Varna Lecture on L^2-Analysis of Minimal Representations

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Abstract Minimal representations of a real reductive group G are the 'smallest' irreducible unitary representations of G. The author suggests a program of global analysis built on minimal representations from the philosophy:
small representation of a group = **large** symmetries in a representation space.
This viewpoint serves as a driving force to interact algebraic representation theory with geometric analysis of minimal representations, yielding a rapid progress on the program. We give a brief guidance to recent works with emphasis on the Schrödinger model.

1 What are Minimal Representations?

Minimal representations of reductive groups G are the 'smallest' infinite dimensional irreducible unitary representations.

The *Weil* (metaplectic, oscillator, the Segal–Shale–Weil, harmonic) *representation*, known by a prominent role in number theory, consists of two minimal representations of the metaplectic group $Mp(n, \mathbb{R})$. The minimal representation of a conformal group $SO(4, 2)$ arises on the Hilbert space of bound states of the Hydrogen atom.

Minimal representations are distinguished among other (continuously many) irreducible unitary representations of G by the following properties that I state loosely.

- 'Smallest' infinite dimensional representations of G.
- One of the 'building blocks' of unitary representations of Lie groups.

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- 'Closest' to the trivial one dimensional representation of G.
- 'Quantization' of minimal nilpotent coadjoint orbits of G.
- Matrix coefficients have a 'slow decay' at infinity.

In algebraic representation theory, there is a distinguished ideal $\mathcal{J}$ introduced by Joseph [14] in the enveloping algebra of a complex simple Lie algebra other than type A (see also [8]). An irreducible representation of a real reductive Lie group G is called *minimal* if its infinitesimal representation is annihilated by $\mathcal{J}$. Thus the terminology 'minimal representations' is defined *inside* representation theory. We remark that not all reductive groups admit minimal representations. Further, minimal representations are not always highest weight modules. Beyond the case of highest weight modules, there has been an active study on minimal representations of reductive groups, in particular, by algebraic approaches, see e.g., [8, 14, 15, 30, 37, 39, 42, 43].

In contrast, my program focuses on global analysis inspired by minimal representations. For this, we switch the viewpoint, led by

Guiding principle 1.1 ([24])

small *representations of a group*
= ***large*** *symmetries in a representation space.*

An extremal case of 'large symmetries' might be stated as

$$\text{dimension of } \Xi < \text{dimension of any non-trivial } G\text{-space} \tag{1.1}$$

when the representation of G is realized on the space of functions on the geometry Ξ. An obvious implication of (1.1) is that G cannot act on Ξ.

The latter point of view, served as a driving force, has brought us to a new line of investigation of geometric analysis modeled on minimal representations. In this program we are trying to dig out new interactions with other areas of mathematics even *outside* representation theory:

- conformal geometry for general pseudo-Riemannian manifolds [21, 31],
- Dolbeault cohomologies on open complex manifolds [23, 30].
- conservative quantities for PDEs [21, 33],
- breaking symmetries and discrete branching laws [32, 34, 36, 38, 39],
- Schrödinger model and the unitary inversion operator [11, 27, 28],
- deformation of the Fourier transform [3],
- geometric quantization of nilpotent orbits [11, 28],
- holomorphic semigroup with a generalized Mehler kernel [3, 26, 27],
- new orthogonal polynomials for fourth order differential operators [9, 10, 29],
- a generalization of the Fock model and Bargmann transforms [12].

The aim of this article is to provide a brief guidance to the rapid progress on our program, [3, 9, 10, 11, 12, 23, 24, 28, 29, 34, 38]. We should mention that in order to avoid an overlap with a recent publication [24], we do not include here some other constructions such as a conformal model of minimal representations (e.g. the construction of the intrinsic conservative quantities for the conformally invariant differential equations). Instead, we highlight an L^2-model (*Schrödinger model*) of the minimal representations and its variant. We apologize for not being able to mention some other important works on minimal representations, e.g., see [8] and references therein. For a comparison of the L^2-model with the conformal model, we refer to [28, Introduction].

2 More Symmetric than Symmetric Spaces

The traditional geometric construction of representations of Lie groups G is given in the following two steps:
Step 1. The group G acts on a geometry X.
Step 2. By the translation, G acts linearly on the space $\Gamma(X)$ of functions (sections of equivariant bundles, or cohomologies, $\cdots$).

Naïvely, the Gelfand–Kirillov dimension of the representation on $\Gamma(X)$ is supposed to be the dimension of X. Thus we may expect that the representation on the function space $\Gamma(X)$ is 'small' if the geometry X itself is small.

First of all, we ask when the geometry X is 'small'.

For this we may begin with the case when G acts transitively on X, or equivalently, X is a homogeneous space G/H. Further, if we compare two homogeneous spaces $X_1 = G/H_1$ and $X_2 = G/H_2$ with $H_1 \subset H_2$, we may think that X_2 is smaller than X_1. Hence 'smaller' representations on $\Gamma(X)$ should be attained if $X = G/H$ where H is a maximal subgroup of G.

Here are two typical settings for real reductive Lie groups G:

- (G, H) is a symmetric pair.
  In this case, the Lie algebra $\mathfrak{h}$ of H is maximal reductive in $\mathfrak{g}$. Analysis on reductive symmetric spaces G/H has been largely developed in particular, since 1950s by the Gelfand school, Harish-Chandra, Shintani, Helgason, Takahashi, Molchanov, Faraut, Flensted-Jensen, Matsuki–Oshima–Sekiguchi, Delorme, van den Ban, Schlichtkrull, among others.
- H is a Levi subgroup of G.
  In this case, there exists a G-invariant polarization on G/H, and its geometric quantization obtained by the combination of the Mackey induction (real polarization) and the Dolbeault cohomologies (complex polarization) produces a 'generic part' of irreducible unitary representations of G. The resulting representations are the 'smallest' if H is a maximal Levi subgroup.

These two typical examples are related: Tempered representations for reductive symmetric spaces (i.e. irreducible unitary representations that contribute to $L^2(G/H)$) are given by the combination of the ordinary and cohomological parabolic inductions. A missing picture in the above two settings is so called 'unipotent representations' including minimal representations.

On the other hand, it is rare but still happens that the representation of G on the function space $\Gamma(X)$ extends to a representation of a group $\widetilde{G}$ which contains G, even when the G-action on the geometry X does not extend to $\widetilde{G}$ (in particular, Step 1 does not work for the whole group $\widetilde{G}$). We discuss this phenomenon in the Schrödinger model of minimal representations when G is a maximal parabolic subgroup (the notation $(G, \widetilde{G})$ here will be replaced by (P, G) in Section 3). Such a phenomenon also occurs when G is reductive. Thus the analysis of minimal representations may be thought of as 'analysis with more symmetries' than the traditional analysis on homogeneous spaces. Here is a typical example:

Example 2.1 ([21, Theorem 5.3]) *The minimal representation of the indefinite orthogonal group $\widetilde{G} = O(p, q)$ ($p + q$:even) is realized in function spaces on symmetric spaces of the subgroups $G = O(p-1, q)$ or $O(p, q-1)$ on which the whole group $\widetilde{G}$ cannot act geometrically.*

Example 2.2 ([34]) *The restriction of the most degenerate principal series representations of $\widetilde{G} = GL(n, \mathbb{R})$ to the subgroup $G = O(p, q)$ ($p + q = n$) reduces to the analysis of the symmetric space of G on which the whole group $\widetilde{G}$ cannot act transitively.*

Further examples and explicit branching rules can be found in [21, 32, 34] where the restriction of minimal representations to subgroups (*broken symmetries*) reduce to analysis on certain semisimple symmetric spaces.

3 Schrödinger Model of Minimal Representations

Any coadjoint orbit of a Lie group is naturally a symplectic manifold endowed with the Kirillov–Kostant–Souriau symplectic form. For a reductive Lie group G, 'geometric quantization' of semisimple coadjoint orbits has been considerably well-understood — this corresponds to the ordinary or cohomological parabolic induction in representation theory, whereas 'geometric quantization' of nilpotent coadjoint orbits is more mysterious (see [4, 12, 23]).

In this section we explain a recent work [11] with Hilgert and Möllers on the L^2-construction of minimal representations built on a Lagrangian subvariety of a real minimal nilpotent orbit, which continues a part of the earlier works [33] with Ørsted, and [28] with Mano.

Suppose that V is a simple Jordan algebra over $\mathbb{R}$. We assume that its maximal Euclidean Jordan subalgebra is also simple. Let G and L be the identity components of the conformal group and the structure group of the Jordan algebra V, respectively. Then the Lie algebra $\mathfrak{g}$ is a real simple Lie algebra and has a Gelfand–Naimark decomposition $\mathfrak{g} = \overline{\mathfrak{n}} + \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{n} \simeq V$ is regarded as an Abelian Lie algebra, $\mathfrak{l} \simeq \mathfrak{str}(V)$ the structure algebra, and $\overline{\mathfrak{n}}$ acts on V by quadratic vector fields.

Let $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$ be a (real) minimal nilpotent coadjoint orbit. By identifying $\mathfrak{g}$ with the dual $\mathfrak{g}^*$, we consider the intersection $V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}}$, which may be disconnected (this happens in the case (3.2) below). Let Ξ be any connected component of $V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}}$. We note that the group L acts on Ξ but G does not. There is a natural L-invariant Radon measure on Ξ, and we write $L^2(\Xi)$ for the Hilbert space consisting of square integrable functions on Ξ. Then we can define a unitary representation on $L^2(\Xi)$ (Schrödinger model) built on a Lagrangian submanifold Ξ in this generality [11], see also [5, 33].

Theorem 3.1 (Schrödinger model) *Suppose $V \not\simeq \mathbb{R}^{p,q}$ with $p+q$ odd.*
1) *Ξ is a Lagrangian submanifold of $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$.*
2) *There is a finite covering group $G^{\sim}$ of G such that $G^{\sim}$ acts on $L^2(\Xi)$ as an irreducible unitary representation.*
3) *The Gelfand–Kirillov dimension of π attains its minimum among all infinite dimensional representations of $G^{\sim}$, i.e.* $\mathrm{DIM}(\pi) = \frac{1}{2} \dim \mathbb{O}_{\min}^{G_{\mathbb{R}}}$.
4) *The annihilator of the differential representation $d\pi$ is the Joseph ideal in the enveloping algebra $U(\mathfrak{g})$ if V is split and $\mathfrak{g}$ is not of type A.*

The simple Lie algebras arisen in Theorem 3.1 are listed as follows:

$$\mathfrak{sl}(2k,\mathbb{R}), \mathfrak{so}(2k,2k), \mathfrak{so}(p+1,q+1), \mathfrak{e}_{7(7)}, \tag{3.1}$$

$$\mathfrak{sp}(k,\mathbb{R}), \mathfrak{su}(k,k), \mathfrak{so}^*(4k), \mathfrak{so}(2,k), \mathfrak{e}_{7(-25)}, \tag{3.2}$$

$$\mathfrak{sp}(k,\mathbb{C}), \mathfrak{sl}(2k,\mathbb{C}), \mathfrak{so}(4k,\mathbb{C}), \mathfrak{so}(k+2,\mathbb{C}), \mathfrak{e}_7(\mathbb{C}), \tag{3.3}$$

$$\mathfrak{sp}(k,k), \mathfrak{su}^*(4k), \mathfrak{so}(k,1). \tag{3.4}$$

Remark 1. In the case where V is an Euclidean Jordan algebra, G is the automorphism group of a Hermitian symmetric space of tube type (see (3.2)) and there are two real minimal nilpotent orbits. The resulting representations π are highest (or lowest) weight modules.

Remark 2. If the complex minimal nilpotent orbit $\mathbb{O}_{\min}^{G_{\mathbb{C}}}$ intersects with $\mathfrak{g}$, then $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$ is equal to $\mathbb{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ or its connected component. We notice that $\mathbb{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ may be an empty set depending on the real form $\mathfrak{g}$. In the setting of Theorem 3.1, this occurs for (3.4). In this case, the representation π in Theorem 3.1 is not a minimal representation as the annihilator of $d\pi$ is not the Joseph ideal, but π is still one of the 'smallest' infinite dimensional representations in the sense that the Gelfand–Kirillov dimension attains its minimum.

Remark 3. There is no minimal representation for any group with Lie algebra $\mathfrak{o}(p+1,q+1)$ with $p+q$ odd, p, $q \geq 3$ (see [43, Theorem 2.13]).

Example 3.2 *Let $V = \mathrm{Sym}(m,\mathbb{R})$. Then $G = Sp(m,\mathbb{R})$ and*

$$V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}} = \{X \in M(m,\mathbb{R}) : X = {}^tX, \mathrm{rank}\, X = 1\}. \tag{3.5}$$

Let $\Xi := \{X \in V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}} : \operatorname{Trace} X > 0\}$. *Then the double covering map (folding map)*

$$\mathbb{R}^m \setminus \{0\} \to \Xi, \quad v \mapsto v^t v$$

induces an isomorphism between $L^2(\Xi)$ *and the Hilbert space* $L^2(\mathbb{R}^m)_{\text{even}}$ *of even square integrable functions on* $\mathbb{R}^m$. *Thus our representation* π *on* $L^2(\Xi)$ *is nothing but the Schrödinger model of the even part of the Segal–Shale–Weil representation of the metaplectic group* $Mp(m, \mathbb{R})$ *[7, 13].*

Example 3.3 *Let* $V = \mathbb{R}^{p,q}$ *with* $p + q$ *even. Then* $\mathfrak{g} = \mathfrak{o}(p + 1, q + 1)$, *and*

$$V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}} = \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\} \setminus \{0\}. \quad (3.6)$$

If $p = 1$, $V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}}$ *consists of two connected components according to the signature of* ξ_1, *i.e. the past and future cones. They yield highest/lowest weight modules. For* $p, q \geq 2$, $V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}}$ *is connected, and our representation* π *on* $L^2(\Xi)$ *is the Schrödinger model of the minimal representation of* $O(p+1, q+1)$ *constructed in [33], which is a neither highest nor lowest weight module.*

As we discussed in Section 2 in contrast to traditional analysis on homogeneous spaces, the group G in our setting is too large to act geometrically on Ξ. This very feature in the Schrödinger model is illustrated by the fact that the Lie algebra $\bar{\mathfrak{n}}$ acts as differential operators on Ξ of second order. They are *fundamental differential operators* [28] in the setting of Example 3.3 (see also Bargmann–Todorov [2]). In [11], these differential operators are said to be *Bessel operators*, and serve as a basic tool to study the Schrödinger model π in the setting of Theorem 3.1.

4 Special Functions to 4th order Differential Operators

Guiding Principle 1.1 suggests that there should exist plentiful functional equations in the representation spaces for minimal representations. Classically, it is well-known that Hermite polynomials form an orthogonal basis for the radial part of the Schrödinger model of the Weil representation [7], whereas Laguerre polynomials arise in the minimal representation of the conformal group $SO(n, 2)$ ([41]).

These classical minimal representations are highest weight modules. However, for more general reductive groups, minimal representations do not always have highest weight vectors, and the corresponding 'special functions' do not necessarily satisfy second order differential equations. We found in [28] that Meijer's G-functions $G_{0\,4}^{2\,0}(x|b_1, b_2, b_3, b_4)$ play an analogous role in the minimal representation of $O(p, q)$. Here Meijer's G-functions $G_{0\,4}^{2\,0}(x|b_1, b_2, b_3, b_4)$ satisfy a fourth order ordinary differential equation

$$\prod_{j=1}^{4}(x\frac{d}{dx} - b_j)u(x) = xu(x).$$

More generally, the following fourth order differential operators

$$\mathcal{D}_{\mu,\nu} := \frac{1}{x^2}((\theta+\nu)(\theta+\mu+\nu) - x^2)(\theta(\theta+\mu) - x^2) - \frac{(\mu-\nu)(\mu+\nu+2)}{2}$$

appear naturally in the Schrödinger model of minimal representations in the setting of Theorem 3.1. Here $\theta = x\frac{d}{dx}$.

The subject of [9, 10, 29] is the study of eigenfunctions of $\mathcal{D}_{\mu,\nu}$ including

- generating functions for eigenfunctions of $\mathcal{D}_{\mu,\nu}$,
- asymptotic behavior near the singularities,
- L^2-eigenfunctions and concrete formulas of L^2-norms,
- integral representations of eigenfunctions,
- recurrence relations among eigenfunctions,
- (local) monodromy.

The L^2-eigenfunctions of $\mathcal{D}_{\mu,\nu}$ arise as K-finite vectors in the Schrödinger model of the minimal representations constructed in Theorem 3.1 in a uniform fashion. These 'special functions' with certain integral parameters yield orthogonal polynomials (the *Mano polynomials* $M_j^{\mu,l}(x)$) satisfying again fourth order differential equations [10], which include Hermite polynomials and Laguerre polynomials as special cases. We note that the fourth order differential equation $\mathcal{D}_{\lambda,\mu}f = \nu f$ reduces to a differential equation of second order when G/K is a tube domain (see (3.2)). See also Kowata–Moriwaki [38] for further analysis of the fundamental differential operators on Ξ.

5 Broken Symmetries and Branching Laws

As indicated in Guiding Principle 1.1, the 'large symmetries' in representation spaces of minimal representations produce also fruitful examples of branching laws which we can expect a simple and detailed study.

Suppose π is a unitary representation of a real reductive Lie group G. We consider π as a representation of a subgroup G' of G, referring it as the restriction $\pi|_{G'}$. In general, the restriction $\pi|_{G'}$ decomposes into a direct integral of irreducible representations of G' (*branching law*). It often happens that the branching law contains continuous spectrum if G' is non-compact. Even worse, each irreducible representation of G' may occur in the branching law with infinite multiplicities. See [20] for such wild examples even when (G, G') is a symmetric pair. In [16, 17], we raised the following:

Program 5.1 *1) Determine the triple (G, G', π) for which the restriction $\pi|_{G'}$ decompose discretely with finite multiplicities.*

2) Find branching laws for (1).

Program 5.1 intends to single out a nice framework of branching problems for which we can expect a detailed and explicit study of the restriction. Concerning Program 5.1 (1) for Zuckerman's derived functor modules π, a necessary and sufficient condition for *discrete decomposition with finite multiplicities* was proved in [17, 19], and a complete classification was given with Oshima [35] when (G, G') is a reductive symmetric pair.

As such, the local theta correspondence with respect to compact dual pairs is a classic example for minimal representations π:

Example 5.2 *Suppose that π is the Weil representation, and that $G' = G'_1 \cdot G'_2$ is a dual pair in $G = Mp(n, \mathbb{R})$ with G'_2 compact. Then the restriction $\pi|_{G'}$ decomposes discretely and multiplicity-freely. The resulting branching laws yield a large part of unitarizable highest weight modules of G'_1 (Enright–Howe–Wallach [6]).*

In order to discuss Program 5.1 for minimal representations, we recall from [17, 18, 19] the general theory. Let K be a maximal compact subgroup of G, T a maximal torus of K, and $\mathfrak{t}$, $\mathfrak{k}$ the Lie algebras of T, K, respectively. We choose the set $\Delta^+(\mathfrak{k}, \mathfrak{t})$ of positive roots, and denote by $\mathfrak{t}_+$ the dominant Weyl chamber in $\sqrt{-1}\mathfrak{t}^*$. We also fix a K-invariant inner product on $\mathfrak{k}$, and regard $\sqrt{-1}\mathfrak{t}^*$ as a subspace of $\sqrt{-1}\mathfrak{k}^*$.

First, suppose that K' is a closed subgroup of K. The group K acts on the cotangent bundle $T^*(K/K')$ of the homogeneous space K/K' in a Hamiltonian fashion. We write

$$\mu : T^*(K/K') \to \sqrt{-1}\mathfrak{k}^*$$

for the momentum map, and define the following closed cone by

$$C_K(K') := \operatorname{Image} \mu \cap \mathfrak{t}_+.$$

Second, let $\operatorname{Supp}_K(\pi)$ be the set of highest weights of finite dimensional irreducible representations of K occurring in a K-module π. The asymptotic K-support $\operatorname{AS}_K(\pi)$ is defined to be the asymptotic cone of $\operatorname{Supp}_K(\pi)$. It is a closed cone in $\mathfrak{t}_+$. There are only finitely many possibilities of $\operatorname{AS}_K(\pi)$ for the restriction $\pi|_K$ of irreducible representations π of G.

The asymptotic cone $\operatorname{AS}_K(\pi)$ tends to be a 'small' subset in $\mathfrak{t}_+$ if π is a 'small' representation. For example,

$$\begin{aligned} \operatorname{AS}_K(\pi) = &\{0\} && \text{if } \dim \pi < \infty, \\ \operatorname{AS}_K(\pi) = &\mathbb{R}_+ \beta && \text{if } \pi \text{ is a minimal representation,} \end{aligned} \tag{5.1}$$

where β is the highest root of the K-module $\mathfrak{p}_{\mathbb{C}} := \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}$. The formula (5.1) holds in a slightly more general setting where the associated variety of π is the closure of a single minimal nilpotent $K_{\mathbb{C}}$-orbit on $\mathfrak{p}_{\mathbb{C}}$ [36]. Concerning Program 5.1, we established an easy-to-check criterion in [18]:

Theorem 5.3 *Suppose G' is a reductive subgroup of G such that $K' := G' \cap K$ is a maximal compact subgroup of G'. If*

$$C_K(K') \cap \mathrm{AS}_K(\pi) = \{0\}, \tag{5.2}$$

then the restriction $\pi|_{G'}$ decomposes discretely into a direct sum of irreducible unitary representations of G' with finite multiplicities.

As was observed in [22], we can expect from the formula (5.1) and from the criterion (5.2) that there is plenty of subgroups G' for which the restriction of the minimal representation of G decomposes discretely and with finite multiplicities. Reductive symmetric pairs (G, G') for which the restriction $\pi|_{G'}$ is (infinitesimally) discretely decomposable for a minimal representation π of G has been recently classified in [36].

6 Generalized Fourier Transform as a Unitary Inversion

In the L^2-model of the minimal representation π of G on $L^2(\Xi)$, the action of the maximal parabolic subgroup P with Lie algebra $\mathfrak{l} + \mathfrak{n}$ is simple, namely, it is given just by translations and multiplications. Let w be the conformal inversion of the Jordan algebra. In light of the Bruhat decomposition

$$G = P \amalg PwP,$$

it is enough to find $\pi(w)$ in order to give a global formula of the G-action on $L^2(\Xi)$. We highlight this specific unitary operator, and set

$$\mathcal{F}_\Xi := c\pi(w), \tag{6.1}$$

where c is a complex number of modulus one (the *phase factor*). We call $\mathcal{F}_\Xi$ the *unitary inversion operator*. We studied in a series of papers [26, 27, 28] with Mano the following:

Problem 6.1 *Find an explicit formula of the integral kernel of $\mathcal{F}_\Xi$.*

The kernel of the Euclidean Fourier transform is given by $e^{-i\langle x,\xi\rangle}$, which is locally integrable. It is plausible that this analytic feature happens if and only if the corresponding minimal representation is of highest weight. Thus we raise the following:

Question 6.2 *Let $(\pi, L^2(\Xi))$ be the L^2-model of a minimal representation π of a simple Lie group $G^\sim$ constructed on a Lagrangian submanifold Ξ of $\mathbb{O}^G_{\min}$ as in Theorem 3.1* [11]. *Are the following two conditions equivalent?*

(i) *The kernel of the unitary inversion operator $\mathcal{F}_\Xi$ is locally integrable.*
(ii) *π is a highest/lowest weight module.*

Here we have excluded the case where the simple Lie algebra $\mathfrak{g}$ is of type A_n (the Joseph ideal is not defined for $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n(\mathbb{C})$). In the case $G = O(p+1, q+1)$ with $p + q$ even > 2, it was proved in [28] that (i) holds if and only if either $\min(p, q) = 1$ (equivalently, (ii) holds) or $(p, q) = (3, 3)$ (equivalently, $\mathfrak{g} = \mathfrak{o}(3, 3) \simeq \mathfrak{sl}(4, \mathbb{R})$ is of type A_3). The implication (ii) $\Rightarrow$ (i) was proved in [12] for tube type, see (8.5). The implication (i) $\Rightarrow$ (ii) is an open problem except for the above mentioned case $\mathfrak{g} = \mathfrak{o}(p + 1, q + 1)$.

When $G = O(p + 1, q + 1)$ (see Example 3.3), $\mathcal{F}_{\Xi}$ intertwines the multiplication of coordinate functions ξ_j $(1 \le j \le p + q)$ with the operators R_j $(1 \le j \le p + q)$ which are mutually commuting differential operators of second order on Ξ (see Bargmann–Todorov [2], [28, Chapter 1]).

This algebraic feature is similar to the classical fact that the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^m}$ intertwines the multiplication operators ξ_j and the differential operators $\sqrt{-1}\partial_j$ $(1 \le j \le m)$ (see Example 3.2). In the setting of Theorem 3.1, $\mathcal{F}_{\Xi}$ intertwines the multiplication of coordinate functions with Bessel operators. Actually, this algebraic feature determines uniquely $\mathcal{F}_{\Xi}$ up to a scalar [11, 28].

Concerning Problem 6.1, the first case is well-known (see [7] for example):

1) $\mathfrak{g} = \mathfrak{sp}(m, \mathbb{R})$.

$\mathcal{F}_{\Xi} =$ the Euclidean Fourier transform on $\mathbb{R}^m$.

Here are some recent results on a closed formula of the integral kernel:

2) $\mathfrak{g} = \mathfrak{o}(p + 1, q + 1)$ (with Mano [27]).

3) The associated Riemannian symmetric space G/K is of tube type (see (8.5)).

We note that minimal representations in the cases 1) and 3) are highest (or lowest) weight modules, whereas minimal representations in the case 2) do not have highest weights when $p, q \ge 2$ and $p + q$ is odd.

Problem 6.1 is open for other cases, in particular, for minimal representations without highest weights except for the case $G = O(p + 1, q + 1)$.

7 SL_2-triple in the Schrödinger Model

On $\mathbb{R}^m$, we set $|x| := (\sum_{j=1}^m x_j^2)^{\frac{1}{2}}$, $E := \sum_{j=1}^m x_j \frac{\partial}{\partial x_j}$ (Euler operator) and $\Delta = \sum_{j=1}^m \frac{\partial^2}{\partial x_j^2}$ (Laplacian). Then it is classically known (e.g., [7, 13]) that the operators

$$\widetilde{h}' := E + \frac{m}{2}, \quad \widetilde{e}' := \frac{\sqrt{-1}}{2}|x|^2, \quad \widetilde{f}' := \frac{\sqrt{-1}}{2}\Delta \tag{7.1}$$

form an $\mathfrak{sl}_2$-triple, namely, the following commutation relation holds:

$$[\widetilde{h}', \widetilde{e}'] = 2\widetilde{e}', \quad [\widetilde{h}', \widetilde{f}'] = -2\widetilde{f}', \quad [\widetilde{e}', \widetilde{f}'] = \widetilde{h}'.$$

On the other hand, we showed in [27] that the following operators

$$\widetilde{h} := 2E + m - 1, \quad \widetilde{e} := 2\sqrt{-1}|x|, \quad \widetilde{f} := \frac{\sqrt{-1}}{2}|x|\Delta \tag{7.2}$$

also forms an $\mathfrak{sl}_2$-triple, i.e., $[\widetilde{h}, \widetilde{e}] = 2\widetilde{e}, \quad [\widetilde{h}, \widetilde{f}] = -2\widetilde{f}, \quad [\widetilde{e}, \widetilde{f}] = \widetilde{h}$.

Further the differential operator

$$D := \frac{1}{2\sqrt{-1}}(-\widetilde{e} + \widetilde{f}) = |x|(\frac{\Delta}{4} - 1)$$

extends to a self-adjoint operator and has only discrete spectra on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ which are given by $\{-(j + \frac{m-1}{2}) : j = 0, 1, 2, \cdots\}$ (see [27]), whereas the *Hermite operator*

$$\mathcal{D} := \frac{1}{2\sqrt{-1}}(-\widetilde{e}' + \widetilde{f}') = \frac{1}{4}(\Delta - |x|^2)$$

extends to a self-adjoint operator and has only discrete spectra on $L^2(\mathbb{R}^m, dx)$ which are given by $\{-\frac{1}{2}(j + \frac{m}{2}) : j = 0, 1, 2, \cdots\}$ (see [7, 13]). Hence, one can define for $\mathrm{Re}\, t \geq 0$:

$$e^{tD} := \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k \quad \text{on } L^2(\mathbb{R}^m, \frac{dx}{|x|}),$$
$$e^{t\mathcal{D}} := \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{D}^k \quad \text{on } L^2(\mathbb{R}^m, dx).$$

They are holomorphic one-parameter semigroups consisting of Hilbert–Schmidt operators for $\mathrm{Re}\, t > 0$, and are unitary operators for $\mathrm{Re}\, t = 0$.

A closed formula for both e^{tD} and $e^{t\mathcal{D}}$ is known. That is, the holomorphic semigroup $e^{t\mathcal{D}}$ has the classical Mehler kernel given by the Gaussian kernel $e^{-|x|^2}$ and reduces to the Euclidean Fourier transform when $t = \sqrt{-1}\pi$ ([13, §5]), whereas the integral kernel of the holomorphic semigroup e^{tD} is given by the I-Bessel function and the special value at $t = \sqrt{-1}\pi$ is by the J-Bessel function (see [27, Theorem A and Corollary B] for concrete formulas).

We can study these holomorphic semigroups by using the theory of discretely decomposable unitary representations (e. g. [16, 17, 18]). Actually, the aforementioned $\mathfrak{sl}_2$-triple arises as the differential action of the Schrödinger model of the minimal representations of $Mp(m, \mathbb{R})$ on $L^2(\mathbb{R}^m, dx)$ and $SO_0(m+1, 2)$ on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$, respectively via

$$\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{sp}(1, \mathbb{R}) \subset \mathfrak{sp}(m, \mathbb{R}),$$
$$\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{so}(1, 2) \ \subset \mathfrak{so}(m+1, 2),$$

for which we write as $d\iota : \mathfrak{sl}(2, \mathbb{R}) \hookrightarrow \mathfrak{g}$.

In both cases, the Lie algebra $\mathfrak{g}$ contains a subalgebra commuting with $\iota(\mathfrak{sl}(2,\mathbb{R}))$, which is isomorphic to $\mathfrak{o}(m)$. Then the minimal representations decompose as the representation of the direct product group $SL(2,\mathbb{R})\times O(m)$ (up to coverings and connected groups) as follows:

$$L^2(\mathbb{R}^m, \frac{dx}{|x|}) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \pi^{SL(2,\mathbb{R})}_{2j+m-1} \boxtimes \mathcal{H}^j(\mathbb{R}^m).$$

$$L^2(\mathbb{R}^m, dx) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \pi^{SL(2,\mathbb{R})}_{j+\frac{m}{2}} \boxtimes \mathcal{H}^j(\mathbb{R}^m),$$

where $\mathcal{H}^j(\mathbb{R}^m)$ denotes the natural representation of $O(m)$ (or $SO(m)$) on the space of harmonic polynomials on $\mathbb{R}^m$ of degree j and $\pi_b^{SL(2,\mathbb{R})}$ stands for the irreducible unitary lowest weight representation of $SL(2,\mathbb{R})$ (or its covering group) with minimal K-type b.

These considerations bring us to interpolate operators occurring two minimal representations of $SO_0(m+1,2)\tilde{}$ and $Sp(m,\mathbb{R})$. For this, we take $a>0$ to be a deformation parameter, and define

$$\widetilde{h}_a := \frac{2}{a}E + \frac{m+a-2}{a}, \quad \widetilde{e}_a := \frac{\sqrt{-1}}{a}|x|^a, \quad \widetilde{f}_a := \frac{\sqrt{-1}}{a}|x|^{2-a}\Delta.$$

The operators (7.1) in the Weil representation corresponds to the case $a=2$, and the operators (7.2) for $SO_0(m+1,2)\tilde{}$ corresponds to the case $a=1$. They extend to self-adjoint operators on the Hilbert space $L^2(\mathbb{R}^m, |a|^{a-2}dx)$, form an $\mathfrak{sl}_2$-triple, and lift to a unitary representation of the universal covering group $SL(2,\mathbb{R})\tilde{}$ of $SL(2,\mathbb{R})$ for every $a>0$. The Hilbert space decomposes into a multiplicity-free discrete sum of irreducible unitary representations of $SL(2,\mathbb{R})\tilde{}\times O(m)$ as follows:

$$L^2(\mathbb{R}^m, |x|^{a-2}dx) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \pi^{SL(2,\mathbb{R})}_{\frac{2j+m-2}{a}+1} \boxtimes \mathcal{H}^j(\mathbb{R}^m).$$

The discrete decomposition of $\mathfrak{sl}_2$-modules becomes a tool to generalize the study of the unitary inversion operator $\mathcal{F}_{\Xi}$ and the holomorphic semigroup in [26, 27] to the following settings:

- Dunkl operators (with Ben Saïd and Ørsted [3]),
- Conformal group of Euclidean Jordan algebras (with Hilgert and Möllers [12]).

8 Quantization of Kostant–Sekiguchi Correspondence

In this section we discuss Theorem 3.1 in a special case where V is Euclidean, equivalently, G/K is a tube domain, and explain a recent work [12] with Hilgert, Möllers, and Ørsted on the construction of a new model (a Fock-type model) of minimal representations with highest weights and a generalization of the classical Segal–Bargmann transform, which we called a 'geometric quantization' of the Kostant–Sekiguchi correspondence. In the underlying idea, the discretely decomposable restriction of $\mathfrak{sl}(2,\mathbb{R})$, which appeared in [26], plays again an important role.

We recall (e.g., [7, 13]) that the classical Fock space $\mathcal{F}(\mathbb{C}^m)$ is a Hilbert space in the space $\mathcal{O}(\mathbb{C}^m)$ of holomorphic functions defined by

$$\mathcal{F}(\mathbb{C}^m) := \{f \in \mathcal{O}(\mathbb{C}^m) : \int_{\mathbb{C}^m} |f(z)|^2 e^{-|z|^2} dz < \infty\},$$

and that the Segal–Bargmann transform is a unitary operator

$$\mathcal{B} : L^2(\mathbb{R}^m) \xrightarrow{\sim} \mathcal{F}(\mathbb{C}^m), \quad u \mapsto (\mathcal{B}u)(z) := \int_{\mathbb{R}^m} K_{\mathcal{B}}(x,z) f(x) dx,$$

with the kernel

$$K_{\mathcal{B}}(x,z) := \exp(-\frac{1}{2}\langle z,z\rangle + 2\langle z,x\rangle - \langle x,x\rangle).$$

From a representation theoretic viewpoint, the classical Segal–Bargmann transform intertwines the two models of the Weil representation of the metaplectic group $Mp(m,\mathbb{R})$, namely, the Schrödinger model on $L^2(\mathbb{R}^m)$ and the Fock model on $\mathcal{F}(\mathbb{C}^m)$.

In order to find a natural generalization of this classical theory, we begin by examining how one may rediscover the classical Fock model. Our idea is to use the action of $\mathfrak{sl}_2$, more precisely, a 'holomorphically extended representation' of an open semigroup of $SL(2,\mathbb{C})$ rather than a unitary representation of $SL(2,\mathbb{R})$ itself. For this, we take a standard basis of $\mathfrak{sl}(2,\mathbb{R})$ as

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{8.1}$$

They satisfy the following Lie bracket relations: $[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$. We set

$$\begin{aligned} k :=& i(-e+f) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ c_1 :=& \begin{pmatrix} 1 & -i \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & \frac{1}{2i} \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{8.2}$$

By a simple matrix computation we have:

$$\exp(-\frac{t}{2}k)|_{t=i\pi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{R}). \tag{8.3}$$

The formula $\mathrm{Ad}(c_1)k = h$ shows that $c_1 \in SL(2,\mathbb{C})$ gives a Cayley transform. Correspondingly, the Bargmann transform may be interpreted as

$$\mathcal{B} = `\pi \circ \iota(c_1)'.$$

The right-hand side is not well-defined. We need an analytic continuation in the Schrödinger model and a lift in the diagram below:

$$\begin{array}{rl} & SL(2,\mathbb{R})^{\sim} \xrightarrow{\iota} G \xrightarrow{\pi} GL(L^2(\Xi)) \\ & \quad\downarrow \\ c_1 \in SL(2,\mathbb{C}) \supset & SL(2,\mathbb{R}) \end{array}$$

To be more precise, we write $w \in G$ for the lift of (8.3) via $d\iota : \mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{g}$. Since the action of the maximal parabolic subgroup P on $L^2(\Xi)$ is given by the translation and the multiplication of functions, it is easy to see what '$\pi(p)$' should look like for $p \in P_{\mathbb{C}}$. Therefore, we could give an explicit formula for the (generalized) Bargmann transform $\mathcal{B} =$ '$\pi \circ \iota(c_1)$' if we know the closed formula of the unitary inversion:

$$\mathcal{F}_{\Xi} = \mathcal{F}(w) \equiv \mathcal{F} \circ \iota \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Of course, this is not a rigorous argument, and $\pi(p)$ does not leave $L^2(\Xi)$ invariant. However, the formula (8.2) suggests what the function space $\pi \circ \iota(c_1)(L^2(\Xi))$ ought to be, and led us to an appropriate generalization of the classical Fock space as follows:

$$\mathcal{F}(\mathbb{O}_{\min}^{K_{\mathbb{C}}}) := \{F \in \mathcal{O}(\mathbb{O}_{\min}^{K_{\mathbb{C}}}) : \int_{\mathbb{O}_{\min}^{K_{\mathbb{C}}}} |F(z)|^2 \widetilde{K}_{\lambda-1}(|z|) d\nu(z) < \infty\}. \tag{8.4}$$

Here $\mathbb{O}_{\min}^{K_{\mathbb{C}}}$ is the minimal nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$ which is the counterpart of the minimal (real) nilpotent coadjoint orbit $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$ in $\mathfrak{g}^* \simeq \mathfrak{g}$ under the Kostant–Sekiguchi correspondence [40], see Figure 8.1. Thus the generalized Fock space $\mathcal{F}(\mathbb{O}_{\min}^{K_{\mathbb{C}}})$ is a Hilbert space consisting of L^2-holomorphic functions on the complex manifold $\mathbb{O}_{\min}^{K_{\mathbb{C}}}$ against the measure given by a renormalized K-Bessel function $\widetilde{K}_{\lambda-1}(|z|)d\nu(z)$ (see the comments after (8.5)).

$$\begin{array}{ccc}
\mathfrak{g} & & \mathfrak{p}_{\mathbb{C}} \\
\cup & & \cup \\
\mathbb{O}_{\min}^{G_{\mathbb{R}}} & \overset{\text{Kostant–Sekiguchi}}{\longleftrightarrow} & \mathbb{O}_{\min}^{K_{\mathbb{C}}} \\
\cup \text{ Lagrangian} & & \\
\Xi & &
\end{array}$$

Figure 8.1 (minimal nilpotent orbits in $\mathfrak{g}$ and $\mathfrak{p}_{\mathbb{C}}$)

We recall that Ξ is a Lagrangian submanifold of $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$, and $K_{\mathbb{C}}$ acts holomorphically on $\mathbb{O}_{\min}^{K_{\mathbb{C}}}$. Then as a 'quantization' of the Kostant–Sekiguchi correspondence, we define the generalized Bargmann transform $\mathcal{B} : L^2(\Xi) \to \mathcal{F}(\mathbb{O}_{\min}^{K_{\mathbb{C}}})$ by

$$f \mapsto \Gamma(\lambda) e^{-\frac{1}{2}\operatorname{tr}(z)} \int_{\Xi} \widetilde{I}_{\lambda-1}(2\sqrt{(z|x)}) e^{-\operatorname{tr}(x)} f(x) d\mu(x),$$

whereas the unitary inversion operator $\mathcal{F}_{\Xi}$ is given by

$$(\mathcal{F}_{\Xi} f)(y) = 2^{-r\lambda} \Gamma(\lambda) \int_{\Xi} \widetilde{J}_{\lambda-1}(2\sqrt{(x|y)}) f(x) d\mu(x). \tag{8.5}$$

Here $r = \operatorname{rank} G/K$, $(\ |\ )$ denotes the trace form of the Jordan algebra V, and $\lambda = \frac{1}{2}\dim_{\mathbb{R}} \mathbb{F}$ if $V = \operatorname{Herm}(k, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$, quaternion $\mathbb{H}$, or the octonion $\mathbb{O}$ (and $k = 3$) or $\lambda = \frac{1}{2}(k-2)$ if $V = \mathbb{R}^{1,k-1}$. $\widetilde{J}(t)$, $\widetilde{I}(t)$, and $\widetilde{K}(t)$ are the renormalization of the J-, I-, and K-Bessel function, respectively, following the convention of [28].

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