HARMONIC ANALYSIS ON HOMOGENEOUS MANIFOLDS OF REDUCTIVE TYPE AND UNITARY REPRESENTATION THEORY

Toshiyuki KOBAYASHI*

Institut Mittag-Leffler and University of Tokyo

Abstract. This paper is mostly a survey paper, with few new results. The first half is an exposition of the geometry of various homogeneous manifolds and the construction of standard unitary representations $A_{\mathfrak{q}}(\lambda)$ attached to elliptic orbits. The latter half discusses L^2-harmonic analysis, in particular, discrete series representations for homogeneous manifolds and some applications of the criterion of "admissible restriction" of $A_{\mathfrak{q}}(\lambda)$ to reductive subgroups.

A manifold with a transitive action of a Lie group is said to be a homogeneous manifold. Global analysis on homogeneous manifolds has interacted with various branches of mathematics, such as representation theory, differential geometry, $\mathcal{D}$-modules, functional analysis, algebraic geometry, automorphic forms, combinatorics, integral geometry, and so on. A Lie group is an example of a homogeneous manifold. First of all, we consider analysis on the simplest examples of Lie groups, namely, the torus S^1 and the additive group of real numbers $\mathbb{R}$. Classical harmonic analysis (Fourier series [or Fourier transforms]) is based on an expansion of a function (e.g. in $L^2(S^1)$ [or $L^2(\mathbb{R})$]), into a series [or an integral, respectively] of the form $\sum a_\xi e^{ix\xi}$ [or $\int a_\xi e^{ix\xi} d\xi$]. Here we may regard $x \mapsto e^{ix\xi}$ as a one dimensional irreducible representation of the abelian Lie group S^1 [or $\mathbb{R}$]. In this sense, Fourier series [or Fourier transform] gives an irreducible decomposition of the unitary representations $L^2(S^1)$ of S^1 [or $L^2(\mathbb{R})$ of $\mathbb{R}$]. This insight was realized first by H. Weyl, who obtained the so-called Peter-Weyl theorem, which gives an explicit irreducible decomposition of the Hilbert space of L^2-functions on compact groups (1927). Classification of irreducible (finite dimensional) unitary representations of a connected compact Lie group is known as the highest weight theory of Cartan-Weyl. With these examples as a prototype, unitary representation theory has been developed in various directions maintaining strong links to harmonic analysis on homogeneous manifolds.

A Lie group is said to be real reductive if its complexification is isomorphic to that of a compact Lie group. For example, S^1 and $\mathbb{R}$ have an isomorphic complexification $\mathbb{C}^\times$. $SU(n)$ and $SL(n,\mathbb{R})$ have an isomorphic complexification $SL(n,\mathbb{C})$.

* The author is supported by Institut Mittag-Leffler, the Royal Swedish Academy of Sciences.

Thus, $\mathbb{R}$, S^1, $SL(n,\mathbb{R})$ and $SU(n)$ are real reductive Lie groups. Our concern in this article is with real reductive Lie groups. Here recent developments about the interactive relations between

i) harmonic analysis on homogeneous manifolds of reductive type G/H

$$\longleftrightarrow$$

ii) unitary representation theory of a real reductive Lie group G

are explained from my point of view.

One of the basic problems in (i) is to obtain an explicit formula of the irreducible decomposition of the unitary representation $L^2(G/H)$ of G. To solve this problem, it is important to determine discrete series representations for G/H, which correspond to the discrete spectra in the abstract Plancherel formula. We write $\mathrm{Disc}(G/H)$ for the set of discrete series representations. On the other hand, one of the basic problems in (ii) is to classify the equivalence classes of irreducible unitary representations of G, namely, <u>the unitary dual $\widehat{G}$</u>. Obviously, $\mathrm{Disc}(G/H)$ is a (possibly, empty) subset of $\widehat{G}$. In this sense, the study of (ii) serves as a basic "tool" for problem (i) for each subgroup H. Conversely, (ii) sometimes gives a better understanding for (i). For instance, "new" irreducible unitary representations of G were sometimes found as elements of $\mathrm{Disc}(G/H)$ for certain subgroups H (e.g. [89]; see remarks after Theorem 4.3). The current status of the above basic problems for (i) and (ii) is briefly as follows:

i) There are known three methods for the classification of irreducible admissible representation of G (precisely speaking, the classification of irreducible $(\mathfrak{g}, K)$-modules).*1) However, the classification of the unitary dual $\widehat{G}$ is not yet solved completely except for some special Lie groups.*2)

ii) Discrete series representations for reductive symmetric spaces have been understood fairly well. However, it is an open problem to find a criterion that a homogeneous manifold G/H has a non-empty set of discrete series representations and to classify them, except for reductive symmetric spaces and some few other cases (see §4 and §6).

This article is organized as follows: in the first half, we explain geometry of homogeneous manifolds and construction of irreducible unitary representations attached to elliptic orbits; in the latter half a review of recent results due to the author is given. To be more precise, we explain in §1 the geometry of various homogeneous manifolds of reductive type. This part is particularly expository, and is presented as comprehensively as possible without special knowledge. We shall discuss representations associated to a homogeneous manifold with various kinds of invariant geometric structure, namely, complex structure in §2, (pseudo-)Riemannian structure in §3, symmetric structure in §4 and para-Hermitian structure in §5, respectively. First, we give an exposition on recent results by Schmid, Wong that construct (almost irreducible) representations in the space of Dolbeault cohomology groups over a homogeneous manifolds of reductive type with complex structure as a vast generalization of the Borel-Weil-Bott theorem in §2. Here, we adopt notation with emphasis on the orbit method, and explain its connection with the strategy of Vogan on the study of the unitary dual and derived functor modules defined by Zuckerman. The representations constructed here are often isolated in the unitary dual in the Fell topology and are supposed to be 'useful' for the description of discrete series representations for homogeneous manifolds of reductive type. In §3, we discuss harmonic analysis on Riemannian homogeneous manifolds and the Blattner formula of discrete series representations and then give examples of Sunada's problem on discrete spectra of the Laplacian on a non-compact Riemannian manifold. In §4, we construct discrete series representations for a certain class of homogeneous manifolds of reductive type, which are principal bundles over symmetric spaces with compact fiber. This construction covers all discrete series representations obtained by Harish-Chandra (a group manifold), Flensted-Jensen (a symmetric space), Oshima-Matsuki (a symmetric space), and Schlichtkrull (a principal bundle over a symmetric space with compact fiber). In §6, we introduce a notion (we shall say admissible) for branching rules of unitary representations. This gives an algebraic framework to study a basic problem: "*Find the irreducible decomposition of a unitary representation when restricted to a subgroup.*" Furthermore, we discuss the existence problem of the discrete series representations for some spherical non-symmetric homogeneous manifolds as an application of the criterion of "admissible restrictions".

The author expresses his sincere gratitude to Professor Toshio Oshima, Hisayosi Matumoto, Kyo Nishiyama, Masatoshi Iida, Hideko Sekiguchi for reading this article carefully and providing comments.

1. Homogeneous manifolds of reductive type

In this article, we shall assume that a Lie group is linear, that is, it is realized as a closed subgroup of the general linear group $GL(n,\mathbb{R})$ and sometimes abbreviate an adjective 'linear' . Let $\mathfrak{g}$ be the Lie algebra of the Lie group G and $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g} \otimes \mathbb{C}$ its complexification. Analogous notation is used for other groups denoted by uppercase Roman letters.

A connected closed subgroup G of $GL(n,\mathbb{R})$ is said to be a real reductive Lie group if it is stable under the standard Cartan involution $g \mapsto {}^t g^{-1}$. Moreover, G is said to be a semisimple Lie group if the center $Z(G) := \{g \in G : gx = xg (\forall x \in G)\}$ of G is discrete. Next, suppose that H and G have at most finitely many connected components. The coset space G/H is said to be a homogeneous manifold of reductive type if H and G are realized as closed subgroups of $GL(n,\mathbb{R})$ such that

$GL(n,\mathbb{R}) \supset G \supset H$ are stable under the Cartan involution $g \mapsto {}^t g^{-1}$ (cf. [74]). We have adopted a definition of a real reductive (linear) Lie group by using a realization in $GL(n,\mathbb{R})$ because of its simplicity, but we could also give an intrinsic definition, namely, by the condition that the adjoint representation $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ is completely reducible. We also note that our assumption that G is connected can be relaxed to the one that G is contained in a connected complex Lie group $G_{\mathbb{C}}$ with the Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Example 1.1. The following Lie groups G are real reductive linear Lie groups.
$G = GL(n,\mathbb{R})$, $SU^*(2n)$, $U(p,q)$, $SO^*(2n)$, $SO(p,q)$, $Sp(n,\mathbb{R})$, $Sp(p,q)$.

In view of the natural embedding $GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$, we see that complex semisimple Lie groups such as $G = SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$ are also real reductive Lie groups.

Next, here are examples of homogeneous manifolds of reductive type:

$$GL(2n,\mathbb{R})/GL(n,\mathbb{C}) = \text{the space of complex structures on } \mathbb{R}^{2n},$$
$$U(p,q;\mathbb{F})/U(p-m,q;\mathbb{F}) = \text{the indefinite Stiefel manifold where } \mathbb{F} = \mathbb{R},\ \mathbb{C},\ \mathbb{H},$$
$$Sp(n,\mathbb{R})/U(n) = \text{the upper half space.}$$

Here $\mathbb{H}$ denotes the quaternionic number field and $U(p,q;\mathbb{H}) \simeq Sp(p,q)$.

An involutive automorphism θ of a real reductive (linear) Lie group G (namely, $\theta \in \mathrm{Aut}(G), \theta^2 = \mathrm{id}$) is said to be a Cartan involution if $K := G^\theta = \{g \in G : \theta g = g\}$ is a maximal compact subgroup of G. (If there is no compact factor in G, then G^θ is automatically maximal provided G^θ is compact.) Conversely, given a maximal compact subgroup K of G, there exists a unique involution θ of G such that $K = G^\theta$. From now on, a maximal compact group $K \subset G$ is always supposed to be defined by a Cartan involution $\theta \in \mathrm{Aut}(G)$. Moreover, if G/H is a homogeneous manifold of reductive type, we can and do take a Cartan involution θ of G such that $\theta H = H$. In particular, $\theta_{|H}$ is also a Cartan involution of H, and $K \cap H$ is a maximal compact subgroup of H. Next, the differential of a Cartan involution $\theta \in \mathrm{Aut}(G)$ will be denoted by the same symbol $\theta \in \mathrm{Aut}(\mathfrak{g})$. As $\theta^2 = \mathrm{id}$, the possible eigenvalues of $\theta \in \mathrm{Aut}(\mathfrak{g})$ are both 1 and -1. Correspondingly, we write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the eigenspace decomposition of θ, which is called a Cartan decomposition of the Lie algebra $\mathfrak{g}$. There is a G-invariant non-degenerate symmetric $\mathbb{R}$-bilinear form B on $\mathfrak{g}$, such that

$$B_\theta \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},\ X, Y \mapsto -B(X, \theta Y)$$

is symmetric and positive-definite. If G is a simple Lie group, B coincides with the Killing form of $\mathfrak{g}$ up to positive multiples.

Example 1.2. Suppose G is a real reductive linear Lie group. We realize G as a subgroup of $GL(n,\mathbb{R})$ such that G is stable under $g \mapsto {}^t g^{-1}$. We define $\theta \in \mathrm{Aut}(G)$ by $\theta(g) := {}^t g^{-1}$ $(g \in G)$. Then the differential $\theta \in \mathrm{Aut}(\mathfrak{g})$ is given by $\theta(X) := -{}^t X$ $(X \in \mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R}))$. We set

$$K = O(n) \cap G,$$
$$\mathfrak{k} = \mathfrak{g} \cap \{\text{real skew symmetric matrices}\},$$
$$\mathfrak{p} = \mathfrak{g} \cap \{\text{real symmetric matrices}\},$$
$$B(X,Y) := \mathrm{Trace}(XY) \qquad (X, Y \in \mathfrak{g}),$$
$$B_\theta(X,Y) := \mathrm{Trace}(X\,{}^t Y) \qquad (X, Y \in \mathfrak{g}).$$

It is easy to see that B_θ is positive definite.

The Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of a real reductive Lie algebra $\mathfrak{g}$ can be lifted to a decomposition of a real reductive Lie group G, namely, we have a diffeomorphism:

$$K \times \mathfrak{p} \to G, \ (k, X) \mapsto k \exp(X).$$

For example, if $G = GL(n, \mathbb{R})$, then this decomposition coincides with the polar decomposition

$$GL(n, \mathbb{R}) \simeq O(n) \times \{\text{positive definite symmetric matrices}\},$$

as is well-known in linear algebra. This decomposition theorem for a group manifold is generalized to the case of a homogeneous manifold in the following lemma (e.g. [90], [49]).

Lemma 1.3. *A homogeneous manifold of reductive type G/H is diffeomorphic to the K-homogeneous vector bundle over the compact homogeneous manifold $K/H \cap K$ with typical fiber $\mathfrak{p}/\mathfrak{h} \cap \mathfrak{p}$.*

We enumerate typical examples of homogeneous manifolds of reductive type.

Example 1.4. Let G be a real reductive Lie group and σ an automorphism of G of finite order. Suppose that H is the fixed point subgroup of σ; namely, $G^\sigma = \{g \in G : \sigma g = g\}$ or its open subgroup. Then a homogeneous manifold G/H is of reductive type. In particular, if the order of σ is 2, G/H is called a reductive symmetric space. Moreover, if G is a real semisimple Lie group, it is called a semisimple symmetric space.

Here are examples of semisimple symmetric spaces defined by an automorphism σ of order 2:

$$G \times G/\text{diag}(G) \text{ (group manifold)}, \quad SL(p+q, \mathbb{R})/SO(p, q), \quad SL(n, \mathbb{C})/SL(n, \mathbb{R}).$$

Here are examples of homogeneous manifolds of reductive type defined by an automorphism σ of order 3:

$$G \times G \times G/\text{diag}(G), \quad GL(3, \mathbb{R})/(\mathbb{R}^\times)^3, \quad G_2(\mathbb{C})/SL(3, \mathbb{C}).$$

Example 1.5. For an element X of the Lie algebra $\mathfrak{g}$, we define a subgroup of a Lie group G by

$$L \equiv G(X) := \{g \in G : \text{Ad}(g)X = X\}.$$

The adjoint orbit of G through X is denoted by $\text{Ad}(G)X$ ($\subset \mathfrak{g}$), which is a homogeneous manifold $G/G(X)$. Suppose G is a real reductive Lie group. Then the adjoint representation $\text{Ad}\colon G \to GL(\mathfrak{g})$ is isomorphic to the contragredient representation $\text{Ad}^*\colon G \to GL(\mathfrak{g}^*)$ through the bijection $\mathfrak{g} \simeq \mathfrak{g}^*$ given by an $\text{Ad}(G)$-invariant non-degenerate bilinear form B on $\mathfrak{g}$. Therefore, we can identify the adjoint orbit with the co-adjoint orbit provided G is real reductive. In particular, any adjoint orbit of a real reductive Lie group carries a G-invariant symplectic structure induced from that on the coadjoint orbit. If X is semisimple (namely, if $\text{ad}(X) \in \text{End}(\mathfrak{g})$

is a semisimple linear transformation), then $\mathrm{Ad}(G)X \simeq G/G(X)$ is said to be a semisimple orbit, which is a homogeneous manifold of reductive type. Moreover, if all the eigenvalues of $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ are purely imaginary (resp. real), then the semisimple orbit $\mathrm{Ad}(G)X$ is called an elliptic orbit (a hyperbolic orbit, respectively).

As a special case of Example (1.5), we review how complex structure is defined on the elliptic orbit $\mathrm{Ad}(G)X \simeq G/G(X)$. Let

$$\begin{aligned} \mathfrak{l} \equiv \mathfrak{g}(X) &:= \{Y \in \mathfrak{g} : [X, Y] = 0\}, \quad \mathfrak{l}_{\mathbb{C}} \equiv \mathfrak{g}_{\mathbb{C}}(X) = \mathfrak{l} \otimes \mathbb{C}, \\ \mathfrak{u} \equiv \mathfrak{u}(X) &:= \text{ the direct sum of eigenspaces } (\subset \mathfrak{g}_{\mathbb{C}}) \text{ with positive eigenvalues} \\ &\qquad\qquad \text{of } \sqrt{-1}\,\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}}), \\ \mathfrak{q} \equiv \mathfrak{q}(X) &:= \mathfrak{g}_{\mathbb{C}}(X) + \mathfrak{u}(X) \ (\subset \mathfrak{g}_{\mathbb{C}}). \end{aligned}$$

As X is an elliptic element, there exists a Cartan involution θ satisfying $\theta X = X$. In particular, we have $\theta\mathfrak{q} = \mathfrak{q}$. The parabolic subalgebra $\mathfrak{q}$ is said to be a *θ-stable parabolic subalgebra* of $\mathfrak{g}_{\mathbb{C}}$, with emphasis on this property (e.g. [101]). Let Q be a connected complex subgroup of $G_{\mathbb{C}}$ with the Lie algebra $\mathfrak{q}$. Then we have $G \cap Q = G(X)$ and $\mathfrak{g} + \mathfrak{q} = \mathfrak{g}_{\mathbb{C}}$, so that an elliptic orbit $\mathrm{Ad}(G)X$ is embedded into an open subset of the flag manifold $G_{\mathbb{C}}/Q$:

$$\mathrm{Ad}(G)X \simeq G/G(X) \subset G_{\mathbb{C}}/Q \quad \text{(generalized Borel embedding).}$$

Hence, a G-invariant complex structure on the elliptic orbit $\mathrm{Ad}(G)X \simeq G/G(X)$ is defined as an open subset of the complex homogeneous manifold $G_{\mathbb{C}}/Q$. We note that if we replace X by $-X$, we have $G(X) = G(-X) (= L)$ so that $G/G(X)$ is diffeomorphic to $G/G(-X)$. Note that the complex structure on $G/L \simeq G/G(X)$ is complex conjugate to that on $G/L \simeq G/G(-X)$. In general, there are finitely many G-invariant complex structures on the elliptic orbit G/L. Each G-invariant complex structure is given by a choice of a θ-stable parabolic subalgebra with fixed real Levi subgroup $L\,(= G(X))$. Geometric quantizations corresponding to different polarizations which are not K-conjugate to one another yield different 'series' of unitary representations of G (§2).

Example 1.6 (indefinite complex projective space). The natural representation of the general linear group $G_{\mathbb{C}} = GL(n, \mathbb{C})$ induces a transitive action on the space of complex lines in $\mathbb{C}^n$. This means that the projective space $\mathbb{P}^{n-1}\mathbb{C}$ is realized as a homogeneous manifold of $G_{\mathbb{C}}$. Let Q be a maximal parabolic subgroup of $G_{\mathbb{C}}$, consisting of matrices whose $(j, 1)$-components $(2 \leq j \leq n)$ are 0. Then Q is the isotropy subgroup at $[1 : 0 : \cdots : 0] \in \mathbb{P}^{n-1}\mathbb{C}$ and we have a biholomorphic map $G_{\mathbb{C}}/Q \simeq \mathbb{P}^{n-1}\mathbb{C}$. We fix p, q with $p + q = n$ $(p, q \geq 1)$, and define an indefinite Hermitian metric on $\mathbb{C}^n$ with the signature (p, q) by

$$(z, w) := z_1\overline{w_1} + \cdots + z_p\overline{w_p} - z_{p+1}\overline{w_{p+1}} - \cdots - z_n\overline{w_n} \quad (z, w \in \mathbb{C}^n).$$

Let $\mathbb{P}^{p-1,q}\mathbb{C}$ be the set of complex lines contained in the positive cone $\{z \in \mathbb{C}^n : (z, z) > 0\} \cup \{0\}$. Clearly, $\mathbb{P}^{p-1,q}\mathbb{C}$ is an open set of $\mathbb{P}^{n-1}\mathbb{C}$. The indefinite unitary group $G := U(p,q)\,(\subset G_{\mathbb{C}} = GL(n,\mathbb{C}))$ acts on $\mathbb{P}^{p-1,q}\mathbb{C}$ transitively with isotropy subgroup $L \simeq U(1) \times U(p-1,q)$. Thus we obtain an open embedding

$$G/L = U(p,q)/U(1) \times U(p-1,q) \simeq \mathbb{P}^{p-1,q}\mathbb{C} \subset \mathbb{P}^{n-1}\mathbb{C} \simeq G_{\mathbb{C}}/Q.$$

The case $p = q = 1$ corresponds to a well-known example that "the Poincaré plane is embedded in the upper half plane of $\mathbb{P}^1\mathbb{C}$". Among all elliptic orbits of $U(p,q)$, $\mathbb{P}^{p-1,q}$ has the lowest dimension. We shall deal with the unitary representations of G realized on $\mathbb{P}^{p-1,q}$ in Example 2.5 in §2.

A homogeneous manifold of reductive type always carries a G-invariant pseudo-Riemannian metric induced from the bilinear form B. Furthermore, some of them carry some other geometric structures such as G-invariant complex structure, G-invariant symplectic structure, G-invariant Riemannian metric and so on. Here are subclasses of homogeneous manifolds of reductive type, which we shall need later.

$^\exists$ G-invariant Riemannian metric

$^\exists$ G-invariant symplectic structure

$^\exists$ G-invariant complex structure

SSS SO

RSS HO EO

SGM PHSS

HSS

Figure 1.7

SSS : semisimple symmetric spaces,
RSS : Riemannian symmetric spaces,
SO : semisimple orbits,
EO : elliptic orbits,
HO : hyperbolic orbits,
SGM : semisimple group manifolds,
PHSS : para-Hermitian symmetric spaces,
HSS : Hermitian symmetric spaces.

2. Unitary representations of a real reductive Lie group

The classification of irreducible unitary representations of a real reductive Lie group is still an open problem, but some of important series of unitary representations have been constructed in a systematic way.

From the view point of representation theory, a good insight into the unitary dual was given by Vogan's strategy ([104],[106]), which is based on functors inducing from representations of smaller groups to those of larger groups (preserving irreducibility and unitarity) playing the role of a weaver's warp and which enables us to concentrate on the end of the warp ('small unitary representations' ≈ *unipotent representations* of smaller groups). One extremal case of the functors (= "the weaver's warp") is classical parabolic induction which constructs principal series representations. The opposite extrem is cohomological parabolic induction, which is a generalization of the Borel-Weil-Bott theorem. Main properties of cohomological parabolic induction have been investigated since the late 70's ([101], [104], [45], [113]). Translation functors ([120]) and coherent continuation of representations are also important in Vogan's strategy which play the role of "woof" of the unitary dual. In this section, we explain the family of standard unitary representations which are obtained by taking the Zuckerman's derived functor as 'a weaver's warp', and by taking one dimensional representations as 'the edge of the warp'. Our exposition here follows the geometric construction due to Schmid and Wong ([117]), which was the original model of Zuckerman's algebraic construction. This construction is regarded as the geometric quantization of an elliptic orbit equipped with a complex structure as explained in §1.

Before entering rigorous arguments, we look over both the more general framework and very special examples. First, we recall the orbit method in the unitary representation theory of Lie groups. Let us consider the contragredient representation $\mathrm{Ad}^*\colon G \to GL(\mathfrak{g}^*)$ of the adjoint representation of G, $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$. This non-unitary finite dimensional representation often has a surprising intimate relation with the unitary dual $\widehat{G}$, the equivalence classes of irreducible representations of G (which are infinite dimensional in general). We define an equivalent relation in $\mathfrak{g}^*$ by

$$\lambda \sim \mu \Leftrightarrow {}^\exists g \in G,\ \mathrm{Ad}^*(g)\lambda = \mu.$$

The set of equivalence classes is denoted by $\mathfrak{g}^*/G$, the set of coadjoint orbits.

For example, if $G = \mathbb{R}^n$, then $\mathfrak{g}^*/G \simeq \mathbb{R}^n$, and we have a natural bijective correspondence between the unitary dual $\widehat{G}$ and $\sqrt{-1}\mathfrak{g}^*/G \simeq \sqrt{-1}\mathbb{R}^n$. This abelian example is generalized to the case of nilpotent Lie groups by Kirillov. The following theorem played an initial role in the orbit method by Kirillov-Kostant in the '60 s ([58]).

Theorem 2.1 ([44]). *Let G be a connected and simply connected nilpotent Lie group. Then there is a natural bijection between the coadjoint orbit $\mathfrak{g}^*/G$ and the unitary dual $\widehat{G}$.*

There is known a generalization of Theorem (2.1) for a solvable group G of the exponential type. On the other hand, known examples suggest that if G is a real reductive Lie group (an opposite extremal to a solvable Lie group), then the set of coadjoint orbits $\mathfrak{g}^*/G$ (more precisely its subset with some integral conditions) still gives a fairly good approximation of the unitary dual $\widehat{G}$. In particular, the series of unitary representations $\Pi(G,\lambda)$ that we shall define in this section are interpreted as representations attached to an integral elliptic orbit $\mathrm{Ad}^*(G)\lambda \in \mathfrak{g}^*/G$. For this reason, our formulation uses the notation $\Pi(G,\lambda)$ with emphasis on the orbit method. An advantage in this formulation is that we have an overview of the set of (irreducible) unitary representations constructed in this section as a subset of the unitary dual $\widehat{G}$, in terms of $\mathfrak{g}^*/G$. A disadvantage is that we leave aside a precise description of unitary representations with singular parameter (cf. Problem (2.10)).

Next, we consider a very special example. The space of holomorphic functions $\mathcal{O}(\mathbb{P}^1\mathbb{C})$ on $\mathbb{P}^1\mathbb{C}$ consists of constant functions by Liouville's theorem. Because $\mathbb{P}^1\mathbb{C}$ is a homogeneous manifold of $SU(2)$, this gives the trivial one dimensional representation of $SU(2)$ on the representation space on $\mathcal{O}(\mathbb{P}^1\mathbb{C})$. On the other hand, the space of holomorphic functions $\mathcal{O}(\mathcal{H})$ on the Poincaré plane $\mathcal{H}$ is infinite dimensional. Because $\mathcal{H}$ is a homogeneous manifold of $SL(2,\mathbb{R})$, this gives an infinite dimensional representation of $SL(2,\mathbb{R})$ on $\mathcal{O}(\mathcal{H})$, which is almost irreducible (in fact, there are two irreducible subquotients). Furthermore, the Hardy space, which is a subspace of $\mathcal{O}(\mathcal{H})$, gives rise to an irreducible unitary representation of $SL(2,\mathbb{R})$. Though there is an apparent difference between $\mathcal{O}(\mathbb{P}^1\mathbb{C})$ and $\mathcal{O}(\mathcal{H})$ (or Hardy space), namely dimension, they play essentially the same role in constructing (almost) irreducible representations of $SU(2)$ and $SL(2,\mathbb{R})$, respectively. More generally, if we consider holomorphic sections of equivariant holomorphic line bundles over $\mathbb{P}^1\mathbb{C}$ or $\mathcal{H}$, we obtain a family of representations with line bundle parameters. This is a prototype of a standard construction of unitary representations, of which we will give an exposition in the following. We note that $\mathcal{H}$ is an elliptic orbit of $SL(2,\mathbb{R})$ and $\mathcal{H} \subset \mathbb{P}^1\mathbb{C}$ is the Borel embedding. However, these examples are too special in the sense that all representations obtained have highest weight vectors. For more general construction of irreducible unitary representations, we need to take cohomology groups in higher dimensions into account. We introduce a rigorous definition as follows:

Suppose that G is a real reductive Lie group. Since there is an $\mathrm{Ad}(G)$-invariant non-degenerate bilinear form B on $\mathfrak{g}$, we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ as G-modules via the G-invariant $\mathbb{R}$-linear bijection $\mathfrak{g}^* \ni \lambda \mapsto X_\lambda \in \mathfrak{g}$ (here $B(X_\lambda, Y) = \lambda(Y)$, ${}^\forall Y \in \mathfrak{g}$). Through this identification, the notions defined for $\mathfrak{g}$ in §1 are translated into those for $\mathfrak{g}^*$. For example, we say $\sqrt{-1}\lambda \in \mathfrak{g}^*$ is elliptic if $X_{-\sqrt{-1}\lambda} \in \mathfrak{g}$ is elliptic. As in §1, an elliptic element $X_{-\sqrt{-1}\lambda}$ gives rise to a reductive subgroup $L = G(X_{-\sqrt{-1}\lambda})$ and a θ-stable parabolic subalgebra $\mathfrak{q} \equiv \mathfrak{q}(X_{-\sqrt{-1}\lambda}) = \mathfrak{l}_\mathbb{C} + \mathfrak{u} \equiv \mathfrak{g}_\mathbb{C}(X_{-\sqrt{-1}\lambda}) + \mathfrak{u}(X_{-\sqrt{-1}\lambda})$, for which we also write $L = G(\lambda)$ and $\mathfrak{q}(\lambda) = \mathfrak{g}_\mathbb{C} + \mathfrak{u}(\lambda)$, respectively.

Then the coadjoint orbit $\mathrm{Ad}^*(G)\lambda \simeq G/G(\lambda)$ carries a G-invariant complex structure by the parabolic subalgebra $\mathfrak{q}(\lambda)$. We define $\rho(\mathfrak{u}) \in \sqrt{-1}\mathfrak{g}(\lambda)^*$ by

$$\langle 2\rho(\mathfrak{u}), Y\rangle := \mathrm{Trace}(\mathrm{ad}(Y)_{|\mathfrak{u}(\lambda)}) \quad (Y \in \mathfrak{g}(\lambda)).$$

We say $\sqrt{-1}\lambda \in \mathfrak{g}^*$ is integral, or the orbit $\mathrm{Ad}^*(G)\lambda$ is integral, if the one dimensional representation of Lie algebra $\lambda + \rho(\mathfrak{u})\colon \mathfrak{g}(\lambda) \to \mathbb{C}$ lifts to the character $\chi_{\lambda+\rho(\mathfrak{u})}\colon G(\lambda) \to \mathbb{C}^\times$. The character $\chi_{\lambda+\rho(\mathfrak{u})}$ will be simply denoted by $\mathbb{C}_{\lambda+\rho(\mathfrak{u})}$. Let $G \underset{G(\lambda)}{\times} \mathbb{C}_{\lambda+\rho(\mathfrak{u})}$ be the G-equivariant holomorphic line bundle over $G/G(\lambda)$ associated to the character $\mathbb{C}_{\lambda+\rho(\mathfrak{u})}$. Then we have a natural G-action on the Dolbeault cohomology group $H^j_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ ($j \in \mathbb{N}$) with coefficients in the G-equivariant holomorphic line bundle $G \underset{G(\lambda)}{\times} \mathbb{C}_{\lambda+\rho(\mathfrak{u})} \to G/G(\lambda)$. This module is finite dimensional by the theorem due to Kodaira and Serre if $G/G(\lambda)$ is compact. On the other hand, it is in general infinite dimensional (possibly zero) if $G/G(\lambda)$ is non-compact. If $G/G(\lambda)$ is non-compact, there arise analytic difficulties, concerning the closed range property of the $\bar\partial$ operator in the Dolbeault complex and consequently it is not clear whether the cohomology group is Hausdorff or not. As was given in Lemma (1.3), the homogeneous manifold $G/G(\lambda)$ carries the fiber bundle structure $\mathfrak{p}/\mathfrak{g}(\lambda) \cap \mathfrak{p} \to G/G(\lambda) \to K/G(\lambda) \cap K$. In extremal cases, we have
(i) The fiber is one point $\Leftrightarrow$ the complex manifold $G/G(\lambda)$ is compact.
(ii) The base space is one point $\Leftrightarrow$the complex manifold $G/G(\lambda)$ is a Stein manifold.
There are no analytic difficulties concerning the closed range property of $\bar\partial$ in these special (opposite extremal) cases. In fact, the resulting representations of G are finite dimensional representations of a compact Lie group, which coincide with the construction of the Borel-Weil-Bott theorem in the case (i); holomorphic discrete series representations which coincide with the Harish-Chandra construction in the case (ii) (Example (2.7); note that cohomology appears only in the degree $j = 0$). Taking the above special cases (i),(ii) as a prototype, many people have tried to overcome the analytic difficulties in a general case, and also studied algebraic analogue of the corresponding representations. Geometric construction of discrete series representation for a group manifold (see Example 2.8; also see §3 for the definition of discrete series representations) was first carried out by Schmid in the 70's under the assumption that $G(\lambda)$ is compact and later by Wong in the general case where $G(\lambda)$ is not necessarily compact.

Theorem 2.2 ([13],[93],[94],[34],[77],[29],[101],[117]). *Suppose that G is a real reductive Lie group and $G/G(\lambda) \simeq \mathrm{Ad}^*(G)\lambda \subset \sqrt{-1}\mathfrak{g}^*$ is an integral elliptic orbit. Let S be the complex dimension of the flag manifold $K/G(\lambda) \cap K$.*

1) *The Dolbeault cohomology group $H^j_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ is naturally equipped with a Fréchet topology as a quotient space of the Dolbeault complex with C^∞ coefficients. Thus a continuous representation of G is naturally defined on $H^j_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})$.*

2) $H^j_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})}) = 0$ *if $j \neq S$.*

3) *The underlying $(\mathfrak{g}, K)$-module of $H^S_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ is isomorphic to the*

Zuckerman derived functor module $\mathcal{R}^S_{\mathfrak{q}(\lambda)}(\mathbb{C}_\lambda)$ *as* $(\mathfrak{g}, K)$*-modules.*

4) *(generalized Blattner formula)*

$$H^S_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})_{|K}$$
$$\simeq \sum_{j,m} (-1)^{S-j} H^j_{\bar\partial}(K/G(\lambda)\cap K, S^m(\mathfrak{u}(\lambda)/\mathfrak{u}(\lambda)\cap\mathfrak{k}) \otimes \mathbb{C}_{\lambda+\rho(\mathfrak{u})}).$$

Let us explain briefly the terminologies used in Theorem (2.2). We say that (ρ, V) is a (continuous) representation of G if V is a complete locally convex linear topological space over $\mathbb{C}$ and if $\rho\colon G \to GL(V)$ is a homomorphism such that $G \times V \to V$, $(g, X) \mapsto \rho(g)X$ is continuous. One might expect an intimate relation between a continuous representation of Lie group and its "differential representation $d\rho\colon \mathfrak{g} \to \mathrm{End}(V)$" , as an analogue of finite dimensional cases. For its justification, we encounter the following two problems:

(i) Is "the differential representation" well-defined if $\dim V = \infty$?

(ii) Conversely, does a given differential representation of the Lie algebra lift to a representation of a Lie group in an appropriate sense ?

If G is a real reductive Lie group, these two problems are settled simultaneously, with the help of representations of a maximal compact subgroup K of G ([26]). To be more precise, suppose a continuous admissible representation (ρ, V) of G is given; we decompose V as a representation of K into irreducible components and define a subspace V_K by the algebraic direct sum of irreducible components. V is said to be admissible (K-admissible in the sense of §6; cf. Example (6.4) (1)) if each K-type occurs with finite multiplicity. Then V_K is a dense subspace of V. Moreover, the differential of ρ, $d\rho\colon \mathfrak{g} \to \mathrm{End}(V_K)$, and its restriction to K, $\rho_{|K}\colon K \to GL(V_K)$, are well-defined on V_K. Thus V_K is endowed with both $\mathfrak{g}$-module and K-module structures with some compatibility conditions. The $(\mathfrak{g}, K)$-module V_K is said to be the underlying $(\mathfrak{g}, K)$ -module of V (or the Harish-Chandra module of V). Let us illustrate $V_K \subset V$ in a simple example where $G = K = S^1$, $V = L^2(S^1)$ (see also Introduction). In this case, each element of V_K is a function which is a finite linear combination of trigonometric functions on S^1. Consequently, each element of V_K is differentiable in a usual sense and also differentiable in the sense of the Fréchet differential in $L^2(S^1)$. V_K is a dense subspace of V by the theory of Fourier series. Conversely, if a $(\mathfrak{g}, K)$-module W is admissible then there exists a topology on W such that a continuous representation of G on the completion $\overline{W}$ of W is defined with $\overline{W}_K \simeq W$. The representation of G on $\overline{W}$ is called a globalization of the $(\mathfrak{g}, K)$-module W. The point is that representation theoretic properties such as composition series and unitarizability of continuous admissible representations of a real reductive Lie group G can be investigated at the level of $(\mathfrak{g}, K)$-modules and that the latter object can be studied by purely algebraic methods. We note that a globalization of a $(\mathfrak{g}, K)$-module is not unique; however, there exists the "maximal" one, known as the maximal globalization, due to Schmid ([95]).

The functor $\mathcal{R}_{\mathfrak{q}}^j$ ($j \in \mathbb{N}$) in Theorem (2.2) (3) is the so-called Zuckerman's derived functor, which is a covariant functor from the category of ($\mathfrak{l}$, $L \cap K$)-modules (strictly speaking, the category of metaplectic ($\mathfrak{l}$, $(L \cap K)^{\sim}$)-modules) to the category of $(\mathfrak{g}, K)$-modules ([101], [104], [45], [113]). We do not go into details of the functor $\mathcal{R}_{\mathfrak{q}}^j$, which gives "an induced representation" from a representation of the subgroup L to that of G on the level of Harish-Chandra modules, and is an algebraic analogue of geometric quantization of a semisimple orbit G/L (see Figure 1.7), which is equipped with a polarization by a particular choice of a parabolic subalgebra $\mathfrak{q}$. Let Q be a parabolic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{q}$. Although our concern here is with a totally complex polarization corresponding to a θ-stable parabolic subalgebra $\mathfrak{q}$, we should mention that Zuckerman's derived functor modules are defined also for more general polarization; in particular, in the opposite extremal case (a totally real polarization) where $Q \cap G$ is a real parabolic subgroup of G, the Zuckerman's derived functor $\mathcal{R}_{\mathfrak{q}}^j$ with $j = 0$ gives an ordinary parabolic induced representation (e.g. [101]).

Returning to our setting where the polarization is totally complex (imaginary), the definition of $\mathcal{R}_{\mathfrak{q}}^j$ due to Zuckerman is modeled on an algebraic analogue (a sort of Taylor series expansion) of Dolbeault cohomology groups. Theorem 2.2 (1) and (3) show that $\mathcal{R}_{\mathfrak{q}}^j$ is the right object for geometric quantization. Furthermore, it is known that $H_{\bar{\partial}}^j(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ is a maximal globalization of $\mathcal{R}_{\mathfrak{q}(\lambda)}^j(\mathbb{C}_\lambda)$, which is a stronger form of (1) of Theorem 2.2 (Wong [117], Kashiwara-Schmid). In (4), S^m denotes the m-th homogeneous symmetric tensor space. The right side of (4) is computed by the Borel-Weil-Bott-Kostant theorem of a compact Lie group. The formula (4) is a generalization of the Blattner conjecture (formula) for discrete series representations. The proof of the generalized Blattner formula is also known for other constructions of representations which are isomorphic at the level of underlying $(\mathfrak{g}, K)$-modules by using Zuckerman's derived functor modules [101] or by using $\mathcal{D}$-modules [10].

One of the most important developments in unitary representation theory in the 80's was that the Zuckerman conjecture was proved by Vogan and Wallach independently. In our formulation it is stated as follows:

Theorem 2.3 ([102],[111]). *Suppose that $\lambda \in \sqrt{-1}\mathfrak{g}^*$ is integral and elliptic. Let $\mathfrak{q} = \mathfrak{q}(\lambda)$ be the corresponding θ-stable parabolic subalgebra. Then there exists a Hermitian inner product on the $(\mathfrak{g}, K)$-module $\mathcal{R}_{\mathfrak{q}}^S(\mathbb{C}_\lambda)$, such that the action of $\mathfrak{g}$ is skew-symmetric.*

We note that the assumption of the parameter here is "fair" in the sense of Vogan ([105]). By Theorem (2.2) and Theorem (2.3), if $\sqrt{-1}\lambda \in \mathfrak{g}^*$ is integral and elliptic, there exists a unitary representation $\Pi(G, \lambda)$ of G whose underlying $(\mathfrak{g}, K)$-module is isomorphic to $H_{\bar{\partial}}^S(G/L, \mathbb{C}_{\lambda+\rho(\mathfrak{u})})_K$.

Replacing λ with $\mathrm{Ad}^*(g)\lambda$, we have a unitary representation $\Pi(G, \mathrm{Ad}^*(g)\lambda)$ which is unitarily equivalent to $\Pi(G,\lambda)$ by an inner automorphism. Consequently, we obtain the map which fits into the philosophy of the orbit method:

$$\begin{aligned} &\{\text{integral elliptic orbits}\} \longrightarrow \{\text{equivalence classes of unitary representations of } G\}, \\ &\mathrm{Ad}^*(G)\lambda \qquad\qquad\qquad \mapsto \Pi(G,\lambda). \end{aligned} \tag{2.4}$$

If the parameter λ is sufficiently regular then $\Pi(G,\lambda)$ is an irreducible unitary representation (cf. Problem 2.10). Furthermore, a recent result due to Vogan ([107]) asserts that *most of the representations* $\Pi(G,\lambda)$ *are isolated in the Fell topology of* $\widehat{G}$.

Here is an example of $\Pi(G,\lambda)$ attached to an elliptic orbit as in Example 1.6 in §1.

Example 2.5. Let $\mathbb{P}^{p-1,q}\mathbb{C} = U(p,q)/U(1)\times U(p-1,q)$ be the indefinite complex projective space in Example 1.6. Then $\mathbb{P}^{p-1,q}\mathbb{C}$ carries the structure of a holomorphic vector bundle over the complex projective space $\mathbb{P}^{p-1}\mathbb{C}$,

$$\mathbb{C}^q \to \mathbb{P}^{p-1,q}\mathbb{C} \to \mathbb{P}^{p-1}\mathbb{C}.$$

In particular, the complex dimension of the base space is $p-1$. Let Ω be the canonical line bundle of $\mathbb{P}^{p-1,q}\mathbb{C}$. Then $H^{p-1}_{\bar\partial}(\mathbb{P}^{p-1,q}\mathbb{C}, \Omega^{\otimes n})$ $(n\in\mathbb{N}_+)$ is an irreducible Fréchet representation of $U(p,q)$ and its unitarization is isomorphic to $\Pi(U(p,q),(2n-1)h)$ in our notation. Here $h\in\sqrt{-1}\mathfrak{g}^*$ is an elliptic element defined by $\langle h, E_{ij}\rangle = \delta_{ij}\,(n\delta_{1j}-1)$ for each matrix unit $E_{ij}\in\mathfrak{g}_{\mathbb{C}}\simeq \mathfrak{gl}(n,\mathbb{C})$. If $q=0$, then $\Pi(U(p,q),(2n-1)h)$ is a polynomial representation of $U(p)$ on the symmetric tensor $S^n(\mathbb{C}^p)$, by the Borel-Weil-Bott theorem. If $p=1$, then $\Pi(U(p,q),(2n-1)h)$ is a holomorphic discrete series representation (Example (2.7)). If $p>1$ and $q>0$, then $\Pi(U(p,q),(2n-1)h)$ is a non-tempered unitary representation of $U(p,q)$ (see [113] for definition). We note that examples introduced at the beginning of this section correspond to the case with $(p,q)=(2,0)$, $(1,1)$ and $n=0$.

In the following Example (2.6) $\sim$ (2.9), we assume G is a real reductive linear Lie group.

Example 2.6 (contribution to the de Rham cohomology groups of locally symmetric spaces). Let Ω be the canonical line bundle of the complex manifold G/L whose complex structure is defined by a θ-stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}_{\mathbb{C}}+\mathfrak{u}$. Then $H^S_{\bar\partial}(G/L,\Omega)$ is an irreducible Fréchet representation of G and its unitarization is unitarily equivalent to $\Pi(G,\rho(\mathfrak{u}))$. The underlying $(\mathfrak{g},K)$-module is isomorphic to the $(\mathfrak{g},K)$-module $A_{\mathfrak{q}}$ (see [109] for notation), which has non-zero $(\mathfrak{g},K)$-cohomology groups due to a result of Vogan-Zuckerman ([109]). Moreover, by using the Matsushima-Murakami isomorphism ([12], [71]), all the unitary representation that contribute to the de Rham cohomology groups of a compact locally Riemannian symmetric space, is one of the following:

$$\{\Pi(G,\rho(\mathfrak{u})) : \mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u} \text{ is a } \theta\text{-stable parabolic subalgebra of } \mathfrak{g}_{\mathbb{C}}\}.$$

For example, if $G = SL(2,\mathbb{R})$, then there are three θ-stable parabolic subalgebras of $\mathfrak{g}$ up to conjugation by $\mathrm{Ad}(G)$. Corresponding to them, there are three unitary representations (the one dimensional trivial representation, holomorphic discrete series, and anti-holomorphic discrete series) that contribute to the de Rham cohomology groups of a closed Riemannian surface with genus ≥ 2.

Example 2.7 (holomorphic discrete series, [26]). Suppose that $\lambda \in \sqrt{-1}\mathfrak{g}^*$ is integral and elliptic. Assume that $G(\lambda)$ coincides with a maximal compact subgroup K of G. We note that this occurs only if $\mathfrak{k}$ has a nontrivial center. Then $G/G(\lambda)$ is a non-compact Hermitian symmetric space and in particular a Stein manifold. Dolbeault cohomology groups are non-zero only if $S = 0$ and the resulting cohomology is the space of global holomorphic sections. If λ is sufficiently regular, K-finite vectors of $H^0(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ are square-integrable. The completion with respect to the L^2-norm is a unitary representation $\Pi(G,\lambda)$. Such representations are called <u>holomorphic discrete representations</u> of G.

Example 2.8 (discrete series for group manifolds (Harish-Chandra); cf. Langlands conjecture; [27], [93], [94], [34], [77]). Let G be a real reductive Lie group. Then we have

$$\mathrm{Disc}(G) = \{\Pi(G,\lambda) : \lambda \text{ is integral and elliptic}, G(\lambda) \text{ is a compact torus}\}.$$

We note that there exists an integral and elliptic element λ such that $G(\lambda)$ is a compact torus if and only if $\operatorname{rank} G = \operatorname{rank} K$. That is, $\mathrm{Disc}(G) \neq \emptyset$ if and only if $\operatorname{rank} G = \operatorname{rank} K$. Geometric construction in the L^2-cohomology group was predicted by Langlands and was proved by Schmid in '76.

Example 2.9 (discrete series for symmetric spaces, [69], [21], [105]). Any discrete series representation π for a semisimple symmetric space G/H is of the form $\pi \simeq \Pi(G,\lambda)$, with a suitable choice of an elliptic element $\lambda \in \sqrt{-1}(\mathfrak{k}/\mathfrak{h} \cap \mathfrak{k})^*$ ($\subset \sqrt{-1}\mathfrak{k}^*$) such that $G(\lambda)/G(\lambda) \cap H$ is a compact torus. We note that the existence for such λ is assured if and only if $\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$. That is, $\mathrm{Disc}(G/H) \neq \emptyset$ if and only if $\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$.

The set of unitary representations treated in Examples (2.7), (2.8) and (2.9) has an inclusive relation in $\widehat{G}$ as

$$\text{Example (2.7)} \subset \text{Example (2.8)} \subset \text{Example (2.9)}.$$

In §4, we shall construct discrete series representations in a more general framework so that it contains Example (2.9) as a special case, building on the method of Flensted-Jensen ([20]) and on vector bundle valued Poisson transforms. We end this section with some open problems related to the unitary representation $\Pi(G,\lambda)$.

Problem 2.10. (1) Find a criterion on λ such that $H^S_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})}) \neq 0$.

(2) Find a criterion on λ such that $H^S_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ is irreducible. Also find

a criterion such that two representations are equivalent to one another.
(3) Construct an explicit inner product in the underlying $(\mathfrak{g}, K)$-module of the Dolbeault cohomology group $H^S_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ which is infinitesimally unitary.

Here are some comments in order. If the parameter λ is sufficiently regular, then $H^S_{\bar\partial}(G/G(\lambda), \mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ is a non-zero irreducible module and is unique up to equivalence. Therefore Problem (2.10)(1) and (2) are interesting only in the case where the parameter is singular (we should note that representations with singular parameter are important and mysterious in the current status of the classification problem of the unitary dual). In order to study Problem (2.10)(2), the theory of $\mathcal{D}$-modules is helpful to some extent (e.g. G is of type A), but general theory still does not give a final answer. Problem(2.10)(1) is also important in the classification of discrete series representations for semisimple symmetric spaces (more generally, for vector bundles over them (§4)). So far, several approaches for Problem 2.10 (1) have been taken by Vogan, Oshima, Matsuki, the author and so on ([100], [101], [48], [68], [70], [105], [51]chapter 4 ,5) in special settings. But the final answer in the general case remains open. Here are some approaches for Problem 2.10 (the following methods 1,2,3 are based on translation principle together with some other ideas). We note that we are interested in the case where $\lambda + \rho(\mathfrak{u}) - 2\rho(\mathfrak{u} \cap \mathfrak{k})$ is not $\Delta^+(\mathfrak{k})$-dominant and where λ is outside the weakly good range in the sense of Vogan [105](the former corresponds to the Blattner parameter and the latter to the Harish-Chandra parameter in the case of discrete series representations for a group manifold).

1. Jantzen-Duflo's τ invariants and wall crossing (Vogan's U_α calculus) (e.g. [100],[101]),
2. explicit description of leading exponents of the asymptotic behavior of spherical functions combined with translation principle ([48]),
3. axiomatic description of the vanishing of $\mathcal{R}^S_{\mathfrak{q}}(\mathbb{C}_\lambda)$ in the coherent continuation between singular parameters which are not equisingular ([51] Chapter 5),
4. a method of an explicit calculation of the Blattner formula ([51] Chapter 4, [70]),
5. analysis on symmetric spaces ([68]).

3. Noncommutative harmonic analysis on Riemannian homogeneous manifolds

Suppose that G/H is a homogeneous manifold of reductive type. A G-invariant pseudo-Riemannian metric on G/H (see §1) induces a G-invariant volume element dx. Let $L^2(G/H)$ be the Hilbert space of square integrable functions on G/H with respect to the measure dx. Then the left translation $G \times L^2(G/H) \ni (g, f(x)) \mapsto f(g^{-1}x) \in L^2(G/H)$ defines a unitary representation G on the representation space $L^2(G/H)$. This representation is called a (quasi) regular representation of G on G/H. In general, quasi-regular representations are not irreducible. We say that π is a discrete series representation of G/H if an irreducible representation π of G is realized in the closed subspace of $L^2(G/H)$. Let $\mathrm{Disc}(G/H)\,(\subset \widehat{G})$ be the set of discrete series representations for G/H. Suppose a unitary representation (τ, V_τ) of H is given. Similarly, we define the Hilbert space $L^2(G/H, V_\tau)$, consisting of L^2-sections for the associated vector bundle $G \underset{H}{\times} V_\tau$ and denote by $\mathrm{Disc}(G/H, \tau) \subset \widehat{G}$ the set of discrete series representations for $L^2(G/H, V_\tau)$. "The Plancherel theorem for G/H" means a formula decomposing the quasi-regular representation $L^2(G/H)$ into irreducible representations of G, and is a fundamental problem in L^2-harmonic analysis on a homogeneous manifold G/H. In the case of a semisimple symmetric space (as a generalization of a group manifold), it is solved by Peter-Weyl (compact group manifolds), Gel'fand-Naimark (classical complex semisimple group manifolds), Harish-Chandra (real semisimple group manifolds), Helgason (Riemannian symmetric spaces), and Oshima (announcement for semisimple symmetric spaces) ([84], [24], [27], [33], [82]). The determination of discrete series plays a very important role in the Plancherel formula in the above cases.

First of all, we consider the case that H is a compact subgroup. Then there is a G-invariant Riemannian metric on G/H induced from the bilinear form B_θ (§1). Because H is compact, we have $L^2(G/H) \subset L^2(G)$; so we can reduce:

$$\begin{aligned} &\text{Plancherel theorem for } L^2(G/H) \\ &\qquad = \text{Plancherel theorem for } L^2(G) \\ &\qquad\quad + \text{an explicit computation of finite dimensional representations.} \end{aligned}$$

Namely, let K be a maximal compact subgroup of G containing H. Then we have

$$L^2(G/H) \simeq \bigoplus_{\tau \in \mathrm{Disc}(K/H)} L^2(G/K, \tau)$$

by induction by stages on induced representations. In particular, discrete series representations are given by

$$\mathrm{Disc}(G/H) = \bigcup_{\tau \in \mathrm{Disc}(K/H)} \mathrm{Disc}(G/K, \tau) \ (\subset \mathrm{Disc}(G)).$$

On the other hand, in view of the description of (generic) tempered representations in terms of cuspidal parabolic induction (see [113] for example) which are the support of the Plancherel measure on a group manifold G ([27]), the continuous spectrum on $L^2(G/H)$ is also determined by studying a similar problem about discrete spectrum for smaller groups (Levi subgroups). Therefore, based on the Plancherel theorem of $L^2(G)$ due to Harish-Chandra, we can obtain an abstract Plancherel theorem of $L^2(G/H)$ if we determine $\mathrm{Disc}(K/H)$ and $\mathrm{Disc}(G/K, \tau)$ (and similar problems for smaller groups). From Frobenius reciprocity, $\mathrm{Disc}(K/H)$ is obtained by computation of finite dimensional representations, and $\mathrm{Disc}(G/K, \tau)$ is also reduced to computations of finite dimensional representations by the Blattner formula (Theorem (2.2)(4)) and the Borel-Weil-Bott-Kostant theorem ([57]). Actual calculation of such finite dimensional representations is complicated in general, (the reader might compare a direct proof by this approach and that in §6 of the fact $\mathrm{Disc}(SO(1,2q)/U(q)) \neq \emptyset \Leftrightarrow q \in 2\mathbb{N}$, which is a special case of Example 6.8) but even simple examples sometimes give nice models of the spectral geometry of a Riemannian manifold.

Here are some general facts about $\mathrm{Disc}(G/K,\tau)$ ($\subset \mathrm{Disc}(G)$):

Theorem 3.1. *Suppose that K is a maximal compact subgroup of a real reductive Lie group G, and (τ, V_τ) is a finite dimensional representation of K.*

(1) $\mathrm{Disc}(G/K,\tau)$ *is a finite set. Moreover,* $\bigcup_{\tau\in\widehat{K}} \mathrm{Disc}(G/K,\tau) = \mathrm{Disc}(G)$.
(2) *If* $\dim V_\tau = 1$, *then* $\mathrm{Disc}(G/K,\tau)$ *consists of (anti-) holomorphic discrete series representations of G.*
(3) *If τ is the trivial one dimensional representation of K, then* $\mathrm{Disc}(G/K,\tau) = \emptyset$.

Here are some comments on Theorem 3.1. We can give an upper estimate of $\#\mathrm{Disc}(G/K,\tau)$ by using the Dirac inequality for unitary representations due to Parthasarathy (this inequality reflects non-negativity of the Laplacian; [12],[83]). (3) follows from the Blattner formula (see Theorem (2.2)(4)). We shall also mention another approach of (3) in the next section.

According to the condition on τ such that (1) $\tau \in \widehat{K}$ is arbitrary; (2) $\dim V_\tau = 1$ ([97]); or (3) $\tau = \mathbf{1}$, the set of discrete series representations $\mathrm{Disc}(()G/K,\tau)$ is remarkably different. There is another interpretation of (2) with emphasis on the uniquely embedded property into principal series, but we do not go into details here.

Example 3.2. Let $G = Sp(1,n) \supset K = Sp(1) \times Sp(n) \supset T = (S^1)^{n+1}$. Choose a base $\{f_1, \dots, f_{n+1}\}$ of $\sqrt{-1}\mathfrak{t}^*$ such that the root system is represented as

$$\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm f_i \pm f_j, \pm 2f_l : 2 \leq i < j \leq n+1, 1 \leq l \leq n+1\}.$$

Let $\sigma_l \in \widehat{K}$ ($l \in \mathbb{N}$) be the $l+1$ dimensional irreducible representation of $Sp(1) \simeq SU(2)$ which is extended trivially on $Sp(n)$. Then we have

$$\mathrm{Disc}(G/K,\sigma_l) = \{\Pi(G, \lambda_1 f_1 + \lambda_2 f_2) : (\lambda_1,\lambda_2) \in \mathbb{Z}^2, \lambda_1 > \lambda_2 \geq n, \lambda_1 + \lambda_2 = l+1\}.$$

From the view point of global analysis on manifolds, harmonic analysis on a homogeneous manifold deals with very special cases indeed, but it sometimes gives a nice model from an explicit result obtained by representation theory. For example, Theorem (3.1)(1),(3) can be translated into a well-known result: "*The Laplacian for a non-compact Riemannian symmetric space does not admit a discrete spectrum, but the Laplacian acting on sections of the associated vector bundle (for example, the space of differential forms) admit possibly finite discrete spectra*" (the proof requires some argument in representation theory in the case of $\mathbb{R}$-rank $G > 1$). On the other hand, the following problem was posed in a series of research by T.Sunada ([76]):

"Does there exist a simply connected non-compact Riemannian manifold M with the following property ? "

i) M is a universal covering of a compact Riemannian manifold.

ii) There exist point spectra for the Laplacian Δ_M.

The positive answer is given by an interpretation of Example (3.2) ([56]). Suppose G/H is a homogeneous manifold of reductive type. If H is compact, it is known by Borel, Harish-Chandra and Mostow-Tamagawa ([11], [75]) that there is a discrete subgroup Γ of G such that the action of Γ on G/H is properly discontinuous and free so that $\Gamma\backslash G/H$ is a compact manifold.[*3)] Consequently, the manifold G/H equipped with the G-invariant Riemannian metric by B_θ (see §1 for definition) gives an example of a covering manifold of a compact Riemannian manifold. In particular, Example (3.2) gives rise to an answer to the problem of Sunada:

Example 3.3. We equip a manifold $S^2 \times \mathbb{R}^4$ with a Riemannian structure by a diffeomorphism $Sp(1,1)/\mathbb{T} \times Sp(1) \simeq S^2 \times \mathbb{R}^4$ and by B_θ of $Sp(1,1)$ (twisting the Killing form B). Then the continuous spectrum of the Laplacian of the non-compact Riemannian manifold $S^2 \times \mathbb{R}^4$ is given by $[\frac{1}{6}, \infty)$, while the set of point spectra is given by $\{\frac{1}{12}(n^2 + 4mn + m^2 + 3) : n > m > 0, n, m \in \mathbb{N}\}$.

4. Discrete series representations for vector bundles over symmetric spaces

As a generalization of discrete series representations for group manifolds (Harish-Chandra) in the previous section, Flensted-Jensen and Oshima-Matsuki constructed discrete series representations for symmetric spaces. As a further generalization, Schlichtkrull and Kobayashi constructed discrete series representations for vector bundles over symmetric space ([20], [69], [21], [79], [91],[48],[51]) (see Figure 4.1).

[**group manifolds**]		[**symmetric spaces**]		[**vector bundles on symmetric spaces**]
		Oshima-Matsuki type	$\subset$	♣
		$\cup$		$\cup$
Harish-Chandra type	$\subset$	Flensted-Jensen type	$\subset$	Schlichtkrull type

Figure 4.1

In this section, we review the construction of discrete series representations corresponding to ♣ of Figure 4.1 based on Chapter 0 in [51]. Let τ be a finite dimensional unitary representation of a real reductive Lie group H. First, we note that H is locally isomorphic to a direct product $H \approx H_1 \times H_2$, where H_1 is a direct product of a compact group and an abelian group $\mathbb{R}^d$ and $\tau_{|H_2}$ is trivial. We have treated the special case with H_1 compact and $H_2 = 1$ in the previous section, and will treat another special case with $H_1 = \mathbb{R}^d$ in the next section. In this section, we suppose that $H = H_1 \times H_2$ (H_1 is compact) and that $\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$. A typical example of this setting is given by the following indefinite Stiefel manifold G/H_2.

Example 4.2. $G/H_2 = U(p,q;\mathbb{F})/U(p-m,q;\mathbb{F})$, $H_1 = U(m;\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Here the rank condition amounts to $p \geq 2m$.

In the setting as above, the homogeneous manifold G/H_2 is of reductive type, which has a structure of G-equivariant H_1-principal bundle:

$$H_1 \to G/H_2 \to G/H.$$

Because H_1 is compact, we have an inclusion of unitary representations of G

$$L^2(G/H) = L^2(G/H;\mathbf{1}) \hookrightarrow L^2(G/H_2) \simeq \bigoplus_{\tau \in \widehat{H_1}} \dim\tau \cdot L^2(G/H_1 \times H_2; \tau \boxtimes \mathbf{1})$$

as in §3. The principal object of this section is a homogeneous manifold of reductive type G/H_2 (non-symmetric if $H_1 \neq 1$) and our goal is to construct discrete series representation for G/H_2. Note that $\operatorname{Disc}(G/H_2)$ contains $\operatorname{Disc}(G/H) = \operatorname{Disc}(G/H, \mathbf{1})$, the set of discrete series for a symmetric space. In the following construction of discrete series representations for G/H_2, we do not specify $\tau \in \widehat{H_1}$ in $\operatorname{Disc}(G/H_2) = \bigcup_{\tau \in \widehat{H_1}} \operatorname{Disc}(G/H_1 \times H_2, \tau)$, or equivalently, the right action of H_1 on $L^2(G/H_2)$ for simplicity of the exposition. Let $H_\mathbb{C} \subset G_\mathbb{C} \supset K_\mathbb{C}$ be a complexification of $H \subset G \supset K$ and take other real forms $H^r \subset G^r \supset K^r$ of $H_\mathbb{C} \subset G_\mathbb{C} \supset K_\mathbb{C}$ such that H^r is a maximal compact subgroup of G^r. G^r/H^r is said to be the <u>Riemannian dual</u> of G/H (see Example (5.2)). If H is isomorphic to the direct product $H = H_1 \times H_2$, H^r is (locally) isomorphic to the direct product $H^r = H_1^r \times H_2^r$. Moreover if H_1 is compact, then $H_1 = H_1^r$.

Let $P^r = M^r A^r N^r$ be a minimal parabolic subgroup of G^r and $\rho \in (\mathfrak{a}^r)^*$ half the sum of the roots for $\mathfrak{n}^r$. Denote by $\mathcal{A}$ the sheaf of analytic functions and by $\mathcal{B}$ that of hyperfunctions. For $(\delta, V) \in \widehat{M^r}$ and $\nu \in \widehat{A^r}$, we define $\mathcal{F}$ to be the $\mathcal{A}$ or $\mathcal{B}$ valued principal series by

$$\mathcal{F}(G^r/P^r; \delta \otimes \nu) := \{F \in \mathcal{F}(G^r; V) : F(gman) = \delta(m)^{-1} a^{-\nu-\rho} F(g) \quad \text{for } g \in G, man \in M^r A^r N^r\}.$$

Then

$$\Psi \colon \mathcal{B}(G^r/P^r; \delta^* \otimes (-\nu)) \times \mathcal{A}(G^r/P^r; \delta \otimes \nu) \ni (F, v) \mapsto \int_{K^r} \langle F(k), v(k) \rangle \, dk \in \mathbb{C},$$

is a non-degenerate G^r-invariant bilinear map. Fix $v \in \mathcal{A}(G^r/P^r; \delta \otimes \nu)$ such that H_2^r acts on v trivially, and define $\mathcal{P}_v(F)(g) := \Psi(F(g\,\cdot\,), v) = \Psi(F, g\cdot v)$. Then we have a G^r-map

$$\mathcal{P}_v\colon \mathcal{B}(G^r/P^r; \delta^* \otimes (-\nu)) \to \mathcal{A}(G^r/H_2^r),$$

which is a natural generalization of the Poisson transform. If v is cyclic, $\mathcal{P}_v$ is injective. There are finitely many closed orbits of K^r on G^r/P^r, which we write as X_j $(1 \le j \le l)$. For $1 \le j \le l$, we set

$$\mathcal{B}^j_{K^r}(\delta \otimes \nu) := \{F \in \mathcal{B}(G^r/P^r; \delta \otimes \nu) : \operatorname{supp} F \subset X_j\ , F \text{ is a } K^r\text{-finite vector}\}.$$

Theorem 4.3 ([48], [51]). *Assume that*

(a) $\mathcal{A}(G^r/P^r; \delta \otimes \nu)$ *contains an H_2^r-fixed cyclic vector v and*

(b) $\langle \nu, \alpha\rangle > 0 \quad ({}^\forall \alpha \in \Sigma(\mathfrak{n}^r, \mathfrak{a}^r))$.

Then the image of $\mathcal{P}_v\left(\mathcal{B}^j_{K^r}(\delta^* \otimes (-\nu))\right)$ $(1 \le j \le l)$ *by the Flensted-Jensen duality gives discrete series for* G/H_2.

Here the Flensted-Jensen duality is a kind of Weyl's unitary trick based on the holomorphic continuation of (vector valued) functions on symmetric spaces (cf. [21]). Because of our assumption that H_1 is compact, we can prove Theorem 4.3 by the L^2-estimate of $\mathcal{P}_v\left(\mathcal{B}^j_{K^r}(\delta^* \otimes (-\nu))\right)$ similar to the method in [69] (see [48]).

First of all, we consider special cases:

Example 4.4 ([20], [21], [69], [46]; cf. Example (2.8), Example (2.9)).
1) (discrete series representations for group manifolds) Let G' be a real reductive linear group such that $\operatorname{rank} G' = \operatorname{rank} K'$. We put $G = G' \times G'$, $H_2 = H = \operatorname{diag}(G')$, and $H_1 = 1$. Then the discrete series representations constructed in Theorem (4.3) exhausts (Harish-Chandra's) discrete series representations for a group manifold $G/H \simeq G'$. This construction is a special case of a theorem due to Flensted-Jensen, which is quite different from the construction in Example (2.8) (the Langlands conjecture).
2) (discrete series representations for symmetric spaces) Let G/H be a semisimple symmetric space with $\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$. We put $H_1 = 1$ and $H_2 = H$. Then the construction in Theorem (4.3) coincides with that of Oshima-Matsuki, which exhaust all discrete series representations for symmetric spaces ([69]).

The rest of this section is devoted to some comments and related topics of Theorem (4.3).

First, as we saw in Example 4.4, the representations constructed in Theorem 4.3 exhaust all discrete series representations for G/H_2 if $H_1 = 1$. However, this is not always the case if $H_1 \neq 1$, where G/H_2 is a non-symmetric homogeneous manifold ([63], [54]). The classification of $\operatorname{Disc}(G/H_2)$ with $H_1 \neq 1$ is an open problem in higher ranks. (If H_1 is an abelian group, the method in [69] would be effective in the study of $\mathrm{Disc}(G/H_2)$.)

Next, discrete series representations constructed in Theorem (4.3) can be zero for some parameters δ and ν. Because $\mathcal{P}_v$ is injective by the assumption (4.3) (a), it suffices to determine the condition on δ, ν assuring $\mathcal{B}^j_{K^r}(\delta^* \otimes (-\nu)) \neq 0$. This can be done by representation theoretic method described in Problem (2.10) in §2. In fact, there is a duality between $\mathcal{B}^j_{K^r}(\delta^* \otimes (-\nu))$ and a certain Zuckerman's derived functor module, whose vanishing condition can be studied by algebraic methods (see [51] Chapter 4, 5). Moreover, by a standard technique of representation theory, the assumption (a) of Theorem (4.3) (the condition for the injectivity of the vector-valued Poisson transform) is also determined explicitly as inequalities and integral conditions of δ and ν (cf. [51] Chapter 0).

Third, Theorem (4.3) also gives a new result in representation theory. That is, in an opposite way in the second remark as above, L^2-harmonic analysis over homogeneous manifolds also contributes to representation theory. Here we take unitarizability problem as an example (see the method 2 in Problem2.10 for another example): Suppose we are given a representation which is not known to be unitarizable. If it is realized as discrete series representations for a homogeneous manifold with G-invariant measure (like Theorem (4.3)) then it turns out to be unitarizable by the L^2-inner product on G/H. For example, discrete series representations for semisimple symmetric spaces lead to a discovery of "new" irreducible unitary representation as a by-product in the early 80's and gave affirmative support to the Zuckerman conjecture ([89], cf. [1]). Now that the Zuckerman conjecture has been solved (Theorem (2.3)), the unitarizability of representations which are realized as discrete series representations for symmetric spaces (namely, $H_1 = 1$ in Theorem 4.3) also follows from a special case of Theorem 2.3 by an identification with representations associated to elliptic orbits (see Example (2.9)). On the other hand, if $H_1 \neq 1$, discrete series representations constructed in Theorem (4.3) contain singular parameters which are out of the weakly fair range in the sense of Vogan (see [105] for definition) and do not necessarily satisfy the assumptions of the Zuckerman conjecture (Theorem (2.3)). This means that Theorem 4.3 assures the existence of the so-called Wallach set for some series of Zuckerman's derived functor modules which are not highest weight modules ([51] Chapter 2). Here the Wallach set is defined by the set of singular parameters which give unitarizability of standard representations attached to elliptic orbits out of the canonical chambers. Wallach, Rossi, Vergne, Enright, Parthasarathy and Wolf have studied the "Wallach set" by quite different methods for some special family of representations, such as highest weight modules (e.g. coherent continuation of holomorphic discrete series (Example (2.7))) or coherent continuations of discrete series representations ([110], [86], [18], [19]).

Fourth, we recall that there is a distinguishing feature in the existence problem of discrete series representations for Riemannian symmetric spaces between scalar valued ones and vector valued ones (Theorem 3.1). Namely, we have $\mathrm{Disc}(G/K) = \emptyset$

for any non-compact semisimple Lie group G, while $\mathrm{Disc}(G/K, \tau)$ can be a non empty set for some $\tau \in \widehat{K}$ if $\operatorname{rank} G = \operatorname{rank} K$. In the assumption of Theorem (4.3), this difference is reflected in the following way: if $H_1 = 1$, then δ must be the trivial representation of M^r and (b) $\Rightarrow$ (a) (Helgason, Kostant [31], [59])*4); on the other hand, if $H_1 \neq 1$ then the condition (a) and (b) are independent (i.e. do not imply each other) in general. We note that there is known an alternate proof of Theorem (3.1) (3) without using the Blattner formula but based on the study of an asymptotic behavior of spherical functions and on the implication (b) $\Rightarrow$ (a) (in this case $H_1 = 1$).

Fifth, discrete series representations of the Flensted-Jensen type (scalar valued ones) and the Schlichtkrull type (vector valued ones) in the lower row of Figure (4.1) are characterized as a subset of the upper row of Figure (4.1) with the algebraic property that "there exists a unique K-type associated to the symmetric tensor with $m = 0$ in the right side of the Blattner formula (Theorem (2.2)(4))" (it is the minimal K-type ([101]) in most cases), or equivalently, characterized by the analytic property that "$\mathcal{B}^j_{K^r}(\delta \otimes \nu)$ contains distributions of measure class".

Finally we remark that there exists a pair $(G, H) = (G, H_1 \times H_2)$ such that the upper row and the lower one coincide for scalar valued cases and that the lower row is a proper subset of the upper one for vector valued cases in Figure (4.1). This means that it can happen that discrete series representations of the Flensted-Jensen type exhaust $\mathrm{Disc}(G/H)$, while discrete series representations of the Schlichtkrull type do not exhaust $\mathrm{Disc}(G/H_2)$.

5. Harmonic analysis on para-Hermitian symmetric spaces

In the previous section, we have discussed L^2-harmonic analysis on a homogeneous space of reductive type which is a principal bundle over a reductive symmetric space with compact structure group as a generalization of symmetric spaces. In this section, we shall discuss L^2-harmonic analysis on a homogeneous manifold which is a principal bundle over a reductive symmetric space with abelian structure group.

We say a manifold M^{2n} of even dimension has a paracomplex structure if the tangent bundle TM splits into a Whitney direct sum $T^+M \oplus T^-M$ with equidimensional fibers and if $T^{\pm}M$ are completely integrable (see [64]). A pseudo-Riemannian metric g over a paracomplex manifold M is said to be a para-Hermitian metric if $T^{\pm}_x M \subset T_x M$ are maximally totally isotropic subspaces with respect to g for every point x in M. Kaneyuki-Kozai proved that a semisimple symmetric space G/H carries a G-invariant para-Hermitian structure if and only if H has a non-compact center C ([39] Theorem 3.7) and classified irreducible ones based on the latter property. A semisimple symmetric space equipped with G-invariant para-Hermitian structure is said to be a para-Hermitian symmetric space.

For example,

$$SL(p+q,\mathbb{R})/S(GL(p,\mathbb{R})\times GL(q,\mathbb{R})),\ Sp(n,\mathbb{R})/GL(n,\mathbb{R}),\ SO^*(4n)/SU^*(2n)\times\mathbb{R}$$

are para-Hermitian symmetric spaces.

Because the group homomorphism $1\to C\to H\to H/C\to 1$ splits, the character χ of C can be extended to H with trivial action on the remaining factor. We shall write χ also for the character extended to H.

Theorem 5.1. *Suppose that G/H is a semisimple para-Hermitian symmetric space. With the notation as above, the following conditions on (G,H) are equivalent.*

(1) *The Riemannian symmetric space G^r/H^r dual to G/H is a Hermitian symmetric space of the tube type.*

(2) *There is an isomorphism of unitary representations of G*

$$L^2(G/H,\tau\otimes\chi)\simeq L^2(G/H,\tau),$$

for any unitary representation (τ,V_τ) of H and for any unitary character χ of C (extended to H).

Example 5.2. Let $G/H=GL(p+q,\mathbb{R})/GL(p,\mathbb{R})\times GL(q,\mathbb{R})$. Then G/H is a para-Hermitian symmetric space and the Riemannian symmetric space G^r/H^r dual to G/H is isomorphic to $U(p,q)/U(p)\times U(q)$. Therefore, the condition of Theorem (5.1) is equivalent to $p=q$.

Sketch of the proof. (1) $\Rightarrow$ (2) is proved by generalizing the idea of Lipsman, Martin for a similar problem ([65], [67]) where H is a direct product of a compact group and $\mathbb{R}^j$ and by the idea of prehomogeneous vector spaces. We shall explain the proof with emphasis on a geometric idea and avoid representation theoretic terminology. By our assumption, the homogeneous manifold $M=G/H$ has a paracomplex structure with the Whitney sum $TM=T^+M\oplus T^-M$. Let W_+ be the maximal integral manifold along T^+M containing the origin $o=eH$. Let P_+ be the subgroup of G consisting of elements which preserve W_+ under the left action of G on $M=G/H$. Then P_+ contains H as a subgroup and we have $W_+\simeq P_+/H\subset G/H\simeq M$. Considering the induced representation from H to G, we first define a unitary representation of P_+ in the Hilbert space of V_τ-valued L^2-functions on the integral manifold W_+. Similarly, we define a unitary representation P_+ in the space of $V_{\tau\otimes\chi}\simeq V_\tau$-valued L^2-functions on W_+. If the condition (1) is satisfied, then the isotropic linear representation of H on T_oW_+ turns out to be a regular prehomogeneous vector space in the sense of M. Sato ([87]). Then we can construct a unitarily equivalent intertwining map between the two unitary representations of P_+ by using the relative invariants. This proves (1) $\Rightarrow$ (2). Conversely, if (1) is not satisfied, then we can show that (2) is not satisfied by comparing the most continuous part in the Plancherel formula for symmetric spaces in the case of $\tau=\mathbf{1}$ ([79],[81],[5]). □

6. Branching rule of unitary representations and its application to discrete series representations

The subject of this section is taken from [54]. It is a basic problem in L^2-harmonic analysis over a homogeneous manifold G/H to obtain the irreducible decomposition of the induced representation from the trivial representation of H to G, $\mathrm{Ind}(H \uparrow G; \mathbf{1}) \simeq L^2(G/H)$. Similarly, it is one of basic problems in unitary representation theory to obtain the irreducible decomposition of an irreducible representation of G when restricted to H (branching rule of unitary representations; "breaking symmetry" in physics), as is suggested by the Frobenius duality theorem for compact groups.

Suppose that $H \subset G$ are real reductive Lie groups. We compare the status of these problems as follows:

G/H	$L^2(G/H)$ (induced representation from H to G)	$\pi_{\vert H}$ restriction from G to H
G is compact	Peter-Weyl, Cartan-Helgason theorem	branching rule of finite dimensional representations
group manifold	Harish-Chandra's Plancherel theorem	decomposition of tensor product
Riemannian symmetric space	only continuous spectra	only discrete spectra (e.g. Blattner conjecture)
semisimple symmetric space	Oshima-Sekiguchi's Plancherel theorem	?

L^2-harmonic analysis over homogeneous manifolds sometimes arises as a special case of restriction problems. These problems can be equivalent in very special cases. In the following two propositions, we present examples where the decomposition of the tensor product of two irreducible representations (i.e. the restriction of a representation with respect to the symmetric pair $G \times G \supset \mathrm{diag}(G)$) is equivalent to non-commutative harmonic analysis for sections of line bundle over a certain symmetric space.

Proposition 6.1. *Assume that a real reductive Lie group G is the group of automorphisms of a para-Hermitian symmetric space. We write this para-Hermitian symmetric space as a homogeneous manifold G/H. We define subgroups P_+, P_- of G as in §5. Let χ_1, χ_2 be unitary characters of H which are extended from those of the non-compact center of H. Then the tensor product of degenerate principal series representations,* $\mathrm{Ind}(P_+ \uparrow G; \chi_1) \otimes \mathrm{Ind}(P_- \uparrow G; \chi_2)$ *is unitarily equivalent to* $L^2(G/H, \chi_1 \otimes \chi_2)$.

Proposition 6.2 ([85]). *Assume that a real reductive Lie group G is the group of automorphisms of a non-compact Hermitian symmetric space. We write this Hermitian symmetric space as a homogeneous manifold G/K. Let $\lambda_j \in \sqrt{-1}\mathfrak{g}^*$ ($j = 1, 2$) be integral elliptic element which are sufficiently regular and satisfying $\mathfrak{g}(\lambda_j) = \mathfrak{k}$. If $\mathfrak{q}(\lambda_1) \neq \mathfrak{q}(\lambda_2)$, then the tensor product of holomorphic discrete series and anti-holomorphic discrete series $\Pi(G, \lambda_1) \otimes \Pi(G, \lambda_2)$ is unitarily equivalent to $L^2(G/K, \mathbb{C}_{\lambda_1+\lambda_2})$.*

The Plancherel theorem for $L^2(G/H, \chi_1 \otimes \chi_2)$ in Proposition 6.1 reduces to the Plancherel theorem announced by Oshima-Sekiguchi (unpublished; [79],[82]), if $\chi_1 + \chi_2 = 0$ (or for general χ_1, χ_2 provided the assumption on (G, H) in Theorem (5.1) is satisfied). The Plancherel theorem for $L^2(G/K, \mathbb{C}_{\lambda_1+\lambda_2})$ in Proposition 6.2 was obtained by Harish-Chandra, Helgason, Gangolli and so on if $\lambda_1 + \lambda_2 = 0$ in the 60's ([33]), and was obtained by Shimeno, Heckman for general λ_1, λ_2 ([97], [30]) (cf. Theorem (3.1)(3)). As one can see in these two examples, it involves a lot of material to obtain an explicit decomposition of the restriction of a unitary representation $\pi \in \widehat{G}$ to a subgroup G' of G into irreducible representations of G' even if both $\pi \in \widehat{G}$ and (G, G') are quite special cases. Furthermore, there occur various phenomena in the irreducible decomposition of $\pi_{|G'}$ even if $\pi \in \widehat{G}$ is an irreducible unitary representation of G attached to an elliptic orbit and (G, G') is a symmetric pair ([22], [54]). In view of this, we think that it is important to find a nice framework in which the branching rule of unitary representations behaves nicely. We introduce the following definition with emphasis on the possibility of algebraic approaches in the restriction problem.

Definition 6.3 ([54]**).** A unitary representation π of G is said to be <u>G-admissible</u> if π is decomposed into a direct Hilbert sum of irreducible representations of G with finite multiplicities.

Example 6.4. 1) (Harish-Chandra) Let K be a maximal compact subgroup of a real reductive Lie group G. Then $\pi_{|K}$ is K-admissible for any $\pi \in \widehat{G}$ ([26]).

2) (Gelfand, Piatecki-Šapiro) Let Γ be a co-compact discrete subgroup of G. Then $L^2(G/\Gamma)$ is G-admissible ([23]).

3) (Martens) Let $\pi \in \widehat{G}$ be holomorphic discrete series. Let $Z(K)$ be the center of K and suppose $H \supset Z(K)$. Then the restriction $\pi_{|H}$ is H-admissible ([66]).

4) (Howe) Let π be the Weil representation of $G = \widetilde{Sp}$ and (G, H) a dual reductive pair with H containing a compact factor. Then $\pi_{|H}$ is H-admissible ([35]).

In the above examples, the advantage of "admissible restriction" is that we can treat the irreducible decomposition by purely algebraic methods. There has been quite a lot of research, e.g., [29], [101], [66], [37], [38], [41], [1]. We can see allusions to compactness in the assumptions of each setting of Example 6.4. Namely, the subgroup G' is compact in (1), a homogeneous manifold G/Γ is compact in (2), and the unitary representations have a highest weight vector (just like irreducible representations of a compact Lie group) in (3),(4) *5) . The main theme of the first half of this section is to find examples of an admissible restriction $\pi_{|G'}$ in a more general setting, where π does not have a highest weight and where G' is not compact.

Suppose (G, G') is a reductive symmetric pair defined by an involutive automorphism $\sigma \in \mathrm{Aut}(G)$ (see Example 1.4). Take a Cartan involution θ of G, which commutes with σ. We define subspaces of Lie algebra $\mathfrak{k}$ of $K = G^\theta$ by

$$\mathfrak{k}_\pm := \{X \in \mathfrak{k} : \sigma(X) = \pm X\}.$$

We take a maximal abelian subspace $\mathfrak{t}_-$ of $\mathfrak{k}_-$ and extend $\mathfrak{t}_-$ to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Fix positive roots of the restricted root system $\Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_{-\mathbb{C}})$ and choose the set of positive roots $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ of the root system $\Delta(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, which is compatible with $\Sigma^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_{-\mathbb{C}})$. We fix an elliptic orbit ($\subset \sqrt{-1}\mathfrak{g}^*$). It intersects at a single point with the dominant Weyl chamber associated to $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. We write it for $\lambda \in \sqrt{-1}\mathfrak{t}^*$. We define a θ-stable parabolic subalgebra by $\mathfrak{q}(\lambda) = \mathfrak{l}_\mathbb{C} + \mathfrak{u}(\lambda)$, and we define a closed cone of $\sqrt{-1}\mathfrak{t}^*$ by $C_+(\lambda) := \left\{ \sum_{\beta \in \Delta(\mathfrak{u}(\lambda)\cap\mathfrak{p}, \mathfrak{t})} n_\beta \beta : n_\beta \geq 0 \right\}$. Now we consider the restriction of the unitary representation $\Pi(G, \lambda)$ of G with respect to a subgroup G'.

Theorem 6.5 [54]. *In the above setting, if* $C_+(\lambda) \cap \sqrt{-1}\,\mathfrak{t}_-{}^* = \{0\}$, *then* $\Pi(G, \lambda)_{|G'}$ *is* G'*-admissible.*

Similarly, we have the following theorem, which was originally used in a method that manages to control many cancellations occurring in the Blattner formula to find the criterion of the non-vanishing of $\Pi(G, \lambda)$ for singular parameter λ in Proposition 4.1.3 [51] (see the method (4) of Problem 2.10). Also a special case with $K_i \simeq SU(2)$ is also studied by Gross-Wallach (e-mail communication; not published yet) in the study of the Gross-Prasad conjecture [25].

Theorem 6.6 ([51], [54]). *Suppose that a maximal compact subgroup K of a real reductive Lie group G is (locally) isomorphic to the direct product $K_1 \times K_2$. If a reductive subgroup G' of G contains K_1 and if an integral elliptic orbit* $\mathrm{Ad}(G)\lambda$ *intersects with $\sqrt{-1}\mathfrak{k}_1^*$, then the restriction $\Pi(G, \lambda)_{|G'}$ is G'-admissible.*

The proof of these two theorems is not hard; it is based on algebraic properties of $\Pi(G, \lambda)$ (Theorem (2.2)) and upper estimates of K-types. An alternate proof is also given based on algebraic analysis by estimating singularity spectrum ([43]) of distribution characters ([55]). Furthermore, explicit formulas of branching rules are obtained in some cases in the framework of "admissible restrictions" ([50],[54],[55]).

Now we mention some applications of these Theorems. Suppose G' is a subgroup of a Lie group G. In the setting that G acts on a manifold X and G' acts on a manifold X', the representation theoretic counterpart of the G'-equivariant map

$f\colon X' \to X$ is the pull-back of function spaces $f^*\colon \Gamma(X) \to \Gamma(X')$, where there arises naturally the problem of the restriction of representations of G to a subgroup G' through f^*. In particular, the unitary representation $\Pi(G,\lambda)$ attached to elliptic orbits is related intimately to the topology of locally Riemannian symmetric spaces (Example (2.6)) and discrete series representations for symmetric space (Example (2.9), §4). So we may expect that the restriction of $\Pi(G,\lambda)$ to subgroups will give various applications in many settings. Here, we explain one of the simplest applications. The following theorem gives an abstract framework which bridges between harmonic analysis on two homogeneous manifolds through branching rules. In particular, some special cases of branching rules of admissible restriction of irreducible unitary representations (see [50], [99], [36]) clarify the interrelation between discrete spectra of the Laplacian on a Riemannian homogeneous manifold in §3 (Example (3.3)) and discrete series representations for an indefinite Stiefel manifold in §4 (Example (4.2), Theorem (4.3)).

Theorem 6.7 ([55])**.** *Suppose that G is a real reductive Lie group and that H, G' and $H' := H \cap G'$ are closed subgroups which are reductive in G. Let P' be a minimal parabolic subgroup of G'. If $\dim G/H = \dim G'/H' = \dim P'/H' \cap P'$, then there is a bijection*

$$\bigcup_{\pi \in \mathrm{Disc}(G/H)} \mathrm{Disc}(\pi_{|G'}) \simeq \mathrm{Disc}(G'/H')$$

counting with multiplicities.

In the setting of the above theorem, we note that we can construct and classify discrete series representations for a homogeneous manifold G'/H' if we know discrete series representations for G/H and if we know the branching rule of $\pi_{|G'}$ for $\pi \in \mathrm{Disc}(G/H) (\subset \widehat{G})$. Previous to this, very little has been known about the condition on (G,H) that $\mathrm{Disc}(G/H) \neq \emptyset$ except for symmetric spaces. Theorem 6.7 combined with Theorem 6.5 and Theorem 6.6 yields a partial answer for the existence of discrete series representation for real forms of some spherical homogeneous manifolds*6), which were not known by other methods ([54]).

Example 6.8 (discrete series for non-symmetric spherical homogeneous manifolds).

$$\mathrm{Disc}(SU(2p-1,2q)/Sp(p-1,q)) \neq \emptyset \quad (\forall p,q),$$

$$\mathrm{Disc}(SO(2p-1,2q)/U(p-1,q)) \neq \emptyset \text{ if } q \in 2\mathbb{Z},$$

$$\mathrm{Disc}(SO(4,3)/G_2(\mathbb{R})) \neq \emptyset.$$

We end this article with a conjecture on discrete series representations for (general) homogeneous manifolds of reductive type. It is in sharp contrast with vector bundle valued discrete series representations (e.g. Theorem 3.1 (1)). New discrete series representations obtained in §4 and §6 together with well-known cases (Example 2.8, Example 2.9, Example 3.1 (3)) give some evidence for this conjecture.

Conjecture 6.9. *Suppose G/H is a homogeneous manifold of reductive type. If* $\mathrm{Disc}(G/H) \neq \emptyset$*, then there are infinite many discrete series which are not unitarily equivalent (i.e.* $\#\mathrm{Disc}(G/H) = \infty$*).*

Remark 1) (Classification of irreducible $(\mathfrak{g}, K)$-modules.) Three methods are known: i) based on asymptotic behaviors of matrix coefficients due to Langlands, ii) based on minimal K-types due to Vogan and based on Zuckerman's derived functor modules, iii) based on $\mathcal{D}$-module over flag manifold due to Beilinson-Bernstein (for regular parameters) ([62], [47], [101], [8]).

One might separate (ii) into two methods so that one might say there are four methods of classification (cf. [96]).

Remark 2) (Classification of irreducible unitary representations.) The unitary dual $\widehat{G}$ is known if G is $GL(n, \mathbb{R})$, $SU^*(2n)$, or classical complex Lie groups, or if the $\mathbb{R}$-rank is small. On the other hand, $\widehat{G}$ is not classified yet if $G = SO(p,q)$, $SU(p,q)$, $Sp(p,q)$, $Sp(n,\mathbb{R})$, $SO^*(2n)$ (p, q, n are large) (cf. [4], [103], [6], [108] and the references therein).

Remark 3) Many basic questions (e.g. Calabi-Markus phenomenon [15], [116] or criterion of existence of uniform lattice) about discontinuous groups for G/H have not yet found a final answer if H is non-compact for a general Lie group. See ([61], [112], [49], [52], [53], [9], [119]) for recent developments in discontinuous groups for G/H of reductive type.

Remark 4) It leads to the Helgason conjecture (theorem) by means of the Poisson transform at the level of the maximal globalization ([31],[40],[96]).

Remark 5) We remark that not all irreducible $(\mathfrak{g}, K)$-modules have highest weight vectors, if G is a non-compact real reductive Lie group. Classification of irreducible unitary representations with highest weight vectors is known.

Remark 6) Spherical homogeneous spaces are somewhat wider class than symmetric spaces. Compact spherical homogeneous spaces are classified ([60], [14]).

Finally we collect some references in this area, most of which are closely related to the viewpoint in this article. We apologize to the many people whose works are not mentioned here because of the author's ignorance.

Notes in English Version, January, 1996

The English translation of the original manuscript (Sugaku, **46** (1994) 124-143, Math. Soc. Japan) was achieved while the author was a guest at the Institut Mittag-Leffler of the Royal Swedish Academy of Sciences. I am grateful to the staff of the Institute and to the organizers of the special year "Analysis on Lie Groups". Thanks are also due to my friends A.Nilsson and R.Donley for reading the translation carefully.

We mention that P.Delorme announced the Plancherel formula for semisimple symmetric spaces November 1995 (cf. Figure in §6) at Institut Mittag-Leffler. His proof, based on generalized Maass-Selberg relations, gives a different approach from Oshima's announcement. Also the Paley-Wiener theorem for semisimple symmetric spaces was announced by Ban and Schlichtkrull.

The reader will also benefit from the recent textbook of Knapp-Vogan "Cohomological Induction and Unitary Representations", Princeton University Press (1995) and its references for related topics in §2.

T.Knapp kindly informed that he has found the Wallach set in a different setting (cf. remarks in §4) (A.M.S. lectures, August, 1995).

References

[1] J.Adams, *Discrete spectrum of the dual reductive pair* $(O(p,q), Sp(2m))$, Invent.Math. **74** (1984), 449-475.

[2] J.Adams, D.Barbasch and D.Vogan, *The Langlands Classification and Irreducible Characters for Real Reductive Groups*, Progress in Math. **104**, vol. 104, Birkhäuser, 1992.

[3] M.Atiyah and W.Schmid, *A geometric construction of the discrete series for semisimple Lie groups*, Invent. Math. **14** (1977), 1-62.

[4] M.W.Baldoni-Silva and A.Knapp, *Irreducible unitary representations of some groups of real rank two*, Lect. Notes. Math. **1243** (1987), 15-36.

[5] van den Ban and H.Schlichtkrull, *Multiplicities in the Plancherel decomposition for a semisimple symmetric spaces*, Contemp. Math. **145** (1993), 163-180.

[6] D.Barbasch, *The unitary dual for complex classical Lie groups*, Invent. Math. **96** (1989), 103-176.

[7] L.Barchini, A.Knapp and R.Zierau, *Intertwining operators into Dolbeault cohomology representations*, J. Funct. Anal. **106** (1992).

[8] A.Beilinson and J.Bernstein, *Localisation de* $\mathfrak{g}$*-modules*, C. R. Acad. Sci. Paris (1981), 15-18.

[9] Y.Benoist and F.Labourie, *Sur les espaces homogenes modeles de varietes compactes*, (to appear).

[10] F.Bien, $\mathcal{D}$*-modules and Spherical Representations*, Math. Notes **39**, Princeton U.P., 1990.

[11] A.Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111-122.

[12] A.Borel and N.Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, Ann. Math. Stud. **94**, Princeton U.P., 1980.

[13] R.Bott, *Homogeneous vector bundles*, Ann. Math. **66** (1968), 203-248.

[14] M.Brion, *Classification des espaces homogenes spheriques*, Compositio Math. **63-2** (1987), 189-208.

[15] E.Calabi and L.Markus, *Relativistic space forms*, Ann. of Math. **75** (1962), 63-76.

[16] J.T.Chang, *Remarks on localization and standard modules: The duality theorem on a generalized flag variety*, Proc. A.M.S. **117** (1993), 585-591.

[17] D.Collingwood, *Representations of Rank One Lie Groups*, Research notes in Math. **137**, Pitman, 1985.

[18] T.J.Enright, R.Parthasarathy, N.R.Wallach, and J.A.Wolf, *Unitary derived functor module with small spectrum*, Acta.Math. **154** (1985), 105-136.

[19] T.J.Enright and J.A.Wolf, *Continuation of unitary derived functor modules out of the canonical chamber*, Soc. Math. France, Mémoire **15** (1984), 139-156.

[20] M.Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Ann. of Math. **111** (1980), 253-311.

[21] ———, *Analysis on Non-Riemannian Symmetric Spaces*, Conf. Board, **61**, A.M.S., 1986.

[22] I.M.Gelfand and M.I.Graev, *Geometry of homogeneous spaces, representations of groups in homogeneous spaces, and related questions of integral geometry*, Transl. II. Ser., A.M.S. **37** (1964), 351-429.

[23] I.M.Gelfand, M.I.Graev and I.Piatecki-Šapiro, *Representation Theory and Automorphic Functions*, Saunders Math. Books, 1969.

[24] S.G.Gindikin and F.I.Karpelevič, *Plancherel measure of Riemannian symmetric spaces of non-positive curvature*, Soviet Math. Dokl **3** (1962), 962-965.

[25] B.H.Gross and D.Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}, Canad. J. Math. **44** (1992), 974-1002.

[26] Harish-Chandra, *Representations of semi-simple Lie groups*, I, III, Trans.A.M.S. **75**, (1953), 185-243; **76**, (1954), 234-253; IV, Amer.J.Math. **77**, (1955), 743-777.

[27] ———, *Harmonic analysis on real reductive groups*, I, J. Funct. Anal., **19**, (1975), 104-204; II, Invent, Math., **36**, (1976), 1-55; III, Ann. Math., **104**, (1976), 117-201.

[28] H.Hecht, D.Miličic, W.Schmid and J.A.Wolf, *Localization and standard modules for real semisimple Lie groups*, Invent.Math. **90** (1987), 297-332.

[29] H.Hecht and W.Schmid, *A proof of Blattner's conjecture*, Invent. Math. **31** (1976), 129-154.

[30] Heckman, *Lectures on hypergeometric and spherical functions*, Notes at Luminy (1991).

[31] S.Helgason, *A duality for symmetric spaces with applications to group representations* I, II, Adv. in Math. **5**, (1970), 1-154; **22**, (1976), 187-219.

[32] ———, *Differential Geometry, Lie Groups and Symmetric Spaces*, Pure and Appl. Math. **80**, vol. 80, Academic Press, 1978.

[33] ———, *Groups and Geometric Analysis*, Pure and Appl. Math. **113**, vol. 113, Academic Press, 1984.

[34] R.Hotta, *Elliptic complexes on symmetric spaces*, Sugaku **22** (1970), Math. Soc. Japan, 204-216.

[35] R.Howe, *θ-series and invariant theory*, Proc. Symp. Pure Math. **33** (1979), A.M.S., 275-285.

[36] R.Howe and E.Tan, *Homogeneous functions on light cones*, Bull. A.M.S. **28** (1993), 1-74.

[37] H.P.Jakobsen, *Tensor products, reproducing kernels, and power series*, J. Funct. Anal. **31** (1979), 293-305.

[38] H.P.Jakobsen and M.Vergne, *Restrictions and expansions of holomorphic representations*, J. Funct. Anal. **34** (1979), 29-53.

[39] S.Kaneyuki and M.Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math. **8** (1985), 81-98.

[40] M.Kashiwara, A.Kowata, K.Minemura, K.Okamoto, T.Oshima and M.Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. of Math. **107** (1978), 1 - 39.

[41] M.Kashiwara and M.Vergne, *On the Segal-Shale- Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), 1-47.

[42] ———, *K-types and singular spectrum*, Lect. Notes. in Math. **728** (1979), 177-200.

[43] M.Kashiwara, T.Kawai and T.Kimura, *Foundations of Algebraic Analysis*, Princeton Math. Series **37**, 1986.

[44] A.A.Kirillov, *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys **4** (1962), 53-104.

[45] A.Knapp, *Lie Groups, Lie Algebras and Representation Theory*, Math. Notes **34**, Princeton U.P., 1988.

[46] ———, *Representation Theory of Semisimple Groups, an overview based on examples*, Princeton Math. Series **36**, 1986.

[47] A.Knapp and G.J.Zuckerman, *Classification of irreducible tempered representations of semisimple groups*, Ann. Math. **116** (1982), 381-501.

[48] T.Kobayashi, *Construction of discrete series for vector bundles over semisimple symmetric spaces*, master thesis **I** (1987) U. Tokyo; RIMS Kokyuroku, Kyoto University **642** (1988), 134-156.

[49] ———, *Proper action on a homogeneous space of reductive type*, Math. Ann. **285** (1989), 249-263.

[50] ———, *Unitary representations realized in L^2-sections of vector bundles over semisimple symmetric spaces*, Proceedings at the 27-th. Symp. of Functional Analysis and Real Analysis (in Japanese) (1989), Math. Soc. Japan, 39-54.

[51] ———, *Singular Unitary Representations and Discrete Series for Indefinite Stiefel Manifolds $U(p,q;\mathbb{F})/U(p-m,q;\mathbb{F})$*, vol. 462, Memoirs A.M.S. **462**, 1992.

[52] ———, *Discontinuous groups acting on homogeneous spaces of reductive type*, Proceedings of ICM-90 Satellite Conference on Representation Theory of Lie Groups and Lie Algebras, Fuji-kawaguchiko, (1992), World Scientific, 59-75.

[53] ———, *A necessary condition for the existence of compact Clifford-Klein forms of homogeneous spaces of reductive type*, Duke Math. J. **67** (1992), 653-664.

[54] ———, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications*, (to appear in Invent. Math.).

[55] ———, *The Restriction of $A_{\mathfrak{q}}(\lambda)$ to reductive subgroups*, Proc. Acad. Japan **69** (1993), 262-267; II, to apear in Proc. Acad. Japan.

[56] T.Kobayashi, K.Ono and T.Sunada, *Periodic Schrödinger operators on a manifold,*, Forum Mathematicum **1-1** (1989), 69-79.

[57] B.Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. **74** (1961), 329-387.

[58] ———, *Orbits, symplectic structures and representation theory*, Proc. U.S.-Japan Sem. on Diff. Geom., Kyoto (1965).

[59] ———, *Irreducibility of certain series of representations*, Publication of 1971 Summer School in Math., (ed. I.M.Gel'fand), Bolyai-Janos Math.Soc., Budapest.

[60] M.Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compositio Math. **38-2** (1979), 129-153.

[61] R.S.Kulkarni, *Proper actions and pseudo-Riemannian space forms*, Advances in Math. **40** (1981), 10-51.

[62] R.Langlands, *On the classification of irreducible representations of real algebraic groups*, mimeographed notes (1973).

[63] J.S.Li, *On the discrete series of generalized Stiefel manifolds*, Trans. A.M.S. **340** (1993), 753-766.

[64] R.Libermann, *Sur le structures presque paracomplexes*, C.R. Acad. Sci. Paris **234** (1952), 2517-2519.

[65] R.Lipsman, *On the unitary representation of a semisimple Lie group given by the invariant integral on its Lie algebra*, Adv. Math. Suppl. **6**, (1979) 143-158; II, Canad. J. Math. **29**, (1977) 1217-1222..

[66] S.Martens, *The characters of the holomorphic discrete series*, Proc.Nat.Acad.Sci. USA **72** (1975), 3275-3276.

[67] R.Martin, *On the decomposition of tensor products of principal series representations for real-rank one semisimple groups*, Trans. A.M.S. **201** (1975), 177-211.

[68] T.Matsuki, *A description of discrete series for semisimple symmetric spaces* II, Advanced Studies in Pure Math. **14** (1988), 531-540.

[69] T.Matsuki and T.Oshima, *A description of discrete series for semisimple symmetric spaces*, Advanced Studies in Pure Math. **4** (1984), 331-390.

[70] ———, *Existence conditions for discrete series for semisimple symmetric spaces*, RIMS Kokyuroku, Kyoto University **642** (1988), 119-133.

[71] Y.Matsushima and S.Murakami, *On vector bundle valued harmonic forms and automorphic forms on symmetric spaces*, Ann. Math. **78** (1963), 365-416.

[72] H.Matumoto, *On the representations of $U(m,n)$ unitarily induced from derived functor modules*, Preprint.

[73] K.Minemura, *Invariant differential operators and spherical sections on a homogeneous vector bundle*, Tokyo J. Math. **15** (1992), 231-245.

[74] G.D.Mostow, *Self-adjoint groups*, Ann. of Math. **62** (1955), 44-55.

[75] G.D.Mostow and T.Tamagawa, *On the compactness of arithmetically defined homogeneous spaces*, Ann. of Math. **76** (1962), 446-463.

[76] T.Ochiai, *On the spectrum of the Laplacian on a Riemannian manifold – Introduction to works by Professor T.Sunada, the winner of the 15-th. Iyanaga prise*, Sugaku **39** (1987), Math. Soc. Japan, 237-242.

[77] K.Okamoto, *On the Borel-Weil theory*, Sugaku **23** (1971), Math. Soc. Japan, 34-44.

[78] T.Oshima, *Fourier analysis on semisimple symmetric spaces*, Lect. Note Math. **880** (1980), 357-369.

[79] ———, *Harmonic analysis on symmetric spaces*, Sugaku **37** (1985), Math. Soc. Japan, 97-112.

[80] ———, *Asymptotic behavior of spherical functions on semisimple symmetric spaces*, Advanced Studies in Pure Math. **14** (1988), 561-601.

[81] ———, *A realization of semisimple symmetric spaces and construction of boundary value maps*, Advanced Studies in Pure Math. **14** (1988), 603-650.

[82] ———, *A method of harmonic analysis on semisimple symmetric spaces*, Algebraic analysis **II** (1988), 667-680, Academic Press..

[83] R.Parthasarathy, *Criteria for the unitarizability of some highest weight modules*, Proc. Indian Acad. Sci. **89** (1980), 1-24.

[84] F.Peter and H.Weyl, *Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe*, Math. Ann. **97** (1927), 737-755.

[85] J.Repka, *Tensor products of holomorphic discrete series representations*, Can. J. Math. **31** (1979), 836-844.

[86] H.Rossi and M.Vergne, *Analytic continuation of the holomorphic discrete series of a semisimple Lie group.*, Acta Math. **136** (1976), 1 - 59.
[87] H. Rubenthaler, *Espaces prehomogenes de type parabolique*, Lectures on harmonic analysis on Lie groups and related topics (Strasbourg, 1979), Lect. in Math., 14, (1982), Kinokuniya, 189 - 221.
[88] S. Salamanca Riba, *On the unitary dual of some classical Lie groups*, Compo. Math. **68** (1988), 251-303.
[89] H.Schlichtkrull, *The Langlands parameters of Flensted-Jensen's discrete series for semisimple symmetric spaces*, J.Funct.Anal. **50** (1983), 133-150.
[90] ———, *Hyperfunctions and Harmonic Analysis on Symmetric Spaces*, Progress in Math. **49**, Birkhäuser, 1984.
[91] ———, *A series of unitary irreducible representations induced from a symmetric subgroup of a semisimple Lie group*, Invent.Math. **68** (1982), 497-516.
[92] ———, *Eigenspaces of the Laplacian on hyperbolic spaces: composition series and integral transforms*, J. Funct. Anal. **70** (1987), 194-219.
[93] W.Schmid, *Homogeneous complex manifolds and representations of semisimple Lie groups*, Ph.D. Thesis, U.C.Berkeley (1967).
[94] ———, *On a conjecture of Langlands*, Ann. Math. **93** (1971), 1-42.
[95] ———, *Boundary value problems for group invariant differential equations* (1985), Asterisque, hors s'erie, 311-321, Élie Cartan et les mathématiques d'aujourd'hui.
[96] ———, *Construction and classification of irreducible Harish-Chandra modules*, Harmonic Analysis on Reductive Groups, Progress in Math. **101**, Birkhäuser, 1991, pp. 235-275.
[97] N.Shimeno, *The Plancherel formula for spherical functions with a one-dimensional K-type on a simply connected simple Lie group of Hermitian type*, Ph.D. dissertation, Univ. Tokyo (1993).
[98] T.Sunada, *Fundamental groups and the Laplacian.*, Kinokuniya, 1988.
[99] N.Y.Vilenkin and U.Klimyk, *Spectral expansions of some representations of Lie groups*, Ukrain. Math.J. **42** (1990), 116-118.
[100] D.Vogan, *Irreducible characters of semisimple Lie groups.* I, Duke Math.J. **46** (1979), 61-108.
[101] ———, *Representations of Real Reductive Lie Groups*, Progress in Math. **15**, Birkhäuser, 1981.
[102] ———, *Unitarizability of certain series of representations*, Ann. of Math. (1984), 141-187.
[103] ———, *The unitary dual of $GL(n)$ over an archimedean field*, Invent.Math. (1986), 449-505.
[104] ———, *Unitary Representations of Reductive Lie Groups*, Ann. Math. Stud. **118**, Princeton U.P., 1987.
[105] ———, *Irreducibility of discrete series representations for semisimple symmetric spaces*, Advanced Studies in Pure Math. **14** (1988), 191-221.
[106] ———, *Unipotent representations and cohomological induction*, Preprint.
[107] ———, *Isolated unitary representations*, Preprint.
[108] ———, *The unitary dual of G_2*, (to appear in Invent. Math.).
[109] D.Vogan and G.Zuckerman, *Unitary representations with non-zero cohomology*, Comp. Math. **53** (1984), 51-90.
[110] N.R.Wallach, *The analytic continuation of the discrete series.* I, II, Trans. A.M.S. **251** (1979), 1 - 17, 19 - 37.
[111] ———, *On the unitarizability of derived functor modules*, Invent.Math. **78** (1984), 131-141.
[112] ———, *Two problems in the theory of automorphic forms*, Open Problems in Representation Theory, pp. 39-40, Katata, 1986.
[113] ———, *Real Reductive Groups* I, II, Pure and Appl. Math. **132**, vol. 132, Academic Press, 1988, 1992.
[114] G.Warner, *Harmonic Analysis on Semisimple Lie Groups* I, II, Springer Verlag, Berlin, 1972.
[115] H.Weyl, *Theorie der Darstellung kontinuierlicher Gruppen* I, II, III, Math. Z., **23**, (1925), 271-309 **24**, (1925), 328-376, 377-395.

[116] J.A.Wolf, *The Clifford-Klein space forms of indefinite metric*, Ann. of Math. **75** (1962), 77-80.
[117] H.Wong, *Dolbeault cohomologies and Zuckerman modules associated with finite rank representations*, Ph.D. dissertation, Harvard University (1991).
[118] H.Yamashita, *Criteria for the finiteness of restriction of* $U(\mathfrak{g})$*-modules to subalgebras and applications to Harish-Chandra modules*, Proc. Japan Acad. **68**, (1992) 316-321..
[119] R.J.Zimmer, *Discrete groups and non-Riemannian homogeneous spaces*, (Preprint).
[120] G.Zuckerman, *Tensor products of finite and infinite dimensional representations of semisimple Lie groups*, Ann. of Math. **106** (1977), 295-308.

Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Meguro, Tokyo 153 Japan

current address:
Institut Mittag-Leffler
Auravägen 17
S-182 62 Djursholm, Sweden
E-mail address: toshi@ms.u-tokyo.ac.jp