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Hidden Symmetries and Spectrum of the Laplacian on an Indefinite Riemannian Manifold

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Dedicated to Toshikazu Sunada on the occasion of his 60th birthday

Abstract. Inspired by Sunada's problem, we find a six dimensional, non-compact Γ-periodic Riemannian manifold that admits countably many discrete spectra of the Laplacian. This manifold also carries a three dimensional complex structure with *indefinite* Kähler metric. We observe a *hidden symmetry* in the sense that the automorphism group of the indefinite Kähler metric is larger than the group of Riemannian isometries. This very symmetry breaks a path to the theory of discontinuous groups for non-Riemannian manifolds and the theory of discrete decomposable branching laws of unitary representations.

1. Introduction

For a complete Riemannian manifold X, we denote by Δ the self-adjoint extension of the Laplace–Beltrami operator on X, and by $\mathrm{Spec}(X, \Delta)_d$ the set of its discrete spectra, namely, the set of those eigenvalues for which there exist L^2-eigenfunctions of Δ.

A simply-connected Riemannian manifold X is called Γ-*periodic* if X is an isometric covering of some compact Riemannian manifold M. Here, Γ stands for the fundamental group $\pi_1(M)$. Equivalently, X is a simply-connected Riemannian manifold admitting a fixed-point-free discontinuous group Γ of isometries such that the quotient space $\Gamma \setminus X$ (*Clifford–Klein form*) is compact.

It was probably in the summer of l987 when Kaoru Ono came to me, asking what I thought about the following problem. This problem was referred to in the citation [**26**] of Professor T. Sunada, the awardee of Iyanaga prize in March, 1987.

Problem 1 (T. Sunada). Does there exist any non-compact Γ-periodic Riemannian manifold X such that $\mathrm{Spec}(X, \Delta)_d \neq \emptyset$?

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According to T. Ochiai [**26**], no example of such X was known at that time. For instance, the standard flat Riemannian manifold $\mathbb{R}^n$ is Γ-periodic with $\Gamma \simeq \mathbb{Z}^n$, and $\operatorname{Spec}(\mathbb{R}^n, \Delta)_d = \emptyset$. Likewise, any Riemannian symmetric space X of non-compact type is Γ-periodic by a theorem of A. Borel [**3**], and $\operatorname{Spec}(X, \Delta)_d = \emptyset$ by the Plancherel-type theorem for the Fourier–Helgason transform [**11**].

Soon after, we realized that this is not always the case for more general Riemannian manifolds, and discovered certain non-compact Γ-periodic Riemannian manifolds X such that $\operatorname{Spec}(X, \Delta)_d \neq \emptyset$. We then wrote a letter to Professor Sunada. The joint work [**22**] was thus started. We had two different proofs for these counterexamples, namely, by using the Atiyah–Singer index theorem or unitary representation theory. My original counterexample to Problem 1 was the following:

Theorem 2. *There exists a six dimensional, non-compact, simply-connected Riemannian manifold X such that $\# \operatorname{Spec}(X, \Delta)_d = \infty$.*

More precisely, X is defined to be an S^2-bundle over the quarternionic unit disk (see (2.3)), and all discrete spectra for X are found to be

$$\operatorname{Spec}(X, \Delta)_d = \{\frac{1}{12}(a^2 + 4ab + b^2 + 3) \colon a > b \geq 1, a + b \text{ odd}\}.$$

We did not include Theorem 2 in the joint paper [**22**] because its proof relies heavily on infinite dimensional representation theory.

On the other hand, it turns out that this six dimensional manifold X is interesting of its own right. It carries not only a Γ-periodic Riemannian structure but also a three dimensional complex structure with *indefinite* Kähler metric. The transformation group $\widetilde{G}$ of biholomorphic and indefinite Kähler isometries is much larger than the group G of Riemannian isometries. Thus, this indefinite Kähler structure may be thought of as a *hidden symmetry.*

Later on, this example has become a driving force of the following three unexpectedly far-reaching general theories:

(A) discontinuous groups for non-Riemannian manifolds [**1, 12, 18, 25, 33**],
(B) discretely decomposable restriction of unitary representations [**14, 15, 16, 19**],
(C) vanishing theorem for modular symbols [**21**].

This paper is dedicated to Professor Sunada on the occasion of his 60th birthday. The subject of this paper is:

1) To give an explanation of the original example (Theorem 2) and its proof in a way as clear and elementary as possible.
2) To clarify the motivations of (A) and (B) in connection with 'strange phenomena' arising from hidden symmetries and Theorem 2.

This paper is organized as follows: Sections 1 and 2 are devoted to 1), and Section 3 explains the hidden symmetry of our manifold X. Then, (A) and (B) are discussed in Sections 4 and 5, respectively.

2. Six dimensional Riemannian manifold X

In this section, we define the six dimensional non-compact manifold X as was mentioned in Introduction. We shall endow X with a Riemannian structure on which the de Sitter group acts isometrically (see §2.4).

2.1. Unit disk in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Let $\mathbb{H}$ be the quarternionic number field $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$, and $\mathbb{P}^1\mathbb{H}$ the quarternionic projective line, that is, the set of equivalence classes of $(\zeta_1, \zeta_2) \in \mathbb{H}^2 \setminus \{0\}$, where the equivalence relation is given by the right $\mathbb{H}$-action:

$$(\zeta_1, \zeta_2) \sim (\zeta_1 a, \zeta_2 a) \quad \text{for some } a \in \mathbb{H}^\times. \tag{2.1}$$

For a better understanding of the quarternionic unit disk $D_{\mathbb{H}}$, we also consider simpler objects, namely, the unit disks $D_{\mathbb{R}}$ and $D_{\mathbb{C}}$, simultaneously. In what follows, we give two models for each, namely, the *unit ball model* and the *projective model.*

$$\begin{aligned}
D_{\mathbb{R}} &:= \{x \in \mathbb{R} : |x| < 1\} && \text{(unit ball model)}\\
&\simeq \{[x_1 : x_2] \in \mathbb{P}^1\mathbb{R} : |x_1| > |x_2|\} && \text{(projective model)},\\
D_{\mathbb{C}} &:= \{z = x + iy \in \mathbb{C} : |z| < 1\} && \text{(unit ball model)}\\
&\simeq \{[z_1 : z_2] \in \mathbb{P}^1\mathbb{C} : |z_1| > |z_2|\} && \text{(projective model)},\\
D_{\mathbb{H}} &:= \{\zeta = x + iy + ju + kv : |\zeta| < 1\} && \text{(unit ball model)}\\
&\simeq \{[\zeta_1 : \zeta_2] \in \mathbb{P}^1\mathbb{H} : |\zeta_1| > |\zeta_2|\} && \text{(projective model)}.
\end{aligned}$$

2.2. Definition and topology of X. In this subsection, we introduce our key object, namely, a six dimensional real manifold X as an S^2-bundle over $D_{\mathbb{H}}$.

We consider another equivalence relation on $\mathbb{H}^2 \setminus \{0\}$ given by

$$(\zeta_1, \zeta_2) \sim (\zeta_1 a, \zeta_2 a) \quad \text{for } a \in \mathbb{C}^\times. \tag{2.2}$$

The equivalence relation (2.1) is stronger than the equivalence relation (2.2), and consequently, we have a fibration

$$\pi : (\mathbb{H}^2 \setminus \{0\})/\mathbb{C}^\times \longrightarrow (\mathbb{H}^2 \setminus \{0\})/\mathbb{H}^\times \simeq \mathbb{P}^1\mathbb{H}$$

with typical fiber $\mathbb{H}^\times/\mathbb{C}^\times \simeq S^2$.

As an open subset of $(\mathbb{H}^2 \setminus \{0\})/\mathbb{C}^\times$, we define

$$\begin{aligned}
X &:= \pi^{-1}(D_{\mathbb{H}})\\
&= \{(\zeta_1, \zeta_2) \in \mathbb{H}^2 : |\zeta_1| > |\zeta_2|\}/\mathbb{C}^\times.
\end{aligned} \tag{2.3}$$

Then, we again have an S^2-bundle:

$$S^2 \to X \to D_{\mathbb{H}}. \tag{2.4}$$

We see from (2.4) that the total space X is a non-compact, simply-connected, six dimensional manifold.

2.3. X as a homogeneous space. For $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, we let $\mathbb{F}^n$ be the n-dimensional vector space with right $\mathbb{F}$-action. Then, any endomorphism of $\mathbb{F}^n$ commuting with the right $\mathbb{F}$-action is given by the left multiplication of $M(n;\mathbb{F})$.

Let $n = p + q$. We consider the quadratic form on $\mathbb{F}^n$ defined by the matrix

$$I_{p,q} := \begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{p} & \\ & \overbrace{-1 \ \ddots \ -1}^{q} \end{pmatrix}.$$

Then the group consisting $\mathbb{F}$-linear transforms preserving this quadratic form is given by

$$U(p,q;\mathbb{F}) := \{g \in M(n;\mathbb{F}) : {}^t\overline{g} I_{p,q} g = I_{p,q}\}.$$

The group $U(p,q;\mathbb{F})$ is a classical group called an *indefinite unitary group* over $\mathbb{F}$, and usually written as $O(p,q)$, $U(p,q)$, and $Sp(p,q)$, respectively for $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. We write $U(p;\mathbb{F})$ for $U(p,0;\mathbb{F})$ and $Sp(p)$ for $Sp(p,0)$, etc. We note that there are group isomorphisms $Sp(1) \simeq SU(2)$, $SU(1,1) \simeq SL(2,\mathbb{R})$, and $Sp(1,1) \simeq Spin(4,1)$ (de Sitter group).

In light of the projective model in §2.1, the indefinite unitary group $U(1,1;\mathbb{F})$ acts naturally on $D_{\mathbb{F}}$. It is easy to see that this action is transitive. As a homogeneous space, we have

$$D_{\mathbb{F}} \simeq U(1,1;\mathbb{F})/(U(1,\mathbb{F}) \times U(1,\mathbb{F})),$$

that is,

$$\begin{aligned}
&D_{\mathbb{R}} \simeq O(1,1)/(O(1)\times O(1)),\\
&D_{\mathbb{C}} \simeq U(1,1)/(U(1)\times U(1)) \qquad (\simeq SL(2,\mathbb{R})/SO(2)),\\
&D_{\mathbb{H}} \simeq Sp(1,1)/(Sp(1)\times Sp(1)) \quad (\simeq Spin(4,1)/Spin(4)).
\end{aligned}$$

Bearing these classic objects in mind, we now consider the S^2-bundle $X \to D_{\mathbb{H}}$ given in (2.4). We set

$$\begin{aligned}
G &:= U(1,1;\mathbb{F}) = Sp(1,1),\\
K &:= Sp(1)\times Sp(1),\\
H &:= U(1)\times Sp(1).
\end{aligned}$$

Since the left $M(2,\mathbb{H})$-action on $\mathbb{H}^2$ commutes with the right $\mathbb{H}$-action, the S^2-bundle $X \to D_{\mathbb{H}}$ is G-equivariant. Moreover, because G acts transitively on the base space $D_{\mathbb{H}} \simeq G/K$ and K acts transitively on the fiber $S^2 \simeq K/H$, we conclude that the G-action on the total space X is also transitive. As a homogeneous space, we have $X \simeq G/H$. Hence, the fibration $S^2 \to X \to D_{\mathbb{H}}$ (see (2.4)) has the following group theoretic expression

$$K/H \to G/H \to G/K,$$

which is derived from the inclusion $H \subset K \subset G$.

2.4. Riemannian structure on X. In this subsection, we give a Riemannian structure on X.

Retain the notation of §2.3, and let θ be the Cartan involution of the Lie algebra $\mathfrak{g}$ of G, corresponding to the maximal compact subgroup K. Let $B(\cdot,\cdot)$ be the Killing form of $\mathfrak{g}$. Then, the bilinear form $B_\theta := -B(\theta\cdot,\cdot)$ is K-invariant and positive definite on $\mathfrak{g}$. In particular, it induces an H-invariant inner product on the quotient vector space $\mathfrak{g}/\mathfrak{h}$ because $H \subset K$. Identifying $\mathfrak{g}/\mathfrak{h}$ with the tangent space $T_o(G/H)$ at $o = eH \in G/H$, we define a Riemannian metric on $X \simeq G/H$ by left G-translations. This is well-defined because the isotropy subgroup H acts on $T_o(G/H)$ as orthogonal transformations. The resulting Riemannian metric is complete. By definition, G acts on X as isometries of this Riemannian metric. From now on, we shall regard X as a Riemannian manifold by this metric.

Let us add some few words by comparing with the (well-known) Riemannian structure on the base space $D_{\mathbb{H}}$. As in the case of $X \simeq G/H$, we can define a

G-invariant Riemannian metric on $D_{\mathbb{H}} \simeq G/K$ by the left translation of the K-invariant inner product on $\mathfrak{g}/\mathfrak{k} \simeq T_o(G/K)$ induced from B_θ. This metric is the following Riemannian metric ds^2 on $D_{\mathbb{H}}$ (see (2.5)) multiplied by 24.

$$ds^2 = \frac{dx^2}{(1-x^2)^2} \qquad \text{on } D_{\mathbb{R}},$$

$$ds^2 = \frac{dx^2+dy^2}{(1-x^2-y^2)^2} \qquad \text{on } D_{\mathbb{C}},$$

$$ds^2 = \frac{dx^2+dy^2+du^2+dv^2}{(1-x^2-y^2-u^2-v^2)^2} \quad \text{on } D_{\mathbb{H}}. \tag{2.5}$$

Here, we have listed the cases $D_{\mathbb{R}}$ and $D_{\mathbb{C}}$ as well in order to pay our attention to the following classic facts:

1) ($\mathbb{F} = \mathbb{R}$) The Riemannian metric ds^2 on $D_{\mathbb{R}}$ is induced from the standard Euclidean metric on $\mathbb{R}$ through the diffeomorphism

$$D_{\mathbb{R}} \simeq \mathbb{R}, \quad x = \tanh s \leftrightarrow s.$$

2) ($\mathbb{F} = \mathbb{C}$) ds^2 on $D_{\mathbb{C}}$ is the (usual) Poincaré metric.

3) ($\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$) In all the three cases, the group $U(1,1;\mathbb{F})$ acts on $D_{\mathbb{F}}$ as isometries, and $D_{\mathbb{F}}$ becomes a Riemannian symmetric space.

3. Sketch of Proof of Theorem 2

Unlike $D_{\mathbb{H}} \simeq G/K$, the homogeneous Riemannian manifold $X \simeq G/H$ introduced in Section 2 is *not* a Riemannian symmetric space. Correspondingly, we cannot apply the well-established theory of global analysis on Riemannian symmetric spaces (see Helgason [**11**] and references therein) to the proof of Theorem 2. In fact, Theorem 2 presents a different phenomenon that never appears in the Riemannian symmetric case.

On the other hand, since the isotropy subgroup H is compact, the proof of Theorem 2 can be carried out by a combination of existing techniques in the infinite dimensional representation theory.

The purpose of this section is to give an explanation of these techniques, and provides a sketch of the proof for Theorem 2.

3.1. Discrete series representations. We begin with some general notations (see [**17**] for more details).

For a group G, we denote by $\widehat{G}$ the set of unitary equivalence classes of irreducible unitary representations of G. $\widehat{G}$ is called the *unitary dual* of G.

For a measure space X, we write $L^2(X)$ for the Hilbert space consisting of square integrable (complex valued) functions. If G acts on X in a measure preserving fashion, then we have naturally a unitary representation of G on $L^2(X)$ (*regular representation*) because

$$\int_X |f(g^{-1}x)|^2 d\mu(x) = \int_X |f(x)|^2 d\mu(x).$$

We say $\pi \in \widehat{G}$ is a *discrete series representation* for X if

$$\mathrm{Hom}_G(\pi, L^2(X)) \neq 0,$$

where Hom_G denotes the set of continuous G-intertwining operators. This terminology generalizes the original case (e.g. [**9, 10, 29**]) where X itself is a group manifold G equipped with a (left) Haar measure. We write $\mathrm{Disc}_G(X)$ for the set of discrete series representations. $\mathrm{Disc}_G(X)$ is a (possibly, empty) subset of $\widehat{G}$.

From now on, suppose we are in the setting of Section 2. In particular, $G = Sp(1,1) \simeq Spin(4,1)$ and $X \simeq G/H$ where $H = U(1) \times Sp(1) \simeq U(1) \times SU(2)$.

3.2. Γ-periodicity of X. This subsection shows that our Riemannian manifold X is Γ-periodic, i.e., it is an isometric covering of some compact Riemannian manifold.

We take a cocompact discrete subgroup Γ in G. The existence of such Γ is known. In fact, Borel [**3**] constructed Γ as an arithmetic subgroup of a reductive linear group, while Vinberg [**32**] ($n = 3, 4$) and Gromov–Piatetski-Shapiro [**7**] (n: general) constructed Γ as a non-arithmetic subgroup of $SO(n,1)$. Here, we note that our group G is locally isomorphic to the de Sitter group $SO(4,1)$.

Any cocompact discrete subgroup of G is finitely generated. By a lemma of Selberg [**30**], we can find a torsion-free subgroup of finite index. Therefore, we may and do assume that Γ is torsion-free. Then, Γ acts properly discontinuously, fixed point freely, and cocompactly on X. On the other hand, as G acts isometrically on X, so does any subgroup. Hence, X is a Γ-periodic manifold.

3.3. Description of discrete series representations for X. The G-invariant Riemannian metric on X (see §2.4) induces a G-invariant measure on X. Thus, we have a unitary representation of G on $L^2(X)$. In this subsection, we explain about how to find $\mathrm{Disc}_G(X)$ explicitly, in our specific setting $X \simeq G/H$.

Step 1. Characterization of $\mathrm{Disc}_G(X)$ as a subset of $\mathrm{Disc}_G(G)$.

Since H is compact, we can regard $L^2(X)$ as a subspace of $L^2(G)$, namely, the subspace $L^2(G)^H$ consisting of right H-invariant L^2-functions on G. Therefore, we have

$$\mathrm{Disc}_G(X) = \{\pi \in \mathrm{Disc}_G(G) : \pi \text{ contains a non-zero } H\text{-fixed vector}\}.$$

Step 2. Description of $\mathrm{Disc}_G(G)$.

The Harish-Chandra theory gives a precise description of $\mathrm{Disc}_G(G)$ for a reductive group manifold G. One of the known geometric constructions of discrete series representations is an infinite-dimensional generalization of the Borel–Weil–Bott theory. This was conjectured by Langlands and proved by Schmid [**29**]. This geometric construction (and its further generalization is useful for the description of $\mathrm{Disc}_G(X)$ (see Proposition 3 and Fact 7 below, see also [**17**, §2]).

Step 3. Branching laws (actual computation).

By a *branching law* we mean the irreducible decomposition formula of the restriction of π of a group G to its subgroup H. In light of the inclusive relation $G \supset K \supset H$ in our setting, this is divided into the following two substeps:

3-1) ($G \downarrow K$) Branching laws of infinite dimensional representations of the non-compact group G (use the solution to the Blattner conjecture [**10**]).

3-2) ($K \downarrow H$) Branching laws of finite dimensional representations of the compact group K (an easy part).

We note that an actual computation of the substep 3-1) is usually hard because it involves many cancellations (e.g. [**13**]), however, we can carry it out for small reductive groups like our G.

Combining Steps 1, 2 and 3, we see that $\mathrm{Disc}_G(X)$ consists of countably many irreducible unitary representations. Here is a precise description:

Proposition 3 (see [**17**, Example 3.2]).

$$\mathrm{Disc}_G(X) = \{\pi_{a,b} \in \widehat{G} : (a,b) \in \mathbb{Z}^2,\ a > b \geq 1,\ a+b \in 2\mathbb{Z}+1\}.$$

Loosely, $\pi_{a,b}$ is realized in the L^2-cohomology for a G-equivariant holomorphic line bundle parametrized by (a,b) over a complex manifold G/T where T is a two dimensional toral subgroup of G.

For the convenience to experts in representation theory of semisimple Lie groups, we note that $\pi_{a,b}$ is a discrete series representation of G with Harish-Chandra parameter (a,b) and with Blattner parameter $(a,b-1)$. It has also a K-type $(a+b-1,0)$. Here, we have identified $\widehat{K}$ with $\mathbb{N}^2$ by using the Cartan–Weyl highest weight theory. The underlying $(\mathfrak{g},K)$-module of $\pi_{a,b}$ is isomorphic to $W_+(a-b,a+b-1)$ with the notation as in [**14**, §6].

3.4. Spectrum of the Laplacian. This subsection discusses discrete spectrum of the Laplacian Δ on X.

The regular representation on $L^2(G)$ is a unitary representation of the direct product group $G \times G$ given by $f(x) \mapsto f(g_1^{-1} x g_2)$. Harish-Chandra's Plancherel formula gives an explicit irreducible decomposition of $L^2(G)$ for real reductive linear Lie groups [**9**]. It is of the form

$$L^2(G) \simeq \sum_{\pi \in \mathrm{Disc}_G(G)}^{\oplus} \pi \widehat{\otimes} \pi^{\vee} \oplus (\text{continuous spectrum}),$$

where $\pi^{\vee}$ denotes the contragredient representation of π, $\widehat{\otimes}$ is the Hilbert completion of the tensor product representation, and $\sum^{\oplus}$ is the Hilbert direct sum.

In our specific setting, we use the following two observations from §3.3:

1) $L^2(X) \simeq L^2(G)^H \subset L^2(G)$.
2) For $\pi \in \mathrm{Disc}_G(G)$, the space of H-fixed vectors in $\pi^{\vee}$ is at most of one dimension. Moreover, it is non-zero iff $\pi \in \mathrm{Disc}_G(X)$.

Therefore, we have the Plancherel-type formula:

$$L^2(X) \simeq \sum_{\pi \in \mathrm{Disc}_G(X)}^{\oplus} \pi \oplus (\text{continuous spectrum}). \tag{3.1}$$

In order to find the spectrum of the Laplacian Δ on $L^2(X)$, the key formula is

$$\Delta = -C_G + \frac{2}{3}\widetilde{\Delta}_{S^2}. \tag{3.2}$$

Here, C_G is a second order differential operator (*Casimir operator*) on X which is induced from the Casimir element of the enveloping algebra $U(\mathfrak{g})$, and $\widetilde{\Delta}_{S^2}$ is a left G-invariant differential operator that comes from the (normalized) Laplacian on the fiber S^2 (recall (2.4) for the S^2-bundle structure of X).

The point here is that C_G acts on irreducible unitary representations as scalars thanks to Schur's lemma for unitary representations, and that $\widetilde{\Delta}_{S^2}$ acts on spherical harmonics (along the fiber S^2) of degree k by the scalar $k(k+1)$. It should be noted that, unlike the symmetric case [**11**], the Casimir operator does not coincide with the Laplacian Δ. The eigenvalues of C_G and $\widetilde{\Delta}_{S^2}$ for $\pi_{a,b}$ (and its realization in $L^2(X)$) amount to:

- C_G acts on $\pi_{a,b}$ as the scalar $\frac{1}{12}(a^2+b^2-5)$.
- $\widetilde{\Delta}_{S^2}$ acts on the image of $\pi_{a,b}$ into $L^2(X)$ as the scalar $\frac{1}{4}((a+b)^2-1)$.

Therefore, it follows from the formula (3.2) that the Laplacian Δ acts on the representation space of $\pi_{a,b}$ in $L^2(X)$ as the scalar $\frac{1}{12}(a^2+4ab+b^2+3)$. In other words, the discrete series representation $\pi_{a,b}$ is entirely contained in the L^2-eigenspace of the Laplacian Δ for the eigenvalue $\frac{1}{12}(a^2+4ab+b^2+3)$. On the other hand, there is no contribution to the discrete spectrum of Δ from continuous spectrum in the Plancherel-type formula (3.1). This completes the proof of Theorem 2.

By a similar argument, we see that the continuous spectrum of the Laplacian Δ for $L^2(X)$ is given by $[\frac{1}{6},\infty)$.

4. Hidden symmetry of X

As we mentioned in Introduction, this is not the end of the story but the beginning. We shall consider other geometric structures on the same manifold X, and the corresponding hidden symmetries.

4.1. Hidden symmetry. We set

$$Y := \{(z_1,z_2,z_3,z_4)\in\mathbb{C}^4 : |z_1|^2+|z_2|^2>|z_3|^2+|z_4|^2\}/\mathbb{C}^\times. \tag{4.1}$$

As an open subset of $\mathbb{P}^3\mathbb{C}$, Y becomes naturally a three dimensional complex manifold. The indefinite unitary group

$$\widetilde{G} := U(2,2)$$

acts biholomorphically on Y.

By the isomorphism $\mathbb{H}\simeq\mathbb{C}^2$ as right $\mathbb{C}$-modules, we have from the expression (2.3) of X an obvious diffeomorphism

$$X\simeq Y,$$

and an inclusion $G\subset\widetilde{G}$. In summary, we have the following key scheme:

$$\begin{array}{ccccc} G & = & Sp(1,1) & \curvearrowright & X \\ \cap & & & & \wr\!| \\ \widetilde{G} & = & U(2,2) & \curvearrowright & Y \end{array}$$

Diagram 4.1

As G acts transitively on X, so does $\widetilde{G}$ on Y. Then, Y is represented as a homogeneous space

$$Y\simeq\widetilde{G}/\widetilde{H}, \tag{4.2}$$

where $\widetilde{H} := U(1)\times U(1,2)$.

4.2. Indefinite Kähler structure on Y. We endow $\mathbb{C}^4 = \mathbb{C}^{2+2}$ with the standard indefinite Hermitian form

$$dz_1 d\overline{z_1} + dz_2 d\overline{z_2} - dz_3 d\overline{z_3} - dz_4 d\overline{z_4}.$$

This form induces an indefinite Hermitian structure, to be denoted by h, of signature $+--$ on an open subset Y of $\mathbb{P}^3\mathbb{C}$ by formulas analogous to formulas for the Fubini–Study metrics. Clearly, $\widetilde{G}$ acts isometrically on (Y, h). The real part of h gives rise to an indefinite Riemannian metric on Y of signature $++----$, and the imaginary part of h gives a symplectic form on Y. In accordance with the standard terminology for the positive definite case, we shall call h an *indefinite Kähler metric*, and (Y, h) as such an *indefinite Kähler manifold*. (The same notion is also called $\frac{1}{2}$ Kähler in [**2**], and pseudo-Kähler in [**31**].)

We pin down the geometry and symmetries of X and Y in Diagram 4.1.

PROPOSITION 4. *1) X has a Riemannian structure, on which G acts isometrically.*

2) Y has a three dimensional complex manifold structure and an indefinite Kähler metric of signature $+--$, on which $\widetilde{G}$ acts isometrically.

We note that the one dimensional center $\widetilde{Z} := \{\mathrm{diag}(a, a, a, a) : |a| = 1\}$ of $\widetilde{G}$ acts trivially on Y. Therefore, the actual transformation group of Y is $\widetilde{G}/\widetilde{Z}$, a non-compact semisimple Lie group of dimension 15.

From a group theoretic viewpoint, we may ask when the following isomorphism occurs:

$$G/H \simeq \widetilde{G}/\widetilde{H} \qquad \text{(hidden symmetry)} \tag{4.3}$$

(see Diagram 4.1). This is explicitly determined in [**14**, §5] by means only of Lie algebras in the general setting of homogeneous spaces of reductive type. See [**14**, Example 5.2] for the list of such isomorphisms (4.3).

5. Discontinuous groups beyond Riemannian settings

One of interesting outcomes of the hidden symmetries in Diagram 4.1 is the existence of 'large' discontinuous groups for non-Riemannian homogeneous spaces. In this section, we explain an example of cocompact discontinuous groups of isometries for the indefinite Riemannian manifold Y, and then analyze its meaning from the viewpoint of non-compact transformation groups.

5.1. Trick and Theorem. We have already seen that a discrete subgroup Γ of G acts properly discontinuously on X. This is obvious from the general fact: the action of any discrete group of isometries for a Riemannian manifold is automatically properly discontinuous.

On the other hand, as we saw in Diagram 4.1, we have a diffeomorphism $X \simeq Y$, through which Γ acts biholomorphically and isometrically on the indefinite Kähler manifold Y. Then, clearly we have:

THEOREM 5. *The indefinite Kähler manifold Y is Γ-periodic. That is, there exists a compact complex manifold with indefinite Kähler metric such that its universal covering manifold is biholomorphic and isometric to Y.*

Here, we have used the terminology 'Γ-periodic' in an obvious manner for more general geometric structures than the Riemannian case.

5.2. Discontinuous groups for indefinite Riemannian manifolds. The geometric meaning of Theorem 5 is interesting in view of the following general fact: the isometric action of a discrete group is *not* always properly discontinuous for an *indefinite* Riemannian manifold.

For example, the cerebrated Calabi–Markus phenomenon [**4**] for Lorentz manifolds asserts that the isometric action of an infinite discrete group on a relativistic spherical space form (i.e. a complete Lorentz manifold with constant positive sectional curvature) is never properly discontinuous. Thus, Theorem 5 is the opposite extreme from the Calabi–Markus phenomenon because our indefinite Kähler manifold Y admits a cocompact isometric discontinuous group.

5.3. Example: discontinuous groups for $\mathbb{P}^{p-1,q}\mathbb{C}$. The existence problem of 'large' discontinuous groups of isometries on indefinite Riemannian manifold M is in general a hard problem even for the case where M is a well-known homogeneous space. In this subsection, we illustrate this by a higher dimensional generalization of Y as follows.

We fix positive integers p, q and define an open subset of the projective space $\mathbb{P}^{p+q-1}\mathbb{C}$ by

$$\mathbb{P}^{p-1,q}\mathbb{C} := \{(z_1, \dots, z_{p+q}) \in \mathbb{C}^{p+q} : |z_1|^2 + \dots + |z_p|^2 > |z_{p+1}|^2 + \dots + |z_{p+q}|^2\}/\mathbb{C}^\times.$$

Then, Y corresponds to the case $(p, q) = (2, 2)$. As in the case of Y, $\mathbb{P}^{p-1,q}\mathbb{C}$ carries an indefinite Kähler metric of signature $(p-1, q)$, on which $U(p, q)$ acts isometrically. Furthermore, $\mathbb{P}^{p-1,q}\mathbb{C}$ is homotopic to the projective space $\mathbb{P}^{p-1}\mathbb{C}$, and consequently, is simply-connected. We ask:

QUESTION 6. Is the indefinite Kähler manifold $\mathbb{P}^{p-1,q}\mathbb{C}$ Γ-periodic?

So far, the following results have been achieved

1) $p > q$ $\quad$ $\mathbb{P}^{p-1,q}\mathbb{C}$ is not Γ-periodic.
2) $p = 1$ $\quad$ $\mathbb{P}^{p-1,q}\mathbb{C}$ is Γ-periodic.
3) $p = 2$ and q even $\quad$ $\mathbb{P}^{p-1,q}\mathbb{C}$ is Γ-periodic.

1) is obtained as a special case of the general criterion of the Calabi–Markus phenomenon [**12**], 2) follows from a theorem of Borel [**3**] ($\mathbb{P}^{0,q}\mathbb{C}$ is nothing but a complex hyperbolic space), and 3) is proved by using a generalization of Diagram 4.1 (see [**12**]). For other parameters (p, q), Question 6 has not been solved.

5.4. Manifolds with locally homogeneous geometric structures. In light of the expression $Y \simeq \widetilde{G}/\widetilde{H}$ as a homogeneous space, we can interpret Theorem 5 as an existence theorem of compact manifolds locally modeled on homogeneous spaces.

The existence problem of cocompact discontinuous groups for Riemannian symmetric spaces was established by Borel [**3**] in the early 1960s, while that for general non-Riemannian homogeneous spaces is relatively new. In fact, Theorem 5 was the first example of compact complex manifolds that are modeled on indefinite Kähler semisimple symmetric spaces. In the last two decades, various approaches have been employed for this problem, including

- criterion of proper actions (see Benoist [**1**], Kobayashi [**12**])
- unitary representation theory (see Margulis [**25**])
- ergodic actions, Ratner's theory (see Zimmer [**33**])

See [**18, 23**] for a survey on recent developments in this area.

6. Discretely decomposable restriction of unitary representations

Another interesting outcome of the hidden symmetries in Diagram 4.1 is discretely decomposable restriction of unitary representations.

In this section, we compare spectra of two Laplacians Δ and $\Box$ on non-compact Γ-periodic (indefinite) Riemannian manifolds $X \simeq Y$, and state a strange phenomenon (see Theorem 8) about the non-existence of continuous spectrum in connection with Theorem 2. Then we analyze its meaning from the modern viewpoint of unitary representation theory. The idea here is to forget the geometry in the original specific example.

6.1. Laplacian on indefinite Riemannian manifold Y. We begin with Y as an indefinite Riemannian manifold of signature $++----$ (see §4.1), from which we have a $\widetilde{G}$-invariant measure on Y, and the Laplacian $\Box$ on Y. We remark that our Laplacian $\Box$ is not elliptic but ultra-hyperbolic. Explicitly, $\Box$ is computed and normalized as follows. First, let $\Box_{\mathbb{C}^{2,2}}$ be the generalized wave operator on $\mathbb{C}^4$

$$\Box_{\mathbb{C}^{2,2}} := -\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} - \frac{\partial^2}{\partial z_2 \partial \overline{z_2}} + \frac{\partial^2}{\partial z_3 \partial \overline{z_3}} + \frac{\partial^2}{\partial z_4 \partial \overline{z_4}},$$

where $\frac{\partial^2}{\partial z_j \partial \overline{z_j}} := \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}$ for $z_j = x_j + iy_j$. Second, let $[z,z] := |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2$ and $Y_1 := \{z \in \mathbb{C}^4 : [z,z] = 1\}$. Then $Y \simeq Y_1/S^1$. We identify $f \in C^\infty(Y)$ with an S^1-invariant function on Y_1, and let $\widetilde{f}(z) := f([z,z]^{-\frac{1}{2}} z)$ for $z \in \mathbb{C}^4$ such that $[z,z] > 0$. Then $\Box f$ is the pull-back of the function $\Box_{\mathbb{C}^{2,2}} \widetilde{f}$ to Y.

Since $\Box$ commutes with the isometric action of $\widetilde{G}$ on Y, we have a natural representation of $\widetilde{G}$ on each eigenspace of $\Box$. The following result is known by representation theoretic methods:

Fact 7 (see [**5**] and [**28**, Lemma 9]).
1) The ultra-hyperbolic operator $\Box$ has a self-adjoint extension on $L^2(Y)$.
2) There are countably many discrete spectra of $\Box$.
We set

$$L^2\text{-}\mathcal{S}ol(\Box, \lambda) := \{f \in L^2(Y) : \Box f = \lambda f\}.$$

Then, $L^2\text{-}\mathcal{S}ol(\Box, \lambda) \neq 0 \iff \lambda \in \{\lambda_n := 4(n-2)(n+1) : n = 1, 2, 3, \dots\}$.
3) For each n, $L^2\text{-}\mathcal{S}ol(\Box, \lambda_n)$ is an infinite dimensional closed subspace of $L^2(Y)$, on which $\widetilde{G}$ acts irreducibly.

We shall write ϖ_n for the irreducible unitary representation of $\widetilde{G}$ on $L^2\text{-}\mathcal{S}ol(\Box, \lambda_n)$. For the convenience to experts in representation theory, we note that the underlying $(\mathfrak{g}, K)$-module of ϖ_n is isomorphic to $V_0(2n-1, 1)$ in the notation of [**14**, §6]. In particular, it has an infinitesimal character $\frac{1}{2}(2n-1, 1, -1, -2n+1)$ in the Harish-Chandra parameter and a minimal K-type parameter $(n, -n, 0, 0)$.

6.2. Two Laplacians Δ and $\Box$. We have two self-adjoint differential operators of second order on the non-compact manifold $X \simeq Y$ (see Diagram 4.1):

- Δ: the (elliptic) Laplacian for the Riemannian manifold X
- $\Box$: the (ultra-hyperbolic) Laplacian for the indefinite Riemannian manifold Y.

It turns out that these two operators Δ and $\Box$ commute with each other. Then, a natural question is to find joint eigenspace decompositions of Δ and $\Box$. Then I discovered the following strange phenomenon:

Theorem 8 (1988). *For every $n = 1, 2, \ldots$, L^2-$\mathcal{S}ol(\Box, \lambda_n)$ decomposes discretely into a direct sum of countably many eigenspaces of the Laplacian Δ.*

Each eigenspace of Δ on X is infinite dimensional (see §3.4). Thus, crude information of the decomposition in Theorem 8 is the following dimensional formula:

$$\infty = \infty + \infty + \cdots, \tag{6.1}$$

where there is no term like $\int \infty d\mu$ in the right-hand side corresponding to the non-existence of continuous spectrum.

For a representation theoretic meaning of (6.1), we recall that $\Box$ commutes with the isometric action of $\widetilde{G}$ on Y, and Δ commutes with the isometric action of G on X. However, Δ does not commute with $\widetilde{G}$ (*broken symmetry*). Hence, we can interpret Theorem 8 as a theorem about the restriction of the irreducible representation ϖ_n of $\widetilde{G}$ to its subgroup G. In fact, we have the following branching law from $\widetilde{G}$ to G.

$$\varpi_n\big|_G \simeq \sum_{a-b=2n-1}^{\oplus} \pi_{a,b}. \tag{6.2}$$

Here we recall $\pi_{a,b}$ is an (infinite dimensional) irreducible unitary representation of G, and $\pi_{a,b} \not\simeq \pi_{a',b'}$ if $(a,b) \neq (a',b')$. Hence the branching law (6.2) is discretely decomposable and multiplicity-free.

6.3. Trick of the proof. As we saw in Theorem 2, an interesting feature of the Riemannian manifold X is that it is Γ-periodic but $\#\operatorname{Spec}(X,\Delta)_d = \infty$. Furthermore, Theorem 8 shows even the non-existence of continuous spectrum in eigenspaces of $\Box$ in $L^2(X)$.

A simple and geometric trick to prove Theorem 8 is based on the S^2-fibration (2.4). The key formula is

$$\Box = -24\Delta + 12\widetilde{\Delta}_{S^2}. \tag{6.3}$$

Therefore, the representation space of $\pi_{a,b}$ in $L^2(X)$ is entirely contained in the eigenspace of $\Box$ with the eigenvalue

$$(a-b)^2 - 9 = -2(a^2 + 4ab + b^2 + 3) + 3((a+b)^2 - 1).$$

The left-hand side amounts to $4(n-2)(n+1)$ if $a - b = 2n - 1$.

6.4. Idea of generalization: forgetting the geometric setting. Theorem 8 brings us to a strange (but very nice) phenomenon in unitary representation theory. Once we find one example, then we might expect to find a rich family of objects of similar nature if we formalize and analyze the problem properly. This subsection discusses briefly how to transform Theorem 8 into a reasonable question in representation theory. The point is to forget all about our previous geometric settings such as $X \simeq Y$.

We begin with a general setting:

$\widetilde{G} \supset G$: locally compact groups,

ϖ: irreducible unitary representation of $\widetilde{G}$.

The theory of von-Neumann algebras assures that there exists a measure μ on $\widehat{G}$ and a measurable function (*multiplicity*) $m_\varpi : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ such that the restriction

$\varpi|_G$ is decomposed into a direct integral of irreducible unitary representations of G:

$$\varpi|_G \simeq \int_{\widehat{G}}^{\oplus} m_\varpi(\pi)\pi d\mu(\pi) \qquad \text{(branching law).} \tag{6.4}$$

Now we consider (6.4) for reductive Lie groups. Here, we recall that building blocks of Lie groups are simple Lie groups, or slightly more generally, reductive Lie groups. Reductive symmetric pairs $(\widetilde{G}, G)$ are classic examples of pair of reductive groups. Our previous example $(\widetilde{G}, G) = (U(2,2), Sp(1,1))$ is the case. The pairs $(GL(n,\mathbb{C}), GL(n,\mathbb{R}))$ and $(GL(n,\mathbb{R}), O(p,n-p))$ are also the case (see Berger [**2**] for the infinitesimal classification of reductive symmetric pairs).

In contrast to the well-developed global analysis on reductive symmetric spaces $\widetilde{G}/G$ [**6, 11, 13**] (equivalently, analysis on the induced representation of finite dimensional representations from G to $\widetilde{G}$), it is notorious that the branching laws may behave very badly even for reductive symmetric pairs $(\widetilde{G}, G)$ if G is non-compact: the multiplicity $m_\varpi(\pi)$ may take ∞, and the support of the measure μ may not be a countable set (the branching law (6.4) may involve continuous spectra).

From this viewpoint, Theorem 8 gives us a promising example in branching problems by showing a non-trivial triple $(\widetilde{G}, G, \varpi)$ such that the branching law is discretely decomposable and with finite multiplicities.

By forgetting all the previous geometric setting and the trick (6.3), we can raise the following problem:

Problem 9 (see [**14**]). Find a triple $(\widetilde{G}, G, \varpi)$ such that the restriction $\varpi|_G$ decomposes discretely with finite multiplicities.

This problem was substantially resolved in [**15**] for reductive Lie groups by using a powerful machinery of micro-local analysis. A sufficient condition for the discretely decomposable restriction $\varpi|_G$ is given roughly in the following form:

$$\text{(the cone determined by } G) \cap \text{(the cone determined by } \varpi) = \{0\}.$$

The former cone is given as the image of a certain momentum map in symplectic geometry, whereas the latter cone is a polytope generated by a finite subset which can be studied also by algebraic representation theory (see [**16, 20**]). Beyond the original geometric setting, the formula (6.2) has been generalized to branching laws of 'small' unitary representations with respect to reductive symmetric pairs by the author [**14**], Gross–Wallach [**8**], and Ørsted–Speh [**27**] in three ways.

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