DISCONTINUOUS GROUPS AND CLIFFORD-KLEIN FORMS OF PSEUDO-RIEMANNIAN HOMOGENEOUS MANIFOLDS

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Abstract

Let G be a Lie group and H its subgroup. A Clifford-Klein form of a homogeneous manifold G/H is the double coset space $\Gamma\backslash G/H$ if Γ is a subgroup of G acting properly discontinuously and freely on G/H. For example, any closed Riemann surface M with genus ≥ 2 is biholomorphic to a compact Clifford-Klein form of the Poincaré plane $G/H = PSL(2,\mathbb{R})/SO(2)$. On the other hand, there is *no* compact Clifford-Klein form of the hyperboloid of one sheet $G/H = PSL(2,\mathbb{R})/SO(1,1)$. Even more, there is no infinite discrete subgroup of G which acts properly discontinuously on G/H (the Calabi-Markus phenomenon). As is observed in the second example, not all discrete subgroup of G can act properly discontinuously on a homogeneous manifold G/H if H is noncompact. We discuss recent developments in the theory of discontinuous groups acting on a homogeneous manifold G/H where G is a real reductive Lie group and H is a noncompact reductive subgroup. Geometric ideas of various methods together with a number of examples are presented regarding fundamental problems: "which homogeneous manifolds G/H admit properly discontinuous actions of infinite discrete subgroups of G ?", and "which homogeneous manifolds G/H admit compact Clifford-Klein forms ?"

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§0. Introduction to Clifford-Klein forms

0.1. Homogeneous manifolds

First, we consider the setting:

$$G \supset H.$$

Here G is a Lie group and H is a closed subgroup of G. Then the coset space G/H equipped with the quotient topology carries a C^∞-manifold structure so that the natural quotient map

$$\pi \colon G \to G/H$$

is a C^∞-map. We say G/H is a *homogeneous manifold.*

0.2. Clifford-Klein forms

Second, we consider a more general setting:

$$\Gamma \subset G \supset H.$$

Here G is a Lie group, H is a closed subgroup and Γ is a discrete subgroup of G. Then one might ask

Question 0.2. *Does the double coset space $\Gamma\backslash G/H$ equipped with the quotient topology carry a C^∞-manifold structure so that the natural quotient map*

$$\varpi \colon G \to \Gamma\backslash G/H$$

is a C^∞-map ?

If *yes*, we say $\Gamma\backslash G/H$ is a *Clifford-Klein form of the homogeneous manifold G/H.*

Unfortunately, the action of a discrete subgroup Γ on G/H is not always properly discontinuous (see Definition 1.3.1) when H is not compact, and the quotient topology is not necessarily Hausdorff. Thus, we cannot always expect an affirmative answer to Question 0.2. This is the main difficulty of our subject. However, leaving this question aside for a moment, we first discuss the geometric aspect of Clifford-Klein forms in this section.

0.3. Clifford-Klein forms from the view point of geometry

From the view point of differential geometry, the important point in considering a Clifford-Klein form $\Gamma\backslash G/H$ is that we have a local diffeomorphism $p\colon G/H \to \Gamma\backslash G/H$, with the following commutative diagram:

$$\begin{array}{ccc} & G & \\ \pi \swarrow & & \searrow \varpi \\ G/H & \underset{p}{\longrightarrow} & \Gamma\backslash G/H. \end{array}$$

Therefore any G-invariant local structure (e.g. affine, complex, symplectic, Riemannian or pseudo-Riemannian, ...) on a homogeneous manifold G/H induces the same kind of structure on a Clifford-Klein form $\Gamma\backslash G/H$. In other words, the double coset space $\Gamma\backslash G/H$ is a manifold enjoying the same local properties as G/H, as long as the discrete subgroup Γ allows an affirmative answer to Question 0.2.

Conversely, let us start from a differentiable manifold M endowed with some local structure $\mathcal{T}$ and then explain how a Clifford-Klein form arises. Let $\widetilde{M}$ be the universal covering manifold of M. Then the local structure $\mathcal{T}$ is also defined on $\widetilde{M}$ through the covering map $\widetilde{M} \to M$. We set

$$G \equiv \mathrm{Aut}(\widetilde{M}, \mathcal{T}) := \{\varphi \in \mathrm{Diffeo}(\widetilde{M}) : \varphi \text{ preserves the structure } \mathcal{T}\}.$$

We note that G is the group of isometries if $\mathcal{T}$ is a Riemannian structure; the group of biholomorphic automorphisms if $\mathcal{T}$ is a complex structure.

We fix a point $o \in \widetilde{M}$ and write $\bar{o} \in M$ for its image of the covering map $\widetilde{M} \to M$. Then the fundamental group $\pi_1(M, \bar{o})$ acts effectively on $\widetilde{M}$ as the covering automorphism. We write Γ for the image of the injection $\pi_1(M, \bar{o}) \hookrightarrow G = \mathrm{Aut}(\widetilde{M}, \mathcal{T})$. That is, Γ is a discrete subgroup of G in the compact open topology, which is isomorphic to $\pi_1(M, \bar{o})$. Then we have a natural diffeomorphism $M \simeq \Gamma\backslash\widetilde{M}$.

Assume $G = \mathrm{Aut}(\widetilde{M}, \mathcal{T})$ is small enough to be a Lie group (see a textbook by Shoshichi Kobayashi [Ko72] for some sufficient conditions) and is large enough to act transitively on $\widetilde{M}$. Then $\widetilde{M}$ is represented as a homogeneous manifold $\widetilde{M} \simeq G/H$ with the isotropy subgroup H at $o \in \widetilde{M}$. Consequently, M is naturally represented as a Clifford-Klein form of G/H:

$$\begin{array}{ccc} \widetilde{M} & \simeq & G/H \\ {\scriptstyle \pi_1(M)}\downarrow & & \downarrow{\scriptstyle \Gamma} \\ M & \simeq & \Gamma\backslash G/H \end{array}$$

We remark that Question 0.2 has automatically an affirmative answer in this case.

We shall give a number of examples of simply connected manifolds $\widetilde{M} \simeq G/H$ equipped with some local structure in §0.4; and Clifford-Klein forms of these manifolds in §0.5.

0.4. Examples of homogeneous manifolds

First, we present some examples where $G = \mathrm{Aut}(\widetilde{M}, \mathcal{T})$ acts transitively on a simply connected manifold $\widetilde{M}$ so that $\widetilde{M}$ is represented as a homogeneous manifold of G.

Example 0.4.1. Let

$$\begin{cases} \widetilde{M} := \mathbb{R}^n, \\ \mathcal{T} := \text{ the canonical affine connection } \nabla. \end{cases}$$

Then $\mathrm{Aut}(\mathbb{R}^n, \nabla) \simeq GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ is the affine transformation group, acting transitively on $\mathbb{R}^n$ by

$$(g, b) \cdot x := gx + b \qquad \text{for } (g, b) \in GL(n, \mathbb{R}) \ltimes \mathbb{R}^n,\ x \in \mathbb{R}^n.$$

The isotropy subgroup at the origin $0 \in \mathbb{R}^n$ is isomorphic to $GL(n, \mathbb{R})$ and therefore the manifold $\mathbb{R}^n$ is represented as the homogeneous manifold $GL(n, \mathbb{R}) \ltimes \mathbb{R}^n / GL(n, \mathbb{R})$.

Example 0.4.2. Let

$$\begin{cases} \widetilde{M} := \mathbb{R}^n, \\ \mathcal{T} := \text{standard Riemannian structure } g. \end{cases}$$

Then $\mathrm{Aut}(\mathbb{R}^n, g) \simeq O(n) \ltimes \mathbb{R}^n$ is the Euclidean motion group, acting transitively on $\mathbb{R}^n$ by

$$(g, b) \cdot x := gx + b \qquad \text{for } (g, b) \in O(n) \ltimes \mathbb{R}^n,\ x \in \mathbb{R}^n.$$

Thus $\mathbb{R}^n$ is represented as the homogeneous manifold $O(n) \ltimes \mathbb{R}^n / O(n)$.

Example 0.4.3. Let

$$\begin{cases} \widetilde{M} := \left\{ x \in \mathbb{R}^{n+1} : {x_1}^2 + \cdots + {x_n}^2 - {x_{n+1}}^2 = 1 \right\} \quad (n \geq 3), \\ \mathcal{T} := \text{ the Lorentz metric } g \text{ induced from } {dx_1}^2 + \cdots + {dx_n}^2 - {dx_{n+1}}^2. \end{cases}$$

Let $O(n, 1)$ be the indefinite orthogonal group preserving the quadratic form ${x_1}^2 + \cdots + {x_n}^2 - {x_{n+1}}^2$. Then $O(n, 1)$ acts transitively and isometrically on the hyperboloid of one sheet $\widetilde{M}$, and it coincides with the group of isometries of $\widetilde{M}$. Thus, $\mathrm{Aut}(\widetilde{M}, g) = O(n, 1)$ and $\widetilde{M} \simeq O(n, 1)/O(n-1, 1)$.

Example 0.4.4. Let

$$\begin{cases} \widetilde{M} := \mathbb{C}, \\ \mathcal{T} := \text{ the standard complex structure } J. \end{cases}$$

Then the group of biholomorphic automorphisms is given by $\mathrm{Aut}(\mathbb{C}, J) \simeq \mathbb{C}^\times \ltimes \mathbb{C}$, a semi-direct product of $\mathbb{C}^\times$ and $\mathbb{C}$ with $\mathbb{C}$ normal. The action of $\mathbb{C}^\times \ltimes \mathbb{C}$ on $\mathbb{C}$ is given by the complex affine transformation:

$$(a, b) \cdot z = az + b \quad \text{ for } \ (a, b) \in \mathbb{C}^\times \ltimes \mathbb{C}.$$

This action is obviously transitive so that the complex plane $\mathbb{C}$ is represented as the homogeneous manifold $\mathbb{C}^\times \ltimes \mathbb{C}/\mathbb{C}^\times$.

Example 0.4.5. Let

$$\begin{cases} \widetilde{M} := \mathbb{C}P^1 \qquad \text{(the complex projective space)}, \\ \mathcal{T} := \text{ the standard complex structure } J. \end{cases}$$

The natural action of $SL(2, \mathbb{C})$ on $\mathbb{C}^2 \setminus \{0\}$ induces a transitive action of $PSL(2, \mathbb{C})$ on the projective space $\mathbb{C}P^1 = \mathbb{C}^2 \setminus \{0\}/\mathbb{C}^\times \simeq \mathbb{C} \cup \{\infty\}$, which is given by the linear fractional transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \quad \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I_2\}.$$

Then $PSL(2, \mathbb{C}) \simeq \mathrm{Aut}(\mathbb{C}P^1, J)$, the group of biholomorphic automorphisms. The projective space $\mathbb{C}P^1$ is represented as a homogeneous manifold $PSL(2, \mathbb{C})/B$ where B is a Borel subgroup of $PSL(2, \mathbb{C})$.

Example 0.4.6. Let

$$\begin{cases} \widetilde{M} := \mathcal{H} \equiv \{z \in \mathbb{C} : \mathrm{Im} z > 0\} \quad \text{the Poincaré plane}, \\ \mathcal{T} := \text{ the standard complex structure } J. \end{cases}$$

Then the group of biholomorphic automorphisms is given by $\mathrm{Aut}(\mathcal{H}, J) \simeq PSL(2, \mathbb{R})$, where the action of $PSL(2, \mathbb{R})$ is also defined by the linear fractional transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \quad \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I_2\}.$$

This action is transitive so that the Poincaré plane $\mathcal{H}$ is represented as the homogeneous manifold $PSL(2,\mathbb{R})/SO(2)$.

0.5. Examples of Clifford-Klein forms

Next, we present a number of examples of Clifford-Klein forms of the homogeneous manifolds given in Example 0.4. We also explain that these examples are closely related to the following interesting topics:

Examples 0.5.1,0.5.2 $\cdots$ the Auslander Conjecture on the fundamental group of compact complete affine manifolds.

Example 0.5.4 $\cdots\cdots\cdots$ the Calabi-Markus phenomenon for relativistic spherical forms.

Example 0.5.5 $\cdots\cdots\cdots$ the uniformization theorem of Riemannian surfaces due to Klein, Poincaré and Koebe.

Example 0.5.1 (see Example 0.4.1)**.** An *affine* manifold M is a manifold which admits a torsion free affine connection whose curvature tensor vanishes. It is said to be *complete* if every geodesic can be defined on all time intervals. Then it is known that the universal covering of any complete affine manifold $M = M^n$ is isomorphic to $(\mathbb{R}^n, \nabla)$ as affine manifolds. Therefore it follows from Example 0.4.1 that M can be represented as a Clifford-Klein form

$$M \simeq \Gamma \backslash GL(n,\mathbb{R}) \ltimes \mathbb{R}^n / GL(n,\mathbb{R}),$$

where Γ is a discrete subgroup of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$ which is isomorphic to the fundamental group $\pi_1(M)$.

Example 0.5.2 (see Example 0.4.2)**.** Retain the notation in Example 0.5.1. If M^n is a Riemannian manifold and if M^n is a complete affine manifold (see Example 0.5.1) for the Levi-Civita connection, then the universal covering of M is isometric to $\mathbb{R}^n$ endowed with the standard Riemannian metric g. Therefore it follows from Example 0.4.2 that M is represented as another Clifford-Klein form

$$M \simeq \Gamma \backslash O(n) \ltimes \mathbb{R}^n / O(n)$$

with $\Gamma \subset O(n) \ltimes \mathbb{R}^n$.

Similarly, if M^n is a Lorentz manifold (namely, M carries a pseudo-Riemannian metric of type $(n-1,1)$), and if M is a complete affine manifold for the Levi-Civita connection, then M is reduced to be a Clifford-Klein form

$$M \simeq \Gamma \backslash O(n-1,1) \ltimes \mathbb{R}^n / O(n-1,1)$$

with $\Gamma \subset O(n-1,1) \ltimes \mathbb{R}^n$.

Remark 0.5.3 (the Auslander Conjecture). Regarding to Example 0.5.1, 0.5.2, we mention the Auslander conjecture which asserts that *the fundamental group π_1 of any compact complete affine manifold is virtually solvable* (see [Au64], [Mi77], [Ma83] and references therein). In view of Example 0.5.1, this is equivalent to the conjecture that *a discrete group Γ is virtually solvable if $\Gamma \backslash GL(n,\mathbb{R}) \ltimes \mathbb{R}^n / GL(n,\mathbb{R})$ is a compact Clifford-Klein form of a homogeneous manifold $GL(n,\mathbb{R}) \ltimes \mathbb{R}^n / GL(n,\mathbb{R}) \simeq \mathbb{R}^n$.* Auslander's conjecture remains open except for some special cases such as $\Gamma \subset O(n) \ltimes \mathbb{R}^n$ (Bieberbach's theorem, see [Ra72], Corollary 8.26), $\Gamma \subset O(n-1,1) \ltimes \mathbb{R}^n$ ([GK84], see also [To90] for a generalization to rank one groups). The geometric meaning of $\Gamma \subset O(n) \ltimes \mathbb{R}^n$ (or $\Gamma \subset O(n-1,1) \ltimes \mathbb{R}^n$) is that the affine connection is the Levi-Civita connection of the standard Riemannian (or Lorentz) metric (see Example 0.5.2).

Example 0.5.4 (a relativistic spherical space form; see Example 0.4.3).

In the physics of relativistic cosmology, the space-time continuum is taken to be a Lorentz manifold M^4. A relativistic spherical space form is a complete Lorentz manifold M^n for $n \geq 3$ with constant sectional curvature $K = +1$. Any relativistic spherical space form M is represented as a Clifford-Klein form:

$$M \simeq \Gamma \backslash O(n,1)/O(n-1,1),$$

where Γ is a discrete subgroup of $O(n,1)$ which is isomorphic to the fundamental group $\pi_1(M)$. We shall see that Γ must be a finite group in §2 (the Calabi-Markus phenomenon [CM62]).

Example 0.5.5 (the uniformization theorem of Riemann surfaces; see Example 0.4.4, 0.4.5 and 0.4.6). The uniformization theorem of Riemann surfaces due to Klein, Poincaré and Koebe asserts that a simply connected Riemann surface M is biholomorphic to one of the following complex manifolds:

$$\mathbb{C}, \quad \mathbb{C}P^1, \quad \text{or} \ \mathcal{H}$$

We recall that the Riemann mapping theorem is the special case obtained by assuming M to be a domain of $\mathbb{C}$. We refer the reader to [Sp57], [AhSa60] for details.

The point here is that any of simply connected Riemann surfaces $\mathbb{C}$, $\mathbb{C}P^1$, and $\mathcal{H}$ has a transitive transformation group of biholomorphic automorphisms, as we have seen in Examples 0.4.4, 0.4.5 and 0.4.6, respectively. Consequently, any connected Riemann surface M is represented as a Clifford-Klein form

$$M \simeq \Gamma \backslash G/H,$$

where (G, H) is $(\mathbb{C}^\times \ltimes \mathbb{C}, \mathbb{C}^\times)$, $(PSL(2, \mathbb{C}), B)$ or $(PSL(2, \mathbb{R}), SO(2))$, and Γ is a discrete subgroup of G which is isomorphic to $\pi_1(M)$.

In particular, suppose M is compact. The rank of the first homology group $H_1(M)$ is always even, and it is denoted by $2g$. Then g is said to be the *genus* of the Riemann surface M, and such a Riemann surface is usually denoted by M_g. Then the universal covering of M_g is uniquely determined up to biholomorphic automorphism by the genus g. In fact, we have

$$\begin{aligned}
&M_g \simeq \Gamma\backslash PSL(2,\mathbb{C})/B, &&\Gamma = \pi_1(M_g) = \{e\}, &&(g = 0),\\
&M_g \simeq \Gamma\backslash \mathbb{C}^\times \ltimes \mathbb{C}/\mathbb{C}^\times, &&\Gamma = \pi_1(M_g) \simeq \mathbb{Z}^2, &&(g = 1),\\
&M_g \simeq \Gamma\backslash PSL(2,\mathbb{R})/SO(2), &&\Gamma = \pi_1(M_g), &&(g \geq 2).
\end{aligned}$$

§1. Discontinuous actions (discrete and continuous version)

Let G be a Lie group, H a closed subgroup and Γ a discrete subgroup of G. In §0, we have defined that a *Clifford-Klein form* of a homogeneous manifold G/H is the double coset space $\Gamma\backslash G/H$ if it is Hausdorff and carries naturally a C^∞ structure (see Question 0.2). This condition is satisfied if Γ acts properly discontinuously and freely on G/H (see Definition 1.3.1), as we shall see in Lemma 1.3.2. The purpose of this section is to understand properly discontinuous actions by exhibiting a number of typical bad features of the action of noncompact groups.

1.1. Clifford-Klein form of $PSL(2,\mathbb{R})/SO(2)$

First we recall Example 0.5.5. Let M_g be a closed Riemann surface with genus ≥ 2. The universal covering manifold of M_g is biholomorphic to the the Poincaré plane $\mathcal{H}$. So M_g is represented as a compact Clifford-Klein form $\Gamma\backslash PSL(2,\mathbb{R})/SO(2)$ of the Poincaré plane $PSL(2,\mathbb{R})/SO(2)$, where Γ is an infinite discrete subgroup of $PSL(2,\mathbb{R})$ isomorphic to $\pi_1(M_g)$.

There are two directions of generalization of this classical example, namely,

i) generalization to higher dimensions (e.g. Theorem 4.2).

ii) generalization to noncompact isotropy subgroups.

Here we are interested in the latter direction.

1.2. Clifford-Klein form of $PSL(2,\mathbb{R})/SO(1,1)$

There is *no* compact Clifford-Klein form of the hyperboloid of one sheet $PSL(2,\mathbb{R})/SO(1,1)$. This fact can be proved in various ways, which are of importance as typical examples of later arguments. We indicate here four different proofs, which we shall elaborate in the following sections.

i) A direct calculation (§1).

Assume that Γ is a discrete subgroup of $PSL(2,\mathbb{R})$ such that $\sharp\Gamma = \infty$. Then we can show that the action of Γ on $PSL(2,\mathbb{R})/SO(1,1)$ is never properly discontinuous, and the corresponding quotient space $\Gamma\backslash PSL(2,\mathbb{R})/SO(1,1)$ is not Hausdorff. We consider first the most non-trivial case where Γ consists of unipotent elements. For example, let us consider the case where Γ is a discrete subgroup of $PSL(2,\mathbb{R})$ generated by a single unipotent element. Then the quotient topology of $\Gamma\backslash PSL(2,\mathbb{R})/SO(1,1)$ is locally Hausdorff but non-Hausdorff, as we shall explain in this section (see Example 1.6.2 and Exercise 1.6.3 (iii), (v)). On the other hand, if Γ does not contain a nilpotent element (still we assume $\sharp\Gamma = \infty$), then the action of Γ has an accumulating point in $PSL(2,\mathbb{R})/SO(1,1)$ (see Definition 1.3.4) so that $\Gamma\backslash PSL(2,\mathbb{R})/SO(1,1)$ is not Hausdorff, too (see Example 1.6.1 (ii) and Exercise 1.6.3 (iv),(v)).

ii) Calabi-Markus phenomenon (§2).
We will give a necessary and sufficient condition in terms of the ranks of G and H for a homogeneous space G/H of reductive type to admit an infinite discontinuous group (see Theorem 2.5 for a criterion of the so-called Calabi-Markus phenomenon; see also Example 0.5.4). The idea of proof is to study an analogous problem in a continuous setting and to look at the infinite points by means of the Cartan decomposition.

iii) Hirzebruch's proportionality principle (§3).
A Clifford-Klein form M of $PSL(2,\mathbb{R})/SO(1,1)$, the hyperboloid of one sheet, carries an indefinite-Riemannian metric of type $(1,1)$ induced from the Killing form. This metric gives rise to a non-vanishing vector field on M. If M were compact and orientable, this would imply the vanishing of the Euler-Poincaré class of M thanks to a theorem of Poincaré-Hopf. On the other hand, the Euler-Poincaré class of a "compact real form" $S^2 \simeq SU(2)/SO(2)$ does not vanish. This leads to a contradiction by a generalized Hirzebruch's proportionality principle (see Corollary 3.8). The idea here leads to a necessary condition for the existence of compact Clifford-Klein forms (see Corollary 3.12.1 and Example 3.12.2).

iv) Semisimple orbits, invariant complex structure (§4).
The hyperboloid of one sheet is realized as a semisimple orbit of the adjoint representation of $PSL(2,\mathbb{R})$. We shall see in §4 that a semisimple orbit having a compact Clifford-Klein form must be isomorphic to an elliptic orbit, so that it carries an invariant complex structure (see Corollary 4.12). But this is not the case for the hyperboloid of one sheet.

1.3. Properly discontinuous action

A distinguished feature in the setting §1.2 is that the isotropy subgroup $H \simeq SO(1,1)$ is noncompact. As a consequence, the action of a discrete subgroup Γ on G/H is not automatically properly discontinuous. Here we recall the definition of properly discontinuous actions:

Suppose that a discrete group Γ acts continuously on a locally compact Hausdorff topological space X. For a subset S of X, we put

$$\Gamma_S := \{\gamma \in \Gamma : \gamma S \cap S \neq \emptyset\}.$$

Note that if S is a singleton $\{p\}$ ($p \in X$) then Γ_S is nothing but the isotropy subgroup at p. In general Γ_S is not a subgroup.

Definition 1.3.1. The action of Γ on X is said to be:
i) *properly discontinuous* if Γ_S is a finite subset for any compact subset S of X,
ii) *free* if $\Gamma_{\{p\}}$ is trivial for any $p \in X$.

Then we have the following standard fact:

Lemma 1.3.2. *Suppose that a discrete group Γ acts on a [C^∞, Riemannian, complex, ...] manifold X properly discontinuously and freely [and smoothly, isometrically, holomorphically, ...]. Equipped with the quotient topology, $\Gamma\backslash X$ is then a Hausdorff topological space, on which a manifold structure is uniquely defined so that*

$$\pi\colon X \to \Gamma\backslash X$$

is locally homeomorphic [diffeomorphic, isometric, biholomorphic, ...].

Although this lemma is well-known, we give a sketch of its proof in order that the readers get used to the definition of properly discontinuous actions.

Sketch of proof.
1) (Hausdorff) (We use only the assumption that the action is properly discontinuous.) Take $x, y \in X$ so that $\pi(x) \neq \pi(y)$. We want to find neighbourhoods V, W of X such that $x \in V$, $y \in W$ and $\pi(V) \cap \pi(W) = \emptyset$. First, we take relatively compact neighbourhoods V_1 and W_1 of X such that $x \in V_1$, $y \in W_1$. As the action of Γ on X is properly discontinuous, $\Gamma_{V_1 \cup W_1} = \{\gamma \in \Gamma : \gamma(V_1 \cup W_1) \cap (V_1 \cup W_1) \neq \emptyset\}$ is a finite set, say, $\{\gamma_1, \cdots, \gamma_k\}$. Second, we take neighbourhoods V and W such that $x \in V \subset V_1$, $y \in W \subset W_1$ and $\gamma_j V \cap W = \emptyset$ $(j = 1, \cdots, k)$. Then we have $\Gamma V \cap \Gamma W = \emptyset$, that is, $\pi(V) \cap \pi(W) = \emptyset$, which we wanted to prove.
2) (manifold structure) The proof is quite similar as that of (1). In fact, by using the assumption that the action is properly discontinuous and free, we can find a neighbourhood $V \subset X$ at each point $x \in X$ such that

$$\{\gamma \in \Gamma : \gamma V \cap V \neq \emptyset\} = \{e\}.$$

Such an open set V is homeomorphic to $\pi(V) \subset \Gamma\backslash X$, and these sets form a basis for the open sets giving local charts of $\Gamma\backslash X$. $\square$

Here is a necessary condition that the action of Γ on X is properly discontinuous.

Lemma 1.3.3. *Suppose a discrete group Γ acts properly discontinuously on a locally compact, Hausdorff topological space X. Then we have:*

1) *There is no accumulating point of the action of* Γ *on* X.

2) *Each* Γ*-orbit is a closed subset of* X.

Here we recall:

Definition 1.3.4. For each element x of X, the Γ-orbit through x is a subset of X given by

$$\Gamma \cdot x := \{\gamma x : \gamma \in \Gamma\}.$$

We say $y\,(\in X)$ is *an accumulating point of the* Γ*-orbit* $\Gamma \cdot x$ if $\sharp(U \cap \Gamma \cdot x) = \infty$ for any neighbourhood U of y in X. We say $y\,(\in X)$ is *an accumulating point of the action of* Γ *on* X, if there exists $x \in X$ such that y is an accumulating point of the orbit $\Gamma \cdot x$.

Proof of Lemma 1.3.3.

1) We want to show that the action of Γ on X is not properly discontinuous if there exists an accumulating point of the action of Γ on X. Suppose that $y \in X$ is an accumulating point of the action of Γ on X. That is, there exists $x \in X$ such that $\sharp(U \cap \Gamma \cdot x) = \infty$ for any neighbourhood U of y. We choose U to be relatively compact and define $S := U \cup \{x\}$ and $\Gamma' := \{\gamma \in \Gamma : \gamma x \in U\}$. Then we have $\sharp\Gamma' = \infty$ because $\sharp(U \cap \Gamma \cdot x) = \infty$. Then we have $\sharp\Gamma_S = \infty$ because

$$\Gamma_S = \{\gamma \in \Gamma : \gamma S \cap S \neq \emptyset\} \supset \Gamma'.$$

Hence the action of Γ on X is not properly discontinuous.

2) We want to show that the action of Γ on X is not properly discontinuous if there exists a non-closed orbit $\Gamma \cdot x$. Take $y \in \overline{\Gamma \cdot x} \setminus \Gamma \cdot x$ and relatively compact, open neighbourhoods U_j $(j = 1, 2, \cdots)$ of y such that $U_1 \supset U_2 \supset \cdots$ and that $\cap_j U_j = \{y\}$. Then we can take $\gamma_j \in \Gamma$ such that $\gamma_j \cdot x \in U_j$ for each $j \in \mathbb{N}$ because $\Gamma \cdot x \cap U_j \neq \emptyset$. Then we have $\sharp\{\gamma_j : j = 1, 2, \cdots\} = \infty$ because $\cap_j U_j = \{y\}$. Finally, we put $S := U_1 \cup \{x\}$, and we have $\sharp\Gamma_S = \infty$ because

$$\Gamma_S \supset \cup_j \{\gamma \in \Gamma : \gamma \cdot x \in U_j\} \supset \{\gamma_j : j = 1, 2, \cdots\}.$$

Hence the action of Γ on X is not properly discontinuous. □

1.4. Discontinuous groups acting on homogeneous manifolds

Definition 1.4. A discrete subgroup Γ of G is said to be *a discontinuous group acting on* G/H if the action of Γ on G/H is properly discontinuous.

If Γ is a discontinuous group acting on G/H and if the action of Γ on G/H is free, then the double coset space $\Gamma\backslash G/H$ carries a natural C^∞-manifold structure from Lemma 1.3.2 so that $\Gamma\backslash G/H$ is a Clifford-Klein form of the homogeneous manifold G/H. Thus we have an affirmative answer to Question 0.2 in this case.

Here, the point is that Γ is assumed to be a subgroup of G so that $\Gamma\backslash G/H$ inherits any G-invariant (local) structure on the homogeneous manifold G/H. One should keep in mind the essential difference between the action on G/H of a discrete group $\Gamma \subset G$ (our case) and a discrete group $\Gamma \subset \mathrm{Diffeo}(G/H)$. For example, if $G/H = SL(2,\mathbb{R})/SO(1,1)$ (a hyperboloid of one sheet), then G/H is diffeomorphic to $S^1 \times \mathbb{R}$ (a cylinder), which admits a properly discontinuous action of $\mathbb{Z}\,(\subset \mathrm{Diffeo}(G/H))$ along the direction of $\mathbb{R}$. But this action does not come from $SL(2,\mathbb{R})$ and is not isometric with respect to a natural indefinite Riemannian metric on the hyperboloid.

1.5. Remark

In this lecture, proper discontinuity is essentially important, but freeness is less important. In fact, if a discrete group Γ acts on a manifold X properly discontinuously, then the isotropy group $\Gamma_{\{p\}}$ at $p \in X$ is not necessarily trivial indeed but it is always finite. Correspondingly, $\Gamma\backslash X$ is not necessarily a smooth manifold but still has a nice structure called *V-manifold* in the sense of Satake [Sa56] or called an *orbifold* in the sense of Thurston (see also [Car60]). Moreover, if Γ acts properly discontinuously on X and if $\Gamma' \subset \Gamma$ is a torsion free subgroup (i.e. $x \in \Gamma'$, $x^n = e\,(n \geq 1) \Rightarrow x = e$), then the action of Γ' on X is properly discontinuous and free. In view of this, the following result due to Selberg is quite useful.

Theorem 1.5.1 ([Sel60] Lemma 8)**.** *A finitely generated matrix group has a torsion free subgroup of finite index.*

1.6. Examples and Exercises

Example 1.6.1. Suppose a discrete group $\Gamma := \mathbb{Z}$ acts on a manifold $X := \mathbb{R}$ in the following two different manners:

i) $\Gamma \times X \to X, \quad (n, x) \mapsto x + n.$

ii) $\Gamma \times X \to X, \quad (n, x) \mapsto 2^n x.$

The action in (i) is properly discontinuous and free. The resulting quotient manifold $\Gamma\backslash X$ is diffeomorphic to S^1.

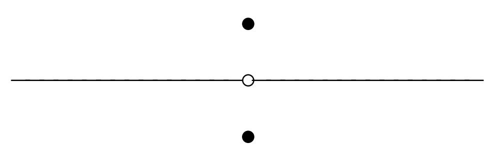
On the other hand, the action in (ii) is *not* properly discontinuous because $\Gamma_S = \mathbb{Z}$ is not a finite set if we take $S := [0,1]$, the unit interval in $\mathbb{R}$. The resulting quotient space $\Gamma\backslash X$ has a non-Hausdorff topology. That is, $\Gamma\backslash X$ is homeomorphic to $S^1 \cup \{\text{point}\} \cup S^1$ which is topologized to be connected !

In Example (1.6.1)(ii), we easily see that the action of Γ is not properly discontinuous because the origin 0 is an accumulation point of the action of Γ. The next example is more subtle without accumulation points.

Example 1.6.2. Suppose a discrete group $\Gamma := \mathbb{Z}$ acts on a manifold $X := \mathbb{R}^2 \setminus \{0\}$ in the following manner:

$$\Gamma \times X \to X, \quad (n, (x,y)) \mapsto (2^n x, 2^{-n} y).$$

Then this action is *not* properly discontinuous. In fact, let $B_\varepsilon(x,y)$ be a ball of radius ε with the center (x,y) and we put $S := \overline{B_{\frac{1}{2}}(1,0)} \cup \overline{B_{\frac{1}{2}}(0,1)}$. Then it is an easy exercise to see that $\Gamma_S \equiv \{\gamma \in \mathbb{Z} : \gamma S \cap S \neq \emptyset\}$ is equal to $\mathbb{Z}$. We note that there is no accumulation point of Γ. In fact every Γ-orbit is closed in X. The resulting quotient topology of $\Gamma\backslash X$ is not Hausdorff, though it is *locally Hausdorff* in the sense that one can find a Hausdorff neighbourhood of each point of $\Gamma\backslash X$. A picture of a similar topology as that of $\Gamma\backslash X$ is illustrated by the following one dimensional example:



Exercise 1.6.3. In the setting as in Example 1.6.2, prove the following:

i) $\Gamma_S = \mathbb{Z}$ if $S = \overline{B_{\frac{1}{2}}(1,0)} \cup \overline{B_{\frac{1}{2}}(0,1)}$.

ii) Let $G = SL(2,\mathbb{R})$ act naturally on $\mathbb{R}^2 \setminus \{0\}$ from the left. This action is transitive and the isotropy group at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$ so that we have a diffeomorphism:

$$G/H \simeq \mathbb{R}^2 \setminus \{0\}.$$

Let $L := \left\{ \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} : y > 0 \right\}$, $L_{\mathbb{Z}} := \left\{ \begin{pmatrix} 2^x & 0 \\ 0 & 2^{-x} \end{pmatrix} : x \in \mathbb{Z} \right\} \simeq \mathbb{Z}$.

Via the isomorphisms $G/H \simeq \mathbb{R}^2 \setminus \{0\}$ and $L_{\mathbb{Z}} \simeq \Gamma \, (= \mathbb{Z})$, the action of $L_{\mathbb{Z}}$ on G/H coincides with that of Γ on $\mathbb{R}^2 \setminus \{0\}$ in Example 1.6.2.

Thus the action of $L_{\mathbb{Z}}$ on G/H is not properly discontinuous, and the quotient topology on $L_{\mathbb{Z}}\backslash G/H$ is not Hausdorff.

iii) Let $H_{\mathbb{Z}} := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z} \right\}$. Then the action of $H_{\mathbb{Z}}$ on G/L is not properly discontinuous, and the quotient topology on $H_{\mathbb{Z}}\backslash G/L$ is not Hausdorff.

iv) The action of $L_{\mathbb{Z}}$ on G/L has an accumulating point and is not properly discontinuous, too.

v) We define a quadratic form on $\mathbb{R}^2$ by $Q\begin{pmatrix} x \\ y \end{pmatrix} := xy$. The polarization of Q gives an indefinite metric of signature $(1,1)$. Then L is the identity component of

$$\left\{ g \in GL(2,\mathbb{R}) : Q(g\begin{pmatrix} x \\ y \end{pmatrix}) = Q\begin{pmatrix} x \\ y \end{pmatrix} \ \text{ for any } x, y \in \mathbb{R} \right\} \simeq O(1,1).$$

Thus $G/L \simeq SL(2,\mathbb{R})/SO_0(1,1)$.

Hint: The topology of $H_{\mathbb{Z}}\backslash G/L$ has a similar feature of Example 1.6.2, while that of $L_{\mathbb{Z}}\backslash G/L$ is similar to Example 1.6.1(ii). The actions in (ii) and in (iii) are in a kind of duality, between the action of L on G/H and that of H on G/L. That is, in the setting

$$L_{\mathbb{Z}} \subset L \subset G \supset H \supset H_{\mathbb{Z}}$$

with $L/L_{\mathbb{Z}}$ and $H/H_{\mathbb{Z}}$ compact, the action of $L_{\mathbb{Z}}$ on G/H is properly discontinuous if and only if that of $H_{\mathbb{Z}}$ on G/L is properly discontinuous (see Lemma 1.11.3(1), see also Observation 1.9, [Bou60] Chapitre 3).

1.7. Basic problems in a discrete setting

Suppose G is a Lie group and H is a closed subgroup. Here are the basic problems in the theory of discontinuous groups acting on a homogeneous manifold G/H:

Problem 1.7.

1) *Which homogeneous manifold G/H admits an infinite discontinuous group ?*
2) *Which homogeneous manifold G/H admits a compact Clifford-Klein form ?*

(1) should be a first step in the study of discontinuous groups acting on homogeneous manifolds. The existence of a compact Clifford-Klein form

in (2) would be interesting not only from the view point of geometry but also from that of harmonic analysis and representation theory (see Open problems 10 in §5). We will study (1) in §2, and (2) in §4 (partly also in §2 and §3).

1.8. Proper actions — as a continuous analogue of properly discontinuous actions

In general, the study of a discrete group is quite difficult. Our approach is to approximate the action of discrete groups by that of connected Lie groups. For this purpose, it is crucial to find a continuous analogue of a properly discontinuous action:

Definition 1.8 (see [Pa61])**.** Suppose that a locally compact topological group L acts continuously on a locally compact topological space X. For a subset S of X, we define a subset of L by

$$L_S = \{\gamma \in L : \gamma S \cap S \neq \emptyset\}\,.$$

The action of L on X is said to be *proper* if and only if L_S is compact for every compact subset S of X.

Compared with the definition of proper discontinuity (Definition 1.3), compactness in Definition 1.8 has now replaced by finiteness. We note that the action of L on X is properly discontinuous if and only if the action of L on X is proper and L is discrete, because a discrete and compact set is finite.

1.9. Observation

The following elementary observation is a bridge between the action of a discrete group and that of a connected group.

Observation 1.9 ([Ko89a], Lemma 2.3)**.** *Suppose a Lie group L acts on a locally compact space X. Let Γ be a cocompact discrete subgroup of L. Then*
1) *The L-action on X is proper if and only if the Γ-action is properly discontinuous.*
2) *$L\backslash X$ is compact if and only if $\Gamma\backslash X$ is compact.*

Proof. 1) Suppose Γ acts properly discontinuously. Take a compact subset C in L so that $L = C \cdot \Gamma$ and $C = C^{-1}$. Then for any compact subset S in X, $L_S = \{g \in L : g \cdot S \cap S \neq \emptyset\} \subset C \cdot \Gamma_{CS}$. Because Γ_{CS} is a finite set from the assumption, we conclude that the action of L is proper. In view

of $\Gamma_S = \Gamma \cap L_S$, the 'only if' part follows immediately from the definition.
2) Suppose $L\backslash X$ is compact. We take an open covering $X = \bigcup_\alpha U_\alpha$ by relatively compact sets U_α. Then $X = \bigcup_\alpha LU_\alpha$ gives an open covering of $L\backslash X$. By the compactness of $L\backslash X$, we can choose finitely many U_j's among $\{U_\alpha\}$ so that $X = \bigcup_j L \cdot U_j = \bigcup_j (\Gamma \cdot C) \cdot U_j = \bigcup_j \Gamma \cdot (C \cdot U_j)$, showing $\Gamma\backslash X$ is compact. The converse statement is clear. $\square$

1.10. Problems in a continuous analogue

In view of Observation 1.9, we pose the following analogous problems in a continuous setting.

Problem 1.10. *Let G be a Lie group and H and L closed subgroups.*
1) *Find the criterion on the triplet (L, G, H) such that the action of L on G/H is proper ?*
2) *Find the criterion on the triplet (L, G, H) such that the double coset $L\backslash G/H$ is compact in the quotient topology ?*

We will give a complete answer to Problem (1.10) in terms of Lie algebras in the following cases:
i) Problem 1.10(1) when G is reductive (see §2)
ii) Problem 1.10(2) when the groups G, H, L are real reductive (see §4).

In preparation for the next lecture we introduce in the rest of this lecture some notations which are useful for a further study of Problem 1.10(1).

1.11. Relations $\sim$ and $\pitchfork$

Suppose that H and L are subsets of a locally compact topological group G.

Definition 1.11.1 ([Ko94b] **Definition 2.1.1).** We denote by $H \sim L$ in G if there exists a compact set S of G such that $L \subset SHS^{-1}$ and $H \subset SLS^{-1}$. Here $SHS^{-1} := \{ahb^{-1} \in G : a, b \in S, h \in H\}$. Then the relation $H \sim L$ in G defines an equivalence relation.

We say the pair (H, L) is *proper* in G, denoted by $H \pitchfork L$ in G, iff $SHS^{-1} \cap L$ is relatively compact for any compact set S in G.

The above definition is motivated by the following:

Observation 1.11.2. *Let H and L be closed subgroups of G, and Γ a discrete subgroup of G.*

1) *The action of L on the homogeneous manifold G/H is proper if and only if $H \pitchfork L$ in G.*
2) *The action of Γ on the homogeneous manifold G/H is properly discontinuous if and only if $H \pitchfork \Gamma$ in G.*

Here are some elementary properties of the relations $\sim$, $\pitchfork$:

Lemma 1.11.3 ([Ko94b])**.** *Suppose G is a locally compact topological group and that H, H' and L are subsets of G.*

1) *$H \pitchfork L$ if and only if $L \pitchfork H$.*
2) *If $H \sim H'$ and if $H \pitchfork L$ in G, then $H' \pitchfork L$ in G.*

Now we are ready to give a reformulation of Problem 1.10 (1) as:

Problem 1.11.4 (a reformulation of Problem 1.10 (1)). *Let G be a Lie group and H and L subsets of G. Find the criterion on the pair (L, H) (or on the pair of their equivalence classes with respect to $\sim$) such that $L \pitchfork H$ in G.*

1.12. Property (CI)

If a discrete group Γ acts on X properly discontinuously, then every isotropy subgroup is finite and every Γ orbit is closed (Lemma 1.3.3). The latter condition corresponds to the fact that each point is closed in the quotient topology of $\Gamma\backslash X$. In general, the converse implication does not hold (cf. Example 1.6.2).

We have a similar picture in a continuous setting. In fact, let H, L be closed subgroups of a locally compact topological group G. If L acts properly on G/H, then any L-orbit $LgH \simeq L/L \cap gHg^{-1} \subset G/H$ is a closed subset and each isotropy subgroup $L \cap gHg^{-1}$ is compact. In general, the converse implication is not true. However, we focus our attention on the latter property, that is, each isotropy subgroup is compact.

Definition 1.12.1 ([Ko90a,92a])**.** Suppose that H and L are subsets of a locally compact topological group G. We say that the pair (L, H) has *the property* (CI) in G if and only if $L \cap gHg^{-1}$ is compact for any $g \in G$.

Here (CI) stands for that the action of L has a *compact isotropy subgroup* $L \cap gHg^{-1}$ at each point $gH \in G/H$, or stands for that L and gHg^{-1} has a *compact intersection* $(g \in G)$ (see also [Lip94]).

If $H \pitchfork L$ in G, then the pair (L, H) has the property (CI) in G. The point is to understand how and to what extent the property (CI) implies the proper action.

Problem 1.12.2 ([Ko90a,92a] Open Problem 2). *For which Lie groups, does the following equivalence (1.3) hold ?*

(1.12.3) $\quad H \pitchfork L$ *in* $G \Leftrightarrow$ *the pair* (L, H) *has the property (CI) in* G.

In the next section, we shall see that the equivalence (1.12.3) holds if G, H, L are real reductive algebraic groups (see Theorem (2.9.1)). Recently, R. Lipsman [Lip94] pointed out that it is likely that the equivalence (1.12.3) holds as well if G is a simply connected nilpotent group. There are some further cases, especially in the context of a continuous analogue of the Auslander conjecture (see Remark 0.5.3), where the equivalence (1.12.3) is known to hold. See Example 5 and Proposition (A.2.1) in [Ko90a,92a]; Theorem (3.1) and Theorem (5.4) in [Lip94]. However, we should note that the equivalence (1.12.3) does not always hold. For instance, if $G = KAN$ is an Iwasawa decomposition of a real reductive group G and if we put $L := A$ and $H := N$, then (L, H) has the property (CI) in G, while $L \not\pitchfork H$ in G as we saw in Exercise 1.6.3 when $G = SL(2, \mathbb{R})$.

§2. Calabi-Markus phenomenon

2.1. Calabi-Markus phenomenon

Let G be a Lie group and H a closed subgroup. In the second lecture, we focus our attention on the first basic problem:

Problem 2.1 (see Problem (1.7)(1)). *Which homogeneous manifold G/H admits an infinite discontinuous group ?*

We first note that if M is a compact manifold then there is no infinite discrete group ($\subset$ Diffeo(M)) acting on M properly discontinuously. So, we might expect that the answer to Problem 2.1 should be related to certain aspect of *compactness* of a homogeneous manifold G/H with respect to the transformation group $G \subset$ Diffeo(M). One observes that a criterion given in the reductive case (Theorem 2.5) has such an aspect.

Our main interest here is the reductive case. However, we include a quick review of some other typical cases as well.

i) relativistic spherical space form (§2.2),
ii) G/H with H compact (§2.3),
iii) G/H with G solvable (§2.4),
iv) G/H with G reductive (§2.5-).

2.2. Relativistic spherical space form

In 1962, Calabi and Markus discovered a surprising phenomenon on the fundamental group π_1 of a Lorentz manifold with constant curvature:

Theorem 2.2.1 ([CM62]). *Every relativistic spherical space form is non-compact and has a finite fundamental group π_1 (see Example 0.5.4 for definition).*

As we saw in Example 0.5.4, the Calabi-Markus theorem is reformulated in a group language as:

Theorem 2.2.2. *If $n \geq 3$, then there does not exist a discrete subgroup Γ of $O(n,1)$ acting on $O(n,1)/O(n-1,1)$ properly discontinuously and freely such that the fundamental group $\pi_1(\Gamma\backslash G/H)$ is infinite.*

We note that if $n = 2$, then G/H is diffeomorphic to a cylinder and has the fundamental group $\simeq \mathbb{Z}$. Theorem 2.2.2 is also reformulated as:

Theorem 2.2.3. *Any discontinuous group acting (see Definition 1.4) on a homogeneous manifold $SO(n,1)/SO(n-1,1)$ $(n \geq 2)$ is finite.*

This result will be generalized in §2.5 for homogeneous spaces of reductive type (Definition 2.6.2).

Setting the main subject aside for a moment, we mention a related problem in non-Riemannian geometry based on the following observation:

Observation 2.2.4.
i) *Any complete Riemannian manifold M^n ($n \geq 2$) with constant sectional curvature $+1$ is compact with a finite fundamental group.*
ii) *Any complete Lorentz manifold M^n ($n \geq 3$) with constant sectional curvature $+1$ is non-compact with finite fundamental group.*

We recall again in a group language that $M \simeq \Gamma \backslash O(n+1)/O(n)$ in the first case, while $M \simeq \Gamma \backslash O(n,1)/O(n-1,1)$ in the second case. A classical theorem due to Myers may be interpreted as a "perturbation" of Riemannian metric in the first statement:

Theorem 2.2.5 ([My41]). *If the Ricci curvature of a complete Riemannian manifold M satisfies $K(U,U) \geq c > 0$ for all unit vectors, then M is compact with finite fundamental group.*

The author does not know a result concerning a "perturbation" of the Lorentz metric in Observation 2.2.4 (ii). So we pay an attention on the following problem in pseudo-Riemannian geometry:

Problem 2.2.6. *Suppose M is a complete Lorentz manifold (or more generally a complete manifold equipped with an indefinite Riemannian metric).*

Find a sufficient condition in terms of a local property of M which assures that M is noncompact with a finite fundamental group? In particular, is there a sufficient condition given by some positiveness of the curvature of M?

2.3. H is compact

Suppose that H is compact. Then it follows immediately from the definition of properly discontinuous actions that

$$\begin{aligned} &\text{The action of } \Gamma \text{ on } G/H \text{ is properly discontinuous} \\ \Leftrightarrow\ &\Gamma \text{ is a discrete subgroup of } G. \end{aligned}$$

Moreover, if Γ is a torsion free discrete subgroup of G, then the action of Γ on G/H is properly discontinuous and free whenever H is compact. Therefore we have the following:

Proposition 2.3. *If G is a non-compact connected Lie group and H is a compact subgroup, then there always exists an infinite discrete subgroup Γ of G acting on G/H properly discontinuously and freely on G/H.*

Proof. It suffices to show that any noncompact Lie group G contains an infinite torsion free discrete subgroup.

Assume first that G is a noncompact semisimple Lie group, we write $G = KAN$ for an Iwasawa decomposition. We note that G is noncompact if and only if $\mathbb{R}\text{-rank}\, G = \dim A > 0$. So we can take a lattice of A, which is isomorphic to $\mathbb{Z}^{\dim A}$, and is an infinite and torsion free discrete subgroup of G.

Assume second that G is a noncompact solvable Lie group. Let $\widetilde{G}$ be the universal covering group of G, and Z the kernel of the covering map $\widetilde{G} \to G$. It follows from Theorem 1 and Remark of [Ch41] that we find $(0 \le)\ r \le n = \dim G$ and a basis $X_1, \dots, X_n$ of the Lie algebra of G which has the following properties:

i) $\mathbb{R}^n \to \widetilde{G}, \quad (t_1, \dots, t_n) \mapsto \exp(t_1 X_1) \dots \exp(t_n X_n)$ is a surjective homeomorphism.

ii) Z is isomorphic to a free abelian group generated by $\exp X_1, \dots, \exp X_r$.

We note that G is noncompact if and only if $r < n$. Therefore, if G is noncompact, we put $\Gamma := \{\exp(nX_{r+1}) \in G : n \in \mathbb{Z}\}$, which is an infinite and torsion free discrete subgroup of G.

Now, we suppose G is a noncompact connected Lie group. Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{s}$ is a Levi decomposition, where $\mathfrak{s}$ is the radical of $\mathfrak{g}$ and $\mathfrak{l}$ is a semisimple Lie algebra. We write L, S for the analytic subgroups of G with Lie algebra $\mathfrak{l}, \mathfrak{s}$ respectively. Then L and S are closed subgroups of G. If G is noncompact, then L or S must be noncompact. In either case, we can find an infinite and torsion free discrete subgroup of G. □

2.4. Solvable groups

In the case of simply connected solvable homogeneous spaces, it turns out that the Calabi-Markus phenomenon does not occur. The following result is proved based on a structural result of a simply connected solvable groups due to Chevalley [Ch41].

Theorem 2.4 ([Ko93] Theorem 2.2; see also [Lip94])**.** *Suppose G is a solvable Lie group and H is a proper closed subgroup of G. Then there exists a discrete subgroup Γ of G acting on G/H properly discontinuously and freely such that the fundamental group $\pi_1(\Gamma\backslash G/H)$ is infinite.*

2.5. Reductive cases

The Calabi-Markus phenomenon in the reductive case has been studied by Calabi, Markus, Wolf, Kulkarni and Kobayashi, and it has been settled completely in terms of a rank condition:

Theorem 2.5 ([CM62; Wo62; Wo84; Ku81; Ko89]). *Let G/H be a homogeneous space of reductive type. The following statements are equivalent:*
1) *Any discontinuous group acting on G/H is finite.*
2) $\mathbb{R}$-rank $G = \mathbb{R}$-rank H.

We will explain some terminology used here in the following section §2.6, and then give some examples and a sketch of the proof in §2.7 and 2.8.

2.6. Homogeneous spaces of reductive type

We set up some notation of real reductive groups. Good references for what we need are [He78] Chapter VI and [War72] Chapter I. Let G be a connected real linear reductive Lie group.

Fix a maximal compact subgroup K in G. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition of the Lie algebra of G. Then the homogeneous space G/K equipped with a G-invariant Riemannian metric is said to be a *Riemannian symmetric space*. We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Then $\mathfrak{a}$ is said to be *a maximally split abelian subspace for* G. If we want to emphasize the group G, we write $\mathfrak{a}_G$ for $\mathfrak{a}$. We write A for the corresponding connected Lie subgroup. We define:

$$\begin{aligned}
\Sigma(\mathfrak{g}, \mathfrak{a}) &: \text{ the restricted root system,} \\
W_G &: \text{ the Weyl group associated to } \Sigma(\mathfrak{g}, \mathfrak{a}), \\
\operatorname{rank} G &:= \text{ the dimension of any maximal semisimple abelian subspace of } \mathfrak{g}, \\
c\text{-rank}\, G &:= \operatorname{rank} K, \\
\mathbb{R}\text{-rank}\, G &:= \dim \mathfrak{a} = \operatorname{rank} G/K, \\
\mathrm{d}(G) &:= \dim \mathfrak{p} = \dim G/K.
\end{aligned}$$

We note that

$$c\text{-rank}\, G \ \le\ \operatorname{rank} G \ge\ \mathbb{R}\text{-rank}\, G \ \le\ \mathrm{d}(G).$$

Example 2.6.1. Let $G = SO_0(p,q)\ (p \ge q)$ be the identity component of the indefinite orthogonal group of signature (p,q). Then $K \simeq SO(p) \times SO(q)$, $A \simeq \mathbb{R}^q$, c-rank $G = [\frac{p}{2}] + [\frac{q}{2}]$, rank $G = [\frac{p+q}{2}]$, $\mathbb{R}$-rank $G = q$, $\mathrm{d}(G) = pq$, and $W_G \simeq \mathfrak{S}_q \ltimes (\mathbb{Z}/2\mathbb{Z})^q$. Here $\mathfrak{S}_q$ denotes the q-th symmetric group.

Definition 2.6.2. Suppose that H is a closed subgroup in G with at most finitely many connected components. If there exists a Cartan involution of G which stabilizes H, then H is said to be *reductive in* G and G/H is said to be *a homogeneous space of reductive type.*

If G/H is a homogeneous space of reductive type, then G and H have a realization in $GL(n, \mathbb{R})$ such that $H \subset G \subset GL(n, \mathbb{R})$ are closed subgroups and that $H = {}^tH$ and $G = {}^tG$. Here ${}^tG := \{{}^tg : g \in G\}$, the transposed set of G in $GL(n, \mathbb{R})$.

Example 2.6.3. Suppose G is a real reductive linear group.

i) If H is compact, then G/H is a homogeneous space of reductive type.

ii) ([Yo38]) If H is semisimple, then G/H is a homogeneous space of reductive type.

iii) Let σ be an involutive automorphism of G. If H is an open subgroup of $G^\sigma := \{g \in G : \sigma g = g\}$, then G/H is a homogeneous space of reductive type. The homogeneous space G/H is said to be a *reductive symmetric space.* If G is semisimple, G/H is said to be a *semisimple symmetric space.*

iv) If $X \in \mathfrak{g}$ is a semisimple element, namely, $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ is semisimple, then the semisimple orbit $G/Z_G(X) \simeq \mathrm{Ad}(G) \cdot X\ (\subset \mathfrak{g})$ is a homogeneous space of reductive type, where $Z_G(X) := \{g \in G : \mathrm{Ad}(g)X = X\}$.

v) ([Mos55]) If $G_0 \supset G_1 \supset \cdots \supset G_n$ and if G_{i-1}/G_i $(1 \le i \le n)$ are all homogeneous spaces of reductive type, then so is G_0/G_n.

Suppose G/H is a homogeneous space of reductive type. There is a non-degenerate $\mathrm{Ad}(H)$-invariant bilinear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. (Actually, there exists B which is $\mathrm{Ad}(G)$-invariant.) Because $\mathfrak{h}$ is θ-stable, the restriction of B to $\mathfrak{h}$ is also non-degenerate. Therefore B induces a non-degenerate $\mathrm{Ad}(H)$-invariant bilinear form $\mathfrak{g}/\mathfrak{h} \simeq T_o(G/H)$, the tangent space of G/H at $o = eH \in G/H$. It induces a G-invariant (indefinite)-Riemannian metric on G/H by the left translation. The signature of this metric is $(\dim(\mathfrak{p}/\mathfrak{p} \cap \mathfrak{h}), \dim(\mathfrak{k}/\mathfrak{k} \cap \mathfrak{h}))$.

In analogy with the polar coordinate in the Euclidean space $\mathbb{R}^n$, there is a polar coordinate in a Riemannian symmetric space G/K. In a group language, this means that a real reductive linear Lie group G has a Cartan decomposition

$$G = K\,A\,K.$$

In this decomposition, there is an element $a(g) \in A$, unique up to conjugation by W_G such that $g \in Ka(g)K$ for each $g \in G$.

Definition 2.6.4 ([Ko89a], [Ko94b]). For each subset L of G, we define:

$$\begin{aligned} A(L) &:= A \cap KLK = \{w \cdot a(g) : w \in W_G, g \in L\} \subset A, \\ \mathfrak{a}(L) &:= \log A(L) \qquad\qquad\qquad\qquad\qquad\quad \subset \mathfrak{a}. \end{aligned}$$

Here $\log\colon A \to \mathfrak{a}$ is the inverse of the diffeomorphism $\exp\colon \mathfrak{a} \to A$.

If L is a subgroup of G which is reductive in G, then we can take a maximal compact subgroup K of G such that $L \cap K$ is also a maximal compact subgroup of a reductive Lie group L. Let $\mathfrak{a}_L$ be a maximally split abelian subspace for L. Then there exists an element g of G such that $\operatorname{Ad}(g)\mathfrak{a}_L \subset \mathfrak{a}_G$. Then $\mathfrak{a}(L)$ is a finite union of subspaces in $\mathfrak{a}_G$:

$$\mathfrak{a}(L) = W_G \cdot \operatorname{Ad}(g)\mathfrak{a}_L \ \subset \mathfrak{a}_G.$$

In this case, the notation $\mathfrak{a}(L)$ coincides with that in [Ko89a] as a subset of $\mathfrak{a}_G/W_G$.

Remark 2.6.5. Several remarks are in order.

i) If G/H is a homogeneous space of reductive type, then both G and H are real reductive Lie groups. Note that the converse statement is not always true.

ii) We avoid the terminology *reductive homogeneous space* which is usually used in the following sense: the Lie algebra $\mathfrak{g}$ may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{h}$ and a H-stable subspace $\mathfrak{m}$ (see for example, [KoN69], Chapter X §2). This notion is wider than that of homogeneous spaces of reductive type in Definition (2.6.2). In particular, neither G nor H itself is required to be reductive in this usual definition of a reductive homogeneous space.

2.7. Examples of the Calabi-Markus phenomenon

We present here some examples of Theorem 2.5.

Example 2.7.1. Let $G/H = SO(p+1,q)/SO(p,q)$. Then $\mathbb{R}\text{-rank}\, G = \min(p+1,q)$ and $\mathbb{R}\text{-rank}\, H = \min(p,q)$. Therefore, we have

$$\begin{aligned} &\text{there is no infinite discontinuous group acting on } G/H \\ \Leftrightarrow\quad & \mathbb{R}\text{-rank}\, G = \mathbb{R}\text{-rank}\, H \\ \Leftrightarrow\quad & q \le p. \end{aligned}$$

The case $q = 1$ is the original result of Calabi-Markus (see Theorem 2.2.3).

Example 2.7.2.
i) Any para-Hermitian symmetric space does not admit an infinite discontinuous group (see [Lib52] for the definition of para-complex structure; see also [KaKo85] for the definition and a classification of irreducible para-Hermitian symmetric spaces).
ii) Any hyperbolic orbit does not admit an infinite discontinuous group. Here, we say a semisimple orbit $G/Z_G(X) \simeq \mathrm{Ad}(G)X$ (see Example 2.6.3 (iv)) is a *hyperbolic orbit* if $X \in \mathfrak{g}$ is a hyperbolic element, that is, if all eigenvalues of $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ are real. We note that a para-Hermitian symmetric space is a reductive symmetric space that can be realized as a hyperbolic orbit.

Example 2.7.3. There does not exist an infinite discontinuous group acting on the following homogeneous manifolds;

reductive symmetric spaces

$$GL(n,\mathbb{C})/GL(n,\mathbb{R}), \quad SO(2n+1,\mathbb{C})/SO(2n,\mathbb{C}), \quad U(m,n)/O(m,n),$$

para-Hermitian symmetric spaces

$$Sp(n,\mathbb{R})/GL(n,\mathbb{R}), \quad SO^*(4n)/SU^*(2n)\times\mathbb{R},$$

hyperbolic orbits

$$GL(n_1+\cdots+n_k,\mathbb{R})/GL(n_1,\mathbb{R})\times\cdots\times GL(n_k,\mathbb{R}).$$

On the other hand, there exists a discontinuous group which is isomorphic to $\mathbb{Z}^n$ acting on the following homogeneous manifolds;

$$Sp(2n,\mathbb{R})/U(n,n),\ GL(2n,\mathbb{R})/GL(n,\mathbb{C}),\ O(2m,2n)/U(m,n)\,(n\le m).$$

2.8. Sketch of proof of the criterion for the Calabi-Markus phenomenon

First we explain a continuous analogue, that is, Problem 1.10 (1) for the criterion for a proper action in a general reductive setting (Theorem 2.9.1). The criterion for the Calabi-Markus phenomenon (Theorem 2.5) is obtained as a very special case of the continuous analogue, combined with Observation 1.9.

2.9. Criterion of proper actions in a continuous setting

Theorem 2.9.1 ([Ko89a]). *Let G/H, G/L be homogeneous spaces of reductive type. Then the following four conditions are equivalent:*
1) *L acts on G/H properly.*

1)′ *H acts on G/L properly.*

1)″ *$H \pitchfork L$ in G.*

2) *The triplet (L, G, H) has the property (CI). That is, for any $g \in G$, $L \cap gHg^{-1}$ is compact.*

2)′ $\mathfrak{a}(L) \cap \mathfrak{a}(H) = \{0\}$.

See §1.11 for the definition of $\pitchfork$; §1.12 for the property (CI); and Definition 2.6.4 for $\mathfrak{a}(L)$.

In the above theorem, (1) $\Leftrightarrow$ (1)′ $\Leftrightarrow$ (1)″, (2) $\Leftrightarrow$ (2)′ and (1) $\Rightarrow$ (2) are easy. The non-trivial part is the implication (2) $\Rightarrow$ (1). The proof of (2) $\Rightarrow$ (1) is divided into two steps:

i) Reduction to the case where H and L are abelian. This is an easy step that can be proved by using the Cartan decomposition.

ii) Proof of the abelian case. This is done by looking at the infinite points in a Riemannian symmetric space based on some structural results on parabolic subgroups and nilpotent elements.

Similar techniques lead us to a generalization to the case where H and L are not necessarily reductive:

Theorem 2.9.2 ([Ko94b]**).** *Let H, L are subsets of a real reductive linear Lie group G.*

1) *$H \pitchfork L$ in G $\Leftrightarrow$ $\mathfrak{a}(H) \pitchfork \mathfrak{a}(L)$ in $\mathfrak{a}$.*

2) *$H \sim L$ in G $\Leftrightarrow$ $\mathfrak{a}(H) \sim \mathfrak{a}(L)$ in $\mathfrak{a}$.*

If H and L are reductive in G, then it is easy to see that Theorem 2.9.2 (1) implies Theorem 2.9.1. If $G = GL(n, \mathbb{R})$ and $H = GL(m, \mathbb{R})$, then Theorem 2.9.2 (1) implies a result of Friedland ([Fr94] Theorem (3.1)). For more details, we refer to [Ko89a], [Ko94b].

2.10. Examples

Example 2.10. Let $G = SO(2m, 2n)$, $L = U(m, n) \subset G$ and $H = SO(p, q) \simeq 1_{2m-p} \times SO(p, q) \times 1_{2n-q} \subset G$. Here we assume $0 < q \leq p$, $0 < n \leq m$, $p \leq 2m$ and $q \leq 2n$ for simplicity. Then with a suitable coordinate, we can identify $\mathfrak{a}_G$ with $\mathbb{R}^{2n}$ and $W_G \simeq \mathfrak{S}_{2n} \ltimes (\mathbb{Z}/2\mathbb{Z})^{2n}$ in $GL(\mathfrak{a}_G)$. Up to the conjugacy by an element of W_G, we have

$$\mathfrak{a}(L) = \{(a_1, a_1, a_2, a_2, \ldots, a_n, a_n) : a_i \in \mathbb{R}\ (1 \leq i \leq n)\},$$
$$\mathfrak{a}(H) = \{(b_1, b_2, \ldots, b_q, 0, \ldots, 0) : b_i \in \mathbb{R}\ (1 \leq i \leq q)\}.$$

So the condition (2)′ in Theorem 2.9.1 amounts to $q = 1$. Therefore we conclude that $U(m, n)$ acts on $SO(2m, 2n)/SO(p, q)$ properly if and only if $q = 1$.

2.11. Exercise

Exercise 2.11. *Let $G/H = SO(2p,2q)/U(i,j)$ $(i \le p, j \le q, i \le j)$. Prove that if $i < \min(p,q)$ then there exists a discontinuous group $\Gamma\,(\subset G)$ acting on G/H such that Γ is isomorphic to $\pi_1(M_g)$.*

Hint: Show first that there is a subgroup of G which is isomorphic to $PSL(2,\mathbb{R})$ that acts properly on G/H (use Theorem 2.9.1). Then use the fact that there exists a discrete subgroup Γ of $PSL(2,\mathbb{R})$ such that $\Gamma \simeq \pi_1(M_g)$ (see Example 0.5.5).

2.12. Historical Notes

Reductive Case: For the implication (1) $\Rightarrow$ (2) in Theorem 2.5, E.Calabi and L.Markus first proved in the case $G/H = SO(n+1,1)/SO(n,1)$ in [CM62] (see Theorem 2.2.1). Then J.Wolf extended their result to the case $G/H = SO(p+1,q)/SO(p,q)\,(q \le p)$ in [Wo62]. After finding some other similar results in symmetric spaces of rank one (e.g. [Wo64]), he finally obtained the sufficiency of the real rank condition in the case of semisimple symmetric spaces in the 60's (see [Wo84]). His idea is also applicable to our more general setting. On the other hand, the proof of the necessity of the real rank condition given there was incomplete because of some confusion with the definition of properly discontinuous actions (see Example 1.6.2 for the difference between local Hausdorff topology and Hausdorff topology).
The converse implication (2) $\Rightarrow$ (1) in Theorem 2.5 is more difficult because we have to show the existence of an infinite discontinuous group acting on G/H if $\mathbb{R}\text{-rank}\,H < \mathbb{R}\text{-rank}\,G$. It took about twenty years before the converse implication was first proved in the rank one case $G/H = SO(p+1,q)/SO(p,q)$ $(q > p)$ by R.Kulkarni ([Ku81] Theorem 5.7). The method there is based on a study of quadratic form of type $(p+1,q)$. The general case is due to T.Kobayashi [Ko89a] as an application of Theorem 2.9.1 with $\dim L = 1$.
Solvable Case: It was proved by Kobayashi that there always exist a Clifford-Klein form of a homogeneous manifold G/H with infinite fundamental group (Theorem 2.4) if G is a solvable group and $H \neq G$ ([Ko93]). Lipsman made a further study of properly discontinuous actions and the property (CI) in [Lip94].
General Case: It is still an open problem to find a criterion on (G,H) for a general Lie group G such that a homogeneous space G/H admits an infinite discontinuous group.

§3. Generalized Hirzebruch's proportionality principle and its application

3.1. Hirzebruch's proportionality principle

In 1956, Hirzebruch proved

Theorem 3.1([Hi56]). *Let D be a bounded Hermitian symmetric domain, Γ a torsion free discrete cocompact subgroup of the automorphism group $Aut(D)$ of D, and M the compact Hermitian symmetric space dual to D. Then there is a real number $A = A(\Gamma)$ such that $c^\alpha(\Gamma\backslash D)[\Gamma\backslash D] = Ac^\alpha(M)[M]$ for any c^α, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index and $c^\alpha = c_1^{\alpha_1} \cup \cdots \cup c_k^{\alpha_k}$ is a monomial of Chern classes.*

In the third lecture, we shall clarify this principle by eliminating unnecessary conditions. The setting is generalized as follows:

Hermitian symmetric spaces $\longrightarrow$ homogeneous spaces of reductive type

tangent bundles $\longrightarrow$ homogeneous vector bundles.

characteristic numbers $\longrightarrow$ characteristic classes

As an application, we find an obstruction to the existence of compact Clifford-Klein forms of homogeneous spaces of reductive type by means of the Euler-Poincaré class (see Corollary 3.12.1).

3.2. Sketch of idea

It has been a classical and standard technique, in particular, in representation theory, to compare two objects through a holomorphic continuation: For instance,

i) Weyl's unitary trick — finite dimensional representations of reductive groups

iii) Flensted-Jensen duality — eigenspaces on semisimple symmetric spaces (e.g. [Fl86])

The argument in §3 lies in the same line:

iii) (generalized) Hirzebruch's proportionality principle — characteristic classes of homogeneous spaces of reductive type.

Let us explain the idea briefly in the case of Theorem 3.1, taking $G = SL(2,\mathbb{R})$ as an example. In this case

$$D = SL(2,\mathbb{R})/SO(2) \qquad \text{(the Poincaré upper half plane)},$$
$$M = SU(2)/SO(2)\,(\simeq \mathbb{C}P^1) \qquad \text{(the projective space)}.$$

There are two ways to compare the two manifolds D (or $\Gamma\backslash D$) and M:

i) $D \subset M$ (the Borel embedding),

ii) $D \subset SL(2,\mathbb{C})/SO(2,\mathbb{C}) \supset M$ (complexification).

The proportionality principle for Hermitian symmetric spaces (Theorem 3.1) can be proved based on the Borel embedding (i) as well as based on the complexification (ii). However, we will see that the argument using (ii) has a wider application, even when there is no natural map between D and M. Returning to our special example, we note that $M_{\mathbb{C}} := SL(2,\mathbb{C})/SO(2,\mathbb{C})$ is a common complexification of $M = SU(2)/SO(2)$ and $D = SL(2,\mathbb{R})/SO(2)$ in (ii), if we *forget* the original complex structures and look upon M and D simply as real manifolds. Now, we can compare differential forms which represent characteristic classes of two manifolds D and M through the holomorphic continuation on $M_{\mathbb{C}}$. This is the main idea of the proof of a generalized Hirzebruch's proportionality principle, which we discuss in §3.3 and §3.4.

3.3. Complexification and associated Riemannian spaces of compact type

Let us introduce the setting to realize the idea in §3.2. Let G be a connected real reductive linear group and H a connected subgroup reductive in G. We assume that there is a connected complex reductive Lie group $G_{\mathbb{C}}$ and a connected closed subgroup $H_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ respectively, such that $G \subset G_{\mathbb{C}}$. Take a Cartan involution θ of G such that $\theta H = H$. We write the corresponding Cartan decomposition as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let G_U be a connected subgroup of $G_{\mathbb{C}}$ with a Lie algebra $\mathfrak{g}_U := \mathfrak{k} + \sqrt{-1}\mathfrak{p}$, and H_U a connected subgroup of $G_{\mathbb{C}}$ with a Lie algebra $\mathfrak{h}_U := \mathfrak{h} \cap \mathfrak{k} + \sqrt{-1}\mathfrak{h} \cap \mathfrak{p}$. Then G_U is a maximal compact subgroup of $G_{\mathbb{C}}$. Similarly, H_U is that of $H_{\mathbb{C}}$. The homogeneous manifold G_U/H_U is said to be an *associated Riemannian space of compact type*. In summary, we have the following setting:

$$\begin{array}{ccccc} G & \subset & G_{\mathbb{C}} & \supset & G_U \\ \cup & & \cup & & \cup \\ H & \subset & H_{\mathbb{C}} & \supset & H_U. \end{array}$$

Remark 3.3.1. The argument in this section is still valid for H with finitely many connected components, if we replace $H_{\mathbb{C}}$ by $H_{\mathbb{C}}(H \cap K)$ and H_U by $H_U(H \cap K)$.

Example 3.3.2.

i) Let $G/H = SL(n,\mathbb{R})/SO(p,n-p)$. Then $G_\mathbb{C}/H_\mathbb{C} = SL(n,\mathbb{C})/SO(n,\mathbb{C})$ and $G_U/H_U = SU(n)/SO(n)$.
ii) Let $G/H = GL(n,\mathbb{R})/(\mathbb{R}^\times)^n$. Then $G_\mathbb{C}/H_\mathbb{C} = GL(n,\mathbb{C})/(\mathbb{C}^\times)^n$ and $G_U/H_U = U(n)/U(1)^n$.
iii) Let $G/H = Sp(2n,\mathbb{R})/Sp(n,\mathbb{C})$. Then $G_\mathbb{C}/H_\mathbb{C} = Sp(2n,\mathbb{C})/Sp(n,\mathbb{C}) \times Sp(n,\mathbb{C})$ and $G_U/H_U = Sp(2n)/Sp(n) \times Sp(n)$.

Example 3.3.3. The above assumption (i.e. closedness of $H_\mathbb{C}$) is satisfied for (G,H) in Example 2.6.3 (i), (ii), (iii), (iv). We recall that G/H is a reductive symmetric space in Example 2.6.3 (iii); G/H is a semisimple orbit in Example 2.6.3 (iv). The corresponding associated Riemannian space of compact type G_U/H_U is:

$$\begin{aligned} G_U/H_U &= \text{ a compact symmetric space} \quad \text{for Example 2.6.3 (iii)},\\ G_U/H_U &= \text{ a generalized flag variety} \quad \text{for Example 2.6.3 (iv)}. \end{aligned}$$

3.4. A homomorphism between cohomology rings

In the setting of §3.3, suppose a discrete subgroup Γ of G acts on G/H properly discontinuously and freely, so that $\Gamma\backslash G/H$ is a smooth manifold. (We do not require that $\Gamma\backslash G/H$ is compact. Γ is allowed to be the trivial subgroup $\{e\}$.)

We define

$$M_U := G_U/H_U \subset M_\mathbb{C} := G_\mathbb{C}/H_\mathbb{C} \supset M_\mathbb{R} := G/H.$$

Then in order to compare characteristic classes of the two manifolds M_U and $M_\mathbb{R}$ we use the following restriction maps:

$$\begin{array}{ccc} & \mathcal{O}\left(\bigwedge^p TM_\mathbb{C}\right) & \\ \swarrow \text{rest.} & & \searrow \text{rest.} \\ \mathcal{A}\left(\bigwedge^p_\mathbb{R} TM_U\right) & & \mathcal{A}\left(\bigwedge^p_\mathbb{R} TM_\mathbb{R}\right). \end{array}$$

The groups G_U and G acting on M_U and $M_\mathbb{R}$ respectively have the common complexification $G_\mathbb{C}$, which acts holomorphically on $M_\mathbb{C}$. Because G_U is compact, we can find a G_U invariant differential form as a representative of an arbitrary element of the de Rham cohomology group $H^*(G_U/H_U;\mathbb{C})$. This element extends to a holomorphic differential form on $G_\mathbb{C}/H_\mathbb{C}$, and then we can restrict to G/H and to $\Gamma\backslash G/H$. Thus, the above diagram induces a homomorphism on the cohomology level:

Theorem 3.4. *We have a natural* $\mathbb{C}$*-algebra homomorphism*

$$\eta\colon H^*(G_U/H_U;\mathbb{C}) \to H^*(\Gamma\backslash G/H;\mathbb{C}).$$

If $\Gamma\backslash G/H$ *is compact and* H *is connected, then* η *is injective.*

The last statement follows from the Poincaré duality.

3.5. Real homogeneous vector bundles

We review the definition of an associated vector bundle. Let $\rho\colon H \to GL_\mathbb{R}(V)$ be a representation of H on a real vector space V. Suppose a discrete subgroup Γ acts on a homogeneous space G/H properly discontinuously. Associated to the principal H-bundle

$$H \to \Gamma\backslash G \to \Gamma\backslash G/H,$$

is a real homogeneous vector bundle over $\Gamma\backslash G/H$

$${}^\Gamma E := \Gamma\backslash G \underset{\rho}{\times} V$$

defined by the set of equivalence classes with respect to the action of H, that is,

$$\begin{aligned}&(\overline{g}, v), (\overline{g'}, v') \in \Gamma\backslash G \times V \text{ are equivalent}\\ &\qquad\qquad\Leftrightarrow \overline{g} = \overline{g'}h \text{ and } v = \rho(h^{-1})v' \text{ for some } h \in H.\end{aligned}$$

The projection on the first component $\Gamma\backslash G \times V \to \Gamma\backslash G$ gives rise to the projection ${}^\Gamma E \equiv \Gamma\backslash G \underset{\rho}{\times} V \to \Gamma\backslash G/H$ with typical fiber V. Similarly, an associated real vector bundle $E_U := G_U \underset{\rho_U}{\times} V_U$ over G_U/H_U is defined if a representation $\rho_U\colon H_U \to GL_\mathbb{R}(V_U)$ is given.

3.6. Weyl's unitary trick

The idea of Weyl's unitary trick is a holomorphic continuation of finite dimensional representations. Here, we set up some notation of finite dimensional representations that fits into Hirzebruch's proportionality principle.

Let $H \subset H_{\mathbb{C}} \supset H_U$ be as in §3.3, namely, $H_{\mathbb{C}}$ is a complex reductive Lie group, H is a real form and H_U is a compact real form. Let

$$\rho\colon H \to GL_{\mathbb{R}}(V),$$
$$\rho_U\colon H_U \to GL_{\mathbb{R}}(V_U)$$

be finite dimensional representations over $\mathbb{R}$. We say that *the complexifications of ρ and ρ_U are isomorphic* if there are

i) a complex vector space $V_{\mathbb{C}}$

ii) a holomorphic representation $\rho_{\mathbb{C}}\colon H_{\mathbb{C}} \to GL(V_{\mathbb{C}}, \mathbb{C})$

iii) isomorphisms $\psi\colon V \otimes \mathbb{C} \overset{\sim}{\to} V_{\mathbb{C}}$ and $\psi_U\colon V_U \otimes \mathbb{C} \overset{\sim}{\to} V_{\mathbb{C}}$

such that the following diagram commutes:

$$\begin{array}{ccccc} H & \overset{\iota}{\hookrightarrow} & H_{\mathbb{C}} & \hookleftarrow & H_U \\ \rho\big\downarrow & & \big\downarrow\rho_{\mathbb{C}} & & \big\downarrow\rho_U \\ GL_{\mathbb{R}}(V) & \underset{\psi_\sharp}{\hookrightarrow} & GL_{\mathbb{C}}(V_{\mathbb{C}}) & \underset{\psi_{U\sharp}}{\hookleftarrow} & GL_{\mathbb{R}}(V_U). \end{array}$$

3.7. Hirzebruch's proportionality principle

Theorem 3.7([KoO90]). *Retain the setting as in §3.3. Let Γ be any discrete subgroup of G acting on G/H freely and properly discontinuously from the left. Suppose that the complexifications of $\rho\colon H \to GL_{\mathbb{R}}(V)$ and $\rho_U\colon H_U \to GL_{\mathbb{R}}(V_U)$ are isomorphic. Then the i-th Pontryagin class satisfies*

$$\eta(p_i(E_U)) = p_i({}^{\Gamma}E) \in H^{4i}(\Gamma\backslash G/H; \mathbb{R}).$$

In particular, if there is a relation $\sum a_\alpha p^\alpha(E_U) = 0$ in $H^*(G_U/H_U; \mathbb{R})$, then the equation $\sum a_\alpha p^\alpha({}^{\Gamma}E) = 0$ in $H^*(\Gamma\backslash G/H; \mathbb{R})$ holds. Here $\alpha = (\alpha_1, \dots, \alpha_k)$ is a multi-index and $p^\alpha = p_1^{\alpha_1} \cup \dots \cup p_k^{\alpha_k}$ is a monomial of Pontryagin classes.

Sketch of Proof. We take an invariant connection (the canonical connection of the second kind on G/H in the sense of [No54]) on the principal bundles $G \to G/H$, $G_U \to G_U/H_U$, respectively. Then the curvature forms are represented in terms of the Lie algebras. By the Chern-Weil theory (see [D78], [KoN69], [We80]), characteristic classes are represented by using curvature forms. Now, we have the theorem by the usual Weyl's unitary trick for finite dimensional representations and by using the holomorphic continuation through η. $\square$

We note that in the case of Riemannian symmetric spaces (in particular, Hermitian symmetric spaces), we can prove the theorem without using a holomorphic continuation. That is, we pull back directly the curvature forms via the $\mathbb{R}$-linear bijection $\mathfrak{g}_U/\mathfrak{h}_U \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h}$. This map is different from η in Theorem 3.4 only by a constant multiple *depending on* degrees.

3.8. Tangent bundle

The characteristic classes of a manifold are, by definition, those of the tangent bundle. The tangent bundle $T(\Gamma\backslash G/H)$ is associated to the adjoint representation

$$\mathrm{Ad}_{|H}\colon H \to GL_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}),$$

that is,

$$T(\Gamma\backslash G/H) \simeq \Gamma\backslash G \underset{\mathrm{Ad}_{|H}}{\times} \mathfrak{g}/\mathfrak{h},$$

and similarly

$$T(G_U/H_U) \simeq G_U \underset{\mathrm{Ad}_{|H_U}}{\times} \mathfrak{g}_U/\mathfrak{h}_U.$$

Since $\mathrm{Ad}_{|H}\colon H \to GL_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$ and $\mathrm{Ad}_{|H_U}\colon H_U \to GL_{\mathbb{R}}(\mathfrak{g}_U/\mathfrak{h}_U)$ have isomorphic complexifications, we have now relations of Pontryagin classes (of the tangent bundle) between G_U/H_U and $\Gamma\backslash G/H$ as follows.

Corollary 3.8. *In the same setting as Theorem 3.7, we have*

$$\eta\,(p_i(G_U/H_U)) = p_i(\Gamma\backslash G/H) \in H^{4i}(\Gamma\backslash G/H;\mathbb{R}).$$

Furthermore, if H is connected, then we have a relation of Euler-Poincaré classes:

$$\eta\,(\chi(G_U/H_U)) = \chi(\Gamma\backslash G/H) \in H^{n}(\Gamma\backslash G/H;\mathbb{R}).$$

We note that both $\Gamma\backslash G/H$ and G_U/H_U are orientable if H is connected.

3.9. Complex homogeneous vector bundles

Let $\rho\colon H_{\mathbb{C}} \to GL_{\mathbb{C}}(V_{\mathbb{C}})$ be a holomorphic representation on a finite dimensional vector space $V_{\mathbb{C}}$ over $\mathbb{C}$. Associated to the principal H-bundle

$$H \to \Gamma\backslash G \to \Gamma\backslash G/H,$$

we define a homogeneous complex vector bundle over $\Gamma\backslash G/H$ by

$$^{\Gamma}E_{\mathbb{C}} := \Gamma\backslash G \underset{\rho|H}{\times} V_{\mathbb{C}}.$$

Similarly, we define a homogeneous complex vector bundle over G_U/H_U by

$$E_{U\mathbb{C}} := G_U \underset{\rho|H_U}{\times} V_{\mathbb{C}}.$$

3.10. Chern classes

Theorem 3.10.1 ([KoO90]). *Retain the setting as above. Let Γ be any discrete subgroup of G acting on G/H freely and properly discontinuously from the left. Then the i-th Chern class satisfies*

$$\eta(c_i(E_{U\mathbb{C}})) = c_i(^{\Gamma}E_{\mathbb{C}}) \in H^{2i}(\Gamma\backslash G/H; \mathbb{R}).$$

In particular, if there is a relation $\sum a_\alpha c^\alpha(E_U) = 0$ in $H^*(G_U/H_U;\mathbb{R})$, then the equation $\sum a_\alpha c^\alpha(^{\Gamma}E) = 0$ in $H^*(\Gamma\backslash G/H;\mathbb{R})$ holds. If $\Gamma\backslash G/H$ is compact and H is connected, the converse statement also holds.

Exercise 3.10.2. Formulate an analogous result to Corollary 3.8 in the case of Chern classes of homogeneous manifolds.
(Hint: The assumption will be that G/H is an elliptic orbit. See §4.12. See also [KoO90] Corollary 4.)

3.11. Examples

Example 3.11.1. We consider a semisimple symmetric space $G/H = SO(p,q)/SO(p-1,q)$. All Pontryagin classes of any Clifford-Klein form $\Gamma\backslash G/H$ of a homogeneous manifold G/H vanish in $H^*(\Gamma\backslash G/H;\mathbb{R})$ because we know the corresponding result holds for $G_U/H_U \simeq S^{p+q-1}$. In particular, all Pontryagin classes of a Riemannian manifold of constant sectional curvature vanish (see also [Su76]). We mention that there exist compact Clifford-Klein forms of $SO(p,q)/SO(p-1,q)$ if $(p,q) = (1,n), (2,2n), (4,4n), (8,8)$ $(n \in \mathbb{N})$ (see Example 4.13.1).

Example 3.11.2. We endow $\mathbb{C}^{p+q+1}$ with an (indefinite) Hermitian metric of type $(p+1,q)$, that is,

$$(z,z) := z_1\overline{z_1} + \cdots + z_{p+1}\overline{z_{p+1}} - z_{p+2}\overline{z_{p+2}} - \cdots - z_{p+q+1}\overline{z_{p+q+1}}.$$

Let $X(p,q)$ be the open subset of the projective space $\mathbb{C}P^{p+q}$, which consists of the complex lines on which the restriction of the indefinite Hermitian metric is positive definite. Then $U(p+1,q)$ acts transitively on $X(p,q)$ so that we have a diffeomorphism

$$X(p,q) \simeq U(p+1,q)/U(1) \times U(p,q) =: G/H$$
$$\subset\ G_U/H_U := U(n+1)/U(1) \times U(n) \simeq \mathbb{C}P^n.$$

Here we put $n = p+q$. We note that $X(n,0) = \mathbb{C}P^n \simeq G_U/H_U$ and that $X(0,n)$ is the dual Hermitian symmetric domain of the noncompact type (ref. [He78] for the terminology). Let Γ be a discrete subgroup of $U(p+1,q)$ acting on $X(p,q)$ freely and properly discontinuously so that $\Gamma\backslash X(p,q)$ is a Clifford-Klein form of $X(p,q)$. Then we have a relation among Chern classes:

$$c_j(\Gamma\backslash X(p,q)) = (\prod_{l=0}^{j-1} \frac{n+1-l}{n+1})\, c_1(\Gamma\backslash X(p,q))^j \quad (1 \le j \le n).$$

This follows from the corresponding fact for

$$X(n,0) = \mathbb{C}P^n \simeq U(n+1)/U(1) \times U(n),$$

that is, the total Chern class $c(\mathbb{C}P^n) = 1 + c_1(\mathbb{C}P^n) + \cdots + c_n(\mathbb{C}P^n)$ of the complex projective space $\mathbb{C}P^n$ is given by

$$c(\mathbb{C}P^n) \equiv (1+x)^{n+1} \mod x^{n+1},$$

where x is the first Chern class of the hyperplane section bundle (see [BoHi58] 15.1, [MiSt74] Theorem 14.10). If $\Gamma\backslash X(p,q)$ is a compact Clifford-Klein form of $X(p,q)$, then $c_j(\Gamma\backslash X(p,q)) \neq 0$ for any j with $1 \le j \le n$. There exists a compact Clifford-Klein form of $X(0,n), X(n,0)$ (Riemannian case) and $X(1,2r)$ (see Corollary 4.7), whereas any discrete subgroup acting properly discontinuously on $X(p,q)$ with $p \ge q$ is finite (see Theorem 2.5).

Example 3.11.3. Let M be a compact Clifford-Klein form of a complex manifold $SO(p,q+2)/SO(p,q) \times SO(2)$. Then the Chern class $c_j(M)$, for any j with $1 \le j \le p+q$, of a complex manifold M does not vanish, because we know that the corresponding result holds for the Hermitian symmetric space $SO(n+2)/SO(n) \times SO(2)$. There exists a compact Clifford-Klein form for $(p,q) = (n,0)$ $(n \in \mathbb{N})$ and $(4,1)$ (see Corollary 4.7).

3.12. Compact Clifford-Klein form

We mentioned in §1.2 a third proof that $PSL(2,\mathbb{R})/SO(1,1)$ does not admit a compact Clifford-Klein form by using a non-vanishing vector field. Now we are ready to state a generalization of this result to a higher dimensional setting:

Corollary 3.12.1. *Let (G, H) be as in §3.3. Denote by K a maximal compact subgroup of G such that $H \cap K$ is also a maximal compact subgroup of H. If* $\operatorname{rank} G = \operatorname{rank} H$ *and* $\dim K/H \cap K$ *is odd, then G/H admits no uniform lattice, that is, there exists no compact Clifford-Klein form of G/H.*

Sketch of Proof. We may and do assume H is connected. Then G/H admits a G-invariant orientation so that a Clifford-Klein form $\Gamma\backslash G/H$ is an orientable manifold. The tangent bundle $T(\Gamma\backslash G/H)$ splits according to the $H \cap K$ module decomposition $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p}$. Then $\chi(\Gamma\backslash G/H) = 0$ because $\dim_{\mathbb{R}} \mathfrak{q} \cap \mathfrak{k} = \dim K/H \cap K$ is odd. On the other hand, as H is of maximal rank in G, so is H_U in G_U. As H is connected, so is H_U. Therefore G_U/H_U is a compact orientable manifold with non-vanishing Euler number $\chi(G_U/H_U)$ ([HoS40]). Now it follows from Theorem 3.4 that the fact $\chi(G_U/H_U) \neq 0$ contradicts to that $\chi(\backslash G/H) = 0$ if $\Gamma\backslash G/H$ is compact. Hence $\Gamma\backslash G/H$ cannot be compact. □

Example 3.12.2 (see §1.2)**.** We know that the hyperboloid of one sheet $G/H = SL(2, \mathbb{R})/SO(1, 1)$ does not admit a compact Clifford-Klein form. This fact was explained by using non-vanishing vector fields in §1.2 (3), which is a geometric idea of Corollary 3.12.1. In this case, we have $\operatorname{rank} G = \operatorname{rank} H = 1$ and $\dim K/H \cap K = 1$, so that the assumptions in Corollary 3.12.1 are satisfied.

Exercise 3.12.3 (see also Example 4.13.1)**.** Prove that a semisimple symmetric space $SO(i + k, j + l)/SO(i, j) \times SO(k, l)$ does not have a compact Clifford-Klein form if exactly one element among i, j, k, l is even.

At the end of this section, we pose the following conjecture:

Conjecture 3.12.4 (see [Ko89b] Conjecture 6.4)**.** *Let G/H be a homogeneous space of reductive type. It is likely to hold the inequality*

$$\operatorname{rank} G + \operatorname{rank}(H \cap K) \geq \operatorname{rank} H + \operatorname{rank} K$$

if G/H admits a compact Clifford-Klein form.

The case with $\operatorname{rank} G = \operatorname{rank} H$ can be proved based on Corollary 3.8 and on an argument of the cohomological dimension of a discrete group, as a slight improvement of Corollary 3.12.1 (see [Ko89a] Corollary 5). If H is compact or if G/H is a group manifold, then the above inequality is obviously satisfied. As far as the author knows, the above inequality holds for all homogeneous spaces of reductive type that are proved to admit compact Clifford-Klein forms (cf. Corollary 4.7).

3.13. Notes

1) Most material in this section is taken from [KoO90].
2) To establish the holomorphic continuation, we calculate characteristic forms in terms of Lie algebras. Similar calculations are also carried out in [Bo67], [CGW76] for Riemannian symmetric spaces.
3) There exist different generalizations of Hirzebruch's proportionality principle, due to D.Mumford [Mu77] for the noncompact Clifford-Klein forms of Hermitian symmetric spaces, J.L.Dupont-W.Kamber [DK93] for the secondary characteristic numbers.
4) F.Labourie informed me that Corollary 3.12.1 is also valid for a manifold modeled on a homogeneous space G/H (by e-mail, 1994).
5) A similar idea (at least implicitly) to Corollary 3.12.1 is used in [Ku81] based on the Gauss-Bonnet theorem in the case of rank 1 symmetric spaces $G/H = SO(p+1,q)/SO(p,q)$ (cf. Exercise (3.12.3)).

§4. Compact Clifford-Klein forms of non-Riemannian homogeneous spaces

In the fourth lecture, we focus our attention on compact Clifford-Klein forms of homogeneous spaces of reductive type. §4 is organized as follows:

i) §4.1 - §4.9 Homogeneous spaces with compact Clifford-Klein forms.

ii) §4.10 - §4.12 Homogeneous spaces without compact Clifford-Klein forms.

4.1. Uniform lattice, compact Clifford-Klein form

Definition 4.1. Suppose G is a Lie group and H is a closed subgroup. If a discrete subgroup $\Gamma \subset G$ satisfies the following two conditions:
(4.1.1)(a) Γ acts properly discontinuously and freely on G/H,
(4.1.1)(b) $\Gamma \backslash G/H$ is compact,
then Γ is said to be a *uniform lattice for the homogeneous space* G/H. Then the double coset space $\Gamma \backslash G/H$ is a *compact Clifford-Klein form of the homogeneous space* G/H.

Suppose there exists a G-invariant measure on G/H. (This is the case for a homogeneous space of reductive type.) Then it induces a measure on a Clifford-Klein form $\Gamma \backslash G/H$. The discrete subgroup Γ is said to be a *lattice for the homogeneous space* G/H provided both (4.1.1)(a) and
(4.1.2)(b) $\Gamma \backslash G/H$ is of finite volume
are satisfied.

If H is compact (in particular $H = \{e\}$ or $H = K$), the above definition coincides with a usual one.

4.2. Riemannian cases

If the isotropy subgroup H is compact, then any discrete subgroup of G acts properly discontinuously on G/H as we saw in §2.3. This case, reffered to as the *Riemannian case*, allows a compact Clifford-Klein form. We recall the following important theorem due to Borel, Harish-Chandra, Mostow-Tamagawa.

Theorem 4.2([Bo63],[BoHa62],[MoTa62]). *Suppose G is a real linear reductive Lie group and H is a compact subgroup of G. Then G/H admits compact Clifford-Klein forms. Furthermore, G/H has noncompact Clifford-Klein forms which have finite volume if G contains a noncompact semisimple factor.*

We shall not prove this theorem, and instead refer the reader to [Ra72], [Zi84] as well as original papers. We shall mention, however, some typical examples.

i) A compact Riemann surface $M_g\,(g \geq 2)$ is a compact Clifford-Klein form of the Poincaré plane $PSL(2,\mathbb{R})/SO(2)$ (see Example 0.5.5).

ii) If $\Gamma = \mathbb{Z}^n \subset G = \mathbb{R}^n \supset H = \{0\}$, then $\Gamma\backslash G/H \simeq S^1 \times \cdots \times S^1$ is a compact Clifford-Klein form of $G/H \simeq \mathbb{R}^n$.

iii) If $\Gamma = GL(n,\mathbb{Z}) \subset G = GL(n,\mathbb{R}) \supset H = \{e\}$ $(n \geq 2)$, then $\Gamma\backslash G/H$ is a noncompact Clifford-Klein form of $G/H \simeq GL(n,\mathbb{R})$ with finite volume.

4.3. Moore's Ergodicity Theorem

A simple remark here is that if Γ is a uniform lattice for a group manifold $G = G/\{e\}$, then the quotient topology of the double coset space $\Gamma\backslash G/H$ is always compact. However, if H is non-compact, the double coset space $\Gamma\backslash G/H$ does not have a good topology in general. In fact, the action of Γ on G/H is not properly discontinuous. We leave it to the reader as an easy exercise:

Exercise 4.3.1. Prove that the action of Γ on G/H is not properly discontinuous if Γ is a uniform lattice of G and if H is noncompact.

Furthermore, the action of Γ on G/H can be ergodic:

Theorem 4.3.2 ([Moo66], see also [Zi84] Chapter 2)**.** *Let G be a non-compact simple linear Lie group. If $\Gamma \subset G$ is a lattice and $H \subset G$ is a closed subgroup, then*

$$\textit{The action of } \Gamma \textit{ is ergodic on } G/H \quad \Leftrightarrow \quad H \textit{ is noncompact.}$$

Here, we recall that the action of Γ is said to be ergodic if every Γ-invariant measurable set is either null or conull.

Thus, a uniform lattice for G/H must be *smaller* than a uniform lattice for G in some sense. In this respect, the cohomological dimension of an abstract group is a nice measure for the 'size' of a discrete group (see [Ko89a] Corollary 5.5). We use it in the proof of Theorem 4.9 and Theorem 4.10. Basic references of the cohomological dimension of groups for which we need are [Ser71] and [Bi76].

4.4. Indefinite-Riemannian case

Theorem 4.4 ([Ko89a] §4)**.** *Suppose that G is a real reductive linear group and that H and L are both reductive in G. If the triplet (G, L, H)*

satisfies both of the following conditions

$$\text{(4.4)(a)} \qquad \mathfrak{a}(L) \cap \mathfrak{a}(H) = \{0\},$$

$$\text{(4.4)(b)} \qquad \mathrm{d}(L) + \mathrm{d}(H) = \mathrm{d}(G),$$

then G/H admits compact Clifford-Klein forms. Furthermore, G/H admits noncompact Clifford-Klein forms which have finite volume if L contains a noncompact semisimple factor.
So does G/L because a symmetric role of H and L.

4.5. Trivial examples (I) – Riemannian cases revisited

Suppose G/H is of reductive type. We know that there exists compact Clifford-Klein forms of G/H if H is compact (Theorem 4.2). This fact is explained in the context of Theorem 4.4 as follows.

Suppose H is compact. Then we have $\mathfrak{a}(H) = \{0\}$ and $\mathrm{d}(H) = 0$. If we take $L = G$, then the conditions (4.4)(a) and (b) are satisfied. Thus, Theorem 4.2 (Riemannian case) is a special case of Theorem 4.4. But this is rather stupid because we shall use Theorem 4.2 as a starting point for the proof of Theorem 4.4.

4.6. Trivial examples (II) – group manifold cases

Let G' be a noncompact real reductive Lie group. Suppose $(G, H) = (G' \times G', \operatorname{diag} G')$ so that $G/H \simeq G'$ is a group manifold. This case is trivial because it is obvious that G/H admits a compact Clifford-Klein form by Theorem 4.2. But it is instructive to see how Theorem 4.4 is applied in the group manifold case because $H \simeq G'$ is noncompact.

Let $\mathfrak{g}' = \operatorname{Lie} G' = \mathfrak{k}' + \mathfrak{p}'$ be a Cartan decomposition and $\mathfrak{a}_{G'} \subset \mathfrak{p}'$ be a maximally abelian subspace. We write $W_{G'}$ for the Weyl group of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. Then we have

$$\mathfrak{a}_G = \mathfrak{a}_{G'} \oplus \mathfrak{a}_{G'},$$
$$W_G = W_{G'} \times W_{G'}.$$

Let us take $L := G' \times 1$ (or $1 \times G'$). Then we have

$$\mathfrak{a}(H) = \{(X, wX) : X \in \mathfrak{a}_{G'},\ w \in W_{G'}\}$$
$$\mathfrak{a}(L) = \{(X, 0) : X \in \mathfrak{a}_{G'}\}$$
$$\mathrm{d}(H) = \mathrm{d}(L) = \mathrm{d}(G'), \quad \mathrm{d}(G) = 2 \dim G'.$$

Hence the conditions (4.4)(a) and (b) are satisfied.

4.7. Examples of indefinite-Riemannian homogeneous spaces that have compact Clifford-Klein forms

As a corollary of Theorem 4.4 we have

Corollary 4.7. *The following homogeneous spaces (six series and six isolated ones) admit compact Clifford-Klein forms. Also they admit non-compact Clifford-Klein forms of finite volume.*

1) *a)* $SU(2,2n)/Sp(1,n)$, *b)* $SU(2,2n)/U(1,2n)$,
2) *a)* $SO(2,2n)/U(1,n)$, *b)* $SO(2,2n)/SO(1,2n)$,
3) *a)* $SO(4,4n)/Sp(1,n)$, *b)* $SO(4,4n)/SO(3,4n)$,
4) *a)* $SO(8,8)/SO(8,7)$, *b)* $SO(8,8)/Spin(8,1)$,
5) *a)* $SO(4,4)/SO(4,1) \times SO(3)$, *b)* $SO(4,4)/Spin(4,3)$,
6) *a)* $SO(4,3)/SO(4,1) \times SO(2)$, *b)* $SO(4,3)/G_{2(2)}$.

Here, (a) and (b) in each line forms a pair $(G/H, G/L)$ which satisfies the assumptions (4.4) (a) and (b) of Theorem 4.4. For example, the first line means $(G,H,L) = (SU(2,2n), Sp(1,n), U(1,2n))$ satisfies (4.4)(a) and (b).

The signature of the indefinite metric on G/H (and also on Clifford-Klein forms) induced from the Killing form is given by $(4n, 3n^2 - 2n)$, $(4n, 2)$, $(2n, n^2 - n)$, $(2n, 1)$, $(12n, 7n^2 - 4n + 3)$, $(4n, 3)$, $(8, 7)$, $(56, 28)$, $(12, 3)$, $(4, 3)$, $(8, 2)$, and $(4, 3)$, respectively.

4.8. Sketch of Proof

The idea of Theorem 4.4 is illustrated by the abelian case. Let $G = \mathbb{R}^n \supset H = \mathbb{R}^k$ $(n \geq k)$. Take a complementary subspace $L \simeq \mathbb{R}^{n-k}$ of H in G and choose a lattice $\Gamma \simeq \mathbb{Z}^{n-k}$ of L. Then

$$\Gamma \backslash G/H \simeq \mathbb{Z}^{n-k} \backslash \mathbb{R}^n / \mathbb{R}^k \simeq S^1 \times \cdots \times S^1$$

is a compact Clifford-Klein form of G/H. In this case (4.4)(a) is satisfied because $L \simeq \mathbb{R}^{n-k}$ is complementary to $H \simeq \mathbb{R}^k$, and (4.4)(b) is satisfied because $\mathrm{d}(L) + \mathrm{d}(H) = (n-k) + k = n = \mathrm{d}(G)$.

For the general case, the proof of Theorem 4.4 is divided into the following three steps.

i) The criterion for proper actions (continuous analogue of discontinuous groups) (Theorem 2.9.1),
ii) The criterion for compact quotient (continuous analogue of uniform lattice) (Theorem 4.9),
iii) Existence of uniform lattice in the Riemannian case (Theorem 4.2).

If both (4.4)(a) and (4.4)(b) are satisfied, then L acts properly on G/H by Theorem 2.9.1 and the quotient topology of $L\backslash G/H$ is compact by Theorem 4.9. On the other hand, Theorem 4.2 assures the existence of a torsion free cocompact discrete subgroup Γ of L. Then Γ turns out to be a uniform lattice for G/H thanks to Observation 1.9. This shows the first half of Theorem 4.4. Similarly, a torsion free co-volume finite discrete subgroup Γ of L is also a lattice for G/H.

4.9. Continuous analogue

A continuous analogue of a compact Clifford-Klein form $\Gamma\backslash G/H$ of a homogeneous manifold is a compact double coset space $L\backslash G/H$ in the quotient topology where L, H are closed subgroups of G such that L acts properly on G/H (see Definition 1.8).

Theorem 4.9. *Let G be a real reductive linear Lie group, H and L closed subgroups which are reductive in G. Under the equivalent conditions in Theorem 2.9.1 (i.e. $L \pitchfork H$ in G in the sense of Definition 1.11.1), the following two conditions are equivalent:*

(4.9)(a) *$L\backslash G/H$ is compact in the quotient topology.*

(4.9)(b) $\mathrm{d}(L) + \mathrm{d}(H) = \mathrm{d}(G)$.

A flavor of the proof. We recall that $\mathrm{d}(G) = \dim_{\mathbb{R}} \mathfrak{p}$ if we write a Cartan decomposition as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. In view of the Cartan decomposition of Lie group $G \simeq K \times \exp(\mathfrak{p})$, we may regard

(4.9.1) $\mathrm{d}(G) =$ the dimension of "noncompact part" of G.

A homogeneous space of reductive type G/H has a vector bundle structure over a compact manifold $K/H \cap K$ with typical fiber $\mathfrak{p}/\mathfrak{p} \cap \mathfrak{h}$ (e.g. [Ko89a] Appendix). In view of $\dim_{\mathbb{R}} \mathfrak{p}/\mathfrak{p} \cap \mathfrak{h} = \mathrm{d}(G) - \mathrm{d}(H)$, we may regard:

(4.9.2) $\mathrm{d}(G) - \mathrm{d}(H) =$ the dimension of "noncompact part" of G/H.

Now, one might expect that

(4.9.3)

$\mathrm{d}(G) - \mathrm{d}(H) - \mathrm{d}(L) =$ the dimension of "noncompact part" of $L\backslash G/H$,

and that (4.9.3) would lead to Theorem 4.9. However, (4.9.3) is not always "true" if we do not assume that $L \pitchfork H$ in G. (We remark that the dimension of "noncompact part" of $L\backslash G/H$ is not defined yet.) The simplest and illustrative observation is when $G = \mathbb{R}^n$ and H, L are vector subspaces of G with dimension k, l, respectively. Then

$$\mathrm{d}(G) - \mathrm{d}(H) - \mathrm{d}(L) = n - k - l,$$

$$L\backslash G/H \simeq \mathbb{R}^{n-k-l+\dim(H\cap L)}.$$

Therefore, if we assume $\dim(H \cap L) = 0$ (or equivalently $L \pitchfork H$ in G), then we have

$$\mathrm{d}(G) - \mathrm{d}(H) - \mathrm{d}(L) = 0 \Longleftrightarrow L\backslash G/H \text{ is compact.}$$

This explains why the assumption $L \pitchfork H$ is necessary in Theorem 4.9, and how (4.9)(a) and (4.9)(b) are related.

For the general case where G is a real reductive linear group, the underlying idea is similar but we need some more work. This is carried out in a frame work of a discrete analogue of Theorem 4.9, where L is replaced by an arithmetic subgroup Γ of L and $\mathrm{d}(L)$ is replaced by the cohomological dimension of Γ. See [Ko89a] for details. □

4.10. Necessary conditions for the existence of compact Clifford-Klein forms

For the existence of compact Clifford-Klein forms of homogeneous spaces of reductive type, we have presented two necessary conditions so far:

i) Calabi-Markus phenomenon (Theorem 2.5)

ii) Hirzebruch's proportionality principle (Corollary 3.12.1)

Here we give a third necessary condition for the existence of compact Clifford-Klein forms of homogeneous spaces:

Theorem 4.10. *Let G/H be a homogeneous space of reductive type. G/H does not admit a compact Clifford-Klein form if there exists a closed subgroup L reductive in G satisfying the following two conditions:*

(4.10)(a) $$\mathfrak{a}(L) \subset \mathfrak{a}(H),$$

(4.10)(b) $$\mathrm{d}(L) > \mathrm{d}(H).$$

The proof of Theorem 4.10 is similar to that of Theorem 4.4 and Theorem 4.9, by using the cohomological dimension of an abstract group together with Theorem 2.9.1. See [Ko92b] for details.

For an application of Theorem 4.10, we need to find a suitable L satisfying (4.10)(a),(b), provided G/H is given. This is done systematically for some typical homogeneous space of reductive type in §4.11 and §4.12.

4.11. Semisimple symmetric spaces

In order to apply Theorem 4.10 to a semisimple symmetric space, we need some results on the root system for semisimple symmetric spaces. First of all, we give a brief review of the notion of ϵ-families introduced by T.Oshima and J.Sekiguchi [OS84].

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$, σ an involution of $\mathfrak{g}$, θ a Cartan involution of $\mathfrak{g}$ commuting with σ. Then $\sigma\theta$ is also an involution of $\mathfrak{g}$ because $(\sigma\theta)^2 = \sigma^2\theta^2 = 1$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{h} + \mathfrak{q} = \mathfrak{h}^a + \mathfrak{q}^a$ be direct sum decompositions of eigenspaces with eigenvalues ± 1 corresponding to θ, σ and $\sigma\theta$ respectively. The symmetric pair $(\mathfrak{g}, \mathfrak{h}^a)$ is said to be the *associated symmetric pair of* $(\mathfrak{g}, \mathfrak{h})$. Note that $\mathfrak{h}^a = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}$ and that $(\mathfrak{h}^a)^a = \mathfrak{h}$. Fix a maximally abelian subspace $\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}$ of $\mathfrak{p} \cap \mathfrak{q}$. Then $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) \equiv \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p},\mathfrak{q}})$ satisfies the axiom of root system (see [Ro] Theorem 5, [OS84] Theorem 2.11) and is called the *restricted root system of* $(\mathfrak{g}, \mathfrak{h})$. As $\sigma\theta \equiv \mathrm{id}$ on $\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}$, we have a direct sum decomposition of the root space $\mathfrak{g}(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}, \lambda) = (\mathfrak{g}(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}, \lambda) \cap \mathfrak{h}^a) + (\mathfrak{g}(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}, \lambda) \cap \mathfrak{q}^a)$. We define a map

$$(m^+, m^-)\colon \Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) \to \mathbb{N} \times \mathbb{N}, \tag{4.11.1}$$

by $m^+(\lambda) := \dim(\mathfrak{g}(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}, \lambda) \cap \mathfrak{h}^a)$, $m^-(\lambda) := \dim(\mathfrak{g}(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}, \lambda) \cap \mathfrak{q}^a)$. Note that if $(\mathfrak{g}, \mathfrak{h})$ is a Riemannian symmetric pair, then $\mathfrak{h}^a = \mathfrak{g}$ and $m^- \equiv 0$. A map $\varepsilon\colon \Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) \cup \{0\} \to \{1, -1\}$ is said to be a *signature of* $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}})$ if ε is a semigroup homomorphism with $\varepsilon(0) = 1$. It is easy to see that a signature is determined by its restriction to Ψ, a fundamental system for $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}})$ and that any map $\Psi \to \{1, -1\}$ is uniquely extended to a signature.

To a signature ε of $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}})$, we associate an involution σ_ε of $\mathfrak{g}$ defined by $\sigma_\varepsilon(X) := \varepsilon(\lambda)\sigma(X)$ if $X \in \mathfrak{g}(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}; \lambda)$, $\lambda \in \Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) \cup \{0\}$. Then σ_ε defines a symmetric pair $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$. The set

$$F((\mathfrak{g}, \mathfrak{h})) := \{(\mathfrak{g}, \mathfrak{h}_\varepsilon) : \varepsilon \text{ is a signature of } \Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}})\}$$

is said to be an *ε-family of symmetric pairs* ([OS84] §6). Among an ε-family, there is a distinguished symmetric pair called *basic* characterized by,

$$m^+(\lambda) \geq m^-(\lambda) \text{ for any } \lambda \in \Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) \text{ such that } \frac{\lambda}{2} \notin \Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}).$$

It is known that there exists a basic symmetric pair of $F = F((\mathfrak{g}, \mathfrak{h}))$ unique up to isomorphisms ([OS84] Proposition 6.5).

Example 4.11.1.

i) $\{(\mathfrak{sl}(n, \mathbb{R}), \mathfrak{so}(p, n-p)) : 1 \leq p \leq n\}$ is an ε-family with $(\mathfrak{sl}(n, \mathbb{R}), \mathfrak{so}(n))$ basic. This family is an example of the so-called K_ε-family, which is a special case of an ε-family.

ii) For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (a quaternionic number field), we write $U(p, q; \mathbb{F})$ for $O(p, q), U(p, q)$ and $Sp(p, q)$, respectively. We fix $r < q < p$. Then

$$\{(\mathfrak{u}(r, p+q-r; \mathbb{F}), \mathfrak{u}(k, p-k; \mathbb{F}) + \mathfrak{u}(r-k, q-r+k; \mathbb{R}) : 0 \leq k \leq r\}$$

is an ε-family with $(\mathfrak{u}(r, p+q-r; \mathbb{F}), \mathfrak{u}(r, p-r; \mathbb{F}) + \mathfrak{u}(q; \mathbb{R}))$ basic.

Now we are ready to state an application of Theorem 4.10 to symmetric spaces:

Corollary 4.11.2 ([Ko92b] Theorem 1.4). *If a semisimple symmetric space G/H admits a compact Clifford-Klein form, then the associated symmetric pair $(\mathfrak{g}, \mathfrak{h}^a)$ is basic in the ε-family $F((\mathfrak{g}, \mathfrak{h}^a))$.*

Sketch of Proof. We apply Theorem 4.10 with "H":$= H^a_\varepsilon$ and "L":$= H^a$. Then the assumptions (4.10)(a) and (4.10)(b) follow from the following lemma (see [Ko92b] Lemma 4.5.2). □

Lemma 4.11.3. *With notations as above, let $(\mathfrak{g}, \mathfrak{h})$ be basic in the ε-family $F = F((\mathfrak{g}, \mathfrak{h}))$ and $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$ be not basic in F. Then we have*

1) $\mathfrak{a}(H^a) = \mathfrak{a}({H_\varepsilon}^a)$.
2) $\mathrm{d}(H^a) > \mathrm{d}({H_\varepsilon}^a)$.

Here are some examples of symmetric pair $(\mathfrak{g}, \mathfrak{h})$ such that the associated pair $(\mathfrak{g}, \mathfrak{h}^a)$ is basic in its ε-family.

Example 4.11.4. As usual, we write $(\mathfrak{g}, \mathfrak{k})$ for a Riemannian symmetric space if $\mathfrak{g}$ is a Lie algebra of noncompact reductive Lie group G. In the following table $\mathbb{F}$ denotes $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

$(\mathfrak{g}, \mathfrak{h})$	$(\mathfrak{g}, \mathfrak{h}^a)$ is basic
$(\mathfrak{g}, \mathfrak{g})$	$(\mathfrak{g}, \mathfrak{k})$
$(\mathfrak{g}, \mathfrak{k})$	$(\mathfrak{g}, \mathfrak{g})$
$(\mathfrak{g} + \mathfrak{g}, \operatorname{diag} \mathfrak{g})$	$(\mathfrak{g} + \mathfrak{g}, \operatorname{diag} \mathfrak{g})$
$(\mathfrak{g}_\mathbb{C}, \mathfrak{k}_\mathbb{C})$	$(\mathfrak{g}_\mathbb{C}, \mathfrak{g})$
$(\mathfrak{u}(p, q; \mathbb{F}), \mathfrak{u}(m, q; \mathbb{F}) + \mathfrak{u}(p-m; \mathbb{F}))$	$(\mathfrak{u}(p, q; \mathbb{F}), \mathfrak{u}(m; \mathbb{F}) + \mathfrak{u}(p-m, q; \mathbb{F}))$

A stupid remark is that the associated symmetric pair $(\mathfrak{g}, \mathfrak{h}^a)$ of a Riemannian symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is $(\mathfrak{g}, \mathfrak{g})$ which is obviously basic, and that of a group manifold $(\mathfrak{g}' + \mathfrak{g}', \operatorname{diag} \mathfrak{g}')$ is again a group manifold which is also basic. This is consistent with the fact that both Riemannian symmetric spaces and group manifolds admit compact Clifford-Klein forms (cf. §4.5 and §4.6). See [Ko92b] Table (4.4) (and also [Ko90b] Table (5.4.3)) for a list of semisimple symmetric spaces which are proved by this method not to admit compact Clifford-Klein forms.

4.12. Semisimple orbits

Another typical example of a homogeneous space of reductive type is a semisimple orbit $\operatorname{Ad}(G) \cdot X \simeq G/Z_G(X)$, where $X \in \mathfrak{g}$ is a semisimple element (Example 2.6.3 (iv)). We apply Theorem 4.10 to semisimple orbits.

Corollary 4.12 (see [Ko90a,92a], [Ko92b]; [BL92]). *Let G be a real reductive linear Lie group, and X a semisimple element of $\mathfrak{g}$. If $G \cdot X \simeq G/Z_G(X)$ admits a compact Clifford-Klein form, then the orbit $G \cdot X$ carries a G-invariant complex structure.*

Before giving a sketch of proof, we remark that a simple group G cannot be a complex group if there is a nonzero semisimple element $X \in \mathfrak{g}$ such that $\operatorname{Ad}(G) \cdot X \simeq G/Z_G(X)$ admits a compact Clifford-Klein form. In fact, since $\operatorname{rank} G = \operatorname{rank} Z_G(X)$, we have $\mathbb{R}\text{-rank}\, G = \mathbb{R}\text{-rank}\, Z_G(X)$ if G is a complex reductive Lie group. Then there is no infinite discontinuous group acting on $G/Z_G(X)$ by Theorem 2.5 (the Calabi-Markus phenomenon). Therefore $G/Z_G(X)$ does not admit a compact Clifford-Klein form. This means that G itself is not a complex group. Nevertheless Corollary 4.12 asserts that the homogeneous space $G/Z_G(X)$ carries a G-invariant complex structure. This is because $G/Z_G(X)$ is realized as an elliptic orbit, which is a crucial point of the proof.

Sketch of Proof. Let $X = X_e + X_h$ be a decomposition of a semisimple element $X \in \mathfrak{g}$, where X_e is elliptic and X_h is hyperbolic such that $[X_e, X_h] = 0$. Applying Theorem 4.10 with $L := Z_G(X_e)$ and $H := Z_G(X)$, we obtain Corollary 4.12. An alternative proof is given in [BL92] based on symplectic structure. □

Let X be an elliptic element of $\mathfrak{g}$. We make here a quick review that an elliptic orbit $\operatorname{Ad}(G) \cdot X \simeq G/Z_G(X)$ has a rich geometric structure. It is well-known that an elliptic orbit $G/Z_G(X)$ carries a G-invariant complex structure via the generalized Borel embedding into the generalized flag variety $G_U/Z_{G_U}(X)$:

$$G/Z_G(X) \subset G_U/Z_{G_U}(X).$$

See for example [KoO90] Appendix for a proof. (This embedding is a usual Borel embedding if $G/Z_G(X)$ is a Hermitian symmetric space, equivalently, if $G/Z_G(X)$ is a symmetric space and if $Z_G(X)$ is compact (see [He78]). $G/Z_G(X)$ is said to be '*dual manifolds of Kähler C-space*' in the sense of Griffiths-Schmid if $Z_G(X)$ is compact; $G/Z_G(X)$ is said to be a '$\frac{1}{2}$-*Kähler symmetric space*' in the sense of Berger if $G/Z_G(X)$ is a symmetric space and if $Z_G(x)$ is noncompact.)
A G-invariant symplectic structure on the elliptic orbit is induced from the one on the coadjoint orbit through the identification $\mathfrak{g} \simeq \mathfrak{g}^*$, which is given by a non-degenerate G-invariant bilinear form B on $\mathfrak{g}$ (e.g. the Killing form if $\mathfrak{g}$ is semisimple). The orbit $G/Z_G(X)$ also carries a G-invariant (indefinite-)Kähler structure induced by B. The indefinite Kähler structure is then compatible with the symplectic structure.

We note that Clifford-Klein forms of an elliptic orbit inherit these structures. Corollary 4.7 asserts that the following homogeneous manifolds

$$U(2, 2n)/U(1) \times U(1, 2n),$$

$$SO(2, 2n)/U(1, n),$$

$$SO(4, 3)/SO(2) \times SO(4, 1)$$

admit compact Clifford-Klein forms. Furthermore, these are realized as elliptic orbits of the adjoint action. So we obtain new examples of compact indefinite-Kähler manifolds.

Finally, it should be in sharp contrast that a hyperbolic orbit does not admit an infinite discontinuous group (Example 2.6.1).

4.13. Examples

We give here a number of examples of homogeneous spaces which are proved not to have (or to have) Clifford-Klein forms by the method in §4. These examples may be helpful to reveal the applications and limitations of various methods known so far (e.g. [Ko88], [Ko89a], [Ko90b], [KoO90], [Ko90a,92a], [Ko92b], [BL92], [Zi94], [LaMZ94], [Ko94c], [Co94]) in studying the existence problem of compact Clifford-Klein forms, which has not yet found a final answer (see also Notes in §4.14).

Example 4.13.1. First we consider a semisimple symmetric space

$$G/H = SO(i+j, k+l)/SO(i,k) \times SO(j,l), \quad (i \le j, k, l).$$

It follows from Theorem 4.10 (or Corollary 4.11) that if G/H admits a compact Clifford-Klein form, then G/H is compact, or H is compact, or $0 = i < l \le j - k$. Moreover, we have $jkl \equiv 0 \mod 2$ as an application of Hirzebruch's proportionality principle in §3 (see Corollary 3.12.1 and a remark after Conjecture 3.12.4).

Conversely, if $(i, j, k, l) = (0, 2n, 1, 1), (0, 4n, 1, 3), (0, 4, 2, 1), (0, 8, 1, 7)$ or if $i = l = 0$, then there exists a compact Clifford-Klein form of G/H (see Corollary 4.7).

Example 4.13.2.

i) A semisimple symmetric space

$$G/H = SO^*(2n)/U(l, n-l)$$

does not admit a compact Clifford-Klein form if $3l \le 2n \le 6l$ and if $n \ge 3$. We explain it in Table 4.13.4 together with similar examples. It admits compact Clifford-Klein forms if $(n, l) = (4, 1)$, $(4, 3)$, $(3, 1)$, $(3, 2)$, $(2, 1)$, $l = 0$ or $l = n$. In fact, we have a local isomorphism $SO^*(8)/U(1,3) \approx SO(2,6)/U(1,3)$, which is proved to admit compact Clifford-Klein forms in Corollary 4.7. It is trivial in the cases $(3,1)$, $(3,2)$ and $(2,1)$ because G/H is then compact. G/H is a Riemannian symmetric space in the cases $l = 0$ or $l = n$ (see Theorem 4.2).

i) A semisimple symmetric space

$$G/H = SO^*(4n)/SO^*(4p+2) \times SO^*(4n-4p-2)$$

does not admit compact Clifford-Klein forms if $1 \le p \le n-2$ owing to Corollary 4.12. We note that $SO^*(8)/SO^*(6) \times SO^*(2)$ (namely, in the case $(n, p) = (2, 1)$) admits compact Clifford-Klein forms. Similarly, there exist compact Clifford-Klein forms in the case $(n, p) = (2, 0)$. In fact we have a local isomorphism $SO^*(8)/SO^*(2) \times SO^*(6) \approx SO(2,6)/U(1,3)$, which is in the list of Corollary 4.7.

Example 4.13.3. Suppose that $G_{\mathbb{C}}/H_{\mathbb{C}}$ is a complex irreducible semisimple symmetric space. It is a conjecture that $G_{\mathbb{C}}/H_{\mathbb{C}}$ admits compact Clifford-Klein forms if and only if $G_{\mathbb{C}}/H_{\mathbb{C}}$ is locally isomorphic to a group manifold. One can prove that $G_{\mathbb{C}}/H_{\mathbb{C}}$ does not have compact Clifford-Klein forms unless $G_{\mathbb{C}}/H_{\mathbb{C}}$ is locally isomorphic to either a group manifold, $SO(2n+2, \mathbb{C})/SO(2n+1, \mathbb{C})$, $SL(2n, \mathbb{C})/Sp(n, \mathbb{C})$ $(n \ge 2)$, or $E_{6,\mathbb{C}}/F_{4,\mathbb{C}}$ (see [Ko92b] Example 1.9).

Example 4.13.4.

G/H does not have a compact Clifford-Klein form		
G	H	range of parameters
$SL(n,\mathbb{R})$	$Sp(l,\mathbb{R})$	$0 < 2l \leq n-2$
$SL(n,\mathbb{C})$	$Sp(l,\mathbb{C})$	$0 < 2l \leq n-1$
$SL(n,\mathbb{C})$	$SO(l,\mathbb{C})$	$0 < l \leq n$
$SO^*(2n)$	$U(l,n-l)$	$3l \leq 2n \leq 6l$, $n \geq 3$
$Sp(n,\mathbb{R})$	$Sp(l,\mathbb{C})$	$0 < 2l \leq n$
$SU^*(2n)$	$SO^*(2l)$	$1 < l \leq n$
$SL(2n,\mathbb{C})$	$SU(p,q)$	$p+q < n$ or $p = q$ $(pq > 0)$
$SL(2n,\mathbb{R})$	$SO(p,q)$	$p+q < n$ or $p = q$ $(pq > 0)$

Table 4.13.4.

These examples except for $Sp(n,\mathbb{R})/Sp(l,\mathbb{C})$ with $n = 2l$ are proved by Theorem 4.10. In applying Theorem 4.10, the choice of "L" is not unique. Here is an example of the choice of "L" for G/H:
$SO(l,n-l)$ for $SL(n,\mathbb{R})/Sp(n,\mathbb{R})$, $U(l,n-l)$ for $SL(n,\mathbb{C})/Sp(l,\mathbb{C})$,
$U([\frac{l}{2}],n-[\frac{l}{2}])$ for $SL(n,\mathbb{C})/SO(l,\mathbb{C})$, $SO^*(4l+2)$ for $SO^*(2n)/U(l,n-l)$,
$U(l,n-l)$ for $Sp(n,\mathbb{R})/Sp(l,\mathbb{C})$, $Sp([\frac{l}{2}],n-[\frac{l}{2}])$ for $SU^*(2n)/SO^*(2l)$,
$U(p,2n-p)$ or $Sp(p,\mathbb{C})$ for $SL(2n,\mathbb{C})/U(p,q)$, and
$SO(p,2n-p)$ or $Sp(p,\mathbb{R})$ for $SL(2n,\mathbb{R})/SO(p,q)$, respectively

The proof for $Sp(2l,\mathbb{R})/Sp(l,\mathbb{C})$ is different. We need to use an argument of Hirzebruch's proportionality principle (see §3; [Ko89a] Example (4.11)).

For more examples of the above type, we refer to Table 4.4 and Table 5.3 in [Ko92b].

Example 4.13.5.

G/H does not have a compact Clifford-Klein form			
G	H	range of parameters	
$(n > m > 1)$		m is even	m is odd
$SL(n,\mathbb{R})$	$SL(m,\mathbb{R})$	$n > \frac{3}{2}m$	$n > \frac{3}{2}m + \frac{3}{2}$
$SU^*(2n)$	$SU^*(2m)$	$n > \frac{3}{2}m - 1$	$n > \frac{3}{2}m + \frac{1}{2}$
$SO^*(2n)$	$SO^*(2m)$	$n > \frac{3}{2}m - 1$	$n > \frac{3}{2}m - \frac{1}{2}$
$Sp(n,\mathbb{R})$	$Sp(m,\mathbb{R})$	$n > \frac{3}{2}m + 1$	$n > \frac{3}{2}m + \frac{3}{2} + \delta_{m3}$
$SL(n,\mathbb{C})$	$SL(m,\mathbb{C})$	$n > \frac{3}{2}m - 1$	$n > \frac{3}{2}m + \frac{1}{2}$
$SO(n,\mathbb{C})$	$SO(m,\mathbb{C})$	$n > \frac{3}{2}m - 1$	$n > \frac{3}{2}m - \frac{1}{2}$
$Sp(n,\mathbb{C})$	$Sp(m,\mathbb{C})$	$n > \frac{3}{2}m$	$n > \frac{3}{2}m + \frac{1}{2} + \delta_{m3}$

Table 4.13.5 (a)

We indicate the proof of Table 4.13.5 (a) for $G = SL(n,\mathbb{R}) \supset H = SL(m,\mathbb{R})$ with m even. We want to show that G/H does not admit a compact Clifford-Klein form if $\frac{2}{3}n > m$. We take a subgroup

$$L := SO(\frac{m}{2}, n - \frac{m}{2}) \subset G.$$

We identify $\mathfrak{a}(G)$ with $\mathbb{R}^n$ by choosing a suitable coordinate of $\mathfrak{a}_G$, so that the action of the Weyl group $W_G \simeq \mathfrak{S}_n$ is given by permutation of coordinates. Then we have

$$\mathfrak{a}(L) = \left\{ (y_1, \dots, y_{\frac{1}{2}m}, -y_1, \dots, -y_{\frac{1}{2}m}, 0, \dots, 0) : y_j \in \mathbb{R}\ (1 \le j \le \frac{1}{2}m) \right\},$$

$$\mathfrak{a}(H) = \left\{ (x_1, \dots, x_m, 0, \dots, 0) : x_j \in \mathbb{R},\ (1 \le j \le m),\ \sum_{j=1}^{m} x_j = 0 \right\},$$

regarded as subsets in $\mathfrak{a}_G/W_G$. Therefore, we have $\mathfrak{a}(L) \subset \mathfrak{a}(H)$.

On the other hand,

$$\begin{aligned} d(L) - d(H) &= \frac{m}{2}(n - \frac{m}{2}) - (\frac{1}{2}m(m+1) - 1) \\ &= \frac{m}{2}(n - \frac{3}{2}m - 1) + 1 > 0. \end{aligned}$$

Now Theorem 4.10 shows that G/H does not admit a compact Clifford-Klein form.

Other cases in Table 4.13.5 (a) are proved similarly from Theorem 4.10. A choice of L for each (G, H) is listed in the following Table.

A choice of L for Table 4.13.5 (a)			
G	H	L	
($n > m > 1$)		m is even	m is odd
$SL(n, \mathbb{R})$	$SL(m, \mathbb{R})$	$SO(\frac{m}{2}, n - \frac{m}{2})$	$SO(\frac{m-1}{2}, n - \frac{m-1}{2})$
$SU^*(2n)$	$SU^*(2m)$	$Sp(\frac{m}{2}, n - \frac{m}{2})$	$Sp(\frac{m-1}{2}, n - \frac{m-1}{2})$
$SO^*(2n)$	$SO^*(2m)$	$U(\frac{m}{2}, n - \frac{m}{2})$	$U(\frac{m-1}{2}, n - \frac{m-1}{2})$
$Sp(n, \mathbb{R})$	$Sp(m, \mathbb{R})$	$U(\frac{m}{2}, n - \frac{m}{2})$	$U(\frac{m-1}{2}, n - \frac{m-1}{2})$
$SL(n, \mathbb{C})$	$SL(m, \mathbb{C})$	$U(\frac{m}{2}, n - \frac{m}{2})$	$U(\frac{m-1}{2}, n - \frac{m-1}{2})$
$SO(n, \mathbb{C})$	$SO(m, \mathbb{C})$	$SO(\frac{m}{2}, n - \frac{m}{2})$	$SO(\frac{m-1}{2}, n - \frac{m-1}{2})$
$Sp(n, \mathbb{C})$	$Sp(m, \mathbb{C})$	$Sp(\frac{m}{2}, n - \frac{m}{2})$	$Sp(\frac{m-1}{2}, n - \frac{m-1}{2})$

Table 4.13.5 (b)

Example 4.13.6.

G/H does not have a compact Clifford-Klein form		
G	H	range of parameters
$(j \geq i > 0, p \geq i, q \geq j)$		
$O(p,q)$	$O(i,j)$	$j \neq q$ or $p > q$ or $(p+1)iq \equiv 1 \mod 2$
$U(p,q)$	$U(i,j)$	$j \neq q$ or $p > q$
$Sp(p,q)$	$Sp(i,j)$	$j \neq q$ or $p > q$

Table 4.13.6 (a)

For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (a quaternionic number field), we write $U(p,q;\mathbb{F})$ for $O(p,q), U(p,q)$ and $Sp(p,q)$, respectively.

Sketch of Proof. To prove Table 4.13.6 (a), we first choose $L = U(i,q;\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$). Then $\mathfrak{a}(H) = \mathfrak{a}(L)$ because $i \leq j \leq q$, and $\mathrm{d}(H) \leq \mathrm{d}(L)$. Here the equality holds if and only if $j = q$. Therefore, it follows from Theorem 4.10 that $U(p,q;\mathbb{F})/U(i,j;\mathbb{F})$ admits a compact Clifford-Klein form only if $j = q$. Now $U(p,q;\mathbb{F})/U(i,q;\mathbb{F})$ admits a compact Clifford-Klein form if and only if so does a reductive symmetric space $U(p,q;\mathbb{F})/U(p-i;\mathbb{F}) \times U(i,q;\mathbb{F})$. Then use Corollary 4.11.2, and we have $p > q$. See Example 4.13.1 for the condition $(p+1)iq \equiv 1 \mod 2$ in the case where $\mathbb{F} = \mathbb{R}$. □

Example 4.13.7.

1) Suppose $G = SL(n,\mathbb{R}) \supset H = SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$ (k-times) ($n \geq 2k > 0$). If $n > 3$, then there does not exist a compact Clifford-Klein form of G/H.
2) Suppose $G = SL(n,\mathbb{C}) \supset H = SL(2,\mathbb{C}) \times \cdots \times SL(2,\mathbb{C})$ (k-times) ($n \geq 2k > 0$). If $n \geq 3$, then there does not exist a compact Clifford-Klein form of G/H. In particular, $SL(3,\mathbb{C})/SL(2,\mathbb{C})$ does not admit a compact Clifford-Klein form ([Ko90a,92a] Example7).

Proof.

1) Take $L = SO(k, n-k)$. Then $\mathfrak{a}(H) = \mathfrak{a}(L)$ and $\mathrm{d}(L) - \mathrm{d}(H) = k(n-k) - 2k = k(n-k-2) > 0$ if $n > 4$ and if $n \geq 2k > 0$. Take $L = Sp(2, \mathbb{R})$ for $n = 4$.
2) Take $L = SU(k, n-k)$. Then $\mathfrak{a}(H) = \mathfrak{a}(L)$ and $\mathrm{d}(L) - \mathrm{d}(H) = 2k(n-k) - 3k = k(2n-2k-3) > 0$ if $n \geq 2k$. □

For more examples of homogeneous spaces without compact Clifford-Klein forms, we refer to [Ko92b] Table 5.3.

4.14. Notes

The existence problem of a compact Clifford-Klein form of a homogeneous space G/H has its origin in the uniformization theorem of Riemann surfaces due to Klein, Poincaré and Koebe (Example 0.5.5). In this case, the homogeneous space is the Poincaré plane $G/H = PSL(2, \mathbb{R})/SO(2)$, which is the simplest example a Riemannian symmetric space of the non-compact type.

At the beginning of the 1960's, the existence problem of a compact Clifford-Klein form of any Riemannian symmetric space was settled affirmatively by Borel, Harish-Chandra, Mostow-Tamagawa ([Bo63], [BoHa62], [MoTa62]; see Theorem 4.2). Contrary to this, around the same time, it was found by Calabi, Markus and Wolf that certain pseudo-Riemannian symmetric spaces (of rank 1) do not admit compact Clifford-Klein forms (Theorem 2.2.1; [CM62], [Wo62], [Wo64], see also Notes in §2.12). Clifford-Klein forms of the real hyperbolic space $SO(p, q)/SO(p-1, q)$ was studied by R.Kulkarni in the begining of 1980's, and in particular the Calabi-Markus phenomenon of this special case was settled [Ku81].

It was in the late 1980's that a systematic study of the existence problem of compact Clifford-Klein forms was basically begun for a general homogeneous space of reductive type, which is a wide class of homogeneous spaces containing Riemannian symmetric spaces, reductive group manifolds, pseudo-Riemannian symmetric spaces and semisimple orbits of the adjoint action. An overview is given in [Ko90a,92a]. A sufficient condition for the existence of compact Clifford-Klein forms (Theorem 4.4) was proved in [Ko89a]. The proof rests on an argument in the continuous setting, namely the criterion for proper actions (Theorem 2.9.1) and the criterion for the compactness of a double coset space (Theorem 4.9). Among six series and six isolated homogeneous spaces that admit compact Clifford-Klein forms in Corollary 4.7, (2-b) and (3-b) (i.e. in the case $SO(p+1, q)/SO(p, q)$) were first proved by Kulkarni ([Ku81]), (4-a), (4-b), (5-a) and (5-b) in [Ko94c], and other cases in [Ko89a] or [Ko90a,92a].

Conversely, necessary conditions for the existence of compact Clifford-Klein forms have been also studied since the late 1980's by various approaches. That is,

i) the Calabi-Markus phenomenon and the criterion for proper actions ([Ko89a], [Fr94], [Ko94b]),
ii) Hirzebruch's proportionality principle ([KoO90]),
iii) Comparison theorem ([Ko92b]),
iv) Construction of symplectic forms ([BL92]),
v) Ergodic theory ([Zi94], [LaMZ94]).

We have explained (i), (ii) and (iii) in §2 (e.g. Theorem 2.5, Theorem 2.9.1, Theorem 2.9.2), §3 (e.g. Corollary 3.12.1) and §4 (e.g. Theorem 4.10, Corollary 4.11.2, Corollary 4.12), respectively. We have not dealt with (iv) and (v) here, which are so different from other methods. However, some of examples obtained so far by other methods (e.g. [BL92], [Zi94], [LaMZ94], [Co94]) are also proved (sometimes in a stronger form) by (i), (ii) and (iii) in the reductive case.

In order to clarify the applications and limitations of various methods, we will examine some typical classes of homogeneous spaces. We note that the setting of the above results due to Benoist, Labourier, Zimmer, Mozes, Corllete ([BL92], [Zi94], [LaMZ94], [Co94]) requires either

$$Z_G(H) \text{ contains } \mathbb{R} \tag{4.14.1}$$

or

$$Z_G(H) \text{ contains a semisimple Lie group with } \mathbb{R}\text{-rank} \geq 2. \tag{4.14.2}$$

First, suppose that G/H is a semisimple symmetric space, which we assume G/H is irreducible for simplicity. We have seen that the methods (i), (ii) and (iii) give rise to necessary conditions for the existence of compact Clifford-Klein forms, namely, Theorem 2.5, Corollary 3.12.1 and Corollary 4.11.2, which are not covered by one another. On the other hand, the assumption (4.14.1) is satisfied if and only if G/H is a para-Hermitian symmetric space which does not admit compact Clifford-Klein forms (Example 2.7.2 (1)) by the Calabi-Markus phenomenon. The assumption (4.14.2) is never satisfied in the symmetric case.

Second, suppose that G/H is a semisimple orbit. Then the assumption (4.14.1) is satisfied. In this case, Benoist-Labourie using the method (iv) gave an alternative proof of Corollary 4.12 (the method (iii)) in a strengthened form ([BL92] Theorem 1).

Typical examples of homogeneous spaces which satisfy the assumption (4.14.2) are those listed in some part of Example 4.13.4, Example 4.13.4 and Example 4.13.6. Zimmer proved that $SL(n,\mathbb{R})/SL(m,\mathbb{R})$ does not admit a compact Clifford-Klein form if $\frac{1}{2}n > m$. His approach is based on Ratner's theorem and ergodic theory ([Zi94] Corollary 1.3). Recently Labourier-Mozes-Zimmer simplified and extended Zimmer's approach to manifolds locally modeled on homogeneous spaces ([LaMZ94]). Their result allows the case $\frac{1}{2}n = m$ in this example. On the other hand, the method (iii) (see §4) shows that $SL(n,\mathbb{R})/SL(m,\mathbb{R})$ does not admit a compact Clifford-Klein form if $\frac{2}{3}n > m$ for even m (or if $\frac{2}{3}n > m+1$ when m is odd), which is a slightly stronger result if n and m are large enough (see Example 4.13.5). Similarly, the indefinite Stiefel manifolds $U(p,q;\mathbb{F})/U(i,j;\mathbb{F})$ $(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$ $(j \geq i > 0,\ p \geq i,\ q \geq j)$ does not admit a compact Clifford-Klein form if $j \neq q$ or if $p > q$ (Example 4.13.6). The special case with $(p,q,i,j) = (n,2,m,1)$, $n > 2m$ and $\mathbb{F} = \mathbb{H}$ implies a result that $Sp(n,2)/Sp(m,1)$ $(n > 2m)$ does not admit a compact Clifford-Klein, which was announced by Corlette in ICM-94 ([Co94] Theorem 12). Corlette's approach apparently differs from other methods (i)-(v).

Since the beginning of the 1990's, the existence problem of compact Clifford-Klein forms of homogeneous spaces with noncompact isotropy subgroups has been observed to have connections with other branches of mathematics. For example, in the classification of certain Anosov flows, Benoist-Foulon-Labourier [BFL92] encountered the non-existence problem of compact Clifford-Klein forms of G/H where G is real reductive and H is a semisimple part of a Levi subgroup of a maximal parabolic subgroup. Also, symplectic geometry [BL92], ergodic theory and Ratner's theory [Zi94], [LaMZ94], harmonic maps [Co94], unitary representation theory [Ko94a] have come to be related with the existence problem of compact Clifford-Klein forms.

Many basic questions about Clifford-Klein forms of non-Riemannian homogeneous spaces have not yet found a final answer. As observed in the recent developments mentioned above, it looks so fascinating that different areas of mathematics seem to be closely related to the existence problem of compact Clifford-Klein forms.

§5. Open problems

At the end of the lecture notes, we collect some open problems.

Open problems 5.

1) (Problem 1.7.2) Find a criterion for the Calabi-Markus phenomenon for a general Lie group.
2) (Problem 1.11.4) Find a criterion for $L \pitchfork H$ in G if L, H are closed subgroups of G.
3) (Problem 1.12.2) In which class of Lie groups, does the following equivalence hold ?

$$H \pitchfork L \text{ in } G \Leftrightarrow \text{ the pair } (L, H) \text{ has the property (CI) in } G.$$

4) (Problem 2.2.6) Is there a sufficient condition on some positiveness of the curvature of a pseudo-Riemannian manifold M (in particular, Lorentz manifold) assuring that M is non-compact with a finite fundamental group ?
5) (Problem 1.7.2) Find a criterion on G/H which admits a compact Clifford-Klein form. Also find a criterion on G/H which admits a non-compact Clifford-Klein form of finite volume.
6) Suppose that G/H is a homogeneous space of reductive type which admits a compact Clifford-Klein form. Is there a subgroup L reductive in G which satisfies the conditions both (4.4)(a) and (4.4)(b) ?
7) (Conjecture 3.12.4) Does the inequality

$$\operatorname{rank} G + \operatorname{rank}(H \cap K) \geq \operatorname{rank} H + \operatorname{rank} K.$$

hold if G/H admits a compact Clifford-Klein form ?
8) Is there a Teichmüller theory for a compact Clifford-Klein form ?
9) If there exists a compact Clifford-Klein form of G/H, then does there also exist a noncompact Clifford-Klein form of G/H of finite volume, and vice versa ?
10) ([Wal86]) Is there an analogue of Eisenstein series which describes the decomposition of $L^2(\Gamma\backslash G/H)$ if $\Gamma\backslash G/H$ is a compact Clifford-Klein form or if $\Gamma\backslash G/H$ is a noncompact Clifford-Klein form of finite volume ?

We have already explained most of these problems. Here are short comments for the convenience of the reader.

Problem (5.1) is solved for the reductive case and the simply connected solvable case as we explained in §2.

Problem (5.2) is solved if G is a a reductive group.

Problem (5.3) is a subproblem for Problem (5.2), and in particular, the property (CI) might be the final solution for a simply connected nilpotent group (the conjecture of Lipsman).

Problem (5.4) is a question in pseudo-Riemannian geometry. It is regarded as a "perturbation" of the Calabi-Markus phenomenon, and should be in a good contrast to a classical theorem of Myers [My41].

Problem (5.5) has been one of the main subjects of this lecture. For example, $SO(i+j,k+l)/SO(i,j) \times SO(k,l)$ remains open (see Example 4.13.1 for partial results obtained so far).

Problem (5.6) and Problem (5.7) are subproblems of Problem (5.5). Problem (5.6) asks if a converse of Theorem (4.4) holds. We should remark that it is not true that the Zariski closure of a uniform lattice for G/H of reductive type does not always satisfy the condition (4.4).

We have found a phenomenon that an analogue of the Weil rigidity does not always hold for semisimple symmetric spaces of higher ranks and that an analogue of the Mostow rigidity does not always hold for semisimple symmetric spaces of higher dimensions [Ko93]. There seems to be a large room to study deformation of uniform lattices, which we pose in Problem (5.8).

Regarding to Problem (5.10), we recall that an existence result of compact Clifford-Klein forms (or noncompact ones of finite volume) of Riemannian symmetric spaces (Theorem 4.2) has opened a theory of Eisenstein series in harmonic analysis on square integrable functions over the double coset space $\Gamma\backslash G/H$. It is natural to expect that an existence result for pseudo-Riemannian symmetric spaces could open a theory of harmonic analysis on such nice double coset spaces.

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There has been some recent progress on Open Problems in §5.

Y.Benoist gave a different proof of Theorem 2.9.2 (1) and proved a non-existence theorem of compact Clifford-Klein forms of some homogeneous spaces such as $SO(4n, \mathbb{C})/SO(4n-1, \mathbb{C})$ (see Example 4.13.3) in [1] (see Open Problem 5.5).

É.Ghys studied a fine structure of the deformation of a lattice for a group manifold $G' \times G'/\operatorname{diag} G'$ with $G' = SL(2, \mathbb{C})$ in [2], and T.Kobayashi studied for which homogeneous manifold of reductive type local rigidity of a uniform lattice fails in [3] (see Open Problem 5.8).

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