

Thus we have proved the formula (4.8) for the action $d\pi_\lambda(E_j)$.

The action of $d\tilde{\pi}_\lambda(E_j)$ is computed in two steps. The first step is to compute the dual action $d\pi^\vee$ reversing the order in the composition of operators and adding sign changes depending on the order of the operator. In the second step we apply the distributional Fourier transform

$$x_j \mapsto -i\partial_{\xi_j}, \, \partial_{x_j} \mapsto -i\xi_j$$

preserving the order of operators in the composition. □

4.3 The case $(G, G') = (SO_o(p,q), SO_o(p,q-1))$.

We realize $G' = SO_o(p, q-1)$ as the subgroup of $G = SO_o(p,q)$ which leaves the basis vector $e_n = {}^t(0, \ldots, 0, 1, 0)$ invariant. We recall $n = p + q - 2$. Then the parabolic subalgebra $\mathfrak{p}$ is $\mathfrak{g}'$-compatible in the sense of Definition 3.3, and therefore $P' := P \cap G'$ becomes a parabolic subgroup of G' with a Levi part $L' := L \cap G'$.

The nilpotent radical $\mathfrak{n}'_-(\mathbb{R}) \simeq \{{}^t X : X = (x_1, \ldots, x_{n-1})\} \simeq \mathbb{R}^{n-1}$, has codimension one in $\mathfrak{n}_-(\mathbb{R})$. We endow $\mathfrak{n}_-(\mathbb{R}) \simeq \mathbb{R}^{p+q-2}$ with the standard flat quadratic form $(p-1, q-1)$, denoted by $\mathbb{R}^{p-1,q-1}$, so that G acts by local conformal transformations on $\mathfrak{n}_-(\mathbb{R})$. The subspace $\mathfrak{n}'_-(\mathbb{R}) \simeq \mathbb{R}^{n-1}$ has signature $(p-1, q-2)$.

According to the recipe of the F-method in Section 2, we begin by finding the L'-module structure on the space Sol (see (2.6) for the definition) in this case.

4.3.1 The space of singular vectors

Recall from (2.7) the isomorphism between the space of singular vectors $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ and the space Sol $\equiv$ Sol$(\mathfrak{g}, \mathfrak{g}'; V^\vee)$ of polynomial solutions to a system of partial differential equations. We are now going to determine the set Sol, and thus we describe completely the set of singular vectors.

For $k \in \mathbb{N}$, we denote by $\mathcal{H}^k(\mathbb{R}^{p-1,q-1})$ the space of harmonic polynomials of degree k, namely, homogeneous polynomials $f(\xi)$ of degree k satisfying

$$(\frac{\partial^2}{\partial \xi_1^2} + \cdots + \frac{\partial^2}{\partial \xi_{p-1}^2} - \frac{\partial^2}{\partial \xi_p^2} - \cdots - \frac{\partial^2}{\partial \xi_{p+q-2}^2})f = 0.$$

Then the indefinite orthogonal group $O(p-1, q-1)$ acts irreducibly on $\mathcal{H}^k(\mathbb{R}^{p-1,q-1})$, which then decomposes

$$\mathcal{H}^k(\mathbb{R}^{p-1,q-1}) \simeq \bigoplus_{j=0}^{k} \mathcal{H}^j(\mathbb{R}^{p-1,q-2}) \tag{4.13}$$

when restricted to the subgroup $O(p-1, q-2)$.

If $C_\ell^\alpha(x)$ is the Gegenbauer polynomial, then $x^\ell C_\ell^\alpha(x^{-1})$ is an even polynomial. Hence we can define another polynomial $\mathcal{C}_\ell^\alpha(s)$ by the relation (see Appendix for more details):

$$x^\ell C_\ell^\alpha(x^{-1}) = \mathcal{C}_\ell^\alpha(x^2). \tag{4.14}$$

We define a homogeneous polynomial $f_K(\xi) \equiv f_{K,\lambda}(\xi_1, \cdots, \xi_n)$ of degree K ($K \in \mathbb{N}$) by

$$f_{K,\lambda}(\xi_1, \cdots, \xi_n) := \xi_n^K \mathcal{C}_K^{-\lambda - \frac{n-1}{2}} \left(-\frac{\epsilon_n \sum_{i=1}^{n-1} \epsilon_i \xi_i^2}{\xi_n^2} \right). \tag{4.15}$$

Since $f_{K,\lambda}$ vanishes when $\lambda \in \{\frac{1-n}{2}, \frac{3-n}{2}, \dots, [\frac{K-1}{2}] + \frac{1-n}{2}\}$ for $K \geq 1$, we renormalize a non-zero element

$$w_K \equiv w_{K,\lambda} \in \mathrm{Pol}[\xi_1, \cdots, \xi_n] \otimes \mathbb{C}_\lambda \tag{4.16}$$

by

$$\begin{aligned} w_{2N,\lambda} \quad &:= \frac{N!}{(-\lambda - \frac{n-1}{2})_N} f_{2N,\lambda}(\xi_1, \cdots, \xi_n) \otimes 1_\lambda, \\ w_{2N+1,\lambda} &:= \frac{N!}{2(-\lambda - \frac{n-1}{2})_{N+1}} f_{2N+1,\lambda}(\xi_1, \cdots, \xi_n) \otimes 1_\lambda, \end{aligned}$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$. We let $A \simeq \mathbb{R}_{>0}$ act on $\mathrm{Pol}[\mathfrak{n}_+] \otimes V^\vee \simeq \mathrm{Pol}[\xi_1, \cdots, \xi_n] \otimes \mathbb{C}_\lambda$ by

$$f \otimes v \mapsto f(a^{-1} \cdot) \otimes a \cdot v \quad \text{for} \quad a > 0.$$

We say $f \otimes v$ has weight μ if this action is given by the multiplication of a^μ. Then the weight of $w_{K,\lambda}$ is $\lambda - K$.

Theorem 4.2. *Let $p \geq 1, q \geq 2, p+q > 4$ and $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}'(\mathbb{R})) = (\mathfrak{so}(p,q), \mathfrak{so}(p, q-1))$. We write 1_λ for a non-zero vector in the one-dimensional vector space $\mathbb{C}_\lambda$ with parameter $\lambda \in \mathbb{C}$.*

(1) *Let $P_j(\lambda)$ $(1 \leq j \leq n)$ be the second-order differential operators defined in (4.7). Then the space* $\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda)$ *(see (2.6)) is given by*

$$\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda) = \{f \otimes 1_\lambda \in \mathrm{Pol}[\xi_1, \cdots, \xi_n] \otimes \mathbb{C}_\lambda : \ P_j(\lambda) f = 0 \, (1 \leq j \leq n-1)\}.$$

(2) *For $\lambda \notin \mathbb{N}$, we have*

$$\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda) = \bigoplus_{K=0}^{\infty} \mathbb{C} w_{K,\lambda}.$$

In particular, the inverse Fourier transform (see (2.7)) gives an L'-isomorphism:

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)^{\mathfrak{n}'_+} \underset{\varphi}{\overset{\sim}{\leftarrow}} \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda) = \bigoplus_{K=0}^{\infty} \mathbb{C} w_{K,\lambda}.$$

(3) *For $\lambda \in \mathbb{N}$, we have*

$$\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda) = \bigoplus_{K=0}^{\infty} \mathbb{C} w_{K,\lambda} \oplus \bigoplus_{j=1}^{\lambda+1} H'_j,$$

where $H'_j \equiv H'_{j,\lambda}$ $(1 \leq j \leq \lambda+1)$ is the subspace of $\mathcal{H}^{\lambda+1}(\mathbb{R}^{p-1,q-1}) \otimes \mathbb{C}_\lambda$ corresponding to the summand $\mathcal{H}^j(\mathbb{R}^{p-1,q-2})$ in (4.13). *In particular, the inverse Fourier transform induces an L'-isomorphism:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)^{\mathfrak{n}'_+} \underset{\varphi}{\overset{\sim}{\leftarrow}} \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda) = \bigoplus_{K=0}^{\infty} \mathbb{C} w_{K,\lambda} \oplus \bigoplus_{j=1}^{\lambda+1} H'_j.$$

Furthermore, for each $j = 1, \ldots, \lambda+1$, the image $\varphi(H'_j)$ is contained in the $\mathfrak{g}'$-submodule generated by the vector $\varphi(w_{\lambda+1-j,\lambda})$.

Proof. We apply the F-method as follows. By (4.7), the equation $d\tilde{\pi}(Z) f = 0$ for $Z \in \mathfrak{n}'_+$ amounts to a system of differential equations

$$P_j(\lambda) f = 0 \ \text{ for } \ j = 1, \ldots, n-1. \tag{4.17}$$

Hence we have shown the first statement.

In order to analyze $\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda)$ explicitly, we first prove that if $\lambda \notin \mathbb{N}$ then any polynomial solution f to (4.17) is $SO_o(p-1, q-2)$-invariant. Since the operators $P_j(\lambda)$ decrease the homogeneity by one, we can assume $f \in \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda)$ to be homogeneous without loss of generality. It follows from (4.17) that

$$(\epsilon_i \xi_i P_j(\lambda) - \epsilon_j \xi_j P_i(\lambda)) f = 0,$$

which amounts to

$$(E - \lambda - 1)(\epsilon_j \xi_j \partial_{\xi_i} - \epsilon_i \xi_i \partial_{\xi_j}) f = 0 \tag{4.18}$$

for all $i, j = 1, \ldots, n-1$. We recall $n = p+q-2$. Hence if $\lambda \neq \deg f - 1$, then $(\epsilon_j \xi_j \partial_{\xi_i} - \epsilon_i \xi_i \partial_{\xi_j}) f = 0$, namely, f is a polynomial invariant under $SO_o(p-1, q-2)$.

Next, let us solve the equation (4.17).

Case 1. $SO_o(p-1, q-2)$-invariant solutions.

As we saw above, this is always the case when $\lambda \notin \mathbb{N}$. Since $p+q-3 \geq 2$, classical invariant theory says that any $SO_o(p-1,q-2)$-invariant, homogeneous polynomial f in n-variables of degree K can be written in the form

$$f(\xi_1,\ldots,\xi_n) = \xi_n^K\, h(t),$$

where

$$t = \frac{\epsilon_n|\xi'|^2}{\xi_n^2},\ |\xi'|^2 = \sum_{i=1}^{n-1} \epsilon_i \xi_i^2$$

and h is a polynomial of degree N (depending on the parity of K, either $K = 2N$ or $K = 2N+1$).

Hence we look for a solution of the form $f = \xi_n^K h(t),\ t = \frac{\epsilon_n|\xi'|^2}{\xi_n^2}$. We get immediately

$$\partial_j f = \xi_n^{K-2} 2\epsilon_n\epsilon_j\xi_j\, h',\ \partial_j^2 f = \xi_n^{K-2}\left(4h''\frac{\xi_j^2}{{\xi_n}^2} + \epsilon_n\epsilon_j 2h'\right),\ j = 1,\ldots,n-1,$$

$$\begin{aligned}
\Box' f &= \epsilon_n\xi_n^{K-2}\left(4h''t + 2(n-1)\,h'\right),\ \partial_{\xi_n} f = \xi_n^{K-1}\left(K\,h - 2\,h't\right),\\
\epsilon_n\partial_{\xi_n}^2 f &= \epsilon_n\xi_n^{K-2}\left(K(K-1)\,h + (-4K+6)\,h't + 4\,h''t^2\right),\\
\Box f &= \epsilon_n\xi_n^{K-2}\left(4t(1+t)\,h'' + (2(n-1) + t(-4K+6))\ h' + K(K-1)\,h\right),\\
(\lambda - E)\partial_j f &= \epsilon_n\epsilon_j\xi_j\xi_n^{K-2}\,(2\lambda - 2K + 2)\,h'.
\end{aligned}$$

Collecting terms together and cancelling the scalar multiple by $\frac{1}{2}\epsilon_n\epsilon_j\xi_j\xi_n^{K-2}$, the partial differential equation (4.17) induces the following ordinary differential equation for $h(t)$ which is independent of j $(1 \leq j \leq n-1)$:

$$R(K, -\lambda - \frac{n-1}{2})h(t) = 0, \tag{4.19}$$

where we define a differential operator $R(l,\alpha)$ of second order by

$$R(l,\alpha) := 4t(1+t)\frac{d^2}{dt^2} + ((6-4l)t + 4(1-\alpha-l))\frac{d}{dt} + l(l-1). \tag{4.20}$$

Hence the resulting system of equations (4.17) (for $j = 1,\ldots,n-1$) reduces to a single ordinary differential equation of second order for $h(t)$.

We set $g(x) := x^K h(-\frac{1}{x^2})$, then $g(x) \in \mathrm{Pol}_K[x]_{\mathrm{even}}$ belongs to

$$\mathrm{Pol}_K[x]_{\mathrm{even}} := \mathbb{C}\text{-span}\Big\{x^{K-2j} : 0 \leq j \leq \Big[\frac{K}{2}\Big]\Big\}.$$

It follows from Lemma 6.1 that $g(x)$ is a scalar multiple of the renormalized Gegenbauer polynomial

$$\widetilde{C}_K^\alpha(x) := \frac{1}{(\alpha)_{[\frac{K+1}{2}]}} C_K^\alpha(x) \quad \text{for} \quad \alpha = -\lambda - \frac{n-1}{2},$$

and $h(t) = \widetilde{C}_K^{-\lambda-\frac{n-1}{2}}(-t)$ up to scalar (see (4.14).) Hence $f(\xi_1,\ldots,\xi_n)$ is a scalar multiple of $\frac{1}{(-\lambda-\frac{n-1}{2})_{[\frac{K+1}{2}]}} f_{K,\lambda}$.

Case 2. $\lambda \in \mathbb{N}$.

We set $k := \lambda + 1$. Suppose f is a homogeneous polynomial. As we saw, if $f \otimes 1_\lambda \in \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda)$, then f is $SO_o(p-1, q-2)$-invariant as far as $\deg f \neq k$. Suppose now $\deg f = k$. Then $(\lambda - E)\partial_{\xi_j} f$ vanishes, because $\partial_{\xi_j} f$ is homogeneous of degree $k - 1\,(= \lambda)$. In view of the formula (4.7) of $P_j(\lambda)$ for homogeneous polynomials f of degree k, $f \otimes 1_\lambda \in \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda)$ if and only if f satisfies the single equation $\Box f = 0$, namely, $f \in \mathcal{H}^k(\mathbb{R}^{p-1,q-1})$. Hence we have shown

$$\begin{aligned} \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda) &= \bigoplus_{K=1}^{\infty} \mathbb{C} w_{K,\lambda} \oplus \mathcal{H}^{\lambda+1}(\mathbb{R}^{p-1,q-1}) \otimes \mathbb{C}_\lambda \\ &= \bigoplus_{K=0}^{\infty} \mathbb{C} w_{K,\lambda} \oplus \bigoplus_{j=1}^{\lambda+1} H'_j. \end{aligned}$$

For $j = 0, \ldots, k\,(= \lambda + 1)$, let M_j be the $\mathfrak{g}'$-submodules in $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ generated by the singular vectors $\varphi(w_{\lambda+1-j,\lambda})$. Finally, let us prove

$$\varphi(H'_j) \subset M_j \quad (j = 0, 1, \cdots, \lambda + 1).$$

This is deduced from the following three claims, which we are going to prove:

- $\varphi(H'_j) \subset M_0 + M_1 + \cdots + M_{\lambda+1}$.
- $\sum_{i=0}^{\lambda+1} M_i$ is a direct sum of $\mathfrak{g}'$-modules.
- Among $\{M_0, M_1, \cdots, M_{\lambda+1}\}$, M_j is the unique $\mathfrak{g}'$-submodule that has the same $\mathfrak{Z}(\mathfrak{g}')$-infinitesimal character with that of the $\mathfrak{g}'$-module generated by $\varphi(H'_j)$.

In the F-method, the Lie algebra $\mathfrak{n}'_-$ acts on $\mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}] \otimes \mathbb{C}_\lambda$ by the multiplication of a linear function of $\xi_1, \cdots, \xi_{n-1}$. Hence we have

$$M_j = \varphi(\mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}] w_{\lambda+1-j,\lambda}).$$

Thus the first claim will be proved if we show

$$\mathrm{Pol}^{\lambda+1}[\xi_1, \cdots, \xi_n] \otimes 1_\lambda \subset \sum_{i=0}^{\lambda+1} \mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}] w_{\lambda+1-i,\lambda}, \tag{4.21}$$

where $\mathrm{Pol}^l[\xi_1, \cdots, \xi_n]$ stands for the space of homogeneous polynomials of degree l. To see (4.21), we observe that the coefficient of $\xi_n^K \otimes 1_\lambda$ in the polynomial $w_{K,\lambda}$ (see (4.16)) is a non-zero multiple of

$$\frac{(\alpha)_K}{(\alpha)_{[\frac{K+1}{2}]}} = (\alpha + [\frac{K+1}{2}]) \cdots (\alpha + K - 2)(\alpha + K - 1) \quad \text{with} \quad \alpha = -\lambda - \frac{n-1}{2},$$

which does not vanish if $K \leq \lambda + 1$. Hence we see by induction on l that $\sum_{K=0}^{l} \mathrm{Pol}[\xi_1, \cdots, \xi_n] w_{K,\lambda}$ coincides with $\sum_{K=0}^{l} \mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}] \xi_n^K \otimes 1_\lambda$ for all $l \leq \lambda + 1$. In particular, (4.21) is proved and the first claim is shown.

To see the second and third claims, let $\mathfrak{j}'$ be a Cartan subalgebra of $\mathfrak{g}' = \mathfrak{so}(n+1, \mathbb{C}) \simeq \mathfrak{so}(p, q-1) \otimes_{\mathbb{R}} \mathbb{C}$. We identify $(\mathfrak{j}')^* \simeq \mathbb{C}^{[\frac{n+1}{2}]}$ via the standard basis $\{e_1, e_2, \cdots, e_{[\frac{n+1}{2}]}\}$ so that $\mathbb{C}_\lambda$ is given by λe_1 ($\lambda \in \mathbb{C}$) and similarly for $\mathfrak{j}^* \simeq \mathbb{C}^{[\frac{n+2}{2}]}$. In the coordinates we have

$$\rho' = (\frac{n-1}{2}, \frac{n-3}{2}, \cdots, \frac{n-1}{2} - [\frac{n-1}{2}]) \in (\mathfrak{j}')^*.$$

Then the $\mathfrak{Z}(\mathfrak{g}')$-infinitesimal character of M_j is given by

$$(\lambda - (\lambda + 1 - j), 0, \cdots, 0) + \rho' = (j + \frac{n-3}{2}, \frac{n-3}{2}, \frac{n-5}{2}, \cdots, \frac{n-1}{2} - [\frac{n-1}{2}])$$

which are distinct for $j = 0, 1, \cdots, \lambda + 1 (= k)$ in $(\mathfrak{j}')^*/W(\mathfrak{g}')$ if $n \geq 3$. Therefore, the sum $M_0 + M_1 + \cdots + M_{\lambda+1}$ is a direct sum if $n \geq 3$.

Let us now consider one fixed summand H'_j for $j = 0, 1, \ldots, k$. Since such f is homogeneous of degree $k (= \lambda + 1)$, the $\mathfrak{Z}(\mathfrak{g}')$-infinitesimal character of the $\mathfrak{g}'$-module generated by $\varphi(H'_j)$ is given by

$$(-1, j, 0, \cdots, 0) + \rho' = (\frac{n-3}{2}, j + \frac{n-3}{2}, \frac{n-5}{2}, \cdots, \frac{n-1}{2} - [\frac{n-1}{2}]),$$

which coincides with that of M_j. Hence we have shown $\varphi(H'_j) \subset M_j$. $\square$

Theorem 4.2 gives a complete description of the set of singular vectors invariant with respect to $SO_o(p-1, q-2)$. In the positive signature, which corresponds to the $p = 1$ case, $SO(q-2)$-invariant singular vectors were found by A. Juhl in [23, Chapter 5] by a heavy combinatorial computation using recurrence relations. Notice that the Juhl's computations depend on the parity of K, and the higher dimensional components of singular vectors are more difficult to detect by algebraic methods. We would like to emphasize that our approach is very different and allows us a uniform treatment.

An important point is that the F-method gives a complete description of the set of singular vectors and its structure as an L'-module. In the second part of Theorem 4.2, we describe also all L'-submodules of higher dimensions, which will be used in the complete description of the composition series when the restriction is not completely reducible (see Theorem 4.10).

4.3.2 Equivariant differential operators for the conformal group

For $0 \leq j \leq N - 1$, we set

$$a_j(\lambda) \equiv a_j^{N,n}(\lambda) := \frac{(-2)^{N-j} N!}{j!(2N-2j)!} \prod_{k=j}^{N-1} (2\lambda - 4N + 2k + n + 1), \tag{4.22}$$

$$b_j(\lambda) \equiv b_j^{N,n}(\lambda) := \frac{(-2)^{N-j} N!}{j!(2N-2j+1)!} \prod_{k=j}^{N-1} (2\lambda - 4N + 2k + n - 1), \tag{4.23}$$

and $a_N(\lambda) = b_N(\lambda) := 1$. Here we have adopted the same notation with Juhl's book [23], i.e. [loc. cit., Corollary 5.1.1] for $a_j(\lambda)$ and [loc. cit., Corollary 5.1.3] for $b_j(\lambda)$ for the convenience of the reader.

Then it follows from (6.8) and (6.9) in Appendix that the formula (4.15) amounts to

$$w_{2N,\lambda} = (\sum_{j=0}^{N} a_j(\lambda)(-|\xi'|^2)^j \xi_n^{2N-2j}) \otimes 1_\lambda, \tag{4.24}$$

$$w_{2N+1,\lambda} = (\sum_{j=0}^{N} b_j(\lambda)(-|\xi'|^2)^j \xi_n^{2N-2j+1}) \otimes 1_\lambda, \tag{4.25}$$

because $\epsilon_n = -1$.

As stated in Theorem 2.4, the homomorphism between the generalized Verma modules of the Lie algebras $\mathfrak{g}'$ and $\mathfrak{g}$ induces an equivariant differential operator acting on local sections of induced homogeneous bundles on the generalized flag manifolds of the Lie groups G' and G. We shall describe these differential operators using the non-compact picture of the induced representation. The restriction from G to N_-P induces the non-compact model of the induced representation by the map

$$\beta : \mathrm{Ind}_P^G(\mathbb{C}_\lambda) \hookrightarrow \mathcal{C}^\infty(N_-) \simeq \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}).$$

Via the injection β, we get the following explicit form of G'-equivariant differential operator by replacing λ with $-\lambda$.

Theorem 4.3. *Let $(G, G') = (SO_o(p, q), SO_o(p, q-1))$, $p \geq 1$, $q \geq 2$, $n = p+q-2 > 2$, and $\lambda \in \mathbb{C}$.*

(1) The singular vectors $\varphi(w_{2N,-\lambda}) \in M_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda)$ in Theorem 4.2 (1) induce (in the non-compact picture) the family $\mathcal{D}_{2N}(\lambda) : \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}) \to \mathcal{C}^\infty(\mathbb{R}^{p-1,q-2})$ of G'-equivariant differential operators given by $\mathcal{D}_{2N}f = (D_{2N}(\lambda)f)|_{x_n=0}$, where $D_{2N}(\lambda)$ is a differential operator of order $2N$ defined as follows:

$$D_{2N}(\lambda) := \sum_{j=0}^{N} a_j(-\lambda)(-\Box')^j (\frac{\partial}{\partial x_n})^{2N-2j}.$$

Here $\Box' = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{p-1}^2} - \frac{\partial^2}{\partial x_p^2} - \cdots - \frac{\partial^2}{\partial x_{n-1}^2}$ is the (ultra)-wave operator on $\mathbb{R}^{p-1,q-2}$. The infinitesimal intertwining property of $\mathcal{D}_{2N}(\lambda)$ is given explicitly as

$$\mathcal{D}_{2N}(\lambda) d\pi_\lambda^G(X) = d\pi_{\lambda+2N}^{G'}(X)\mathcal{D}_{2N}(\lambda) \quad \text{for } X \in \mathfrak{g}'. \tag{4.26}$$

Similarly, the singular vectors $\varphi(w_{2N+1,-\lambda}) \in M_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda)$ in Theorem 4.2 (1) define the family $\mathcal{D}_{2N+1}(\lambda) : \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}) \to \mathcal{C}^\infty(\mathbb{R}^{p-1,q-2})$ of G'-equivariant differential operators induced by

$$D_{2N+1}(\lambda) := \sum_{j=0}^{N} b_j(-\lambda)(-\Box')^j (\frac{\partial}{\partial x_n})^{2N-2j+1}.$$

The operator $\mathcal{D}_{2N+1}(\lambda)$ intertwines $d\pi_\lambda^G$ and $d\pi_{\lambda+2N+1}^{G'}$.

(2) The branching law $\mathcal{H}^k(\mathbb{R}^{p-1,q-1}) \simeq \bigoplus_{j=0}^k \mathcal{H}^j(\mathbb{R}^{p-1,q-2})$ (see (4.13)) is multiplicity-free, and we denote by π_j^k the corresponding orthogonal projections. Note that the $O(p-1, q-1)$-modules $\mathcal{H}^k(\mathbb{R}^{p-1,q-1})$ are isomorphic to the k-th symmetric and trace-free part $\odot_0^k(\mathbb{R}^{p-1,q-1})$ of the defining representation of $O(p-1, q-1)$.

For $-\lambda \in \mathbb{N}$, set $k = |\lambda| + 1$. Then we have G-equivariant differential operator

$$\mathcal{D}_k : \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}) \to \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}) \otimes \mathcal{H}^k(\mathbb{R}^{p-1,q-1}),$$

given by the set $f \otimes 1_{-\lambda}$, $f \in \mathcal{H}^k(\mathbb{R}^{p-1,q-1})$ of singular vectors. In the non-compact picture, the operator $\mathcal{D}_k$ corresponds to the (symmetric) trace-free part of the multiple gradient $\sigma \mapsto \nabla_{(a} \dots \nabla_{b)_0}\sigma$ (number of indices being k).

Moreover, for each $j = 0, 1, 2, \dots, k$, there are G'-equivariant differential operators

$$\mathcal{D}_{k,j} : \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}) \to \mathcal{C}^\infty(\mathbb{R}^{p-1,q-2}) \otimes \mathcal{H}^j(\mathbb{R}^{p-1,q-2});$$

given by the composition $\mathcal{D}_{k,j} = \pi^k_j \circ \mathcal{D}_k$, restricted to $\mathbb{R}^{p-1,q-2}$.

Proof. The first part of the theorem follows immediately from Theorem 4.2 (2) and from the fact that an element $X \in \mathfrak{n}_-(\mathbb{R})$ acts on functions in $\mathcal{C}^\infty(\mathfrak{n}_-(\mathbb{R}))$ by the derivative in the direction X.

The second part follows from Theorem 4.2 (3) and the well-known classification of differential operators on the sphere $S^n \simeq G/P$ equivariant with respect to the action of the conformal group (see, e.g., [42, Sections 8.6–8.9]). □

Remark 4.4. Denote the induced representation $\mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ by $\pi_{\lambda,+}(\equiv \pi_\lambda)$ and denote the representation $\mathrm{Ind}_P^G(\mathrm{sgn} \otimes \mathbb{C}_\lambda) \equiv \mathrm{Ind}_P^G((-1) \otimes \mathbb{C}_\lambda)$ induced from $p \mapsto \epsilon(m)a^\lambda$ by $\pi_{\lambda,-}$ (see (4.5) for notation). They give rise to the same action of the Lie algebra, but on the level of the induced representations we have the following intertwining relation

$$\mathcal{D}_K(\lambda)\pi^G_{\lambda,\epsilon_1}(g') = \pi^{G'}_{\lambda+K,\epsilon_2}(g')\mathcal{D}_K(\lambda), \tag{4.27}$$

where $g' \in G'$ and $\epsilon_1 \cdot \epsilon_2 = (-1)^K$, $K \in \mathbb{N}$.

The results obtained in the first part of the Theorem 4.3 generalize those obtained in [23, Chapter 5] for the positive definite signature. Our proof based on the F-method is completely different from [23], and is significantly shorter even in the $p = 1$ case.

The operator $\mathcal{D}_k$, defined in the second part of Theorem 4.3, is the first BGG operator in the BGG complex corresponding to the G-module given by the k-th symmetric traceless power of its fundamental vector representation. There are also explicit formulae for a majority of operators appearing in the BGG complexes in the compact picture (as well as for their curved versions in the BGG sequences), but the expressions are more complicated and contain many lower order curvature terms (see, [4]).

The second part of Theorem 4.3 is an example of a more general principle, which can be formulated as follows. Every G-equivariant differential operator $\mathcal{D}$, acting between sections of homogeneous bundles over G/P, induces by restriction to G'/P' a G'-equivariant differential operator. Moreover, we may compose G-equivariant differential operators or G'-equivariant differential operators to get G'-equivariant differential operators. This composition may be possible for a discrete set of λ. Such possibilities called factorization identities will be discussed (for densities) in Section 4.3.5 in more details.

4.3.3 The branching rules for Verma modules — generic case

The branching rules for generic parameters are obtained as a special case of the general theory stated in Section 3. The proof does not require the F-method.

Theorem 4.5. *For $\lambda \in \mathbb{C}\backslash\{\frac{1}{2}(k-n) : k = 2,3,4,\cdots\}$, the Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ decomposes as a direct sum of generalized Verma modules of $\mathfrak{g}'$:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}'} \simeq \bigoplus_{b\in\mathbb{N}} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda - b). \tag{4.28}$$

Proof. Apply Theorem 3.6 to the special case where $\mathfrak{g}^\tau = \mathfrak{g}'$ and

$$(\mathfrak{g}, \mathfrak{g}', \mathfrak{p}/\mathfrak{n}_+) \simeq (\mathfrak{so}(n+2,\mathbb{C}), \mathfrak{so}(n+1,\mathbb{C}), \mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C}).$$

Then $l = 1$ and $\nu_1 = -e_1$. Hence we get (4.28) from Theorem 3.6 as the identity in the Grothendieck group for all $\lambda \in \mathbb{C}$. The infinitesimal characters of $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda - b)$ are given by

$$(\lambda - b + \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-5}{2}, \cdots, \frac{n-1}{2} - [\frac{n-1}{2}]),$$

which are all distinct in $(\mathfrak{j}')^*/W(\mathfrak{g}')$ for $b \in \mathbb{N}$ if and only if $2\lambda + n \neq 2,3,4,\cdots$, whence the last statement. $\square$

We shall give in Corollary 4.15 a necessary and sufficient condition on λ for the irreducibility of $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda)$. The branching law in the singular case $\lambda \in \frac{1}{2}(k-n)$ $(k = 2,3,4,\cdots)$ will be treated in Theorem 4.10.

4.3.4 The branching rules — exceptional cases

For integral values of the inducing parameter, the branching law is not always a direct sum decomposition but may involve extensions. To understand this delicate structure, we shall apply the F-method again and use an explicit form of singular vectors. The description of the branching rules for exceptional parameters was earlier studied in a special case corresponding to Juhl's operator in [43].

Let us first notice that the $\mathfrak{g}$-module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \equiv M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ decomposes for all λ into an even and odd part as $\mathfrak{g}'$-modules. In the F-method, we take the Fourier transform of the Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$, and this decomposition is described as the decomposition of $\mathrm{Pol}[\xi_1, \ldots, \xi_n] \otimes \mathbb{C}_\lambda$ into

$$\left(\bigoplus_{k=0}^{\infty} \mathrm{Pol}[\xi_1, \ldots, \xi_{n-1}]\xi_n^{2k}\right) \otimes \mathbb{C}_\lambda \oplus \left(\bigoplus_{k=0}^{\infty} \mathrm{Pol}[\xi_1, \ldots, \xi_{n-1}]\xi_n^{2k+1}\right) \otimes \mathbb{C}_\lambda.$$

By the formula of $d\pi_\lambda(E_j)$ (see Lemma 4.1), it is easy to see that both summands are $\mathfrak{g}'$-submodules.

Any singular vector vector $\varphi(w_{\lambda,K})$ $(K \in \mathbb{N})$ in $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)^{\mathfrak{n}'_+}$ (see Theorem 4.2) generates a $\mathfrak{g}'$-submodule of $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$, which we denote by $V_K \equiv V_K(\lambda)$. Since $\mathfrak{n}_-$ acts freely on $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$, the $\mathfrak{g}'$-module V_K is isomorphic to the $\mathfrak{g}'$-Verma module $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-K})$. We note that the $\mathfrak{g}'$-submodule $\sum_{K\in\mathbb{N}} V_K$ in $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ is not necessarily a direct sum for exceptional parameter λ (see (4.29) below), where two $\mathfrak{g}'$-modules V_K and $V_{K'}$ may have the same $\mathfrak{Z}(\mathfrak{g}')$-infinitesimal character. In order to understand what happens about the $\mathfrak{g}'$-module structure of the $\mathfrak{g}$-module $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ in this case, we apply again the F-method – take the inverse Fourier transform (see (2.3))

$$V_K = \varphi(\mathrm{Pol}[\xi_1, \cdots, \xi_n]w_{K,\lambda})$$

and use an explicit formula for the singular vector $w_{K,\lambda}$ (see (4.16)).

In cases where the submodules V_{2N} and $V_{2N'}$ have the same infinitesimal character, we shall see (due to the knowledge of the explicit form of the singular vectors) that one of them is submodule of the other. In this case, we find a $\mathfrak{g}'$-submodule $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ which allows a non-splitting extension. We shall illustrate it in a number of examples. For this, we begin with explicit formulas of $w_{K,\lambda}$ ($K \leq 4$) as follows (see Appendix for the formula for $\widetilde{C}_K^\lambda(t)$ ($K = 0, 1, \cdots, 4$)):

$$
\begin{aligned}
&w_0 \equiv w_{0,\lambda} = 1 \otimes 1_\lambda, \\
&w_1 \equiv w_{1,\lambda} = \xi_n \otimes 1_\lambda, \\
&w_2 \equiv w_{2,\lambda} = (-(2\lambda + n - 3)\xi_n^2 - |\xi'|^2) \otimes 1_\lambda, \\
&w_3 \equiv w_{3,\lambda} = \xi_n(-\frac{1}{3}(2\lambda + n - 5)\xi_n^2 - |\xi'|^2) \otimes 1_\lambda, \\
&w_4 \equiv w_{4,\lambda} = (\frac{1}{3}(2\lambda + n - 5)(2\lambda + n - 7)\xi_n^4 + 2(2\lambda + n - 5)\xi_n^2|\xi'|^2 + |\xi'|^4) \otimes 1_\lambda,
\end{aligned}
$$

where we recall $|\xi'|^2 = \sum\limits_{i=1}^{n-1} \epsilon_i \xi_i^2$.

We set

$$\lambda_j := \frac{1}{2}(-n + 1 + j). \tag{4.29}$$

Example 4.6. *The case $\lambda = \lambda_1 = \frac{-n+2}{2}$. In this case, all the infinitesimal characters of the $\mathfrak{g}'$-submodules generated by singular vectors w_K ($K \in \mathbb{N}$) are mutually different except those corresponding to w_0 and w_1, which coincide. Due to the fact that the whole $\mathfrak{g}$-module splits into a direct sum of even and odd parts, there is no extension among these $\mathfrak{g}'$-modules and therefore, the branching is the same as in the generic case.*

Example 4.7. *The case $\lambda = \lambda_2 = \frac{-n+3}{2}$. In this case, the infinitesimal characters of $\mathfrak{g}'$-submodules generated by singular vectors w_K ($K \in \mathbb{N}$) coincide only for w_0 and w_2, and all others are mutually different. We compare w_0 and w_2. For $\lambda = \lambda_2$, the first term of $w_2 \equiv w_{2,\lambda}$ vanishes, and w_2 reduces to $-|\xi'|^2 \otimes 1_\lambda$. Hence*

$$\mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}] w_{0,\lambda} \supset \mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}] w_{2,\lambda}$$

for $\lambda = \lambda_2$. In turn, we have $V_0(\lambda_2) \supset V_2(\lambda_2)$.

Thus for $\lambda = \lambda_2 = \frac{-n+3}{2}$, the $\mathfrak{g}'$-submodule $V_0(\lambda_2)$ generated by w_0 contains the unique nontrivial submodule $V_2(\lambda_2)$, generated by w_2. On the other hand, the direct sum M_{02} of the $U(\mathfrak{n}'_-)$-span of $w_0 = 1_\lambda$ and the $U(\mathfrak{n}'_-)$-submodule generated by the vector $\xi_n^2 \otimes 1_\lambda$ is invariant under the action of $\mathfrak{g}'$, and it is a (non-split) extension

$$0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_2) \to M_{02} \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_2 - 2) \to 0. \tag{4.30}$$

All the other infinitesimal characters are mutually different, hence the branching rule is now given by

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_2) \simeq M_{02} \oplus \bigoplus_{b \in \mathbb{N}, b \neq 0,2} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_2 - b).$$

Example 4.8. *The case $\lambda = \lambda_3 = \frac{-n+4}{2}$. In this case, the infinitesimal characters of $\mathfrak{g}'$-submodules generated by singular vectors w_0, w_3 respectively w_1, w_2 coincide, and both characters are different from each other, and differ from all others (which are also mutually different). But again due to the fact that the whole $\mathfrak{g}$-module splits into a direct sum of even and odd parts, the whole branching is again the same as in generic case.*

Example 4.9. *Let $\lambda = \lambda_4 = \frac{-n+5}{2}$. The explicit formula of singular vectors $w_{K,\lambda}$ shows that in this case the vectors shows the first two terms of $w_4 = w_{4,\lambda}$ vanish, and w_4 reduces to $|\xi'|^4 \otimes 1_\lambda$. Hence*

$$\mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}]w_{0,\lambda} \supset \mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}]w_{4,\lambda}$$

for $\lambda = \lambda_4$. In turn, we have $V_0(\lambda_4) \supset V_4(\lambda_4)$. Of course, these two $\mathfrak{g}'$-modules have the same infinitesimal character. Another couple with the same infinitesimal characters (but different from the previous couple) are the two $\mathfrak{g}'$-submodules $V_1(\lambda_4)$ and $V_3(\lambda_4)$ generated by w_1 and w_3, respectively.

Returning to $w_0, w_4 \in \mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}] \otimes 1_\lambda$ for $\lambda = \lambda_4$, we consider

$$M_{04} := \varphi(\mathrm{Pol}[\xi_1, \cdots, \xi_{n-1}]\, \mathbb{C}\text{-span}\{1, \xi_n^2, \xi_n^4\} \otimes 1_\lambda).$$

It turns out that the $U(\mathfrak{n}'_-)$-submodule M_{04} in $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ is $\mathfrak{g}'$-invariant. Clearly, $V_4(\lambda_4) \subset V_0(\lambda_4) \subset M_{04}$. Furthermore, it is possible to show we have a non-splitting exact sequence of $\mathfrak{g}'$-modules:

$$0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4) \to M_{04} \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4 - 4) \to 0. \tag{4.31}$$

Similarly, there is a non-trivial extension

$$0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4 - 1) \to M_{13} \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4 - 3) \to 0 \tag{4.32}$$

of the modules generated by w_1 and w_3, denoted by M_{13}, and the branching rule is

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_4) \simeq M_{04} \oplus M_{13} \oplus \bigoplus_{b \in \mathbb{N}, b \neq 0,1,3,4} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4 - b).$$

We generalize these observations and obtain the following precise description of the extensions among branching laws for integral parameters:

Theorem 4.10. *Recall* (4.29) *for the definition of λ_j.*

(1) *Suppose $j = 2k + 1$ with $k \in \mathbb{N}_+$. Then the branching is the same as in the generic case:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_{2k+1})|_{\mathfrak{g}'} \simeq \bigoplus_{b \in \mathbb{N}} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k+1} - b) \quad \text{(direct sum).} \tag{4.33}$$

(2) *Suppose $j = 2k$ with $k \in \mathbb{N}_+$. Then there exists a $\mathfrak{g}'$-submodule $M_{a,2k-a} \subset M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_{2k})$ for each $a = 0, \ldots, k-1$ with the following two properties: The restriction $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_{2k})$ decomposes into a direct sum of $\mathfrak{g}'$-modules:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_{2k})|_{\mathfrak{g}'} \simeq \bigoplus_{a=0}^{k-1} M_{a,2k-a} \oplus M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - k) \oplus \bigoplus_{b=2k+1}^{\infty} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - b),$$

and there exists a non-split exact sequence of $\mathfrak{g}'$-modules:

$$0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - a) \to M_{a,2k-a} \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - (2k - a)) \to 0. \tag{4.34}$$

In order to give a proof of the theorem, we begin with the following elementary but useful observation.

Lemma 4.11. *Suppose N is a $\mathfrak{g}$-module in the category $\mathcal{O}$, and V_1, V_2 are two submodules of N satisfying the following three conditions:*

1) (character identity) $\mathrm{Ch}(V_1) + \mathrm{Ch}(V_2) = \mathrm{Ch}(N)$,

2) V_1 is irreducible,

3) $\dim \mathrm{Hom}_{\mathfrak{g}}(V_1, V_2) = \dim \mathrm{Hom}_{\mathfrak{g}}(V_1, N) = 1$.

Then there exists a non-split exact sequence of $\mathfrak{g}$-modules:

$$0 \to V_2 \to N \to V_1 \to 0. \tag{4.35}$$

Proof. We note that $V_1 \subset V_2 \subset N$ from the third condition. Since V_1 is irreducible, V_1 is isomorphic to the quotient N/V_2 by the first condition. Thus we have an exact sequence (4.35) of $\mathfrak{g}$-modules. Then the third condition implies that (4.35) does not split. $\square$

The next lemma analyzes a relationship between two singular vectors with the same infinitesimal characters.

Lemma 4.12. *For $a, k \in \mathbb{N}$ such that $a \le 2k$, the Gegenbauer polynomials satisfy:*

$$C_a^{-k}(s) = C_{2k-a}^{-k}(s) \text{ and } \mathcal{C}_a^{-k}(s) = s^{a-k}\mathcal{C}_{2k-a}^{-k}(s).$$

Proof. By using the identity $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$, the polynomial expression (6.4) of the Gegenbauer polynomial can be written as

$$C_a^{-k}(s) = (-1)^a k! \sum_{i=0}^{[\frac{a}{2}]} \frac{(2s)^{a-2i}}{\Gamma(1-a+i+k)i!(a-2i)!}.$$

Switching a with $2k-a$, we have

$$C_{2k-a}^{-k}(s) = (-1)^{2k-a} k! \sum_{i=0}^{[\frac{2k-a}{2}]} \frac{(2z)^{2k-a-i}}{\Gamma(1-k+a+i)i!(2k-a-2i)!},$$

where the terms for $i = 0, \cdots, k-a-1$ vanish if $a < k$. Putting $j = i + a - k$, we have

$$C_{2k-a}^{-k}(s) = (-1)^a k! \sum_{j=0}^{[\frac{a}{2}]} \frac{(2z)^{a-2j}}{\Gamma(1+j)(j+k-a)!(a-2j)!}.$$

Hence $C_a^{-k}(s) = C_{2k-a}^{-k}(s)$. The last assertion follows immediately from the definition (4.14). $\square$

Lemma 4.13. *Let $k \in \mathbb{N}$. We denote by $f_a(\xi) \equiv f_{a,\lambda_{2k}}(\xi)$ ($a \in \mathbb{N}$) the polynomials defined in* (4.15)*. Then for any $a \in \mathbb{N}$ such that $a \le k$, we have*

$$f_{2k-a}(\xi) = (\sum_{i=1}^{n-1} \epsilon_i \xi_i^2)^{k-a} f_a(\xi).$$

Proof. Let $t = \xi_n^{-2} \sum_{i=1}^{n-1} \epsilon_i \xi_i^2$. By (4.15) and Lemma 4.12, we have

$$f_a(\xi) = \xi_n^a \mathcal{C}_a^{-k}(t) = \xi_n^a t^{a-k} \mathcal{C}_{2k-a}^{-k}(t).$$

Hence $f_{2k-a}(\xi) = \xi_n^{2k-a}\mathcal{C}_{2k-a}^{-k}(t) = (\sum_{i=1}^{n-1} \epsilon_i \xi_i^2)^{k-a} f_a(\xi)$. $\square$

Proof of Theorem 4.10. Since there is no extension between two modules with distinct generalized infinitesimal characters, we can collect the terms of the same generalized infinitesimal characters as direct summands.

Let us examine this decomposition in our setting. First we observe that the identity (4.28) holds in the Grothendieck group for all $\lambda \in \mathbb{C}$ by Theorem 3.6. We recall from (4.29) that $\lambda_j = \frac{1}{2}(-n+1+j)$. This shows that generalized $\mathfrak{Z}(\mathfrak{g}')$-infinitesimal characters decomposition of $\mathfrak{g}'$-modules:

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_j)|_{\mathfrak{g}'} = \bigoplus_{\substack{b\in\mathbb{N}\\ 2b\geq j}} N_b,$$

that appear in the direct summands of the restriction $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_j)|_{\mathfrak{g}'}$ are of the form

$$(\frac{j}{2}-b, \frac{n-3}{2}, \frac{n-5}{2}, \dots, \frac{n-1}{2} - [\frac{n-1}{2}]),$$

for some $b \in \mathbb{N}$ with $\frac{j}{2} \leq b$. Correspondingly, in the Grothendieck group, or equivalently, as the character identity, we have

$$N_b = \begin{cases} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_j - b) & (j < b), \\ M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_j - b) + M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_j - j + b) & (\frac{j}{2} < b \leq j), \\ M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_j - b) & (b = \frac{j}{2}). \end{cases}$$

On the other hand, the $\mathfrak{g}'$-module $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_j - b)$ is irreducible for any $b \in \mathbb{N}$ with $\frac{j}{2} \leq b$ by Corollary 4.15 below. Let us consider the $\mathfrak{g}'$-module structure of N_b for $\frac{j}{2} < b \leq j$. It follows from Theorem 4.2 that w_{b,λ_j} and w_{j-b,λ_j} $\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_{\lambda_j})$ generate two $\mathfrak{g}'$-submodules in $\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_{\lambda_j}$, to be denoted by M_b and M_{j-b}, which are isomorphic to $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_j - b)$ and $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_j - j + b)$, respectively.

Furthermore, if $j = 2k$ then by Lemma 4.13 with $b := 2k - a$ on an explicit knowledge of singular vectors we have

$$f_b(\xi) = (\sum_{j=1}^{n-1} \varepsilon_i \xi_i^2)^{b-k} f_{2k-b}(\xi)$$

for $b \geq k$. Thus M_{2k-b} is a submodule of M_b. The theorem follows by application of Lemma 4.11. Here we take V_1 to be M_b and V_2 to be M_{j-b}. $\square$

4.3.5 Factorization identities

Let us return back to the Example 4.7. For $\lambda_2 = \frac{-n+3}{2}$, the action of $\mathfrak{g}'$ on the top two singular vectors w_0 and w_2 generate the $\mathfrak{g}'$-submodules V_0 and V_2 in the $\mathfrak{g}$-module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_2)$, respectively. The second one is a submodule of the first one. The corresponding inclusion is a $\mathfrak{g}'$-homomorphisms ψ, whose dual differential operator is the conformally invariant Yamabe operator. If we denote by ϕ_0 and ϕ_2 the inclusions of V_0 and V_2, respectively, into $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_0)$, we get the relation

$$\phi_2 = \phi_0 \circ \psi.$$

The F-method explains this factorization as

$$f_2(\xi) = (\sum_{i=1}^{n-1} \epsilon_i \xi_i^2) f_0(\xi)$$

by applying Lemma 4.13 with $a = 0$ and $k = 1$.

As another example, let us consider the weight $\lambda_4 = -\frac{n}{2} + \frac{5}{2}$. Then the $\mathfrak{g}'$-submodule V_1 generated by the singular vector w_1 and the $\mathfrak{g}'$-submodule V_3 generated by the singular vector w_3 have the same infinitesimal character. There exists a $\mathfrak{g}'$-homomorphism ψ from V_3 to V_1. The homomorphism ϕ_3 from V_3 to $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_0)$ can be factorized as $\phi_1 \circ \phi$. The F-method explains this factorization as

$$f_3(\xi) = (\sum_{i=1}^{n-1} \epsilon_i \xi_i^2) f_1(\xi)$$

by applying Lemma 4.13 with $a = 1$ and $k = 2$.

Hence for some particular discrete subset of values for λ, there is a possibility to factor an element in $\mathrm{Hom}_{\mathfrak{g}'}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda'), M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))$ as a composition of an element in the space $\mathrm{Hom}_{\mathfrak{g}'}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda'), M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda''))$ and an element in $\mathrm{Hom}_{\mathfrak{g}'}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda''), M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))$. There is also another possibility to factor an element in $\mathrm{Hom}_{\mathfrak{g}'}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda'), M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))$ as a composition of an element in $\mathrm{Hom}_{\mathfrak{g}'}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda'), M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda''))$ and in $\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda''), M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))$.

The fact that such a behaviour can happen only for discrete values of λ is a consequence of classification of homomorphisms of $\mathfrak{g}$-generalized Verma modules. These properties were discovered and used effectively for curved generalizations by A. Juhl (see [23, Chapter 6]) under the name factorization identities. It is not a special feature of this particular example with $G = SO_o(1, n+1)$ but it is a more general fact. It holds not only in Juhl's case (the scalar case) but also in spinor-valued case. If we consider not only differential intertwining operators for the restriction but also continuous intertwining operators ("symmetry breaking operators"), then the factorization identity may hold for continuous values of parameters see [31], [36, Chapters 8, 12]. In the F-method, the factorization identities are derived from the identities of two polynomials in Sol such as the formula given in Lemma 4.13. This viewpoint will be pursued in the second part of the series [32]. See also [35, Sect.9].

In the dual language of differential operators the factorization is described as follows: The first example above expresses the Juhl operator D_2 as the composition of the operator D_0 and the Laplace operator. The second example shows that the operator D_3 is given by the composition of D_1 and the Laplace operator.

4.4 The case $G = G' = SO_o(p, q)$.

We consider an application of the F-method to the special case $G' = G$. In this case, we do not need branching laws. Even in this classical situation, we shall observe that the F-method yields a simple and new independent construction of all differential intertwining operators for $G = SO_o(p, q)$-modules induced from densities.

For $\lambda \in \mathbb{C}$ we recall from Section 4.1 that $\mathbb{C}_\lambda$ the one-dimensional representation of the parabolic subgroup P. We write $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ for $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ as before. With the notation as in Theorem 4.2, we give a classification of all singular vectors via the bijection $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)^{\mathfrak{n}_+} \underset{\varphi}{\overset{\sim}{\leftarrow}}$ $\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}; \mathbb{C}_\lambda)$ by the next proposition.

Proposition 4.14. *Let $\mathfrak{g}(\mathbb{R}) = \mathfrak{so}(p, q)$ and $\lambda \in \mathbb{C}$. Recall $n = p + q - 2$ and $w_{K,\lambda}$ is defined* *by the formula in* (4.16). *Then we have:*

n: even

$$\mathrm{Sol}(\mathfrak{g},\mathfrak{g};\mathbb{C}_\lambda) \simeq \begin{cases} \mathbb{C}w_{0,\lambda} \oplus \mathbb{C}w_{2\lambda+n,\lambda} & \text{if } \lambda+\frac{n}{2} \in \mathbb{N}_+ \ , \\ \mathbb{C}w_{0,\lambda} \oplus \mathbb{C}w_{2\lambda+n,\lambda} \oplus \mathcal{H}^{\lambda+1}(\mathbb{R}^{p-1,q-1}) & \text{if } \lambda \in \mathbb{N}, \\ \mathbb{C}w_{0,\lambda} & \text{otherwise.} \end{cases}$$

n:odd

$$\mathrm{Sol}(\mathfrak{g},\mathfrak{g};\mathbb{C}_\lambda) \simeq \begin{cases} \mathbb{C}w_{0,\lambda} \oplus \mathbb{C}w_{2\lambda+n,\lambda} & \text{if } \lambda+\frac{n}{2} \in \mathbb{N}_+, \\ \mathbb{C}w_{0,\lambda} \oplus \mathcal{H}^{\lambda+1}(\mathbb{R}^{p-1,q-1}) & \text{if } \lambda \in \mathbb{N}, \\ \mathbb{C}w_{0,\lambda} & \text{otherwise.} \end{cases}$$

Proof of Proposition 4.14. Most of the proof was already given in that of Theorem 4.2. In particular, we see

$$\mathcal{H}^{\lambda+1}(\mathbb{R}^{p-1,q-1}) \subset \mathrm{Sol}(\mathfrak{g},\mathfrak{g};\mathbb{C}_\lambda) \quad \text{for all } \lambda \in \mathbb{N}.$$

In view of the obvious inclusion $\mathrm{Sol}(\mathfrak{g},\mathfrak{g};\mathbb{C}_\lambda) \subset \mathrm{Sol}(\mathfrak{g},\mathfrak{g}';\mathbb{C}_\lambda)$, it suffices to determine for which K and λ the vector $w_{K,\lambda}$ belongs to $\mathrm{Sol}(\mathfrak{g},\mathfrak{g};\mathbb{C}_\lambda)$ when $K \in \mathbb{N}_+$. This is equivalent to the condition that $f_{K,\lambda}$ defined in (4.15) is $\mathfrak{so}(p,q)$-invariant. The form of the polynomial $f_{K,\lambda}$, the relation $\epsilon_n = -1$, and the invariance of $f_{K,\lambda}$ with respect to $\mathfrak{so}(p,q)$ imply that $C_K^{-\lambda-\frac{n-1}{2}}(s)$ is a multiple of $(1-s^2)^m$ for some $m \in \mathbb{N}$. This happens if and only if

$$K = 2m \quad \text{and} \quad \lambda + \frac{n}{2} = m.$$

To see this, we verify whether or not $(1-s^2)^m$ satisfies the Gegenbauer differential equation like $C_{2m}^\alpha(s)$ for $\alpha = -\lambda - \frac{n-1}{2}$. Since

$$((1-s^2)\frac{d^2}{ds^2} - (1+2\alpha s^2)\frac{d}{ds} + 4m(m+\alpha)s^2)(1-s^2)^m = 2m(2m+2\alpha-1)s^2(1-s^2)^{m-1},$$

this is zero if and only if $2m+2\alpha-1=0$, namely, $\lambda+\frac{n}{2}=m$. Hence Proposition 4.14 is proved. ☐

Forgetting a concrete description of singular vectors in Proposition 4.14, we still have the following abstract result as a corollary:

Corollary 4.15. *The generalized Verma module $M_\mathfrak{p}^\mathfrak{g}(\lambda)$ is irreducible if and only if $\frac{n}{2}+\lambda \notin \mathbb{N}_+$ (n even); $\lambda \notin \mathbb{N}$ and $\frac{n}{2}+\lambda \notin \mathbb{N}_+$ (n odd).*

If F is a homomorphism from a generalized Verma module $M_\mathfrak{p}^\mathfrak{g}(V)$ to $M_\mathfrak{p}^\mathfrak{g}(\mathbb{C}_\lambda)$, then the image of $1 \otimes V$ by F is an $\mathfrak{l}$-irreducible subspace of $M_\mathfrak{p}^\mathfrak{g}(\lambda)^{\mathfrak{n}_+} = \varphi(\mathrm{Sol}(\mathfrak{g},\mathfrak{g};\mathbb{C}_\lambda))$. In such a way, Proposition 4.14 gives not only a new proof of the well-known classification of all homomorphisms from a generalized Verma module $M_\mathfrak{p}^\mathfrak{g}(\mathbb{C}_\lambda)$ for this specific pair $(\mathfrak{g},\mathfrak{p})$, but also an explicit construction of such homomorphisms by special values of the Gegenbauer polynomials.

In the dual language, the singular vector $w_{0,\lambda}$ gives the identity operator on the induced representations $\mathrm{Ind}_P^G(\mathbb{C}_{-\lambda})$, whereas $w_{2\lambda+n,\lambda}$ and $\mathcal{H}^{\lambda+1}(\mathbb{R}^{p-1,q-1})$ give rise to G-intertwining differential operators

$$\mathrm{Ind}_P^G(\mathbb{C}_{\frac{n}{2}-m}) \to \mathrm{Ind}_P^G(\mathbb{C}_{\frac{n}{2}+m}), \tag{4.36}$$

$$\mathrm{Ind}_P^G(\mathbb{C}_{1-k}) \to \mathrm{Ind}_P^G((-1)^k \otimes \mathcal{H}^k(\mathbb{R}^{p-1,q-1}) \otimes \mathbb{C}_1), \tag{4.37}$$

with $m = \lambda + \frac{n}{2}, k = 1 + \lambda \in \mathbb{N}$, respectively. In the non-compact picture, (4.36) is given by the m-th power of the Laplacian

$$\Box^m : \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}) \to \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}),$$

where $\Box = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{p-1}^2} - \frac{\partial^2}{\partial x_p^2} - \cdots - \frac{\partial^2}{\partial x_{p+q-2}^2}$. For the operator (4.37), we use the fact that the representation of $SO_o(p-1, q-1)$ on $\mathcal{H}^k(\mathbb{R}^{p-1,q-1})$ is self-dual. Then (4.37) is described in the non-compact picture by

$$\mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}) \otimes \mathcal{H}^k(\mathbb{R}^{p-1,q-1}) \to \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}),$$
$$(u(x), f(\xi)) \mapsto f(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{p+q-2}})u.$$

Note that the powers $\Box^m, m = \lambda + \frac{n}{2}$ with $\lambda \leq 0$ are special in the following sense. There exists their 'curved' versions, i.e., on any manifold with a given conformal structure, there are conformally invariant operators of order $m = 1, \ldots, \frac{n}{2}$ with symbol equal to $\Box^m$. These operators were constructed in [17], and they are usually called the GJMS operators. The structure of lower order curvature terms is very complicated and was studied in many publications (see, e.g., [24] and the references in [23]). On the contrary, curved analogues of $\Box^m$ for $m > \frac{n}{2}$ do not exist ([20]).

The series $\mathcal{D}_k$ of operators constructed above are examples of the first BGG operators (equations for the conformal Killing tensors in Theorem 4.3) given by the projection to the symmetric trace-free part of the multiple gradient $\nabla_{(a} \ldots \nabla_{b)_0}\sigma$ (number of indices being k). As the trace-free condition translates by the Fourier transform to the harmonicity condition, the operators $\mathcal{D}_k$ correspond by Proposition 4.14 to $\mathcal{H}^k(\mathbb{R}^{p-1,q-1})$ with $k := \lambda + 1 \in \mathbb{N}_+$.

5 Dirac operators and $Spin_o(p, q)$

In the present section we extend the scalar-valued results considered in Section 4.3 to those for spinor-valued sections. The symmetries for the base manifolds remain the same, given by the pair of Lie groups $(\widetilde{G}, \widetilde{G}') = (Spin_o(p, q), Spin_o(p, q-1))$. The main results are Theorems 5.7 and 5.11.

5.1 Notation

We shall use the same convention as in Section 4. Let $p \geq 1, q \geq 2, n = p+q-2, n = n'+1$, and we suppose that the quadratic form (4.1) on $\mathbb{R}^{p+q} = \mathbb{R}^{n+2}$ is given as in Section 4.1. Let us consider the associated Clifford algebra $\mathcal{C}_{p,q}$. It is generated by an orthonormal basis $e_0, \ldots, e_{p+q-1}$ with the relations

$$e_i^2 = -\epsilon_i \text{ for } i = 1, \ldots, p+q-2 \text{ and } e_0 e_{p+q-1} + e_{p+q-1} e_0 = 1.$$

Let $\mathcal{C}_{p-1,q-1}$ be its subalgebra generated by $e_1, \ldots, e_{p+q-2}$. We realize $Spin(p, q)$ in $\mathcal{C}_{p,q}$ and define $\tilde{G}$ to be the identity component $Spin_o(p, q)$. We write

$$\Pi : \ Spin_o(p, q) \to SO_o(p, q)$$

for the canonical homomorphism, which is a double covering. Via Π, $\tilde{G}$ acts on $\mathbb{R}^{p,q}$ preserving the null cone $\mathcal{N}_{p,q}$ of $\mathbb{R}^{p,q}$ and the projective null cone $\mathbb{P}\mathcal{N}_{p,q}$. We shall keep the notation as in Section 4.1. The subgroup $\tilde{P} \subset \tilde{G}$ is defined as the stabilizer of the chosen null line generated by the vector $(1,0,\dots,0)$, namely, $\tilde{P}=\Pi^{-1}(P)$. According to the Langlands decomposition $P=LN_+=MAN_+$ in $G=SO_o(p,q)$, we have a Langlands decomposition

$$\tilde{P}=\tilde{L}N_+=\tilde{M}AN_+$$

by setting $\tilde{L}:=\Pi^{-1}(L)$ and $\tilde{M}:=\Pi^{-1}(M)\simeq Spin(p-1,q-1)$. Here by a little abuse of notation, we regard A and N_+ as subgroups of $\tilde{G}$.

The Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{p}}$ are isomorphic to $\mathfrak{g}$ and $\mathfrak{p}$, respectively, considered in the case $G=SO_o(p,q)$. We take a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ so that $\mathfrak{h}\subset\mathfrak{l}$. Let us denote by $\mathbb{S}^n_\pm\equiv\mathbb{S}^{p-1,q-1}_\pm$ the irreducible half-spin representations for $\tilde{M}\simeq Spin(p-1,q-1)$ with $n=p+q-2$ even, and $\mathbb{S}^n\equiv\mathbb{S}^{p-1,q-1}$ the spin representation for $\tilde{M}\simeq Spin(p-1,q-1)$ with $n=p+q-2$ odd. We have $\mathbb{S}^n_\pm\simeq\mathbb{S}^{n-1}$ for n even and $\mathbb{S}^n\simeq\mathbb{S}^{n-1}_+\oplus\mathbb{S}^{n-1}_-$ for n odd. By an abuse of notation, we write $\mathbb{S}$ for $\mathbb{S}_\pm$ in the proof, since the differential action is the same. The differential action of the spinor representation is given by

$$\mathfrak{so}(p-1,q-1)\to\mathcal{C}_{p,q},\quad \epsilon_i\epsilon_jE_{ij}-E_{ji}\mapsto-\frac{1}{2}\epsilon_ie_ie_j\quad(1\le i\ne j\le n).$$

Here E_{ij} stands for the matrix unit with 1 at the (i,j)-component, and we recall that $\{X_{ij}=\epsilon_i\epsilon_jE_{ij}-E_{ji}:1\le i<j\le n=p+q-2\}$ forms a basis of $\mathfrak{so}(p-1,q-1)$.

5.2 Representations $d\pi_\lambda$ and $d\tilde{\pi}_\lambda$.

For $\lambda\in\mathbb{C}$, we define the twisted spinor representation $\mathbb{S}_\lambda$ of the Levi factor $\widetilde{L}=\tilde{M}A$ as the outer tensor product $\mathbb{S}\otimes\mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional representation of A with the same normalization as in Section 4.2. The differential representation of $\widetilde{\mathfrak{l}}\simeq\mathfrak{l}$ has the highest weight $(\lambda+\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2})$ (and $(\lambda+\frac{1}{2},\frac{1}{2},\dots,-\frac{1}{2})$ for n even). We extend $\mathbb{S}_\lambda$ to a representation of $\widetilde{P}$ by letting the unipotent radical N_+ act trivially. The (unnormalized) induced representation $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}\mathbb{S}_\lambda$, is denoted by $\pi_{S,\lambda}$, or simply by π_λ. The representation space is identified with $\mathcal{C}^\infty(\tilde{G},\mathbb{S}_\lambda)^{\tilde{P}}$, consisting of smooth functions $F:\widetilde{G}\to\mathbb{S}_\lambda$ subject to

$$F(g\,\tilde{m}an)=a^{-\lambda}\mathbb{S}(m)^{-1}F(g),\quad\text{for all } g\in\tilde{G},\ \tilde{p}=\tilde{m}an\in\tilde{P},$$

where

$$\Pi(\tilde{m}an)=\begin{pmatrix}\epsilon(m)a&\star&\star\\0&m&\star\\0&0&\epsilon(m)a^{-1}\end{pmatrix},\ m\in SO(p-1,q-1),\ a>0,$$

and $\epsilon(m)=+1$ or -1 according to whether or not m belongs to the identity component of $SO_o(p-1,q-1)$.

Similarly, we may treat $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\mathrm{sgn}\otimes\mathbb{S}_\lambda)$ by the condition $F(g\tilde{m}\tilde{a}\tilde{n})=\epsilon(m)^{-1}a^{-\lambda}\mathbb{S}(m^{-1})F(g)$ as in the scalar case (see Remark 4.4), but we omit it because the differential action is the same and the main results hold by a small modification in the signature cases.

Restricting to the open Bruhat cell, we have the non-compact picture $\mathcal{C}^\infty(\mathbb{R}^{p-1,q-1},\mathbb{S}_\lambda)$ of the induced representation $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\mathbb{S}_\lambda)$. In order to calculate the action of $d\pi_\lambda(Z)$ in the non-compact picture for $Z\in\mathfrak{n}_+(\mathbb{R})$, we apply the previous computation in the scalar case. We already observed in (4.11) that if $Z\in\mathfrak{n}_+(\mathbb{R})$ and $X\in\mathfrak{n}_-(\mathbb{R})$ are sufficiently small, then the element $\tilde{m}\in\widetilde{M}$ determined by the condition

$$(\exp Z)^{-1}\exp X\in N_-\tilde{m}AN_+$$

behaves as

$$\operatorname{Id}-\Pi(\tilde{m}) \sim \mathbb{J}{}^tZ \otimes {}^tX\mathbb{J} - X \otimes Z = \sum_{i,j=1}^{n} z_i x_j(\epsilon_i\epsilon_j E_{ij} - E_{ji}), \tag{5.1}$$

up to the first order in $\|Z\|$. The right-hand side of (5.1) acts as the multiplication by the element

$$\begin{aligned} -\frac{1}{2}\sum_{i\neq j}^{n} z_i x_j \epsilon_i e_i e_j &= -\frac{1}{2}\Big((\sum_{i=1}^{n}\epsilon_i z_i e_i)(\sum_{j=1}^{n} x_j e_j) - \sum_{j=1}^{n} z_i x_i \epsilon_i e_i^2\Big) \\ &= -\frac{1}{2}(\underline{z}\,\underline{x} + \sum_{i=1}^{n} x_i z_i) \end{aligned} \tag{5.2}$$

in the corresponding Clifford algebra, where $\underline{x} = \sum_1^n x_i e_i$, and $\underline{z} = \sum_1^n \epsilon_i z_i e_i$.

We define differential operators on $\mathfrak{n}_+(\mathbb{R}) \simeq \mathbb{R}^n$ in the coordinates $(\xi_1, \cdots, \xi_{n-1}, \xi_n)$ by

$$D := \sum_{k=1}^{n} e_k \partial_{\xi_k} = D' + e_n\partial_{\xi_n} \qquad \text{(the Dirac operator on } \mathbb{R}^{p-1,q-1}\text{)}, \tag{5.3}$$

$$E := \sum_{j=1}^{n-1} \xi_j\partial_{\xi_j} + \xi_n\partial_{\xi_n} \qquad \text{(the Euler homogeneity operator)},$$

$$\Box := -D^2 = \Box' - \partial_{\xi_n}^2, \quad \Box' = \sum_{j=1}^{n-1}\epsilon_j\partial_{\xi_j}^2.$$

Summarizing the information obtained so far, we get the following claim as in Lemma 4.1:

Lemma 5.1. *The basis element $E_j \in \mathfrak{n}_+(\mathbb{R})$ $(1 \le j \le n)$ acts on $\mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}_\lambda)$ (i.e., in the non-compact picture for $\mathcal{C}^\infty(\tilde{G}, \mathbb{S}_\lambda)^{\tilde{P}}$) in the coordinates $\{x_1, \cdots, x_n\}$ of $\mathfrak{n}_-(\mathbb{R}) \simeq \mathbb{R}^{p-1,q-1}$ by*

$$d\pi_{S,\lambda}(E_j) = -\frac{1}{2}\epsilon_j|X|^2\partial_{x_j} + x_j(\lambda + \sum_k x_k\partial_{x_k} + \frac{1}{2}) + \frac{1}{2}(\epsilon_j e_j \underline{x}). \tag{5.4}$$

The dual action composed with the Fourier transform is given by

$$d\tilde{\pi}_{S,\lambda}(E_j) = i\left(\frac{1}{2}\epsilon_j\xi_j\Box + (\lambda - E - \frac{1}{2})\partial_{\xi_j} - \frac{1}{2}\epsilon_j e_j D\right). \tag{5.5}$$

Proof. It follows from (5.2) that

$$d\pi_{S,\lambda}(E_j) = d\pi_\lambda(E_j) + \frac{1}{2}(x_j + \epsilon_j e_j \underline{x}),$$

whence the formula (5.4). In turn,

$$d\tilde{\pi}_{S,\lambda}(E_j) = P_j(\lambda) - \frac{i}{2}(\partial_{\xi_j} + \epsilon_j e_j D),$$

whence the formula (5.5) owing to Lemma 4.1. $\Box$

5.3 The space Sol of singular vectors

In the scalar case, we proved in Theorem 4.2 that $\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{C}_\lambda)$ consists of polynomials which are invariant under $\mathfrak{m}' \simeq \mathfrak{so}(n-1, \mathbb{C})$ as far as $\lambda \notin \mathbb{N}$. In the spinor case, we shall consider first such invariant solutions. For this, we work with $\mathcal{C}_n^{\mathbb{C}} (\simeq \mathcal{C}_{p-1,q-1} \otimes_{\mathbb{R}} \mathbb{C})$-valued polynomials in $\xi_1, \cdots, \xi_n$, on which $\widetilde{M} \simeq Spin(p-1, q-1)$ acts as

$$s \mapsto gs(g^{-1}\cdot) \quad \text{for} \quad s \in \mathrm{Pol}[\xi_1, \cdots, \xi_n] \otimes \mathcal{C}_n^{\mathbb{C}}.$$

It is obvious that the following elements

$$\underline{\xi'} := \sum_{j=1}^{n-1} \epsilon_j \, e_j \, \xi_j, \; \underline{\xi_n} := \epsilon_n \, e_n \xi_n, \xi_n$$

belong to $(\mathrm{Pol}[\xi_1, \cdots, \xi_n] \otimes \mathcal{C}_n^{\mathbb{C}})^{\widetilde{M}'}$, and so does any polynomial generated by these three elements. We note $(\underline{\xi'})^2 = -|\xi'|^2 = -\sum_{j=1}^{n-1} \epsilon_j \xi_j^2$ and $(\underline{\xi_n})^2 = -\epsilon_n \xi_n^2 = \xi_n^2$ as $\epsilon_n = -1$. We set

$$t := \epsilon_n \frac{|\xi'|^2}{\xi_n^2}.$$

Then homogeneous polynomials F_K of degree K generated by $\underline{\xi'}, \underline{\xi_n}$ and ξ_n are written as follows: for $K = 2N$ it is of the form

$$F_{2N}(\xi_1, \cdots, \xi_n) = \xi_n^{2N} P(t) + \xi_n^{2N-2} Q(t) \underline{\xi'} \underline{\xi_n} \,, \tag{5.6}$$

where $P(t)$ and $Q(t)$ are polynomials in the variable t, $P(t)$ is of degree N and $Q(t)$ is of degree $N-1$; for $K = 2N+1$, F_K is of the form:

$$F_{2N+1}(\xi_1, \cdots, \xi_n) = \xi_n^{2N} (P(t) \underline{\xi'} + Q(t) \underline{\xi_n}), \tag{5.7}$$

where both $P(t)$ and $Q(t)$ are polynomials of degree N.

Let us consider the question when the spinor $F_K \cdot s_\lambda$, $s_\lambda \in \mathbb{S}_\lambda$, belongs to $\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}_\lambda)$, namely, $F_K \cdot s_\lambda$ is annihilated by the operators

$$-2id\tilde{\pi}_{S,\lambda}(E_j) = \epsilon_j \xi_j \Box - (2E - 2\lambda + 1) \partial_{\xi_j} - \epsilon_j e_j \, D, \; j = 1, \ldots, n-1,$$

where the notation "$\cdot$" in $F_K \cdot s_\lambda$ means the tensor product followed by the Clifford multiplication. This leads us to a system of ordinary differential equations for the polynomials $P(t)$ and $Q(t)$. We shall first treat the case of even homogeneity $K = 2N$.

Lemma 5.2. *Let $\lambda \in \mathbb{C}$, $N \in \mathbb{N}$ and let $s_\lambda \in \mathbb{S}_\lambda$ be a non-zero vector. For polynomials $P(t)$ and $Q(t)$, we set*

$$F_{2N} = \xi_n^{2N} P(t) + \xi_n^{2N-2} Q(t) \underline{\xi'} \underline{\xi_n}.$$

Then $F_{2N} \cdot s_\lambda \in \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}_\lambda)$ if and only if the following system of ordinary differential equations is satisfied:

$$R(2N, -\lambda - \frac{n}{2} + 1) P = 0 \tag{5.8}$$

$$R(2N-1, -\lambda - \frac{n}{2} + 1) Q = 0, \tag{5.9}$$

$$-2 \, N \, P + 2 \, t \, P' + (4N - 2\lambda - n) \, Q - 2 \, t Q' = 0, \tag{5.10}$$

$$2 \, P' - (2N-1) \, Q + 2 \, t Q' = 0. \tag{5.11}$$

The first two equations actually follow from the last two equations.

Proof. By Lemma 5.1, $F_{2N} \cdot s_\lambda \in \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}_\lambda)$ if and only if

$$d\tilde{\pi}_{S,\lambda}(E_j)(F_{2N} \cdot s_\lambda) = 0 \quad \text{for} \quad 1 \le j \le n-1.$$

We set $F_{2N} = v_1 + v_2$ with $v_1 = \xi_n^{2N} P(t)$ and $v_2 = \xi_n^{2N-2} Q(t) \underline{\xi' \xi_n}$. Then

$$\begin{aligned}
\partial_{\xi_j} v_1 =& 2\epsilon_n \epsilon_j \xi_j \xi_n^{2N-2} P'(t), \\
\partial_{\xi_j}^2 v_1 =& 2\epsilon_n \epsilon_j \xi_n^{2N-2} P'(t) + 4\xi_n^{2N-4} \xi_j^2 P''(t), \\
\Box' v_1 =& \epsilon_n \xi_n^{2N-2} (4\, t\, P'' + (2n-2)\, P'), \\
\partial_{\xi_n} v_1 =& \xi_n^{2N-1} (2N\, P - 2\, t\, P'), \\
\partial_{\xi_n}^2 v_1 =& \xi_n^{2N-2} (2N(2N-1)\, P + (-8N+6)\, t\, P' + 4\, t^2\, P''), \\
e_j D' v_1 =& 2\epsilon_n\, e_j \xi_n^{2N-2}\, P' \underline{\xi'}, \\
e_j e_n \partial_{\xi_n} v_1 =& \epsilon_n\, e_j \xi_n^{2N-2} (2N\, P - 2\, t\, P') \underline{\xi_n}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\partial_{\xi_j} v_2 =& \epsilon_n \epsilon_j \xi_n^{2N-4} 2Q' \xi_j \underline{\xi' \xi_n} + \xi_n^{2N-2} \epsilon_j e_j Q(t) \underline{\xi_n}, \\
\Box'\, v_2 =& \xi_n^{2N-4} \epsilon_n (4\, t\, Q'' + (2n+2)\, Q') \underline{\xi' \xi_n}, \\
\partial_{\xi_n} v_2 =& \xi_n^{2N-3} ((2N-1)\, Q - 2\, t\, Q') \underline{\xi' \xi_n}, \\
\partial_{\xi_n}^2 v_2 =& \xi_n^{2N-4} ((2N-1)(2N-2)\, Q + t\, (-8N+10)\, Q' + 4\, t^2\, Q'') \underline{\xi' \xi_n}, \\
-e_j D' v_2 =& -\epsilon_n e_j \xi_n^{2N-2} (2\, t\, Q' + n'\, Q) \underline{\xi_n}, \\
-e_j e_n \partial_{\xi_n} v_2 =& \epsilon_n e_j \xi_n^{2N-2} ((2N-1)\, Q - 2\, t\, Q') \underline{\xi'}.
\end{aligned}$$

Collecting all terms of $d\tilde{\pi}_{S,\lambda}(E_j)(F_{2N} \cdot s_\lambda)$ with respect to the basis

$$\epsilon_j \epsilon_n \xi_j \xi_n^{2N-2}, \;\; \epsilon_j \epsilon_n \xi_j \xi_n^{2N-4} \underline{\xi' \xi_n}, \;\; \epsilon_j\, e_j\, \epsilon_n \xi_n^{2N-2} \underline{\xi'}, \;\; \epsilon_j\, e_j\, \epsilon_n \xi_n^{2N-2} \underline{\xi_n},$$

we see that the the system of equations $d\tilde{\pi}_{S,\lambda}(E_j)(F_{2N} \cdot s_\lambda) = 0$ $(1 \le j \le n-1)$ is equivalent to the four equations in the lemma.

It can be easily checked that a suitable linear combination of the last two equations (5.10) and (5.11) and their differentials implies the first two equations (5.8) and (5.9). For example, the application of $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$ to the equation (5.11) gives

$$2P' = -2tQ' + (2N-1)Q, \quad 2\, P'' = -2\, Q' - 2\, tQ'' + (2N-1)\, Q',$$

and their substitution into the equation (5.10) yields the equation (5.9). Hence the proof of Lemma 5.2 is complete. $\square$

Lemma 5.3. *Let $N \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, and $s_\lambda \in \mathbb{S}_\lambda$ a non-zero vector. We set*

$$\tilde{F}_{2N}(\xi_1, \cdots, \xi_n) = \xi_n^{2N} \widetilde{C}_{2N}^{-\lambda-\frac{n}{2}+1} \left(\frac{|\xi'|^2}{\xi_n^2} \right) + \xi_n^{2N-2} \widetilde{C}_{2N-1}^{-\lambda-\frac{n}{2}+1} \left(\frac{|\xi'|^2}{\xi_n^2} \right) \underline{\xi' \xi_n}. \tag{5.12}$$

Then $\tilde{F}_{2N} \cdot s_\lambda$ is a spinor–valued homogeneous polynomial of $\xi_1, \cdots, \xi_{n-1}$, and ξ_n of degree $2N$, and belongs to $\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}_\lambda)$.

Proof. By Lemma 5.2, it is sufficient to solve the system of ordinary differential equations (5.8)–(5.11) for polynomials $P(t)$ and $Q(t)$.

It follows from Lemma 6.1 in Appendix that the polynomial solutions $P(t)$ and $Q(t)$ are of the form

$$P(t) = A\,\widetilde{C}^{\alpha}_{2N}(-t),\, Q(t) = B\,\widetilde{C}^{\alpha}_{2N-1}(-t) \quad \text{with} \quad \alpha = -\lambda - \frac{n}{2} + 1,$$

for some $A, B \in \mathbb{C}$. Let us show that (5.10) and (5.11) are fulfilled for this pair $(P(t), Q(t))$ if and only if $A = B$. We shall deal with (5.11) below, and omit (5.10) which gives the same conclusion by a similar argument using the formula (6.12) in Appendix.

Suppose $t = -\frac{1}{x^2}$. We note that if two functions $g(x)$ and $h(t)$ are related by the formula

$$g(x) = x^l h(t) \quad (\equiv x^l h(-\frac{1}{x^2})),$$

then $\frac{dx}{dt} = \frac{1}{2}x^3$ and thus

$$h'(t) = \frac{1}{2}(x^{-l+3}g'(x) - lx^2 h(t)). \tag{5.13}$$

Applying (5.13) to

$$g_P(x) := x^{2N}P(t) \quad \text{and} \quad g_Q(x) := x^{2N-1}Q(t),$$

we get

$$2P'(t) = x^{-2N+3}g_P'(x) - 2Nx^2P(t), \tag{5.14}$$

$$-(2N-1)Q(t) + 2tQ'(t) = -x^{-2N+2}g_Q'(x). \tag{5.15}$$

Since $g_P(x) = A\widetilde{C}^{\alpha}_{2N}(x)$ and $g_Q(x) = B\widetilde{C}^{\alpha}_{2N-1}(x)$ with $\alpha = -\lambda - \frac{n}{2}$ (see (4.14) or Lemma 6.1), (5.11) amounts to

$$A((\alpha+N)x\,\widetilde{C}^{\alpha+1}_{2N-1}(x) - N\widetilde{C}^{\alpha}_{2N}(x)) - B\widetilde{C}^{\alpha+1}_{2N-2}(x) = 0$$

by (6.10). Using the identity (6.15), we see that this holds if and only if $A = B$. Thus the proof of Lemma 5.3 is completed.

□

The case of odd homogeneity is contained in the next two lemmas, for which the proof is similar and omitted.

Lemma 5.4. *Let $N \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and $s_\lambda \in \mathbb{S}_\lambda$ a non-zero vector. Then*

$$F_{2N+1} \cdot s_\lambda = \xi_n^{2N}(P(t)\underline{\xi'} + Q(t)\underline{\xi_n}) \cdot s_\lambda$$

is annihilated by $d\tilde{\pi}_{S,\lambda}(E_j)$ $(1 \le j \le n-1)$ if and only if the polynomials $P(t)$ and $Q(t)$ satisfy the following system of ordinary differential equations:

$$R(2N, -\lambda - \frac{n}{2} + 1)P = 0, \tag{5.16}$$

$$R(2N+1, -\lambda - \frac{n}{2} + 1)Q = 0, \tag{5.17}$$

$$(4N - 2\lambda - n + 2)\,P - 2\,t\,P' - (2N+1)\,Q + 2\,tQ' = 0, \tag{5.18}$$

$$2NP - 2\,tP' - 2\,Q' = 0. \tag{5.19}$$

Furthermore, the first two equations follow from the last two equations.

Lemma 5.5. *Let $N \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and $s_\lambda \in \mathbb{S}_\lambda$ a non-zero vector. We set*

$$\tilde{F}_{2N+1}(\xi_1, \cdots, \xi_n) = \xi_n^{2N} \left(\widetilde{\mathcal{C}}_{2N}^{-\lambda-\frac{n}{2}+1} \left(\frac{|\xi'|^2}{\xi_n^2} \right) \underline{\xi'} + (-\lambda - \frac{n}{2} + N + 1) \widetilde{\mathcal{C}}_{2N+1}^{-\lambda-\frac{n}{2}+1} \left(\frac{|\xi'|^2}{\xi_n^2} \right) \underline{\xi_n} \right). \tag{5.20}$$

Then $\tilde{F}_{2N+1} \cdot s_\lambda$ is of homogeneous degree $2N+1$ and is annihilated by $d\tilde{\pi}_\lambda(E_j)(1 \leq j \leq n-1)$, and therefore belongs to $\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}_\lambda)$ *for any $s_\lambda \in \mathbb{S}_\lambda$.*

Remark 5.6. In contrast to the even case in Lemma 5.3, the coefficient $-\lambda - \frac{n}{2} + N + 1\,(= \alpha + N)$ shows up in the odd case in Lemma 5.5 with respect to the renormalized Gegenbauer polynomials. We note

$$\mathcal{C}_{2N}^{\alpha}(-t)\underline{\xi'} + \mathcal{C}_{2N+1}^{\alpha}(-t)\underline{\xi_n} = (\alpha)_N (\widetilde{\mathcal{C}}_{2N}^{\alpha}(-t)\underline{\xi'} + (\alpha + N)\widetilde{\mathcal{C}}_{2N+1}^{\alpha}(-t)\underline{\xi_n}).$$

As in Section 4, the homomorphisms of generalized Verma modules defined by the singular vectors described above induce equivariant differential operators acting on local sections of induced homogeneous vector bundles on the generalized flag manifolds. We describe these differential operators in the non-compact picture of the induced representations.

In the following theorem, we retain the notation of Section 5.1: $\mathbb{S}^n \equiv \mathbb{S}^{p-1,q-1}$ is the spin representation of $Spin(p-1, q-1)$, and $\mathbb{S}^n_\pm \equiv \mathbb{S}^{p-1,q-1}_\pm$ are the half-spin representations when $n = p+q-2$ is even. They are extended to the representations $\mathbb{S}^n_\lambda \equiv \mathbb{S}^{p-1,q-1}_\lambda$, $\mathbb{S}^n_{\pm,\lambda} \equiv \mathbb{S}^{p-1,q-1}_{\pm,\lambda}$, respectively, of the parabolic subgroup $\widetilde{P} = \widetilde{M}AN_+$ with $\widetilde{M} \simeq Spin(p-1, q-1)$ by letting A act as the one-dimensional representation $\mathbb{C}_\lambda$ and N_+ trivially. Then the branching law of the restriction with respect to the pair of the parabolic subgroups $\widetilde{P} \supset \widetilde{P}'$ of $Spin_o(p,q) \supset Spin_o(p, q-1)$ is given as

$$\begin{aligned} &\mathbb{S}^n_\lambda \simeq \mathbb{S}^{n-1}_{+,\lambda} \oplus \mathbb{S}^{n-1}_{-,\lambda}, \quad && n: \text{odd}, \\ &\mathbb{S}^n_{\pm,\lambda} \simeq \mathbb{S}^{n-1}_\lambda, && n: \text{even}. \end{aligned} \tag{5.21}$$

We recall from (4.22) and (4.23) the polynomials $a_j(\lambda) \equiv a_j^{N,n}(\lambda)$ and $b_j(\lambda) \equiv b_j^{N,n}(\lambda)$. Then we have

$$N!\,\widetilde{\mathcal{C}}_{2N}^{-\lambda-\frac{n}{2}+1}(-t) = \sum_{j=0}^{N} a_j(\lambda - \frac{1}{2}) t^j,$$

$$N!\,\widetilde{\mathcal{C}}_{2N-1}^{-\lambda-\frac{n}{2}+1}(-t) = 2N \sum_{j=0}^{N-1} b_j(\lambda - \frac{1}{2}) t^j,$$

$$N!\,\widetilde{\mathcal{C}}_{2N+1}^{-\lambda-\frac{n}{2}+1}(-t) = 2 \sum_{j=0}^{N} b_j(\lambda - \frac{1}{2}) t^j.$$

Therefore, by applying Theorem 2.4 to Lemmas 5.3 and 5.5 with λ replaced by $-\lambda$, we obtain the following theorem:

Theorem 5.7. *Let $(\widetilde{G}, \widetilde{G}') = (Spin_o(p,q), Spin_o(p, q-1))$ and $\lambda \in \mathbb{C}$. We decompose the Dirac operator D on $\mathbb{R}^{p-1,q-1} = \mathbb{R}^{p-1,q-2} \oplus \mathbb{R}^{0,1}$ as*

$$D = D' + \underline{\partial}_n \equiv \sum_{i=1}^{n-1} e_i \frac{\partial}{\partial x_i} + e_n \frac{\partial}{\partial x_n},$$

and write the Laplace–Beltrami operator on $\mathbb{R}^{p-1,q-2}$ *as*

$$\Box' = -(D')^2 = \sum_{i=1}^{n-1} \epsilon_i \frac{\partial^2}{\partial x_i^2}.$$

We introduce a family of $\mathrm{End}(\mathbb{S}^n)$*-valued differential operators* $D_K^{\mathbb{S}}(\lambda)$ *of order* K $(K \in \mathbb{N})$ *by*

$$D_{2N}^{\mathbb{S}}(\lambda) := \sum_{j=0}^{N} a_j(-\lambda - \frac{1}{2})(\Box')^j \frac{\partial^{2N-2j}}{\partial x_n^{2N-2j}} + 2N \sum_{j=0}^{N-1} b_j(-\lambda - \frac{1}{2})(\Box')^j \frac{\partial^{2N-2j-2}}{\partial x_n^{2N-2j-2}} D' \underline{\partial}_n,$$

$$D_{2N+1}^{\mathbb{S}}(\lambda) := \sum_{j=0}^{N} a_j(-\lambda - \frac{1}{2})(\Box')^j \frac{\partial^{2N-2j}}{\partial x_n^{2N-2j}} D' + (-2\lambda - n + 2N + 2) \sum_{j=0}^{N} b_j(-\lambda - \frac{1}{2})(\Box')^j \frac{\partial^{2N-2j}}{\partial x_n^{2N-2j}} \underline{\partial}_n.$$

(1) *The differential operators* $D_K^{\mathbb{S}}(\lambda)$ $(K \in \mathbb{N})$ *induce* $\widetilde{G}'$*-homomorphisms*

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\mathbb{S}_\lambda^n) \to \bigoplus_{\epsilon=\pm} \mathrm{Ind}_{\widetilde{P}'}^{\widetilde{G}'}(\mathbb{S}_{\epsilon,\lambda+K}^{n-1}) \quad \textit{for } n \textit{ odd},$$
$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\mathbb{S}_{\pm,\lambda}^n) \to \mathrm{Ind}_{\widetilde{P}'}^{\widetilde{G}'}(\mathbb{S}_{\lambda+K}^{n-1}) \qquad \textit{for } n \textit{ even},$$

by the following formulas

$$\mathcal{D}_K^{\mathbb{S}}(\lambda) f := (D_K^{\mathbb{S}}(\lambda) f)|_{x_n=0}$$

in the non-compact picture

$$\mathcal{D}_K^{\mathbb{S}}(\lambda) : \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}^n) \to \mathcal{C}^\infty(\mathbb{R}^{p-1,q-2}, \mathbb{S}_+^{n-1} \oplus \mathbb{S}_-^{n-1}) \qquad \textit{for } n \textit{ odd},$$
$$\mathcal{D}_K^{\mathbb{S}}(\lambda) : \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}_\pm^n) \to \mathcal{C}^\infty(\mathbb{R}^{p-1,q-2}, \mathbb{S}^{n-1}) \qquad \textit{for } n \textit{ even},$$

In particular, the following infinitesimal relations are satisfied in both cases:

$$\mathcal{D}_K^{\mathbb{S}}(\lambda) d\pi_{\mathbb{S},\lambda}^{\widetilde{G}}(X) = d\pi_{\mathbb{S},\lambda+K}^{\widetilde{G}'}(X) \mathcal{D}_K^{\mathbb{S}}(\lambda) \quad \textit{for all } X \in \mathfrak{g}'. \tag{5.22}$$

(2) *Conversely, if* $\lambda \in \mathbb{C}$ *satisfies*

$$-\lambda + n - \frac{3}{2} \notin \mathbb{N}_+ \quad \textit{and} \quad -2\lambda + n - 1 \notin \mathbb{N}_+,$$

and if there exist an irreducible finite-dimensional representation W *of* $\widetilde{P'}$ *and a non-trivial differential* $\widetilde{G}'$*-homomorphism* T*, then*

$$W \simeq \begin{cases} \mathbb{S}_{\epsilon,\lambda+K}^{n-1} & (n \textit{ odd}), \\ \mathbb{S}_{\lambda+K}^{n-1} & (n \textit{ even}), \end{cases}$$

for some $K \in \mathbb{N}$ *and* $\epsilon = \pm$ *(*n *odd case) and* T *is given by a scalar multiple of* $\mathcal{D}_K^{\mathbb{S}}(\lambda)$ *in the non-compact picture.*

Remark 5.8. For $n = p + q - 2$ odd, the infinitesimal action $d\pi_{\mathbb{S},\lambda}^{\widetilde{G}}$ of the Lie algebra $\mathfrak{g}$ on the non-compact picture decomposes into a direct sum of two $\mathfrak{g}'$-modules:

$$\mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}^n) \simeq \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}_+^{n-1}) \oplus \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}_-^{n-1}).$$

If $f \in \mathcal{C}^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}_\epsilon^{n-1})$ $(\epsilon = \pm)$, then $\mathcal{D}_K^{\mathbb{S}}(\lambda) f \in \mathcal{C}^\infty(\mathbb{R}^{p-1,q-2}, \mathbb{S}_\delta^{n-1})$ where $\delta = (-1)^K \epsilon$.

Proof of Theorem 5.7. (1) It follows from Lemmas 5.3 and 5.5 that

$$\widetilde{F}_K \cdot s_\lambda \in \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}^n_\lambda) \quad \text{for all } K \in \mathbb{N} \text{ and } s_\lambda \in \mathbb{S}^n_\lambda.$$

Furthermore, since $\widetilde{F}_K$ is an $\mathfrak{m}'$-invariant element in $\mathrm{Pol}[\mathfrak{n}_+] \otimes \mathrm{End}(\mathbb{S}^n_\lambda)$, the subspace $\widetilde{F}_K(\mathbb{S}^n_\lambda)$ (or $\widetilde{F}_K(\mathbb{S}^n_{\pm,\lambda})$ for n odd) belongs to the same $\mathfrak{m}'$-isotypic component, namely,

$$\mathbb{S}^n|_{\mathfrak{m}'} \simeq \mathbb{S}^{n-1}_+ \oplus \mathbb{S}^{n-1}_- \, (n : \text{even}), \quad \text{or} \quad \mathbb{S}^n_\pm|_{\mathfrak{m}'} \simeq \mathbb{S}^{n-1} \, (n : \text{odd}).$$

Now the F-method in Section 2 (with λ replaced by $-\lambda$) leads us to Theorem 5.7 as in the scalar case (Theorem 4.2).

(2) The second statement follows from Theorem 2.4 and Proposition 5.9 below.

□

Proposition 5.9. *(Branching laws $\mathfrak{g} \downarrow \mathfrak{g}'$) We recall $n = p + q - 2$ and $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{so}(n + 2, \mathbb{C}), \mathfrak{so}(n + 1, \mathbb{C}))$.*

(1) In the Grothendieck group of $\mathfrak{g}'$-modules, we have

$$M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{S}^n_\lambda)|_{\mathfrak{g}'} \simeq \bigoplus_{\epsilon=\pm} \bigoplus_{b=0}^{\infty} M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{S}^{n-1}_{\epsilon,\lambda-b}) \quad \textit{for } n \textit{ odd},$$

$$M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{S}^n_{\pm,\lambda})|_{\mathfrak{g}'} \simeq \bigoplus_{b=0}^{\infty} M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{S}^{n-1}_{\lambda-b}) \qquad \textit{for } n \textit{ even},$$

(2) If $\lambda \in \mathbb{C}$ satisfies the following two conditions:

$$\lambda + n - \frac{3}{2} \notin \mathbb{N}_+, \tag{5.23}$$

$$2\lambda + n - 1 \notin \mathbb{N}_+, \tag{5.24}$$

then the first statement gives an irreducible decomposition as $\mathfrak{g}'$-modules.

Proof. (1) We apply Theorem 3.5 with

$$F_\lambda := \begin{cases} \mathbb{S}^n_\lambda \equiv \mathbb{S}^n \otimes \mathbb{C}_\lambda & \text{for } n \text{ odd}, \\ \mathbb{S}^n_{\epsilon,\lambda} \equiv \mathbb{S}^n_\epsilon \otimes \mathbb{C}_\lambda & \text{for } n \text{ even}. \end{cases}$$

Then we have

$$F_\lambda|_{\mathfrak{l}'} \simeq \begin{cases} \mathbb{S}^{n-1}_{+,\lambda} \oplus \mathbb{S}^{n-1}_{-,\lambda} & \text{for } n \text{ odd}, \\ \mathbb{S}^{n-1}_\lambda & \text{for } n \text{ even}. \end{cases}$$

Since the symmetric tensor algebra $S(\mathfrak{n}_-/\mathfrak{n}_- \cap \mathfrak{g}') \simeq \bigoplus_{b \in \mathbb{N}} \mathbb{C}_{-b}$ as a module of $\mathfrak{l}' \simeq \mathfrak{so}(n-1, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$, we have

$$F_\lambda|_{\mathfrak{l}'} \otimes S(\mathfrak{n}_-/\mathfrak{n}_- \cap \mathfrak{g}') \simeq \begin{cases} \bigoplus_{\epsilon=\pm} \bigoplus_{b=0}^{\infty} \mathbb{S}^{n-1}_{\epsilon,\lambda-b} & \text{for } n \text{ odd}, \\ \bigoplus_{b=0}^{\infty} \mathbb{S}^{n-1}_{\lambda-b} & \text{for } n \text{ even}. \end{cases}$$

Here the first statement follows from Theorem 3.5.

(2) Suppose n is odd. Then the $\mathfrak{Z}(\mathfrak{g}')$-infinitesimal character of the $\mathfrak{g}'$-module $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{S}^{n-1}_{\epsilon,\lambda-b})$ is given by

$$(\lambda-b+\frac{n-1}{2},\frac{n}{2}-1,\cdots,\frac{5}{2},\frac{3}{2},\epsilon\frac{1}{2})\in\mathbb{C}^{\frac{n+1}{2}}/W(D_{\frac{n+1}{2}}).$$

They are distinct when b runs over $\mathbb{N}$ and $\epsilon=\pm$ if and only if

$$\lambda-b+\frac{n-1}{2}\neq-(\lambda-b'+\frac{n-1}{2})$$

for all $(b,b')\in\mathbb{N}^2$ with $b\neq b'$, namely, λ satisfies (5.24). Furthermore, the $\mathfrak{g}'$-module $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{S}^{n-1}_{\epsilon,\lambda-b})$ is irreducible if

$$\lambda-b+n+\frac{3}{2}\notin\mathbb{N}_+$$

by (3.2), and in particular, if (5.23) is satisfied.

Therefore, if both (5.23) and (5.24) are fulfilled, then there is no extension among the irreducible $\mathfrak{g}'$-modules $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{S}^{n-1}_{\epsilon,\lambda-b})$, and hence the formula in (1) gives a direct sum of irreducible $\mathfrak{g}'$-modules.

Suppose n is even. Then the $\mathfrak{Z}(\mathfrak{g}')$-infinitesimal character of the $\mathfrak{g}'$-module $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{S}^{n-1}_{\lambda-b})$ is

$$(\lambda-b+\frac{n-1}{2},\frac{n}{2}-1,\cdots,2,1)\in\mathbb{C}^{\frac{n}{2}}/W(B_{\frac{n}{2}}).$$

They are distinct when b runs over $\mathbb{N}$ if and only if (5.24) is satisfied. Furthermore, the condition (3.2) amounts to

$$2(\lambda-b+\frac{n-1}{2})\notin\mathbb{N}_+\quad\text{and}\quad\lambda-b+\frac{n-3}{2}\notin\mathbb{N}_+$$

which are satisfied if λ fulfills (5.23) and (5.24). Thus the second statement also follows for n even.

□

As in the scalar case, we have for $\lambda\in\mathbb{N}+\frac{1}{2}$ an additional set of singular vectors in $\mathrm{Sol}(\mathfrak{g},\mathfrak{g}';\mathbb{S}_\lambda)$. To see this, we retain the notation of Section 5.1 and define the space of monogenic spinors of degree ($j\in\mathbb{N}$) by

$$\mathcal{M}^j(\mathbb{R}^{p-1,q-1},\mathbb{S}^n):=\{s\in\mathrm{Pol}[\xi_1,\cdots,\xi_n]\otimes\mathbb{S}^n:Ds=0,Es=js\},$$

where $D=\sum_{k=1}^n e_k\frac{\partial}{\partial\xi_k}$ is the Dirac operator and $E=\sum_{k=1}^n\xi_k\frac{\partial}{\partial\xi_k}$ is the Euler homogeneity operator. For $n=p+q-2$ even, we also define for $\epsilon=\pm$

$$\mathcal{M}^j(\mathbb{R}^{p-1,q-1},\mathbb{S}^n_\epsilon):=\{s\in\mathrm{Pol}[\xi_1,\cdots,\xi_n]\otimes\mathbb{S}^n_\epsilon:Ds=0,Es=js\}.$$

Then $\mathcal{M}^j(\mathbb{R}^{p-1,q-1},\mathbb{S}^n)$ for n odd (or $\mathcal{M}^j(\mathbb{R}^{p-1,q-1},\mathbb{S}^n_\epsilon)$ for n even) is an irreducible $Spin(p-1,q-1)$-submodule of $\mathrm{Pol}[\xi_1,\cdots,\xi_n]\otimes\mathbb{S}^n$ with highest weight $(j+\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2})$ for n odd (or $(j+\frac{1}{2},\frac{1}{2},\cdots,\epsilon\frac{1}{2})$ for n even), respectively.

Lemma 5.10. *Suppose $\lambda\in-\frac{1}{2}+\mathbb{N}$. Then*

$$\mathcal{M}^{\lambda+\frac{1}{2}}(\mathbb{R}^{p-1,q-1},\mathbb{S}^n)\subset\mathrm{Sol}(\mathfrak{g},\mathfrak{g};\mathbb{S}^n_\lambda),$$
$$\mathcal{M}^{\lambda+\frac{1}{2}}(\mathbb{R}^{p-1,q-1},\mathbb{S}^n_\epsilon)\subset\mathrm{Sol}(\mathfrak{g},\mathfrak{g};\mathbb{S}^n_{\epsilon,\lambda})\quad\textit{for } n \textit{ even}.$$

Proof. Since $D^2 = -\Box$, we have

$$\xi_k \Box - e_k D = -e_k(\mathrm{Id} - \epsilon_k \xi_k e_k D).$$

Hence, by Lemma 5.1, we have

$$d\tilde{\pi}_{S,\lambda}(E_k) = i\left((\lambda - E - \frac{1}{2})\partial_{\xi_k} - \frac{1}{2}e_k(\mathrm{Id} - \epsilon_k\xi_k e_k D)D\right) \quad \text{for} \quad k = 1, \cdots, n.$$

If $s \in \mathcal{M}^{\lambda+\frac{1}{2}}(\mathbb{R}^{p-1,q-1}, \mathbb{S}^n)$, then

$$E(\partial_{\xi_k} s) = (\lambda - \frac{1}{2})\partial_{\xi_k} s \quad (1 \le k \le n), \quad Ds = 0.$$

Therefore, $d\tilde{\pi}_{S,\lambda}(E_k)s = 0$ for all k $(1 \le k \le n)$. Thus the lemma is proved. $\Box$

We also need the branching laws of these modules when restricted from $Spin(p-1, q-1)$ to the subgroup $Spin(p-1, q-2)$, which are given by

$$\mathcal{M}^j(\mathbb{R}^{p-1,q-1}, \mathbb{S}^n) \simeq \bigoplus_{\epsilon=\pm} \bigoplus_{i=0}^{j} \mathcal{M}^i(\mathbb{R}^{p-1,q-2}, \mathbb{S}^{n-1}_\epsilon) \quad \text{for } n \text{ odd}, \tag{5.25}$$

$$\mathcal{M}^j(\mathbb{R}^{p-1,q-1}, \mathbb{S}^n_\epsilon) \simeq \bigoplus_{i=0}^{j} \mathcal{M}^i(\mathbb{R}^{p-1,q-2}, \mathbb{S}^{n-1}) \qquad \text{for } n \text{ even}. \tag{5.26}$$

Let us summarize our results for spinor representation in the following theorem analogous to Theorem 4.2.

Theorem 5.11. *Let $(\widetilde{G}, \widetilde{G}') = (Spin_o(p, q), Spin_o(p, q-1))$, $n = p+q-2$, and $\lambda \in \mathbb{C}$. We recall from Lemmas 5.3 and 5.5 that $\widetilde{F}_K \equiv \widetilde{F}^\lambda_K \in \mathrm{Pol}[\xi_1, \ldots, \xi_n] \otimes \mathcal{C}^{\mathbb{C}}_n$ $(K \in \mathbb{N})$ is defined by*

$$\begin{aligned}
\tilde{F}_{2N} &= \xi_n^{2N}\widetilde{\mathcal{C}}^\alpha_{2N}(-t) + \xi_n^{2N-2}\widetilde{\mathcal{C}}^\alpha_{2N-1}(-t)\underline{\xi'\xi_n}, \\
\tilde{F}_{2N+1} &= \xi_n^{2N}\left(\widetilde{\mathcal{C}}^\alpha_{2N}(-t)\underline{\xi'} + (-\lambda - \frac{n}{2} + N + 1)\,\widetilde{\mathcal{C}}^\alpha_{2N+1}(-t)\underline{\xi_n}\right),
\end{aligned} \tag{5.27}$$

where $\alpha = -\lambda - \frac{n}{2} + 1$ and $t = \frac{\sum_{j=1}^{n-1}\epsilon_j\xi_j^2}{\epsilon_n\xi_n^2}$. Then

(1) *For any $\lambda \in \mathbb{C}$, we have*

$$\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}_\lambda) \supset \bigoplus_{K=0}^{\infty}\{\widetilde{F}_K \cdot s_\lambda : \, s_\lambda \in \mathbb{S}_\lambda\},$$

$$\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}_{\epsilon,\lambda}) \supset \bigoplus_{K=0}^{\infty}\{\widetilde{F}_K \cdot s_\lambda : \, s_\lambda \in \mathbb{S}_{(-1)^K\epsilon,\lambda}\} \quad \textit{for } n \textit{ even.}$$

Moreover, if λ satisfies (5.23) and (5.24), then the above inclusions are the equalities and the inverse Fourier transform (see (2.7)) gives all the singular vectors via the isomorphisms as modules of $\widetilde{L}' \simeq Spin(p-1, q-2) \times \mathbb{R}$:

$$M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{S}_\lambda)^{\mathfrak{n}'_+} \underset{\varphi}{\overset{\sim}{\longleftarrow}} \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}_\lambda) \simeq \bigoplus_{K=0}^{\infty}\{\tilde{F}_K \cdot s_\lambda : \, s_\lambda \in \mathbb{S}_\lambda\},$$

$$M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{S}_{\epsilon,\lambda})^{\mathfrak{n}'_+} \underset{\varphi}{\overset{\sim}{\longleftarrow}} \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}_{\epsilon,\lambda}) \simeq \bigoplus_{K=0}^{\infty}\{\tilde{F}_K \cdot s_\lambda : \, s_\lambda \in \mathbb{S}_{(-1)^K\epsilon,\lambda}\} \quad \textit{for } n \textit{ even.}$$

(2) *For $\lambda \in \mathbb{N} + \frac{1}{2}$, we have $\mathfrak{l}'$-injective maps*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{S}^n_\lambda)^{\mathfrak{n}'_+} \overset{\sim}{\underset{\varphi}{\longleftarrow}} \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}^n_\lambda) \quad \supset \bigoplus_{K=0}^{\infty} \{\tilde{F}_K \cdot s_\lambda : s_\lambda \in \mathbb{S}^n_\lambda\} \oplus \bigoplus_{\epsilon=\pm} \bigoplus_{i=1}^{\lambda+\frac{1}{2}} \mathcal{M}_{i,\epsilon}, \qquad \textit{for } n \textit{ odd},$$

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{S}^n_{\epsilon,\lambda})^{\mathfrak{n}'_+} \overset{\sim}{\underset{\varphi}{\longleftarrow}} \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; \mathbb{S}^n_{\epsilon,\lambda}) \supset \bigoplus_{K=0}^{\infty} \{\tilde{F}_K \cdot s_\lambda : s_\lambda \in \mathbb{S}^n_{(-1)^K\epsilon,\lambda}\} \oplus \bigoplus_{i=1}^{\lambda+\frac{1}{2}} \mathcal{M}_i, \quad \textit{for } n \textit{ even},$$

Here $\mathcal{M}_{i,\epsilon}$ and $\mathcal{M}_i$ are the summands in the branching laws (5.25) and (5.26), respectively.

In the second part [33] of the series, we shall prove the existence of lifts of homomorphisms corresponding to singular vectors in generalized Verma modules induced from spinor representations to homomorphisms of semi-holonomic generalized Verma modules covering them. According to the philosophy of parabolic geometries, [6], we get curved versions of our equivariant differential operators acting on sections of spinor bundles on manifolds with conformal structure.

6 Appendix: Gegenbauer polynomials

In the Appendix we summarize for reader's convenience a few basic conventions and properties of the Gegenbauer polynomials.

The Gegenbauer polynomials are defined in terms of their generating function

$$\frac{1}{(1 - 2xt + t^2)^\alpha} = \sum_{l=0}^{\infty} C_l^\alpha(x) t^l, \tag{6.1}$$

and satisfy the recurrence relation

$$C_l^\alpha(x) = \frac{1}{l} \left(2x(l + \alpha - 1)C_{l-1}^\alpha(x) - (l + 2\alpha - 2)C_{l-2}^\alpha(x)\right) \tag{6.2}$$

with $C_0^\alpha(x) = 1$, $C_1^\alpha(x) = 2\alpha x$. The Gegenbauer polynomials are solutions of the Gegenbauer differential equation

$$\left((1 - x^2)\frac{d^2}{dx^2} - (2\alpha + 1)x\frac{d}{dx} + l(l + 2\alpha)\right) g = 0. \tag{6.3}$$

They are given as special values of the Gaussian hypergeometric series when the series is finite:

$$C_l^\alpha(x) = \frac{(2\alpha)_l}{l!} \, {}_2F_1\left(-l, 2\alpha + l; \alpha + \frac{1}{2}; \frac{1 - x}{2}\right) = \sum_{k=0}^{[\frac{l}{2}]} (-1)^k \frac{\Gamma(l - k + \alpha)}{\Gamma(\alpha)k!(l - 2k)!}(2x)^{l-2k}. \tag{6.4}$$

We renormalize the Gegenbauer polynomials by

$$\widetilde{C}_l^\alpha(x) := \frac{\Gamma(\alpha)}{\Gamma(\alpha + [\frac{l+1}{2}])} C_l^\alpha(x) = \frac{1}{(\alpha)_{[\frac{l+1}{2}]}} C_l^\alpha(x). \tag{6.5}$$

Then $\widetilde{C}_l^\alpha(x)$ is a non-zero solution to (6.3) for all $\alpha \in \mathbb{C}$ and $l \in \mathbb{N}$.

We observe that $x \mapsto x^l C_l^\alpha(x^{-1})$ is an even polynomial of degree $2[\frac{l}{2}]$. Therefore there exists uniquely a polynomial of degree $[\frac{l}{2}]$, to be denoted by $\mathcal{C}_l^\alpha(t)$, such that

$$\mathcal{C}_l^\alpha(x^2) = x^l C_l^\alpha(x^{-1}). \tag{6.6}$$

It follows from (6.4) and (6.6) that

$$\mathcal{C}_l^\alpha(-t) = \sum_{k=0}^{[\frac{n}{2}]} \frac{\Gamma(l-k+\alpha)2^{l-2k}}{\Gamma(\alpha)k!(l-2k)!} t^k. \tag{6.7}$$

Similarly to the renormalization (6.5), we set

$$\widetilde{\mathcal{C}}_l^\alpha(t) = \frac{1}{(\alpha)_{[\frac{l+1}{2}]}} \mathcal{C}_l^\alpha(t).$$

Then

$$\begin{aligned}
\widetilde{\mathcal{C}}_0^\alpha(t) &= 1,\\
\widetilde{\mathcal{C}}_1^\alpha(t) &= 2,\\
\widetilde{\mathcal{C}}_2^\alpha(t) &= 2\,(\alpha+1) - t,\\
\widetilde{\mathcal{C}}_3^\alpha(t) &= 2\,(\frac{2}{3}(\alpha+2) - t),\\
\widetilde{\mathcal{C}}_4^\alpha(t) &= \frac{1}{2}\,(\frac{4}{3}(\alpha+2)(\alpha+3) - 4(\alpha+2)t + t^2).
\end{aligned}$$

In particular, for $\alpha = -\lambda - \frac{n}{2}$

$$N!\,\widetilde{\mathcal{C}}_{2N}^\alpha(-t) = \frac{N!}{(\alpha)_N}\,\mathcal{C}_{2N}^\alpha(-t) = \sum_{j=0}^{N} a_j(\lambda)t^j, \tag{6.8}$$

$$\frac{1}{2}N!\,\widetilde{\mathcal{C}}_{2N+1}^\alpha(-t) = \frac{N!}{2(\alpha)_{N+1}}\,\mathcal{C}_{2N+1}^\alpha(-t) = \sum_{j=0}^{N} b_j(\lambda)t^j, \tag{6.9}$$

where the coefficients $a_j(\lambda) \equiv a_j^{N,n}(\lambda)$ and $b_j(\lambda) \equiv b_j^{N,n}(\lambda)$ are defined in (4.22) and (4.23), respectively.

We recall from (4.20) that

$$R(l,\alpha) := 4t(1+t)\frac{d^2}{dt^2} + ((6-4l)t + 4(1-\alpha-l))\frac{d}{dt} + l(l-1).$$

Lemma 6.1. *Suppose $l \in \mathbb{N}$ and $g(x) = x^l h(-\frac{1}{x^2})$.*

(1) *$h(t)$ satisfies $R(l,\alpha)h(t) = 0$ if and only if $g(x)$ satisfies the Gegenbauer differential equation* (6.3)*.*

(2) *If $h(t)$ is a polynomial of degree $[\frac{l}{2}]$ and satisfies $R(l,\alpha)h(t) = 0$, then g(x) is a scalar multiple of the renormalized Gegenbauer polynomial $\widetilde{C}_l^\alpha(x)$ and $h(t)$ is a scalar multiple of $\widetilde{\mathcal{C}}_l^\alpha(-t)$.*

The Gegenbauer polynomials satisfy the Rodrigues formula

$$C_l^\alpha(x) = \frac{(-2)^l}{l!}\frac{\Gamma(l+\alpha)\Gamma(l+2\alpha)}{\Gamma(\alpha)\Gamma(2l+2\alpha)}(1-x^2)^{-\alpha+1/2}\frac{d^l}{dx^l}\left[(1-x^2)^{l+\alpha-1/2}\right],$$

a basic formula for derivative

$$\frac{d}{dx} C_l^{\alpha}(x) = 2\alpha\, C_{l-1}^{\alpha+1}(x) \tag{6.10}$$

and the following identities:

$$lC_l^{\alpha}(x) - 2\alpha\, xC_{l-1}^{\alpha+1}(x) + 2\alpha C_{l-2}^{\alpha+1}(x) = 0, \tag{6.11}$$

$$-2\alpha\, C_l^{\alpha+1}(x) + (l+2\alpha)\, C_l^{\alpha}(x) + 2\alpha x\, C_{l-1}^{\alpha+1}(x) = 0. \tag{6.12}$$

The formulas (6.10) and (6.11) are restated in terms of the renormalized Gegenbauer polynomials $\widetilde{C}_l^{\alpha}$ as below:

$$\frac{d}{dx}\widetilde{C}_{2N}^{\alpha}(x) = 2(\alpha+N)\, \widetilde{C}_{2N-1}^{\alpha+1}(x), \tag{6.13}$$

$$\frac{d}{dx}\widetilde{C}_{2N+1}^{\alpha}(x) = 2\, \widetilde{C}_{2N}^{\alpha+1}(x), \tag{6.14}$$

$$N\widetilde{C}_{2N}^{\alpha}(x) - (\alpha+N)\, x\widetilde{C}_{2N-1}^{\alpha+1}(x) + \widetilde{C}_{2N-2}^{\alpha+1}(x) = 0, \tag{6.15}$$

$$(2N+1)\, \widetilde{C}_{2N+1}^{\alpha}(x) - 2\, x\widetilde{C}_{2N}^{\alpha+1}(x) + 2\, \widetilde{C}_{2N-1}^{\alpha+1}(x) = 0. \tag{6.16}$$

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