“Geometric Quantization” of Minimal Nilpotent Orbits
— analysis of minimal representations

Toshiyuki Kobayashi

The University of Tokyo
http://www.ms.u-tokyo.ac.jp/~toshi/

Conference in memory of Souriau’s 90th Birthday
25–29 June 2012, Aix-en-Provence, France

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

Some examples of $O(p+1, q+1)$

nilpotent orbits $\subset \partial\{\overline{\text{elliptic orbits}}\}$

nilpotent orbits $\subset \partial\{\overline{\text{hyperbolic orbits}}\}$

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

§1 What are minimal representations?

§2 $\text{Limit} \circ Q = Q \circ \text{Limit}$?

§3 L^2 model of minimal representations
— L^2 functions on Lagrangian
of minimal nilpotent orbits
(arXiv:1106.3621 with J. Hilgert, J. Möllers)

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

§1 What are minimal representations?

§2 Limit $\circ$ Q = Q $\circ$ Limit ?

§3 L^2 model of minimal representations
— unitary inversion operator $\cdots$ $O(p+1, q+1)$
(Memoirs of AMS, 2011, vol 1000, with G. Mano)

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model
— as a “quantization” of the Kostant–Sekiguchi correspondence
(arXiv:1203.5462 with Hilgert, Möllers and Ørsted)

§5 Deformation of Fourier transform

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform
— interpolation of two minimal reps & Dunkl operators
(Compositio Math (2012) (to appear) with Ben Saïd, Ørsted)

What are minimal reps?

Minimal representations of a real reductive group G

What are minimal reps?

Minimal representations of a real reductive group G

Loosely, minimal representations are

- ‘smallest’ infinite dimensional unitary rep. of G

$$G \to \{\text{unitary operators on } \mathcal{H}\}$$

Hilbert space

What are minimal reps?

Minimal representations of a real reductive group G

Algebraically, minimal reps are infinite dim'l irreducible reps whose annihilators are the Joseph ideals in the enveloping alg $U(\mathfrak{g})$

Loosely, minimal representations are

- 'smallest' infinite dimensional unitary rep. of G

What are minimal reps?

Minimal representations of a real reductive group G

Algebraically, minimal reps are infinite dim'l irreducible reps whose annihilators are the Joseph ideals in the enveloping alg $U(\mathfrak{g})$

Loosely, minimal representations are

- 'smallest' infinite dimensional unitary rep. of G
- almost unique among the unitary dual (continuously many)

What are minimal reps?

Minimal representations of a real reductive group G

Algebraically, minimal reps are infinite dim'l irreducible reps whose annihilators are the Joseph ideals in the enveloping alg $U(\mathfrak{g})$

Loosely, minimal representations are

- 'smallest' infinite dimensional unitary rep. of G
- almost unique among the unitary dual (continuously many)
- of matrix coefficients with "slower" decay at infinity
  (cf. Margulis, Oh: properly discontinuous actions)

What are minimal reps?

Minimal representations of a real reductive group G

Algebraically, minimal reps are infinite dim'l irreducible reps whose annihilators are the Joseph ideals in the enveloping alg $U(\mathfrak{g})$

Loosely, minimal representations are

- 'smallest' infinite dimensional unitary rep. of G
- almost unique among the unitary dual (continuously many)
- of matrix coefficients with "slower" decay at infinity
  (cf. Margulis, Oh: properly discontinuous actions)
- one of simplest 'building blocks' of unitary reps.

Building blocks of unitary reps

unitary reps of Lie groups

Building blocks of unitary reps

unitary reps of Lie groups

↑ decomposition into direct integral (Mautner)

irred. unitary reps of Lie groups

Building blocks of unitary reps

unitary reps of Lie groups

↑ decomposition into direct integral (Mautner)

irred. unitary reps of Lie groups

↑ construction (Mackey, Kirillov, Duflo)

irred. unitary reps of reductive groups

Building blocks of unitary reps

Orbit Philosophy à la Kostant–Kirillov–Duflo–Vogan

unitary reps of Lie groups

↑ decomposition into direct integral (Mautner)

irred. unitary reps of Lie groups

↑ construction (Mackey, Kirillov, Duflo)

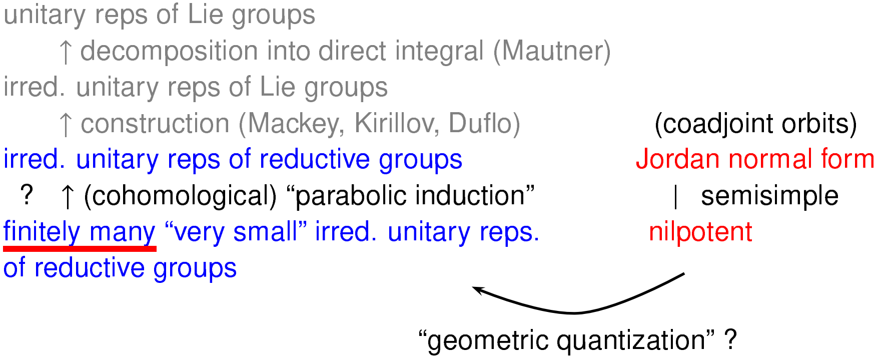
irred. unitary reps of reductive groups

? ↑ “induction functor” (Zuckerman)

finitely many “very small” irred. unitary reps. of reductive groups

Building blocks of unitary reps

Orbit Philosophy à la Kostant–Kirillov–Duflo–Vogan

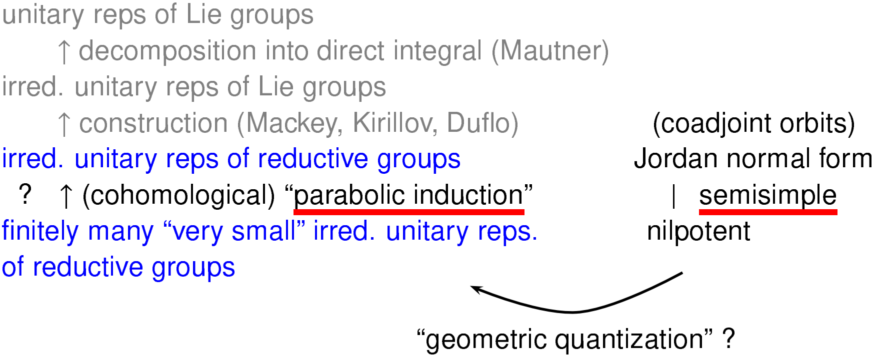


unitary reps of Lie groups
↑ decomposition into direct integral (Mautner)
irred. unitary reps of Lie groups
↑ construction (Mackey, Kirillov, Duflo) (coadjoint orbits)
irred. unitary reps of reductive groups Jordan normal form
? ↑ (cohomological) “parabolic induction” | semisimple
finitely many “very small” irred. unitary reps. nilpotent
of reductive groups

“geometric quantization” ?

Building blocks of unitary reps

Orbit Philosophy à la Kostant–Kirillov–Duflo–Vogan



unitary reps of Lie groups

↑ decomposition into direct integral (Mautner)

irred. unitary reps of Lie groups

↑ construction (Mackey, Kirillov, Duflo)

irred. unitary reps of reductive groups

? ↑ (cohomological) "parabolic induction"

finitely many "very small" irred. unitary reps. of reductive groups

(coadjoint orbits)

Jordan normal form

| semisimple

nilpotent

"geometric quantization" ?

What are minimal reps?

Minimal representations of a real reductive group G

Algebraically, minimal reps are infinite dim'l irreducible reps whose annihilators are the Joseph ideals in $U(\mathfrak{g})$

Loosely, minimal representations are

- 'smallest' infinite dimensional unitary rep. of G
- almost unique among the unitary dual (continuously many)
- of matrix coefficients with "slower" decay at infinity
  (cf. Margulis, Oh: properly discontinuous actions)
- one of 'building blocks' of unitary reps.
- attached to minimal nilpotent coadjoint orbits
  (orbit philosophy à la Kostant–Kirillov–Duflo–Vogan)

What are minimal reps?

Minimal representations of a real reductive group G

Algebraically, minimal reps are infinite dim'l irreducible reps whose annihilators are the Joseph ideals in $U(\mathfrak{g})$

Loosely, minimal representations are

- 'smallest' infinite dimensional unitary rep. of G
- almost unique among the unitary dual (continuously many)
- of matrix coefficients with "slower" decay at infinity
  (cf. Margulis, Oh: properly discontinuous actions)
- one of 'building blocks' of unitary reps.
- attached to minimal nilpotent coadjoint orbits
  (orbit philosophy à la Kostant–Kirillov–Duflo–Vogan)

Minimal reps ⇔ Maximal symmetries

My wish:

Dig out some interesting and (potentially) rich
geometric analysis
inspired by minimal reps.

Study of minimal reps

$\Downarrow$

Try to forget (a part of) rep theory!

Minimal reps $\Leftrightarrow$ Maximal symmetries

My wish:

surprisingly

Dig out some interesting and ~~(potentially)~~ rich

geometric analysis

inspired by minimal reps.

Our concern $\cdots\cdots\cdots$ mainly with simple gp of type D

Cf. Segal–Shale–Weil rep $\cdots$ split simple gp of type C

(e.g. R. Howe $\cdots$ theta correspondence)

Minimal reps $\Leftrightarrow$ Maximal symmetries

My wish:

surprisingly

Dig out some interesting and ~~(potentially)~~ rich geometric analysis inspired by minimal reps.

Viewpoint:

Minimal representation ($\Leftarrow$ group)

$\approx$ Very large symmetries ($\Leftarrow$ rep. space)

Geometric analysis on minimal reps

[1] Fock model and Segal–Bargmann transform for minimal reps $\cdots$
77 pp. arXiv:1203.5462

[2] Minimal representations via Bessel operators $\cdots$ 72 pp. arXiv:1106.3621

[3] Laguerre semigroup and Dunkl operators $\cdots$
72 pp. Compositio Math (2012) (to appear)

[4] Schrödinger model of minimal representations of $O(p,q)$ $\cdots$
Memoirs of Amer. Math. Soc. (2011), no.1000, 132 pp.

[5] Algebraic analysis on minimal representations $\cdots$
Publ. RIMS (2011), 28 pp.

[6] Geometric analysis of small unitary reps of $GL(n,\mathbb{R})$ $\cdots$
J. Funct. Anal. (2011), 39 pp.

[7] Special functions associated to a fourth order differential equation $\cdots$
Ramanujan J. Math (2011) I, II, 50 pp.

[8] Inversion and holomorphic extension $\cdots$
R. Howe 60th birthday volume (2007), 65 pp.

[9] Analysis on minimal representations $\cdots$
Adv. Math. (2003) I, II, III, 110 pp.

Collaborated with S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers, B. Ørsted & M. Pevzner

Kirillov–Kostant–Souriau symplectic form

$G \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}$ adjoint action

Lie group Lie algebra

Kirillov–Kostant–Souriau symplectic form

$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^*$ coadjoint action

Kirillov–Kostant–Souriau symplectic form

$$\begin{tikzcd} & & \mathfrak{g}^* \\ & \arrow[ur, bend left, "\mathrm{Ad}^*"] & \cup \\ G & \curvearrowright & \mathcal{O}_\lambda := \mathrm{Ad}^*(G) \cdot \lambda \end{tikzcd}$$

coadjoint action

coadjoint orbit

symplectic mfd

Kirillov–Kostant–Souriau symplectic form

$$\begin{tikzcd} & \mathfrak{g}^* \\ G \arrow[ur, "\mathrm{Ad}^*"] \arrow[r, phantom, "\curvearrowright"] & \mathcal{O}_\lambda := \mathrm{Ad}^*(G)\cdot\lambda \arrow[u, phantom, "\cup"] \end{tikzcd}$$

coadjoint action

coadjoint orbit

symplectic mfd

<u>Fact</u> $\mathcal{O}_\lambda$ becomes a symplectic manifold by the Kirillov–Kostant–Souriau symplectic form.

$$\begin{tikzcd} \mathfrak{g}\times\mathfrak{g} \arrow[r] \arrow[d] & \mathbb{R},\ (X,Y)\mapsto\lambda([X,Y]) \\ \mathfrak{g}/\mathfrak{g}_\lambda\times\mathfrak{g}/\mathfrak{g}_\lambda \arrow[ur, dashed, "\omega"'] \arrow[d, phantom, "\wr\qquad\wr"] & \\ T_o\mathcal{O}_\lambda\times T_o\mathcal{O}_\lambda & \end{tikzcd}$$

Kirillov–Kostant–Souriau symplectic form

$$\begin{tikzcd} & \mathfrak{g}^* \arrow[d, phantom, "\cup"] \\ \arrow[ur, "\mathrm{Ad}^*"] & {} \end{tikzcd}$$

coadjoint action

$G \curvearrowright \mathcal{O}_\lambda := \mathrm{Ad}^*(G)\cdot\lambda$ coadjoint orbit

symplectic mfd

$\rightsquigarrow$ "Q" (geometric quantization) (?)

$G \curvearrowright \mathcal{H}$

Hilbert space

Fact $\mathcal{O}_\lambda$ becomes a symplectic manifold by the Kirillov–Kostant–Souriau symplectic form.

$$\begin{tikzcd} \mathfrak{g}\times\mathfrak{g} \arrow[r] \arrow[d] & \mathbb{R},\ (X,Y)\mapsto\lambda([X,Y]) \\ \mathfrak{g}/\mathfrak{g}_\lambda \times \mathfrak{g}/\mathfrak{g}_\lambda \arrow[ur, dashed, "\omega"'] \arrow[d, phantom, "\wr \qquad \wr"] & \\ T_o\mathcal{O}_\lambda \times T_o\mathcal{O}_\lambda & \end{tikzcd}$$

Kirillov–Kostant–Souriau symplectic form

$$\begin{tikzcd} & \mathfrak{g}^* \\ G \arrow[ur, "\mathrm{Ad}^*"] \arrow[r, phantom, "\curvearrowright"] & \cup \end{tikzcd}$$

coadjoint action

$G \curvearrowright \mathcal{O}_\lambda := \mathrm{Ad}^*(G)\cdot\lambda$ coadjoint orbit

G-invariant, homogeneous, symplectic mfd

$\rightsquigarrow$ "Q" (geometric quantization) (?)

$G \curvearrowright \mathcal{H}$

unitary, irreducible rep on Hilbert space (?)

Fact $\mathcal{O}_\lambda$ becomes a symplectic manifold by the Kirillov–Kostant–Souriau symplectic form.

$$\begin{tikzcd} \mathfrak{g}\times\mathfrak{g} \arrow[r] \arrow[d] & \mathbb{R}, \quad (X,Y)\mapsto\lambda([X,Y]) \\ \mathfrak{g}/\mathfrak{g}_\lambda \times \mathfrak{g}/\mathfrak{g}_\lambda \arrow[ur, dashed, "\omega"'] & \\ T_o\mathcal{O}_\lambda \times T_o\mathcal{O}_\lambda \arrow[u, phantom, "\wr\qquad\wr"] & \end{tikzcd}$$

Orbit philosophy

Orbit philosophy à la Kirillov–Kostant–Duflo

$$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \quad \text{coadjoint action}$$

Orbit philosophy

Orbit philosophy à la Kirillov–Kostant–Duflo

$$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \quad \text{coadjoint action}$$

$$\boxed{\begin{array}{ccc} \sqrt{-1}\mathfrak{g}^* / \mathrm{Ad}^*(G) & \doteqdot & \widehat{G} \quad \text{(unitary dual)} \\ \{\text{coadjoint orbits}\} & & \{\text{irred. unitary reps of } G\} \end{array}}$$

Orbit philosophy

Orbit philosophy à la Kirillov–Kostant–Duflo

$$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \quad \text{coadjoint action}$$

$$\boxed{\sqrt{-1}\mathfrak{g}^* / \mathrm{Ad}^*(G) \quad \doteqdot \quad \widehat{G} \quad \text{(unitary dual)}}$$

works perfectly for nilpotent group G (Kirillov)

Orbit philosophy

Orbit philosophy à la Kirillov–Kostant–Duflo

$$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \quad \text{coadjoint action}$$

$$\boxed{\sqrt{-1}\mathfrak{g}^* / \mathrm{Ad}^*(G) \quad \doteqdot \quad \widehat{G} \quad \text{(unitary dual)}}$$

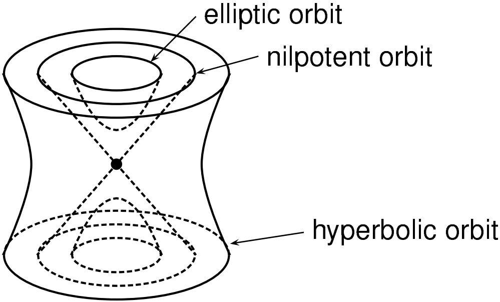
works perfectly for <u>nilpotent group</u> G (Kirillov)
does not work perfectly for <u>reductive group</u> G (still open)

Orbit philosophy

Orbit philosophy à la Kirillov–Kostant–Duflo

$$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \quad \text{coadjoint action}$$

$$\boxed{\sqrt{-1}\mathfrak{g}^* / \mathrm{Ad}^*(G) \quad \doteqdot \quad \widehat{G} \quad \text{(unitary dual)}}$$



$G = SL(2, \mathbb{R})$

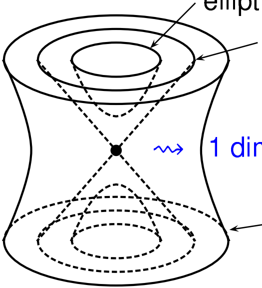
Orbit philosophy

Orbit philosophy à la Kirillov–Kostant–Duflo

$$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \quad \text{coadjoint action}$$

$$\boxed{\sqrt{-1}\mathfrak{g}^* / \mathrm{Ad}^*(G) \quad \overset{Q}{\rightsquigarrow} \quad \widehat{G} \quad \text{(unitary dual) (?)}}$$

“geometric quantization”



elliptic orbit $\rightsquigarrow$ $\boxed{\text{discrete series}}$

nilpotent orbit $\rightsquigarrow$ $\boxed{\ ?\ }$

$\rightsquigarrow$ 1 dim’l trivial rep

hyperbolic orbit $\rightsquigarrow$ principal series

$G = SL(2, \mathbb{R})$

Hyperbolic, elliptic, and nilpotent orbits

$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}, \quad \lambda \leftrightarrow \sqrt{-1}H_\lambda$$

Hyperbolic, elliptic, and nilpotent orbits

$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}, \quad \lambda \leftrightarrow \sqrt{-1}H_\lambda$$

Definition $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \simeq G/G_\lambda$

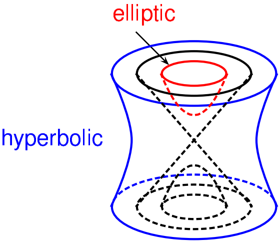
- $\mathcal{O}_\lambda$ is nilpotent $\Leftrightarrow$ $\mathrm{ad}(H_\lambda)$ is nilpotent
- $\mathcal{O}_\lambda$ is semisimple $\Leftrightarrow$ $\mathrm{ad}(H_\lambda)$ is semisimple

Hyperbolic, elliptic, and nilpotent orbits

$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}, \quad \lambda \leftrightarrow \sqrt{-1}H_\lambda$$

Definition $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \simeq G/G_\lambda$

- $\mathcal{O}_\lambda$ is nilpotent $\Leftrightarrow$ $\mathrm{ad}(H_\lambda)$ is nilpotent
- $\mathcal{O}_\lambda$ is semisimple $\Leftrightarrow$ $\mathrm{ad}(H_\lambda)$ is semisimple
  - hyperbolic $\cdots$ all eigenvalues of $\mathrm{ad}(H_\lambda)$ are real
  - elliptic $\cdots$ ———————— are pure imaginary



Geometry of semisimple orbits

$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}, \quad \lambda \leftrightarrow \sqrt{-1}H_\lambda$$

- $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ hyperbolic orbit

Geometry of semisimple orbits

$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}, \quad \lambda \leftrightarrow \sqrt{-1}H_\lambda$$

- $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ hyperbolic orbit

$$\Rightarrow \underset{\mathrm{ad}(H_\lambda)}{\mathfrak{g}} = \underset{-}{\mathfrak{n}_-} + \overbrace{\underset{0}{\mathfrak{g}_\lambda} + \underset{+}{\mathfrak{n}_+}}^{\mathfrak{p}_+} \quad \text{(real polarization)}$$

Geometry of semisimple orbits

$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}, \quad \lambda \leftrightarrow \sqrt{-1}H_\lambda$$

- $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ hyperbolic orbit

  $\Rightarrow \underset{\mathrm{ad}(H_\lambda)}{\mathfrak{g}} = \underset{-}{\mathfrak{n}_-} + \overbrace{\underset{0}{\mathfrak{g}_\lambda} + \underset{+}{\mathfrak{n}_+}}^{\mathfrak{p}_+}$ (real polarization)

  $\Rightarrow \mathcal{O}_\lambda \twoheadrightarrow G/P_+$ (real flag variety)

  Fiber = Lagrangian in $\mathcal{O}_\lambda$

Geometry of semisimple orbits

$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}, \quad \lambda \leftrightarrow \sqrt{-1}H_\lambda$$

- $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ hyperbolic orbit

  $\Rightarrow \underset{\mathrm{ad}(H_\lambda)}{\mathfrak{g}} = \underset{-}{\mathfrak{n}_-} + \overbrace{\underset{0}{\mathfrak{g}_\lambda} + \underset{+}{\mathfrak{n}_+}}^{\mathfrak{p}_+}$ (real polarization)

  $\Rightarrow \mathcal{O}_\lambda \twoheadrightarrow G/P_+$ (real flag variety)

  Fiber = Lagrangian in $\mathcal{O}_\lambda$

- $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ elliptic orbit

Geometry of semisimple orbits

$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}, \quad \lambda \leftrightarrow \sqrt{-1}H_\lambda$$

- $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ hyperbolic orbit

  $\Rightarrow \underset{\mathrm{ad}(H_\lambda)}{\mathfrak{g}} = \underset{-}{\mathfrak{n}_-} + \overbrace{\underset{0}{\mathfrak{g}_\lambda} + \underset{+}{\mathfrak{n}_+}}^{\mathfrak{p}_+}$ (real polarization)

  $\Rightarrow \mathcal{O}_\lambda \twoheadrightarrow G/P_+$ (real flag variety)

  Fiber = Lagrangian in $\mathcal{O}_\lambda$

- $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ elliptic orbit

  $\Rightarrow \underset{\frac{1}{\sqrt{-1}}\mathrm{ad}(H_\lambda)}{\mathfrak{g}_\mathbb{C}} = \underset{-}{\mathfrak{n}_-^\mathbb{C}} + \overbrace{\underset{0}{\mathfrak{g}_\lambda^\mathbb{C}} + \underset{+}{\mathfrak{n}_+^\mathbb{C}}}^{\mathfrak{q}}$ (complex polarization)

Geometry of semisimple orbits

$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}, \quad \lambda \leftrightarrow \sqrt{-1}H_\lambda$$

- $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ hyperbolic orbit

$$\Rightarrow \underset{\mathrm{ad}(H_\lambda)}{\mathfrak{g}} = \underset{-}{\mathfrak{n}_-} + \overbrace{\underset{0}{\mathfrak{g}_\lambda} + \underset{+}{\mathfrak{n}_+}}^{\mathfrak{p}_+} \quad \text{(real polarization)}$$

$\Rightarrow \mathcal{O}_\lambda \twoheadrightarrow G/P_+$ (real flag variety)

Fiber = Lagrangian in $\mathcal{O}_\lambda$

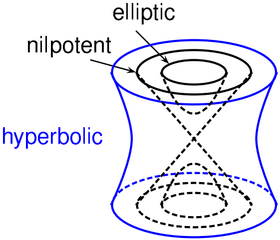
- $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ elliptic orbit

$$\Rightarrow \underset{\frac{1}{\sqrt{-1}}\mathrm{ad}(H_\lambda)}{\mathfrak{g}_{\mathbb{C}}} = \underset{-}{\mathfrak{n}_-^{\mathbb{C}}} + \overbrace{\underset{0}{\mathfrak{g}_\lambda^{\mathbb{C}}} + \underset{+}{\mathfrak{n}_+^{\mathbb{C}}}}^{\mathfrak{q}} \quad \text{(complex polarization)}$$

$\Rightarrow \mathcal{O}_\lambda \underset{\text{Borel embedding}}{\overset{\text{open}}{\subset}} G_{\mathbb{C}}/Q$ (complex flag variety)

Review: geometric quantization of hyperbolic orbits

$$\begin{array}{ccccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} & \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ \text{reductive Lie gp} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$



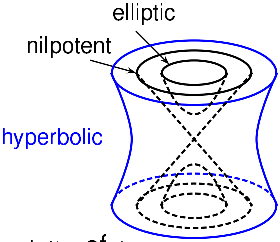
Review: geometric quantization of hyperbolic orbits

Brief summary on classical results

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ hyperbolic orbit

$$\begin{array}{ccccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} & \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$

reductive
Lie gp



$\rightsquigarrow$

$\boxed{\pi_\lambda}$ irred. unitary rep of G

$\mathrm{ad}(H_\lambda)$ defines a real parabolic $\mathfrak{p} = \mathfrak{g}_\lambda + \mathfrak{n}_+$ of $\mathfrak{g}$

$\Rightarrow \mathcal{O}_\lambda \twoheadrightarrow G/P$ (real flag variety)

Lagrangian foliation in $\mathcal{O}_\lambda$

Review: geometric quantization of hyperbolic orbits

Brief summary on classical results

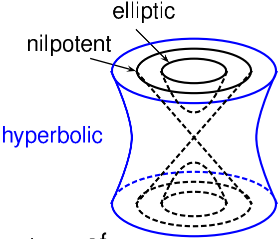
$$\begin{array}{ccccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} & \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$

reductive
Lie gp

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ hyperbolic orbit

$\Downarrow$

$\boxed{\mathcal{L}_\lambda \to G/P}$ G-equiv. line b'dle



$\boxed{\pi_\lambda}$ irred. unitary rep of G

$\mathrm{ad}(H_\lambda)$ defines a real parabolic $\mathfrak{p} = \mathfrak{g}_\lambda + \mathfrak{n}_+$ of $\mathfrak{g}$

$\Rightarrow \mathcal{O}_\lambda \twoheadrightarrow G/P$ (real flag variety)

Lagrangian foliation in $\mathcal{O}_\lambda$

Review: geometric quantization of hyperbolic orbits

Brief summary on classical results

$$\begin{array}{ccccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} & \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ \text{reductive} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \text{Lie gp} & & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ hyperbolic orbit

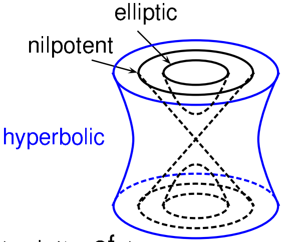
$\Downarrow$

$\boxed{\mathcal{L}_\lambda \to G/P}$ G-equiv. line b'dle

$\Downarrow$

$\boxed{L^2(G/P, \mathcal{L}_\lambda)}$

$\boxed{\pi_\lambda}$ almost irred. unitary rep of G



$\mathrm{ad}(H_\lambda)$ defines a real parabolic $\mathfrak{p} = \mathfrak{g}_\lambda + \mathfrak{n}_+$ of $\mathfrak{g}$

$\Rightarrow \mathcal{O}_\lambda \twoheadrightarrow G/P$ (real flag variety)

Lagrangian foliation in $\mathcal{O}_\lambda$

Review: geometric quantization of hyperbolic orbits

Brief summary on classical results

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ hyperbolic orbit

$\Downarrow$

$\boxed{\mathcal{L}_{\lambda+\rho} \to G/P}$ G-equiv. line b'dle

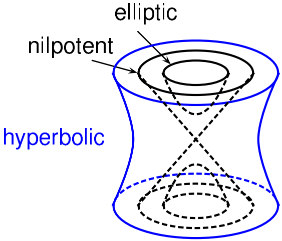
$\Downarrow$

$\boxed{L^2(G/P, \mathcal{L}_{\lambda+\rho})}$

$\boxed{\pi_\lambda}$ almost irred. unitary rep of G

$$\begin{array}{cccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$

reductive
Lie gp



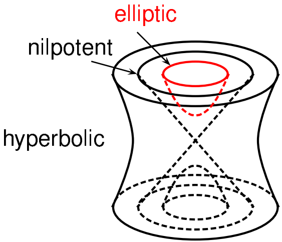
$\mathrm{ad}(H_\lambda)$ defines a real parabolic $\mathfrak{p} = \mathfrak{g}_\lambda + \mathfrak{n}_+$ of $\mathfrak{g}$

$\Rightarrow \mathcal{O}_\lambda \twoheadrightarrow G/P$ (real flag variety)

$$\rho(H) := \frac{1}{2}\,\mathrm{Trace}(\mathrm{ad}(H) : \mathfrak{n}_+ \to \mathfrak{n}_+)$$

Review: geometric quantization of elliptic orbits

$$\begin{array}{ccccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} & \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ \text{reductive} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \text{Lie gp} & & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$



Review: geometric quantization of elliptic orbits

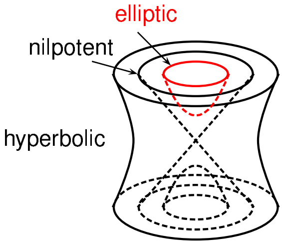
Brief summary on known results (1970s–1990s)

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ elliptic orbit

$$\rotatebox{-90}{$\rightsquigarrow$}$$

$\boxed{\pi_\lambda}$ irred. unitary rep of G

$$\begin{array}{ccccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} & \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ \text{reductive Lie gp} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$



$$\frac{1}{\sqrt{-1}}\mathrm{ad}(H_\lambda) \text{ defines a parabolic subalg } \mathfrak{q} = (\mathfrak{g}_\lambda)_{\mathbb{C}} + \mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$$

$$\Rightarrow \mathcal{O}_\lambda \underset{\text{open}}{\subset} G_{\mathbb{C}}/Q \quad \text{(complex flag variety)}$$

Review: geometric quantization of elliptic orbits

Brief summary on known results (1970s–1990s)

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ elliptic orbit, integral

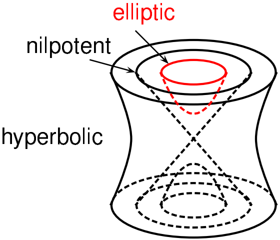
$\Downarrow$ complex structure

$\boxed{\mathcal{L}_\lambda \to \mathcal{O}_\lambda}$ G-equiv. holo. line b'dle

$\boxed{\pi_\lambda}$ irred. unitary rep of G

$$\begin{array}{cccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$

reductive
Lie gp



$$\frac{1}{\sqrt{-1}}\mathrm{ad}(H_\lambda)$$ defines a parabolic subalg $\mathfrak{q} = (\mathfrak{g}_\lambda)_\mathbb{C} + \mathfrak{u} \subset \mathfrak{g}_\mathbb{C}$

$\Rightarrow \mathcal{O}_\lambda \underset{\text{open}}{\subset} G_\mathbb{C}/Q$ (complex flag variety)

Review: geometric quantization of elliptic orbits

Brief summary on known results (1970s–1990s)

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ elliptic orbit, integral

$\Downarrow$ complex structure

$\boxed{\mathcal{L}_\lambda \to \mathcal{O}_\lambda}$ G-equiv. holo. line b'dle

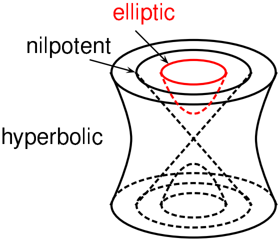
$\Downarrow$

$\boxed{H^*_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda)}$ G-module (Fréchet space)
maximal globalization
(Schmid, Wong '94)

$\boxed{\pi_\lambda}$ irred. unitary rep of G

$$\begin{array}{cccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$

reductive
Lie gp



$$\frac{1}{\sqrt{-1}}\mathrm{ad}(H_\lambda) \text{ defines a parabolic subalg } \mathfrak{q} = (\mathfrak{g}_\lambda)_{\mathbb{C}} + \mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$$

$$\Rightarrow \mathcal{O}_\lambda \underset{\text{open}}{\subset} G_{\mathbb{C}}/Q \quad \text{(complex flag variety)}$$

Review: geometric quantization of elliptic orbits

Brief summary on known results (1970s–1990s)

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ elliptic orbit, integral

$\Downarrow$ complex structure

$\boxed{\mathcal{L}_\lambda \to \mathcal{O}_\lambda}$ G-equiv. holo. line b'dle

$\Downarrow$

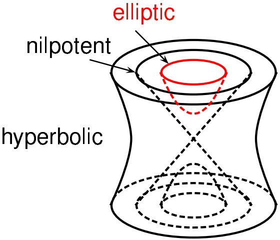
$\boxed{H^*_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda)}$ G-module (Fréchet space)
maximal globalization
(Schmid, Wong '94)

$\cup$ dense

$\boxed{{}^{\exists 1}\pi_\lambda}$ irred. unitary rep of G
(Vogan, Wallach '84)

$$\begin{array}{cccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} & \sqrt{-1}\mathfrak{g}^* & \simeq \quad \sqrt{-1}\mathfrak{g} \\ & & \rotatebox{90}{$\in$} & \quad\ \rotatebox{90}{$\in$} \\ & & \lambda & \leftrightarrow \quad \sqrt{-1}H_\lambda \end{array}$$

reductive
Lie gp



$\dfrac{1}{\sqrt{-1}}\mathrm{ad}(H_\lambda)$ defines a parabolic subalg $\mathfrak{q} = (\mathfrak{g}_\lambda)_{\mathbb{C}} + \mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$

$\Rightarrow \mathcal{O}_\lambda \underset{\text{open}}{\subset} G_{\mathbb{C}}/Q$ (complex flag variety)

Review: geometric quantization of elliptic orbits

Brief summary on known results (1970s–1990s)

$$\begin{array}{ccccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} & \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ & & \Psi & & \Psi \\ & & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$

reductive Lie gp

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ elliptic orbit, integral

$\Downarrow$ complex structure

$\boxed{\mathcal{L}_\lambda \to \mathcal{O}_\lambda}$ G-equiv. holo. line b'dle

$\Downarrow$

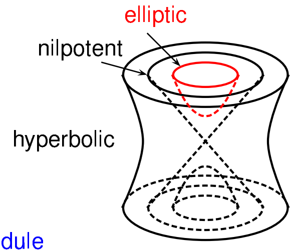
$\boxed{H^*_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda)}$ G-module (Fréchet space) maximal globalization (Schmid, Wong '94)

$\cup$ dense

$\boxed{{}^{\exists 1}\pi_\lambda}$ irred. unitary rep of G (Vogan, Wallach '84)

$\cup$ dense

$A_\mathfrak{q}(\lambda)$ Zuckerman's derived functor module



$$\frac{1}{\sqrt{-1}}\mathrm{ad}(H_\lambda) \text{ defines a parabolic subalg } \mathfrak{q} = (\mathfrak{g}_\lambda)_\mathbb{C} + \mathfrak{u} \subset \mathfrak{g}_\mathbb{C}$$

$$\Rightarrow \mathcal{O}_\lambda \underset{\text{open}}{\subset} G_\mathbb{C}/Q \quad \text{(complex flag variety)}$$

Review: geometric quantization of elliptic orbits

Brief summary on known results (1970s–1990s)

$$\begin{array}{ccccc} G & \overset{\mathrm{Ad}^*}{\curvearrowright} & \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} \\ \text{reductive Lie gp} & & \Cup & & \Cup \\ & & \lambda & \leftrightarrow & \sqrt{-1}H_\lambda \end{array}$$

$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda}$ elliptic orbit, integral

$\Downarrow$ complex structure

$\boxed{\mathcal{L}_{\lambda+\rho} \to \mathcal{O}_\lambda}$ G-equiv. holo. line b'dle

$\Downarrow$

$\boxed{H^*_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_{\lambda+\rho})}$ G-module (Fréchet space)

maximal globalization

(Schmid, Wong '94)

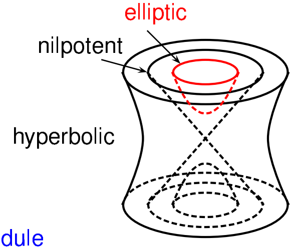
$\cup$ dense

$\boxed{{}^{\exists 1}\pi_\lambda}$ irred. unitary rep of G

(Vogan, Wallach '84)

$\cup$ dense

$A_{\mathfrak{q}}(\lambda-\rho)$ Zuckerman's derived functor module



$\frac{1}{\sqrt{-1}}\mathrm{ad}(H_\lambda)$ defines a parabolic subalg $\mathfrak{q} = (\mathfrak{g}_\lambda)_{\mathbb{C}} + \mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$

$\Rightarrow \mathcal{O}_\lambda \underset{\text{open}}{\subset} G_{\mathbb{C}}/Q$ (complex flag variety)

$\rho(H) := (1/2)\,\mathrm{Trace}(\mathrm{ad}(H) : \mathfrak{u} \to \mathfrak{u})$

Geometric quantization π_λ of elliptic orbits $\mathcal{O}_\lambda$

$$\boxed{\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\cdot\lambda \text{ integral elliptic orbit } \rightsquigarrow \pi_\lambda \text{ unitary rep of } G}$$

λ: sufficiently ‘positive’ $\Longrightarrow \pi_\lambda \neq 0$, irreducible

Geometric quantization π_λ of elliptic orbits $\mathcal{O}_\lambda$

$\mathcal{O}_\lambda = \mathrm{Ad}^*(G) \cdot \lambda$ integral elliptic orbit $\rightsquigarrow$ π_λ unitary rep of G

λ: sufficiently ‘positive’ $\Longrightarrow \pi_\lambda \neq 0$, irreducible

$\mathcal{O}_\lambda \simeq G/G_\lambda$

$G_\lambda = \{g : \mathrm{Ad}^*(g)\lambda = \lambda\}$

G compact $\cdots$ Borel–Weil–Bott construction
G_λ compact torus $\cdots$ π_λ = discrete series
G_λ abelian $\cdots$ π_λ = fundamental series
G_λ maximal compact $\cdots$ π_λ = holomorphic disc. ser. of scalar type

Geometric quantization of coadjoint orbit

G: real reductive groups

$$\begin{tikzcd} \mathfrak{g}^* \supset \mathcal{O} = \mathrm{Ad}^*(G)\lambda \arrow[dd, squiggly, "?"] & \text{semisimple orbit} \\ & \text{“geometric quantization” } Q \\ \widehat{G} \ni \pi & \text{irred. unitary rep of } G \end{tikzcd}$$

Summary (known):

Works fairly well in this case
— combination of hyperbolic and elliptic cases.

Geometric quantization of coadjoint orbit

G: real reductive groups

$\mathfrak{g}^* \supset \mathcal{O}_{\mathrm{min}} = \mathrm{Ad}^*(G)\lambda$ minimal nilpotent orbit

$\wr\!\!\downarrow$? "geometric quantization" Q

$\widehat{G} \ni \pi$ minimal rep of G

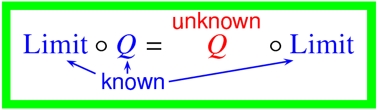
Geometric quantization of coadjoint orbit

G: real reductive groups

$$\begin{tikzcd}[column sep=small] \mathfrak{g}^* \supset \mathcal{O}_{\min} = \mathrm{Ad}^*(G)\lambda \arrow[d, squiggly, "?"] & \text{minimal nilpotent orbit} \\ \widehat{G} \ni \pi & \text{minimal rep of } G \end{tikzcd}$$

“geometric quantization” Q

First idea:



$$\mathrm{Limit} \circ Q = \overset{\text{unknown}}{Q} \circ \mathrm{Limit}$$

Limit set

$\mathfrak{g} \supset \mathcal{O}_\nu$ coadjoint orbits with parameter $\nu > 0$

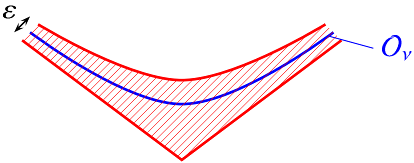
Limit set

$\mathfrak{g} \supset \mathcal{O}_\nu$ coadjoint orbits with parameter $\nu > 0$

Def (limit set)

$$\overline{\bigcup_{\varepsilon>\nu>0} \mathcal{O}_\nu}$$

$\overline{M}$ denotes the closure of M.



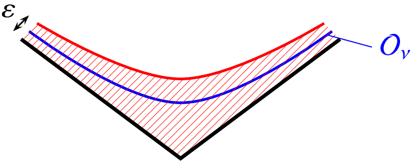
Limit set

$\mathfrak{g} \supset \mathcal{O}_\nu$ coadjoint orbits with parameter $\nu > 0$

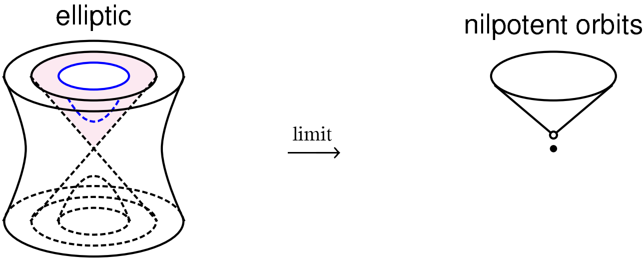
Def (limit set)

$$\lim_{\nu \downarrow 0} \mathcal{O}_\nu := \bigcap_{\varepsilon>0} \overline{\bigcup_{\varepsilon>\nu>0} \mathcal{O}_\nu}$$

$\overline{M}$ denotes the closure of M.

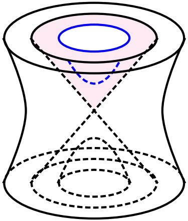


$SL(2,\mathbb{R})$ case



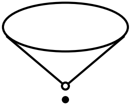
$SL(2,\mathbb{R})$ case

elliptic

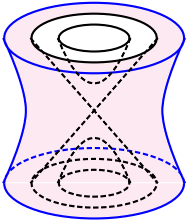


$\xrightarrow{\text{limit}}$

nilpotent orbits

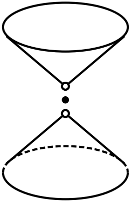


hyperbolic



$\xrightarrow{\text{limit}}$

nilpotent orbits



Indefinite orthogonal group $O(p+1, q+1)$

For $p, q \geq 1$,

$$\begin{aligned} G &= O(p+1, q+1) \\ &= \{g \in GL(p+q+2, \mathbb{R}) : {}^t g \left(\begin{smallmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{smallmatrix}\right) g = \left(\begin{smallmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{smallmatrix}\right) \end{aligned}$$

$\cdots$ real simple Lie group of type D

$\Longleftrightarrow$ $p+q$: even > 2

Example: quantization of minimal elliptic orbits

example of known theory (elliptic orbits of minimal dimension)

$$G = O(p+1, q+1) \qquad (p \geq 1)$$

$$f_1 := E_{12} - E_{21} \in \mathfrak{g} \simeq \mathfrak{g}^*$$

Example: quantization of minimal elliptic orbits

example of known theory (elliptic orbits of minimal dimension)

$$G = O(p+1, q+1) \qquad (p \geq 1)$$

$$f_1 := E_{12} - E_{21} \in \mathfrak{g} \simeq \mathfrak{g}^*$$

$$\mathcal{O}_\nu^{\mathrm{ell}} := \mathrm{Ad}^*(G)(\sqrt{-1}\nu f_1) \simeq G/G_\nu \subset \sqrt{-1}\mathfrak{g}^* \quad (\nu > 0)$$

minimal elliptic orbits

<u>Remark</u> The isotropy group

$$G_\nu \simeq SO(2) \times O(p-1, q)$$

is non-compact if $p > 1$.

Example: quantization of minimal elliptic orbits

example of known theory (elliptic orbits of minimal dimension)

$$G = O(p+1, q+1) \qquad (p \geq 1)$$

$$f_1 := E_{12} - E_{21} \in \mathfrak{g} \simeq \mathfrak{g}^*$$

$$\mathcal{O}_\nu^{\mathrm{ell}} := \mathrm{Ad}^*(G)(\sqrt{-1}\nu f_1) \simeq \underset{\text{minimal elliptic orbits}}{G/G_\nu \subset \sqrt{-1}\mathfrak{g}^*} \quad (\nu > 0)$$

$\rotatebox[origin=c]{-90}{$\rightsquigarrow$}$ geometric quantization

$\pi_\nu \in \widehat{G}$ if $\nu \in \mathbb{Z} + \frac{p+q}{2}$

Quantization of minimal elliptic orbits of $O(p+1, q+1)$

$q = -1$ case

$$\mathcal{O}_\nu^{\mathrm{ell}} \simeq Q_{\mathbb{C}}^{p-1} \quad \text{(complex quadric)} \underset{\text{codim } 1}{\subset} \mathbb{P}^p\mathbb{C}$$

Quantization of minimal elliptic orbits of $O(p+1, q+1)$

$q = -1$ case

$$\mathcal{O}_\nu^{\mathrm{ell}} \simeq Q_{\mathbb{C}}^{p-1} \quad \text{(complex quadric)} \underset{\text{codim } 1}{\subset} \mathbb{P}^p\mathbb{C}$$

Two realizations of $\pi_\nu \in \widehat{G}$, $G = O(p+1, 0) = O(p+1)$

- Borel–Weil–Bott construction on $Q_{\mathbb{C}}^{p-1}$
- Eigenfunction of Laplacian on the sphere S^p
  (spherical harmonics of degree $\nu - \frac{p-1}{2}$)

Quantization of minimal elliptic orbits of $O(p+1, q+1)$

$q = -1$ case

$$\mathcal{O}_\nu^{\text{ell}} \simeq Q_{\mathbb{C}}^{p-1} \quad \text{(complex quadric)} \underset{\text{codim } 1}{\subset} \mathbb{P}^p\mathbb{C}$$

Two realizations of $\pi_\nu \in \widehat{G}$, $G = O(p+1, 0) = O(p+1)$

- Borel–Weil–Bott construction on $Q_{\mathbb{C}}^{p-1}$
- Eigenfunction of Laplacian on the sphere S^p (spherical harmonics of degree $\nu - \frac{p-1}{2}$)

---

q general case $G = O(p+1, q+1)$ noncompact

$$\mathcal{O}_\nu^{\text{ell}} \underset{\text{open}}{\subset} Q_{\mathbb{C}}^{p+q} \underset{\text{codim } 1}{\subset} \mathbb{P}^{p+q+1}\mathbb{C}$$

Quantization of minimal elliptic orbits of $O(p+1, q+1)$

$q = -1$ case

$$\mathcal{O}_\nu^{\mathrm{ell}} \simeq Q_{\mathbb{C}}^{p-1} \quad \text{(complex quadric)} \underset{\text{codim } 1}{\subset} \mathbb{P}^p\mathbb{C}$$

Two realizations of $\pi_\nu \in \widehat{G}$, $G = O(p+1, 0) = O(p+1)$

- Borel–Weil–Bott construction on $Q_{\mathbb{C}}^{p-1}$
- Eigenfunction of Laplacian on the sphere S^p
  (spherical harmonics of degree $\nu - \frac{p-1}{2}$)

---

q general case $G = O(p+1, q+1)$ noncompact

$$\mathcal{O}_\nu^{\mathrm{ell}} \underset{\text{open}}{\subset} Q_{\mathbb{C}}^{p+q} \underset{\text{codim } 1}{\subset} \mathbb{P}^{p+q+1}\mathbb{C}$$

Two realizations of π_ν

- $H^{p-1}(\mathcal{O}_\nu^{\mathrm{ell}}, \mathcal{L}_{\nu+\rho}) \quad \rho = \frac{1}{2}(p+q)$
- L^2-eigenfns of Laplacian on pseudo-Riemannian space form
  $\{x \in \mathbb{R}^{p+q+2} : x_0^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q+1}^2 = 1\}$

Quantization of minimal elliptic orbits of $O(p+1, q+1)$

$q = -1$ case

$$\mathcal{O}_\nu^{\mathrm{ell}} \simeq Q_{\mathbb{C}}^{p-1} \quad \text{(complex quadric)} \underset{\text{codim } 1}{\subset} \mathbb{P}^p\mathbb{C}$$

Two realizations of $\pi_\nu \in \widehat{G}$, $G = O(p+1, 0) = O(p+1)$

- Borel–Weil–Bott construction on $Q_{\mathbb{C}}^{p-1}$
- Eigenfunction of Laplacian on the sphere S^p
  (spherical harmonics of degree $\nu - \frac{p-1}{2}$)

---

q general case $G = O(p+1, q+1)$ noncompact

$$\mathcal{O}_\nu^{\mathrm{ell}} \underset{\text{open}}{\subset} Q_{\mathbb{C}}^{p+q} \underset{\text{codim } 1}{\subset} \mathbb{P}^{p+q+1}\mathbb{C}$$

Two realizations of π_ν

- $H^{p-1}(\mathcal{O}_\nu^{\mathrm{ell}}, \mathcal{L}_{\nu+\rho}) \quad \rho = \frac{1}{2}(p+q)$
- L^2-eigenfns of Laplacian on pseudo-Riemannian space form
  $\{x \in \mathbb{R}^{p+q+2} : x_0^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q+1}^2 = 1\}$

$\mathbf{Limit} \circ Q \overset{?}{=} Q \circ \mathbf{Limit}$

$$G = O(p+1, q+1) \ (p, q \geq 1)$$

<u>Proposition</u> (geometry of coadjoint orbits)

$$\lim_{\nu \downarrow 0} \underset{\text{elliptic}}{\mathcal{O}_\nu^{\text{ell}}} = \underbrace{\mathcal{O}_0^{\text{nilp}} \cup \mathcal{O}_{\text{min}} \cup \{0\}}_{\text{nilpotent orbits}}$$

$\textbf{Limit} \circ Q \overset{?}{=} Q \circ \textbf{Limit}$

$$G = O(p+1, q+1) \ (p, q \geq 1)$$

Proposition (geometry of coadjoint orbits)

$$\lim_{\nu \downarrow 0} \underset{\text{elliptic}}{\mathcal{O}_\nu^{\text{ell}}} = \underbrace{\mathcal{O}_0^{\text{nilp}} \cup \mathcal{O}_{\min} \cup \{0\}}_{\text{nilpotent orbits}}$$

		dimension
$\mathcal{O}_\nu^{\text{ell}}$:	minimal elliptic orbits	$2(p+q)$
$\mathcal{O}_0^{\text{min}}$:	a nilpotent orbit	$2(p+q)$
$\mathcal{O}_{\text{min}}$:	the minimal nilpotent orbit	$2(p+q-1)$

$\mathbf{Limit} \circ Q \overset{?}{=} Q \circ \mathbf{Limit}$

$$G = O(p+1, q+1) \; (p, q \geq 1)$$

Proposition (geometry of coadjoint orbits)

$$\lim_{\nu \downarrow 0} \underset{\text{elliptic}}{\mathcal{O}_\nu^{\text{ell}}} = \underbrace{\mathcal{O}_0^{\text{nilp}} \cup \mathcal{O}_{\text{min}} \cup \{0\}}_{\text{nilpotent orbits}}$$

$\Downarrow$ "geometric quantization"

?

$\mathbf{Limit} \circ Q \overset{?}{=} Q \circ \mathbf{Limit}$

$$G = O(p+1, q+1) \ (p, q \geq 1)$$

Proposition (geometry of coadjoint orbits)

$$\lim_{\nu \downarrow 0} \underset{\text{elliptic}}{\mathcal{O}^{\text{ell}}_{\nu}} = \underbrace{\mathcal{O}^{\text{nilp}}_{0} \cup \mathcal{O}_{\text{min}} \cup \{0\}}_{\text{nilpotent orbits}}$$

$\Downarrow$ “geometric quantization”

Theorem A If $p + q$ is even,

$^{\exists}$non-split exact sequence of G-modules:

$$0 \to \underline{\varpi_{\text{min}}} \to \pi_{-1} \to \underline{\pi_1} \to 0$$

↖ ↗

irreducible unitary rep

ϖ_{min}: minimal representation of $G = O(p+1, q+1)$

$\mathbf{Limit} \circ Q \overset{?}{=} Q \circ \mathbf{Limit}$

$$G = O(p+1, q+1)\ (p, q \geq 1)$$

Proposition (geometry of coadjoint orbits)

$$\lim_{\nu \downarrow 0} \underset{\text{elliptic}}{\mathcal{O}_\nu^{\mathrm{ell}}} = \underbrace{\mathcal{O}_0^{\mathrm{nilp}} \cup \mathcal{O}_{\min} \cup \{0\}}_{\text{nilpotent orbits}}$$

$\Downarrow$ “geometric quantization”

Theorem A If $p + q$ is even,

${}^{\exists}$non-split exact sequence of G-modules:

$$0 \to \varpi_{\min} \to \pi_{-1} \to \pi_1 \to 0$$

$$\pi_\nu \curvearrowright H_{\bar{\partial}}^{p-1}(\mathcal{O}_\nu^{\mathrm{ell}}, \mathcal{L}_{\nu+\rho}), \quad \rho = \frac{1}{2}(p+q)$$

$\mathbf{Limit} \circ Q \overset{?}{=} Q \circ \mathbf{Limit}$

$$G = O(p+1, q+1) \ (p, q \geq 1)$$

Proposition (geometry of coadjoint orbits)

$$\lim_{\nu \downarrow 0} \underset{\text{elliptic}}{\mathcal{O}^{\mathrm{ell}}_{\nu}} = \underbrace{\mathcal{O}^{\mathrm{nilp}}_{0} \cup \mathcal{O}_{\min} \cup \{0\}}_{\text{nilpotent orbits}}$$

$\Downarrow$ "geometric quantization"

Theorem A If $p + q$ is even,

${}^{\exists}$non-split exact sequence of G-modules:

$$0 \to \varpi_{\min} \to \underline{\pi_{-1}} \to \underline{\pi_{1}} \to 0$$

$$\pi_\nu \curvearrowright H^{p-1}_{\bar\partial}(\mathcal{O}^{\mathrm{ell}}_{\nu}, \mathcal{L}_{\nu+\rho}), \quad \rho = \frac{1}{2}(p+q)$$

π_ν is well-understood for $\nu \gg 0$ (Zuckerman, Vogan, Schmid, Wong, ...)

No general theory for π_ν with $\nu < 0$

Limit $\circ\, Q \overset{?}{=} Q \,\circ$ Limit

$$G = O(p+1, q+1)\ (p, q \geq 1)$$

Proposition (geometry of coadjoint orbits)

$$\lim_{\nu \downarrow 0} \underset{\text{elliptic}}{\mathcal{O}_\nu^{\text{ell}}} = \underbrace{\mathcal{O}_0^{\text{nilp}} \cup \mathcal{O}_{\min} \cup \{0\}}_{\text{nilpotent orbits}}$$

$\Downarrow$ “geometric quantization”

Theorem A If $p+q$ is even,

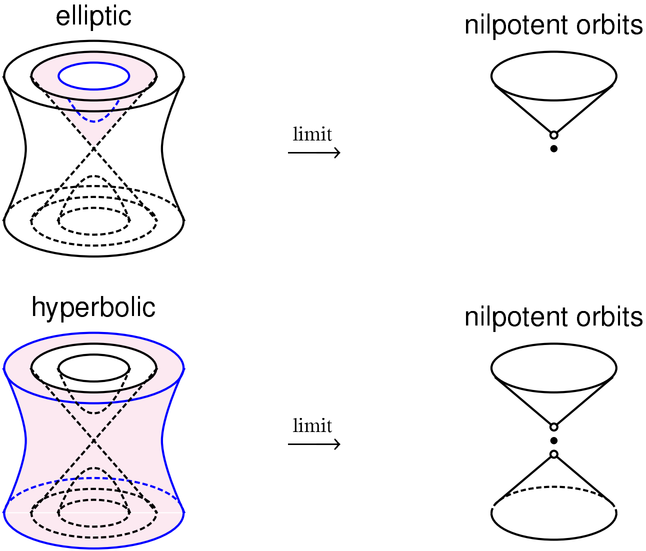
${}^{\exists}$non-split exact sequence of G-modules:

$$0 \to \varpi_{\min} \to \underline{\pi_{-1}} \to \underline{\pi_1} \to 0$$

$$\pi_\nu \curvearrowright H_{\bar\partial}^{p-1}(\mathcal{O}_\nu^{\text{ell}}, \mathcal{L}_{\nu+\rho}), \quad \rho = \tfrac{1}{2}(p+q)$$

$\varpi_{\min}$: minimal representation of $G = O(p+1, q+1)$

$SL(2,\mathbb{R})$ **case**



elliptic

$\xrightarrow{\text{limit}}$

nilpotent orbits

hyperbolic

$\xrightarrow{\text{limit}}$

nilpotent orbits

Quantization of minimal hyperbolic orbits

example of known theory (hyperbolic orbits of minimal dimensions)

$$G = O(p+1, q+1)$$

$$h_0 = E_{0,p+q} + E_{p+q,0} \quad \in \mathfrak{g} \simeq \mathfrak{g}^*$$

$$\boxed{\underset{\text{minimal hyperbolic orb.}}{\mathcal{O}_\nu^{\mathrm{hyp}} := \mathrm{Ad}^*(G)(\sqrt{-1}\nu h_1) \subset \sqrt{-1}\mathfrak{g}^*}}$$

$\Downarrow$ “geometric quantization”

$$\boxed{\mathrm{Ind}_{P_{\max}}^G(\mathbb{C}_\nu) \text{ (induced rep)}}$$

normalization $\cdots$ unitary if $\nu \in \mathbb{R}$

$\Rrightarrow$

$$\boxed{\underset{\mathrm{ad}(h_0)}{\mathfrak{g}} = \underset{-}{\mathfrak{n}_-} + \overbrace{\underset{0}{\mathfrak{g}_{h_0}} + \underset{+}{\mathfrak{n}_+}}^{\mathfrak{p}_{\max}}}$$

$\Lleftarrow$

$$\mathcal{O}_\nu^{\mathrm{hyp}} \to G/P_{\max}$$

Lagrangian foliation

Limit $\circ\, Q \overset{?}{=} Q \,\circ$ Limit (hyperbolic case)

$$G = O(p+1, q+1)\ (p, q \geq 1)$$

<u>Proposition</u> $\displaystyle \lim_{\nu \downarrow 0} \mathcal{O}_\nu^{\mathrm{hyp}} = \mathcal{O}_0^{\mathrm{nilp}} \cup \mathcal{O}_{\min} \cup \{0\}$

$\mathbf{Limit} \circ Q \overset{?}{=} Q \circ \mathbf{Limit}$ (hyperbolic case)

$$G = O(p+1, q+1) \ (p, q \geq 1)$$

Proposition $\displaystyle\lim_{\nu \downarrow 0} \mathcal{O}_\nu^{\mathrm{hyp}} = \mathcal{O}_0^{\mathrm{nilp}} \cup \mathcal{O}_{\min} \cup \{0\}$

$\Downarrow$ "geometric quantization"

Theorem B (K–Ørsted) If $p+q$ is even, then

$^\exists$non-split exact sequence of G-modules:

$$0 \to \varpi_{\min} \to \mathrm{Ind}_{P_{\max}}^{G}(\mathbb{C}_{-1}) \xrightarrow{\widetilde{\Delta}} \mathrm{Ind}_{P_{\max}}^{G}(\mathbb{C}_{1}) \to 0$$

$$\underbrace{\widetilde{\Delta}}_{\text{Yamabe operator}} = \underbrace{\Delta}_{\text{Laplacian}} + c \underbrace{\kappa}_{\text{scalar curvature}}$$

on the pseudo-Riemannian and

$$G/P_{\max} \simeq (S^p \times S^q)/\mathbb{Z}_2$$

Minimal representation of $G = O(p+1, q+1)$ case
$(p, q \geq 1, p+q$ **even**$)$

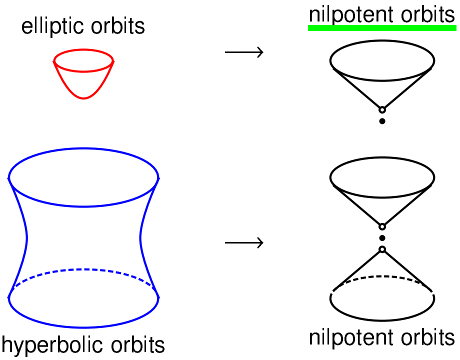
Two geometric constructions of the same rep (minimal reps. $\underline{\varpi_{\min}}$)

$$\begin{array}{ccccccccc}
0 & \to & \underline{\varpi_{\min}} & \to & H_{\bar{\partial}}^{p-1}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{-1+\rho}) & \to & H_{\bar{\partial}}^{p-1}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{1+\rho}) & \to & 0 \\
0 & \to & \underline{\varpi_{\min}} & \to & \mathrm{Ind}_{P_{\max}}^{G}(\mathbb{C}_{-1+\rho}) & \xrightarrow{\widetilde{\Delta}} & \mathrm{Ind}_{P_{\max}}^{G}(\mathbb{C}_{1+\rho}) & \to & 0
\end{array}$$

Minimal representation of $G = O(p+1, q+1)$ case
$(p, q \geq 1, p+q$ **even**$)$

Two geometric constructions of the same rep (minimal reps. $\varpi_{\min}$)

$$\begin{tikzcd}
0 \arrow[r] & \varpi_{\min} \arrow[r] & H^{p-1}_{\bar{\partial}}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{-1+\rho}) \arrow[r] & H^{p-1}_{\bar{\partial}}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{1+\rho}) \arrow[r] & 0 \\
0 \arrow[r] & \varpi_{\min} \arrow[r] & \mathrm{Ind}^G_{P_{\max}}(\mathbb{C}_{-1+\rho}) \arrow[r, "\widetilde{\Delta}"] & \mathrm{Ind}^G_{P_{\max}}(\mathbb{C}_{1+\rho}) \arrow[r] & 0
\end{tikzcd}$$



Minimal representation of $G = O(p+1, q+1)$ case
$(p, q \geq 1,\ p+q$ **even**$)$

Two geometric constructions of the same rep (minimal reps. $\underline{\varpi_{\mathrm{min}}}$)

$$0 \to \underline{\varpi_{\mathrm{min}}} \to H_{\bar\partial}^{p-1}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{-1+\rho}) \to H_{\bar\partial}^{p-1}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{1+\rho}) \to 0$$

$$0 \to \underline{\varpi_{\mathrm{min}}} \to \mathrm{Ind}_{P_{\mathrm{max}}}^{G}(\mathbb{C}_{-1+\rho}) \xrightarrow{\widetilde{\Delta}} \mathrm{Ind}_{P_{\mathrm{max}}}^{G}(\mathbb{C}_{1+\rho}) \to 0$$

$$\lim_{\nu \downarrow 0} \mathcal{O}_{\nu}^{\mathrm{ell}} = \mathcal{O}_{0}^{\mathrm{nilp}} \cup \underline{\mathcal{O}_{\mathrm{min}}} \cup \{0\}$$

$$\lim_{\nu \downarrow 0} \mathcal{O}_{\nu}^{\mathrm{hyp}} = \mathcal{O}_{0}^{\mathrm{nilp}} \cup \underline{\mathcal{O}_{\mathrm{min}}} \cup \{0\}$$

Minimal representation of $G = O(p+1, q+1)$ case
$(p, q \geq 1, p+q$ **even**$)$

Two geometric constructions of the same rep (minimal reps. $\varpi_{\min}$)

$$\begin{array}{ccccccccc}
0 & \to & \varpi_{\min} & \to & H^{p-1}_{\bar\partial}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{-1+\rho}) & \to & H^{p-1}_{\bar\partial}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{1+\rho}) & \to & 0 \\
0 & \to & \varpi_{\min} & \to & \mathrm{Ind}^{G}_{P_{\max}}(\mathbb{C}_{-1+\rho}) & \xrightarrow{\widetilde{\Delta}} & \mathrm{Ind}^{G}_{P_{\max}}(\mathbb{C}_{1+\rho}) & \to & 0
\end{array}$$

$$\begin{array}{llll}
\text{elliptic orbit} & & \mathcal{O}^{\mathrm{ell}} & \\
& & \cap \ \text{open} & \\
& & Q^{p+q}_{\mathbb{C}} & \text{(complex quadric)} \\
& & \cup \ \text{totally real} & \\
\text{hyperbolic orbit} & \mathcal{O}^{\mathrm{hyp}} \twoheadrightarrow & G/P_{\max} &
\end{array}$$

Minimal representation of $G = O(p+1, q+1)$ case
$(p, q \geq 1, p + q$ **even**$)$

Two geometric constructions of the same rep (minimal reps. $\varpi_{\min}$)

$$\begin{tikzcd} 0 \arrow[r] & \varpi_{\min} \arrow[r] & H^{p-1}_{\bar{\partial}}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{-1+\rho}) \arrow[r] & H^{p-1}_{\bar{\partial}}(\mathcal{O}^{\mathrm{ell}}, \mathcal{L}_{1+\rho}) \arrow[r] & 0 \\ 0 \arrow[r] & \varpi_{\min} \arrow[r] & \mathrm{Ind}_{P_{\max}}^{G}(\mathbb{C}_{-1+\rho}) \arrow[r, "\widetilde{\Delta}"] & \mathrm{Ind}_{P_{\max}}^{G}(\mathbb{C}_{1+\rho}) \arrow[r] & 0 \end{tikzcd}$$

geometric quantization of elliptic orbits

geometric quantization of hyperbolic orbits

Minimal representation of $G = O(p + 1, q + 1)$

- $q = 1$
  highest weight module $\oplus$ lowest weight module
  - the bound states of the Hydrogen atom

Minimal representation of $G = O(p+1, q+1)$

- $q = 1$
  highest weight module $\oplus$ lowest weight module
  - the bound states of the Hydrogen atom
- $p = q$
  spherical case
  - $p = q = 3$ case: Kostant (1990), cf. Kazhdan (1990)

Minimal representation of $G = O(p+1, q+1)$

- $q = 1$
  highest weight module $\oplus$ lowest weight module
  - the bound states of the Hydrogen atom
- $p = q$
  spherical case
  - $p = q = 3$ case: Kostant (1990), cf. Kazhdan (1990)
- p, q: general
  non-highest, non-spherical
  - algebraic construction (e.g. dual pair correspondence) (Binegar–Zierau, Howe–Tan, Huang–Zhu)
  - construction by conformal geometry (Theorem B) (K–Ørsted)
  - construction by Dolbeault cohomology (Theorem A)
  - L^2 construction (K–Ørsted, K–Mano)

Minimal representation of $G = O(p+1, q+1)$

- $q = 1$
  highest weight module $\oplus$ lowest weight module
  - the bound states of the Hydrogen atom
- $p = q$
  spherical case
  - $p = q = 3$ case: Kostant (1990), cf. Kazhdan (1990)
- p, q: general
  non-highest, non-spherical
  - algebraic construction (e.g. dual pair correspondence) (Binegar–Zierau, Howe–Tan, Huang–Zhu)
  - construction by conformal geometry (Theorem B) (K–Ørsted)
  - construction by Dolbeault cohomology (Theorem A)
  - L^2 construction (K–Ørsted, K–Mano)

Minimal representation of $G = O(p+1, q+1)$

- $q = 1$
  highest weight module $\oplus$ lowest weight module
  - the bound states of the Hydrogen atom
- $p = q$
  spherical case
  - $p = q = 3$ case: Kostant (1990), cf. Kazhdan (1990)
- p, q: general
  non-highest, non-spherical
  - algebraic construction (e.g. dual pair correspondence)
    (Binegar–Zierau, Howe–Tan, Huang–Zhu)
  - construction by conformal geometry (Theorem B) (K–Ørsted)
  - construction by Dolbeault cohomology (Theorem A)
  - L^2 construction (K–Ørsted, K–Mano)

§1 What are minimal representations?

§2 $\text{Limit} \circ Q = Q \circ \text{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

Geometric quantization of minimal nilpotent orbit

$$\begin{tikzcd} \mathfrak{g}^* \supset \mathcal{O} = \mathrm{Ad}^*(G)\lambda \arrow[d, rightsquigarrow, "?"] & \text{minimal nilp. orbit} \\ \widehat{G} \ni \pi & \text{minimal rep of } G \end{tikzcd}$$

"geometric quantization"

Idea of previous construction

$$\mathrm{Limit} \circ Q = \overset{\text{unknown}}{Q} \circ \mathrm{Limit}$$

known

More direct construction?

Geometric quantization of minimal nilpotent orbit

$$\begin{tikzcd} \mathfrak{g}^* \supset \mathcal{O}_{\min} = \mathrm{Ad}^*(G)\lambda \arrow[d, squiggly, "?"] & \text{minimal nilp. orbit} \\ \widehat{G} \ni \pi & \text{minimal rep of } G \end{tikzcd}$$

"geometric quantization"

Another idea $\Leftarrow G = O(p+1, q+1)$ case

$G \overset{\varpi_{\min}}{\curvearrowright}$ Solutions to Yamabe Laplacian on $G/P_{\max} \doteqdot S^p \times S^q$

Limit of hyperbolic orbits

Geometric quantization of minimal nilpotent orbit

$\mathfrak{g}^* \supset \mathcal{O}_{\min} = \mathrm{Ad}^*(G)\lambda$ minimal nilp. orbit

$\underset{\downarrow}{\wr}$? “geometric quantization”

$\widehat{G} \ni \pi$ minimal rep of G

Another idea $\Leftarrow G = O(p+1, q+1)$ case

$G \overset{\varpi_{\min}}{\curvearrowright}$ Solutions to Yamabe Laplacian on $G/P_{\max} \doteqdot S^p \times S^q \underset{\text{conformal}}{\approx} \mathbb{R}^{p,q} \simeq \overline{\mathfrak{n}}$

$\downarrow$ Fourier transform

Functions on Ξ $\subset \mathfrak{n}$

Geometric quantization of minimal nilpotent orbit

$\mathfrak{g}^* \supset \mathcal{O}_{\min} = \mathrm{Ad}^*(G)\lambda$ minimal nilp. orbit

$\rotatebox[origin=c]{-90}{$\rightsquigarrow$}$? "geometric quantization"

$\widehat{G} \ni \pi$ minimal rep of G

Another idea $\Leftarrow G = O(p+1, q+1)$ case

$G \overset{\varpi_{\min}}{\curvearrowright}$ Solutions to Yamabe Laplacian on $G/P_{\max} \doteqdot S^p \times S^q \underset{\text{conformal}}{\approx} \mathbb{R}^{p,q} \simeq \overline{\mathfrak{n}}$

$\downarrow$ Fourier transform

Functions on Ξ $\subset \mathfrak{n}$

$\cap$ Lagrangian

$\mathcal{O}_{\min}$

Minimal nilpotent orbit $\mathcal{O}_{\min}$

$\mathcal{O}_{\min}$ does not admit G-equivariant Lagrangian foliation

$$\boxed{\underset{\text{Lagrangian}}{\mathcal{O}_{\min} \supset \Xi} \implies \mathcal{O}_{\min} = \bigcup_{g \in G} g\Xi \text{ is } \underline{\text{not}} \text{ disjoint}}$$

Minimal nilpotent orbit $\mathcal{O}_{\min}$

$\mathcal{O}_{\min}$ does not admit G-equivariant Lagrangian foliation

$$\boxed{\mathcal{O}_{\min} \underset{\text{Lagrangian}}{\supset \quad \Xi} \Longrightarrow \mathcal{O}_{\min} = \bigcup_{g\in G} g\Xi \text{ is } \underline{\text{not}} \text{ disjoint}}$$

$$\begin{tikzcd} & {}^{\exists}Z \arrow[dl] \arrow[dr] & \\ \mathcal{O}_{\min} & & G/{}^{\exists}P_{\max} \end{tikzcd}$$

generalized notion of polarization

(Guillemn–Sternberg)

$\cdots$ a family of Lagrangian submfds $\{g\Xi\}$

which are allowed to overlap

Minimal nilpotent orbit $\mathcal{O}_{\min}$

$\mathcal{O}_{\min}$ does not admit G-equivariant Lagrangian foliation

$$\mathcal{O}_{\min} \supset \underset{\text{Lagrangian}}{\Xi} \Longrightarrow \mathcal{O}_{\min} = \bigcup_{g\in G} g\Xi \text{ is } \underline{\text{not}} \text{ disjoint}$$

$$\begin{tikzcd} & {}^{\exists}Z \arrow[dl] \arrow[dr] & \\ \mathcal{O}_{\min} & & G/{}^{\exists}P_{\max} \end{tikzcd}$$

generalized notion of polarization
(Guillemn–Sternberg)

$\cdots$ a family of Lagrangian submfds $\{g\Xi\}$
which are allowed to overlap

Try to amalgamate $gP_{\max}g^{-1} \curvearrowright L^2(g\Xi)$ $(g \in G)$
to get $G \curvearrowright L^2(\Xi)$

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

Ex 1 $V = \mathrm{Symm}(n, \mathbb{R})$

$G = Mp(n, \mathbb{R})$, a double cover of $Sp(n, \mathbb{R})$

Ex 2 $V = \mathbb{R}^{p,q}$

$G = O(p + 1, q + 1)$

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

$\mathfrak{g} \simeq \mathfrak{g}^* \supset \mathcal{O}_{\min}$: minimal nilpotent coadjoint orbit of G

$\Xi := \mathcal{O}_{\min} \cap V$

<u>Observation</u>

$$\begin{array}{ccccc} \mathfrak{g} & & \simeq & & \mathfrak{g}^* \\ \cup & & & & \cup \\ V & & & & \mathcal{O}_{\min} \\ & \supset & & \subset & \\ & & \Xi & & \end{array}$$

L^2-model of minimal rep.

V: simple Jordan algebra with simple max euclidean Jordan subalg
G = (a finite covering of) the conformal group of V

$\mathfrak{g} \simeq \mathfrak{g}^* \supset \mathcal{O}_{\min}$: minimal nilpotent coadjoint orbit of G
$\Xi := \mathcal{O}_{\min} \cap V$

Assume $V \not\simeq \mathbb{R}^{p,q}$ with $p+q$: odd.

Theorem C (with Hilgert, Möllers, arXiv:1106.3621)
1) Ξ is a Lagrangian submanifold of $\mathcal{O}_{\min}$.

L^2-model of minimal rep.

V: simple Jordan algebra with simple max euclidean Jordan subalg
G = (a finite covering of) the conformal group of V

$\mathfrak{g} \simeq \mathfrak{g}^* \supset \mathcal{O}_{\min}$: minimal nilpotent coadjoint orbit of G
$\Xi := \mathcal{O}_{\min} \cap V$

Assume $V \not\simeq \mathbb{R}^{p,q}$ with $p + q$: odd.

Theorem C (with Hilgert, Möllers, arXiv:1106.3621)
1) Ξ is a Lagrangian submanifold of $\mathcal{O}_{\min}$.
2) We get an irreducible unitary representation π of G on $L^2(\Xi)$.

L^2-model of minimal rep.

V: simple Jordan algebra with simple max euclidean Jordan subalg
G = (a finite covering of) the conformal group of V

$\mathfrak{g} \simeq \mathfrak{g}^* \supset \mathcal{O}_{\min}$: minimal nilpotent coadjoint orbit of G
$\Xi := \mathcal{O}_{\min} \cap V$

Assume $V \not\simeq \mathbb{R}^{p,q}$ with $p + q$: odd.

Theorem C (with Hilgert, Möllers, arXiv:1106.3621)
1) Ξ is a Lagrangian submanifold of $\mathcal{O}_{\min}$.
2) We get an irreducible unitary representation π of G on $L^2(\Xi)$.
3) The Gelfand–Kirillov dimension $\mathrm{Dim}(\pi)$ attains its minimum
among all (∞-dim'l) irreducible unitary representations of G.

L^2-model of minimal rep.

V: simple Jordan algebra with simple max euclidean Jordan subalg
G = (a finite covering of) the conformal group of V

$\mathfrak{g} \simeq \mathfrak{g}^* \supset \mathcal{O}_{\min}$: minimal nilpotent coadjoint orbit of G
$\Xi := \mathcal{O}_{\min} \cap V$

Assume $V \not\simeq \mathbb{R}^{p,q}$ with $p+q$: odd.

Theorem C (with Hilgert, Möllers, arXiv:1106.3621)
1) Ξ is a Lagrangian submanifold of $\mathcal{O}_{\min}$.
2) We get an irreducible unitary representation π of G on $L^2(\Xi)$.
3) The Gelfand–Kirillov dimension $\mathrm{Dim}(\pi)$ attains its minimum among all (∞-dim'l) irreducible unitary representations of G.
4) π is a minimal rep of G if V is split and $\mathfrak{g} \not\simeq A_n$.

(minimal rep = the annihilator of $d\pi$ in the enveloping algebra $U(\mathfrak{g})$ is the Joseph ideal)

L^2-model of minimal rep.

V: simple Jordan algebra with simple max euclidean Jordan subalg

G = (a finite covering of) the conformal group of V

$\mathfrak{g} \simeq \mathfrak{g}^* \supset \mathcal{O}_{\min}$: minimal nilpotent coadjoint orbit of G

$\Xi := \mathcal{O}_{\min} \cap V$

Ex 1 $\quad V = Symm(n, \mathbb{R}) \supset \Xi = \{Y \in Symm(n, \mathbb{R}) : \operatorname{rank} Y \leq 1\}$

$G = Mp(n, \mathbb{R})$

$\Longrightarrow$ Schrödinger model of the Weil representation

$G \curvearrowright L^2(\Xi) \simeq L^2(\mathbb{R}^n)_{\mathrm{even}}$

Ex 2 $\quad V = \mathbb{R}^{p,q} \supset \Xi = \{x = (x', x'') \in \mathbb{R}^{p+q} : |x'|^2 - |x''|^2 = 0\}$

$G = O(p+1, q+1)$

$\Longrightarrow G \curvearrowright L^2(\Xi)$ if $p + q$ is even

Simple Jordan algebras V and conformal groups

	V	$\mathfrak{g} = \mathfrak{co}(V)$	$\mathfrak{l} = \mathfrak{str}(V)$
euclidean split	$Sym(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$
	$Herm(n, \mathbb{C})$	$\mathfrak{su}(n, n)$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$
	$Herm(n, \mathbb{H})$	$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n) \oplus \mathbb{R}$
	$\mathbb{R}^{1,n-1}$ $(n \geq 3)$	$\mathfrak{so}(2, n)$	$\mathfrak{so}(1, n-1) \oplus \mathbb{R}$
	$Herm(3, \mathbb{O})$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$
non-euclidean split	$M(n, \mathbb{R})$	$\mathfrak{sl}(2n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$
	$Skew(2n, \mathbb{R})$	$\mathfrak{so}(2n, 2n)$	$\mathfrak{sl}(2n, \mathbb{R}) \oplus \mathbb{R}$
	$\mathbb{R}^{p,q}$ $(p, q \geq 2)$	$\mathfrak{so}(p+1, q+1)$	$\mathfrak{so}(p, q) \oplus \mathbb{R}$
	$Herm(3, \mathbb{O}_s)$	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(6)} \oplus \mathbb{R}$
complex non-split	$Sym(n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$
	$M(n, \mathbb{C})$	$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$
	$Skew(2n, \mathbb{C})$	$\mathfrak{so}(4n, \mathbb{C})$	$\mathfrak{sl}(2n, \mathbb{C}) \oplus \mathbb{C}$
	$\mathbb{C}^n$ $(n \geq 3)$	$\mathfrak{so}(n+2, \mathbb{C})$	$\mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}$
	$Herm(3, \mathbb{O})_{\mathbb{C}}$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}$
non-euclidean non-split	$Sym(2n, \mathbb{C}) \cap M(n, \mathbb{H})$	$\mathfrak{sp}(n, n)$	$\mathfrak{su}^*(2n) \oplus \mathbb{R}$
	$M(n, \mathbb{H})$	$\mathfrak{su}^*(4n)$	$\mathfrak{su}^*(2n) \oplus \mathfrak{su}^*(2n) \oplus \mathbb{R}$
	$\mathbb{R}^{n,0}$ $(n \geq 2)$	$\mathfrak{so}(1, n+1)$	$\mathfrak{so}(n) \oplus \mathbb{R}$

Simple Jordan algebras V and conformal groups

	V	$\mathfrak{g} = \mathfrak{co}(V)$	$\mathfrak{l} = \mathfrak{str}(V)$
euclidean split	$Sym(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$
	$Herm(n, \mathbb{C})$	$\mathfrak{su}(n, n)$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$
	$Herm(n, \mathbb{H})$	$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n) \oplus \mathbb{R}$
	$\mathbb{R}^{1,n-1}$ $(n \geq 3)$	$\mathfrak{so}(2, n)$	$\mathfrak{so}(1, n-1) \oplus \mathbb{R}$
	$Herm(3, \mathbb{O})$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$
non-euclidean split	$M(n, \mathbb{R})$	$\mathfrak{sl}(2n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$
	$Skew(2n, \mathbb{R})$	$\mathfrak{so}(2n, 2n)$	$\mathfrak{sl}(2n, \mathbb{R}) \oplus \mathbb{R}$
	$\mathbb{R}^{p,q}$ $(p, q \geq 2)$	$\mathfrak{so}(p+1, q+1)$	$\mathfrak{so}(p, q) \oplus \mathbb{R}$
	$Herm(3, \mathbb{O}_s)$	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(6)} \oplus \mathbb{R}$
complex non-split	$Sym(n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$
	$M(n, \mathbb{C})$	$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$
	$Skew(2n, \mathbb{C})$	$\mathfrak{so}(4n, \mathbb{C})$	$\mathfrak{sl}(2n, \mathbb{C}) \oplus \mathbb{C}$
	$\mathbb{C}^n$ $(n \geq 3)$	$\mathfrak{so}(n+2, \mathbb{C})$	$\mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}$
	$Herm(3, \mathbb{O})_{\mathbb{C}}$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}$
non-euclidean non-split	$Sym(2n, \mathbb{C}) \cap M(n, \mathbb{H})$	$\mathfrak{sp}(n, n)$	$\mathfrak{su}^*(2n) \oplus \mathbb{R}$
	$M(n, \mathbb{H})$	$\mathfrak{su}^*(4n)$	$\mathfrak{su}^*(2n) \oplus \mathfrak{su}^*(2n) \oplus \mathbb{R}$
	$\mathbb{R}^{n,0}$ $(n \geq 2)$	$\mathfrak{so}(1, n+1)$	$\mathfrak{so}(n) \oplus \mathbb{R}$

L^2-model of minimal rep

V: real Jordan algebra

G: conformal gp of V

$$\mathfrak{g} = \underbrace{V + \mathfrak{str}(V)}_{\text{max parabolic}} + V^\vee$$

L^2-model of minimal rep

V: real Jordan algebra $\quad Symm(n, \mathbb{R})$

G: conformal gp of V

$$\mathfrak{g} = \underbrace{V + \mathfrak{str}(V)}_{\text{max parabolic}} + V^{\vee} \qquad \mathfrak{sp}(n, \mathbb{R})$$

Siegel parabolic

L^2-model of minimal rep

V: real Jordan algebra $\quad Symm(n, \mathbb{R})$

G: conformal gp of V

$$\mathfrak{g} = \underbrace{V + \mathfrak{str}(V)}_{\text{max parabolic}} + V^\vee \qquad \mathfrak{sp}(n, \mathbb{R})$$

Siegel parabolic

Step 1

minimal nilpotent

$$\begin{array}{ccccc} \mathcal{O}^G_{\min} & \supset & \Xi & := & \sqrt{-1}V \cap \mathcal{O}^G_{\min} \\ \cap & & \cap & & \\ \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} & & \end{array}$$

Lagrangian

L^2-model of minimal rep

V: real Jordan algebra	$Symm(n, \mathbb{R})$
G: conformal gp of V	
$\mathfrak{g} = \underbrace{V + \mathfrak{str}(V)}_{\text{max parabolic}} + V^\vee$	$\mathfrak{sp}(n, \mathbb{R})$ Siegel parabolic

Step 1

minimal nilpotent

$$\begin{array}{ccccl} \mathcal{O}^G_{\min} & \supset & \Xi & := & \sqrt{-1}V \cap \mathcal{O}^G_{\min} \quad \text{Lagrangian} \\ \cap & & \cap & & \\ \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} & & \end{array}$$

Step 2 $\quad \mathfrak{g} \curvearrowright C^\infty(\Xi)$

point: $V^\vee$ acts on $C^\infty(\Xi)$ by second order diff. op.

L^2-model of minimal rep

V: real Jordan algebra $\quad Symm(n, \mathbb{R})$

G: conformal gp of V

$$\mathfrak{g} = \underbrace{V + \mathfrak{str}(V)}_{\text{max parabolic}} + V^\vee \qquad \mathfrak{sp}(n, \mathbb{R})$$

Siegel parabolic

Step 1

minimal nilpotent

$$\begin{array}{ccccl} \mathcal{O}^G_{\min} & \supset & \Xi & := & \sqrt{-1}V \cap \mathcal{O}^G_{\min} \quad \text{Lagrangian} \\ \cap & & \cap & & \\ \sqrt{-1}\mathfrak{g}^* & \simeq & \sqrt{-1}\mathfrak{g} & & \end{array}$$

Step 2 $\quad \mathfrak{g} \curvearrowright C^\infty(\Xi)$

point: $V^\vee$ acts on $C^\infty(\Xi)$ by second order diff. op.

Step 3 $\quad$ Lift it to a covering of $G^\sim \curvearrowright L^2(\Xi)$

Orbit method and complementary series

The same construction works for the construction of the complementary series representations for $SO(n,1)$.

$$d\mu \rightsquigarrow d\mu_\lambda$$

a continuous family of measures $\underline{d\mu_\lambda}$ on the Lagrangian manifold Ξ.

$$L^2(\Xi, d\mu) \mapsto L^2(\Xi, d\mu_\lambda)$$

— This gives another geometric model of
the long $\underline{\text{complementary series } \pi_\lambda}$ for $SO(n,1)$ by B. Kostant
Kazhdan's Property (T) is not satisfied for $SO(n,1)$.

Towards a global formula $G \curvearrowright L^2(\Xi)$

$$\boxed{\begin{array}{ccccc} G & \curvearrowright & \mathcal{O}_{\min} & \underset{\text{minimal nilp. orbit}}{\subset} & \sqrt{-1}\mathfrak{g}^* \\ & & \cup\ \text{Lagrangian} & & \\ & & \Xi & & \end{array}}$$

Towards a global formula $G \curvearrowright L^2(\Xi)$

$$\begin{tikzcd} G \arrow[r, phantom, "\curvearrowright"] & \mathcal{O}_{\min} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup", "\text{Lagrangian}"'] & \sqrt{-1}\mathfrak{g}^* \\ & \Xi & \end{tikzcd}$$

minimal nilp. orbit

$\rightsquigarrow$

$G^{\sim} \overset{\pi}{\curvearrowright} L^2(\Xi)$ irreducible unitary rep.

Towards a global formula $G \curvearrowright L^2(\Xi)$

$G \curvearrowright \mathcal{O}_{\min} \subset \sqrt{-1}\mathfrak{g}^*$

minimal nilp. orbit

$\cancel{\curvearrowright}$ $\cup$ Lagrangian

Ξ

$\rightsquigarrow$

$G^{\sim} \overset{\pi}{\curvearrowright} L^2(\Xi)$ irreducible unitary rep.

Towards a global formula $G \curvearrowright L^2(\Xi)$

$$\begin{array}{cccccc} & G & \curvearrowright & \mathcal{O}_{\min} & \underset{\text{minimal nilp. orbit}}{\subset} & \sqrt{-1}\mathfrak{g}^* \\ \text{maximal parabolic} & \cup & \cancel{\curvearrowright} & \cup & \text{Lagrangian} & \\ & {}^{\exists}P & \curvearrowright & \Xi & & \end{array}$$

$$\wr\!\downarrow$$

$$G^{\sim} \overset{\pi}{\curvearrowright} L^2(\Xi) \quad \text{irreducible unitary rep.}$$

Towards a global formula $G \curvearrowright L^2(\Xi)$

$$\begin{array}{lccccc} & G & \curvearrowright & \mathcal{O}_{\min} & \subset & \sqrt{-1}\mathfrak{g}^* \\ & & & & \text{minimal nilp. orbit} & \\ \text{maximal parabolic} & \cup & \cancel{\curvearrowright} & \cup \ \text{Lagrangian} & & \\ & {}^{\exists}P & \curvearrowright & \Xi & & \end{array}$$

$$\wr$$

$$G^{\sim} \overset{\pi}{\curvearrowright} L^2(\Xi) \quad \text{irreducible unitary rep.}$$

Observation

1. $P^{\sim}$-action on $L^2(\Xi)$ is elementary
   (translation and multiplication)
2. $G = P \coprod P^{\exists} w_0 P$ (Bruhat decomposition)

Towards a global formula $G \curvearrowright L^2(\Xi)$

$$\begin{array}{llclll}
 & G & \curvearrowright & O_{\min} & \underset{\text{minimal nilp. orbit}}{\subset} & \sqrt{-1}\mathfrak{g}^* \\
\text{maximal parabolic} & \cup & \cancel{\curvearrowright} & \cup \ \text{Lagrangian} & & \\
 & {}^{\exists}P & \curvearrowright & \Xi & &
\end{array}$$

$$\Big\wr$$

$$G^{\sim} \overset{\pi}{\curvearrowright} L^2(\Xi)$$ irreducible unitary rep.

Observation

1. $P^{\sim}$-action on $L^2(\Xi)$ is elementary
   (translation and multiplication)
2. $G = P \coprod P^{\exists}w_0P$ (Bruhat decomposition)

What is $\pi(w_0)$?

Inversion element

$$G = PGL(2,\mathbb{C}) \underset{\text{Möbius transform}}{\curvearrowright} \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

Inversion element

$$G = PGL(2,\mathbb{C}) \underset{\text{Möbius transform}}{\curvearrowright} \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times,\ b \in \mathbb{C} \right\} \qquad z \mapsto az + b$$

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad z \mapsto -\frac{1}{z}$$

(inversion)

Inversion element

$$G = PGL(2, \mathbb{C}) \quad \underset{\text{Möbius transform}}{\curvearrowright} \quad \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times,\ b \in \mathbb{C} \right\} \qquad z \mapsto az + b$$

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad z \mapsto -\frac{1}{z} \qquad \text{(inversion)}$$

G is generated by P and w.

Inversion element

$$G = PGL(2,\mathbb{C}) \quad \underset{\text{Möbius transform}}{\curvearrowright} \quad \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

$$\doteqdot O(3,1) \qquad\qquad \doteqdot \mathbb{R}^{2,0}$$

$$P = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times,\ b \in \mathbb{C}\right\} \qquad z \mapsto az + b$$

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad z \mapsto -\tfrac{1}{z} \qquad \text{(inversion)}$$

G is generated by P and w.

Inversion element

$$G = PGL(2,\mathbb{C}) \quad \underset{\text{Möbius transform}}{\curvearrowright} \quad \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

$$\doteqdot O(3,1) \qquad\qquad \doteqdot \mathbb{R}^{2,0}$$

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times,\ b \in \mathbb{C} \right\} \qquad z \mapsto az + b$$

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad z \mapsto -\frac{1}{z} \qquad \text{(inversion)}$$

G is generated by P and w.

$$\boxed{\begin{array}{l} G = O(p+1, q+1) = P \coprod PwP \\ w : \ \text{inversion} \\ P = (O(p,q) \cdot \mathbb{R}^\times) \ltimes \mathbb{R}^{p+q} \approx \mathrm{Conf}(\mathbb{R}^{p,q}) \end{array}}$$

Global formula for the L^2-model

$\Xi \subset \mathcal{O}_{\min}^{G}$ (Lagrangian)

Point $G^{\sim}$ cannot act on Ξ, but on $L^2(\Xi)$.

except for $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$, $p + q$ odd

Global formula for the L^2-model

$\Xi \subset \mathcal{O}^G_{\min}$ (Lagrangian)

Point $G^\sim$ cannot act on Ξ, but on $L^2(\Xi)$.

$\mathfrak{g}$-action $d\pi$	on $C^\infty(\Xi)$
$\Vert$	
$\mathrm{str}(V) \oplus V$	diff ops of order ≤ 1
$\oplus$	
$V^\vee$	diff ops of order 2

Global formula for the L^2-model

$\Xi \subset \mathcal{O}^G_{\min}$ (Lagrangian)

Point $G^\sim$ cannot act on Ξ, but on $L^2(\Xi)$.

$\mathfrak{g}$-action $d\pi$	on $C^\infty(\Xi)$
$\parallel$	
$\mathrm{str}(V) \oplus V$	diff ops of order ≤ 1
$\oplus$	
$V^\vee$	diff ops of order 2

Key $\cdots$ unitary inversion operator $\pi(w_0)$ on $L^2(\Xi)$

$G^\sim \ni w_0$: conformal inversion $\mathrm{Ad}(w_0) : V \xrightarrow{\sim} V^\vee$

Global formula for the L^2-model

$\Xi \subset \mathcal{O}_{\min}^G$ (Lagrangian)

Point $G^\sim$ cannot act on Ξ, but on $L^2(\Xi)$.

$\mathfrak{g}$-action $d\pi$ on $C^\infty(\Xi)$

$\|$

$\mathrm{str}(V) \oplus V$ diff ops of order ≤ 1

$\oplus$

$V^\vee$ diff ops of order 2

Key $\cdots$ unitary inversion operator $\pi(w_0)$ on $L^2(\Xi)$

$G^\sim \ni w_0$: conformal inversion $\mathrm{Ad}(w_0) : V \xrightarrow{\sim} V^\vee$

Ex (Schrödinger model of Segal–Shale–Weil rep)

$G^\sim = Mp(n, \mathbb{R})$, $V = Symm(n, \mathbb{R})$

$\pi(w_0) =$ Fourier transform on $\mathbb{R}^n$ (up to phase factor)

Global formula for the L^2-model

$\Xi \subset \mathcal{O}^G_{\min}$ (Lagrangian)

Point $G^\sim$ cannot act on Ξ, but on $L^2(\Xi)$.

$\mathfrak{g}$-action $d\pi$ on $C^\infty(\Xi)$

$$\begin{array}{cl} \mathfrak{g}\text{-action } d\pi & \text{on } C^\infty(\Xi) \\ \| & \\ \mathrm{str}(V) \oplus V & \text{diff ops of order } \leq 1 \\ \oplus & \\ V^\vee & \text{diff ops of order } 2 \end{array}$$

Key $\cdots$ unitary inversion operator $\pi(w_0)$ on $L^2(\Xi)$

$G^\sim \ni w_0$: conformal inversion $\mathrm{Ad}(w_0) : V \xrightarrow{\sim} V^\vee$

Ex (K–Mano (2011) Memoirs of AMS vol. 1000)

$G^\sim = O(p+1, q+1), \quad V = \mathbb{R}^{p,q}$

$\pi(w_0) \cdots$ singular integral by Bessel distribution

Towards a global formula

$p+q$: even > 2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

Towards a global formula

$p + q$: even > 2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

P-action $\cdots$ translation and multiplication

w-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Towards a global formula

$p + q$: even > 2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

P-action $\cdots$ translation and multiplication

w-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Understand $\mathcal{F}_\Xi$ algebraically and analytically

Crucial for a global formula of G-actions,
and should open a beautiful theory.

Towards a global formula

$p+q$: even > 2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

P-action $\cdots$ translation and multiplication

w-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Understand $\mathcal{F}_\Xi$ algebraically and analytically

Cf. Analogous operator for the Weil rep.

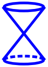
$Mp(n, \mathbb{R}) \curvearrowright L^2(\mathbb{R}^n)$

coincides with Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}$

(up to scalar)!

Algebraic aspects of $\mathcal{F}_\Xi$ on Ξ

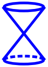
Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi =$ 

Algebraic aspects of $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

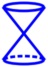
$$\mathcal{F}^4 = \mathrm{id}$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

Algebraic aspects of $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$\mathcal{F}^4 = \mathrm{id}$$

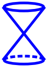
$\mathcal{F}_\Xi$ on $\Xi =$ 

$$\mathcal{F}_\Xi^2 = \mathrm{id}$$

Algebraic aspects of $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$Q_j \mapsto -P_j$$
$$P_j \mapsto Q_j$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

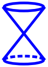
$Q_j = x_j$ (multiplication by coordinates function)

$$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$$

Algebraic aspects of $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$Q_j \mapsto -P_j$$
$$P_j \mapsto Q_j$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

$$Q_j \mapsto R_j$$
$$R_j \mapsto Q_j$$

$Q_j = x_j$ (multiplication by coordinates function)

$$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$$

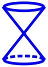
$R_j = {}^{\exists}$second order differential op. on Ξ

Algebraic aspects of $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$\begin{aligned} Q_j &\mapsto -P_j \\ P_j &\mapsto Q_j \end{aligned}$$

$\mathcal{F}_\Xi$ on $\Xi =$



$$\begin{aligned} Q_j &\mapsto R_j \\ R_j &\mapsto Q_j \end{aligned}$$

$Q_j = x_j$ (multiplication by coordinates function)

$$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$$

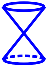
$R_j = {}^{\exists}$second order differential op. on Ξ

Rediscover Bargmann–Todorov's operators (1977)

Algebraic aspects of $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$Q_j \mapsto -P_j$$
$$P_j \mapsto Q_j$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

$$Q_j \mapsto R_j$$
$$R_j \mapsto Q_j$$

$Q_j = x_j$ (multiplication by coordinates function)

$$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$$

$R_j = {}^{\exists}$second order differential op. on Ξ

(Bargmann–Todorov operators)

Notice $\left.\begin{array}{l} Q_1{}^2 + \cdots + Q_p{}^2 - Q_{p+1}{}^2 - \cdots - Q_{p+q}{}^2 = 0 \\ R_1{}^2 + \cdots + R_p{}^2 - R_{p+1}{}^2 - \cdots - R_{p+q}{}^2 = 0 \end{array}\right\}$ on Ξ

Analytic aspects of $\mathcal{F}_\Xi$

$p + q$: even > 2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

w-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

<u>Problem</u> Find an explicit kernel of $\mathcal{F}_\Xi$ explicitly.

Analytic aspects of $\mathcal{F}_\Xi$

$p+q$: even > 2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

w-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Problem Find an explicit kernel of $\mathcal{F}_\Xi$ explicitly.

Cf. Euclidean case $\varphi(t) = e^{-it}$ (one variable)

$\mathcal{F}_{\mathbb{R}^N} f(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy$

$$\mathcal{F}_{\mathbb{R}^N} = \underbrace{\mathcal{F}_{\mathbb{R}^1}}_{\text{one variable}} \circ \text{Radon transform}$$

Explicit formula of $\mathcal{F}_\Xi$ on Ξ

Theorem D (K–Mano, Memoirs AMS, 2011, vol.1000)

Let $G = O(p+1, q+1)$ with $p+q$: even > 2

$$(\mathcal{F}_\Xi f)(x) = \int_\Xi \Phi_{p,q}(\langle x, y\rangle) f(y) dy$$

Explicit formula of $\mathcal{F}_\Xi$ on Ξ

Theorem D (K–Mano, Memoirs AMS, 2011, vol.1000)

Let $G = O(p+1, q+1)$ with $p+q$: even > 2

$$(\mathcal{F}_\Xi f)(x) = \int_\Xi \Phi_{p,q}(\langle x, y\rangle) f(y) dy$$

$q = 1$ case

$$\Phi_{p,1}(t) = 2\pi i (2t)^{-\frac{p-3}{2}} J_{\frac{p-3}{2}}(2\sqrt{2t}) \quad (t > 0)$$

Explicit formula of $\mathcal{F}_\Xi$ on Ξ

Theorem D (K–Mano, Memoirs AMS, 2011, vol.1000)

Let $G = O(p+1, q+1)$ with $p+q$: even > 2

$$(\mathcal{F}_\Xi f)(x) = \int_\Xi \Phi_{p,q}(\langle x, y\rangle) f(y) dy$$

$q = 1$ case

$$\Phi_{p,1}(t) = 2\pi i (2t)^{-\frac{p-3}{2}} J_{\frac{p-3}{2}}(2\sqrt{2t}) \quad (t > 0)$$

Theorem $\mathcal{F}_\Xi$ has a locally integrable kernel if and only if G is $O(p+1, 2)$, $O(2, q+1)$, or $O(3,3) \approx SL(4, \mathbb{R})$.

Explicit formula of $\mathcal{F}_\Xi$ on Ξ

Theorem D (K–Mano, Memoirs AMS, 2011, vol.1000)

Let $G = O(p+1, q+1)$ with $p+q$: even > 2

$$(\mathcal{F}_\Xi f)(x) = \int_\Xi \Phi_{p,q}(\langle x, y\rangle) f(y) dy$$

$q = 1$ case

$$\Phi_{p,1}(t) = 2\pi i(2t)^{-\frac{p-3}{2}} J_{\frac{p-3}{2}}(2\sqrt{2t}) \quad (t > 0)$$

Prop. We have the identities $\mod L^1_{\mathrm{loc}}(\mathbb{R})$

$$\Phi_{p,q}(t) \equiv \begin{cases} 0 & (\min(p,q) = 1) \\ -\pi i \sum_{l=0}^{m-1} \frac{(-1)^l}{2^l(m-l-1)!} \delta^{(l)}(t) & (p, q > 1;\ \text{both even}) \\ -i \sum_{l=0}^{m-1} \frac{l!}{2^l(m-l-1)!} t^{-l-1} & (p, q > 1;\ \text{both odd}) \end{cases}$$

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model
— as a geometric quantization of the Kostant–Sekiguchi correspondence
(arXiv:1203.5462 with Hilgert, Möllers and Ørsted)

§5 Deformation of Fourier transform

Minimal reps with highest weights

So far our minimal reps π are irred. unitary reps without highest weights in general.

Minimal reps with highest weights

So far our minimal reps π are irred. unitary reps
without highest weights in general.

However, some classical such π are highest weight modules, e.g.

$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R}) \quad \cdots \quad \pi$: Segal–Shale–Weil rep.
$\mathfrak{g} = \mathfrak{o}(2, 4) \quad \cdots \quad \pi$: bound states of hydrogen atom.
$\Rightarrow$ simple and detailed analysis

Minimal reps with highest weights

So far our minimal reps π are irred. unitary reps without highest weights in general.

However, some classical such π are highest weight modules, e.g.

$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ $\cdots$ π: Segal–Shale–Weil rep.
$\mathfrak{g} = \mathfrak{o}(2, 4)$ $\cdots$ π: bound states of hydrogen atom.
$\Rightarrow$ simple and detailed analysis

Now we confine ourselves to the case when G/K is a tube domain
($\Rightarrow$ π: highest weight module)

Minimal reps with highest weights

So far our minimal reps π are irred. unitary reps
without highest weights in general.

However, some classical such π are highest weight modules, e.g.

$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ $\cdots$ π: Segal–Shale–Weil rep.
$\mathfrak{g} = \mathfrak{o}(2, 4)$ $\cdots$ π: bound states of hydrogen atom.
$\Rightarrow$ simple and detailed analysis

Now we confine ourselves to the case when G/K is a tube domain
($\Rightarrow$ π: highest weight module)

- Fock model and Segal–Bargmann transform for minimal reps
  (arXiv:1203.5462, joint with J. Hilgert, J. Möllers, B. Ørsted)

Kostant–Sekiguchi correspondence

$$\boxed{\mathfrak{g} + \sqrt{-1}\mathfrak{g} = \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}}$$

Complexified Cartan decomposition

Kostant–Sekiguchi correspondence

$$\boxed{\mathfrak{g} + \sqrt{-1}\mathfrak{g} = \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}}$$

Complexified Cartan decomposition

Kostant–Sekiguchi correspondence

$$\boxed{G \curvearrowright \sqrt{-1}\mathfrak{g}^*} \leftrightarrow \boxed{K_{\mathbb{C}} \curvearrowright \mathfrak{p}_{\mathbb{C}}^*}$$

Bijection between nilpotent orbits

Kostant–Sekiguchi correspondence

$$\boxed{\mathfrak{g} + \sqrt{-1}\mathfrak{g} = \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}}$$

Complexified Cartan decomposition

Kostant–Sekiguchi correspondence

$$\boxed{G \curvearrowright \sqrt{-1}\mathfrak{g}^*} \leftrightarrow \boxed{K_{\mathbb{C}} \curvearrowright \mathfrak{p}^*_{\mathbb{C}}}$$

Bijection between nilpotent orbits

Ex. $G = GL(n, \mathbb{R}) \quad K_{\mathbb{C}} = O(n, \mathbb{C})$

{nilpotents in $M(n, \mathbb{R})\}/GL(n, \mathbb{R}) \simeq$ {nilpotents in $Symm(n, \mathbb{C})\}/O(n, \mathbb{C})$

Geometric quantization of Kostant–Sekiguchi correspondence

$$\begin{tikzcd}
 & \mathbb{O}_{\min}^{G_{\mathbb{C}}} & \\
\sqrt{-1}\mathfrak{g}^* \supset \mathbb{O}_{\min}^{G} \arrow[ur, phantom, "\subset" sloped] \arrow[rr, leftrightarrow, "\text{Kostant–Sekiguchi}"'] & & \mathbb{O}_{\min}^{K_{\mathbb{C}}} \subset \mathfrak{p}_{\mathbb{C}}^* \arrow[ul, phantom, "\subset" sloped]
\end{tikzcd}$$

$$\boxed{\mathfrak{g} + \sqrt{-1}\mathfrak{g} = \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}}$$

Complexified Cartan decomposition

Geometric quantization of Kostant–Sekiguchi correspondence

$$\begin{tikzcd}
 & \mathbb{O}_{\min}^{G_{\mathbb{C}}} & \\
\sqrt{-1}\mathfrak{g}^* \supset \mathbb{O}_{\min}^{G} \arrow[ur, phantom, sloped, "\subset"] \arrow[rr, leftrightarrow, "\text{Kostant–Sekiguchi}"'] & & \mathbb{O}_{\min}^{K_{\mathbb{C}}} \subset \mathfrak{p}_{\mathbb{C}}^* \arrow[ul, phantom, sloped, "\supset"]
\end{tikzcd}$$

Geometric quantization of Kostant–Sekiguchi correspondence

$$\begin{tikzcd}
& \mathbb{O}^{G_{\mathbb{C}}}_{\min} & \\
\sqrt{-1}\mathfrak{g}^* \supset \mathbb{O}^{G}_{\min} \arrow[ur, phantom, sloped, "\subset"] \arrow[rr, leftrightarrow, "\text{Kostant–Sekiguchi}"'] \arrow[d, squiggly] & & \mathbb{O}^{K_{\mathbb{C}}}_{\min} \subset \mathfrak{p}^*_{\mathbb{C}} \arrow[ul, phantom, sloped, "\supset"] \arrow[d, squiggly] \\
\boxed{?} \arrow[rr, phantom, "\text{‘Geometric quantization’}"] & & \boxed{?}
\end{tikzcd}$$

L^2-model of min rep
(Thm C)

Geometric quantization of Kostant–Sekiguchi correspondence

$$\begin{tikzcd}
 & \mathbb{O}^{G_{\mathbb{C}}}_{\min} & \\
\Xi \subset \mathbb{O}^{G}_{\min} \arrow[ur, phantom, sloped, "\subset"] \arrow[rr, leftrightarrow, "\text{Kostant–Sekiguchi}"'] & & \mathbb{O}^{K_{\mathbb{C}}}_{\min} \arrow[ul, phantom, sloped, "\supset"] \\
\text{Lagrangian} \arrow[d, rightsquigarrow] & \text{‘Geometric quantization’} & {} \arrow[d, rightsquigarrow] \\
L^2(\Xi) & & \boxed{?}
\end{tikzcd}$$

L^2-model of min rep
(Thm C)

Geometric quantization of Kostant–Sekiguchi correspondence

$$\begin{tikzcd}
 & \mathbb{O}^{G_{\mathbb{C}}}_{\min} & \\
\Xi \subset \mathbb{O}^{G}_{\min} \arrow[ur, phantom, "\subset" sloped] \arrow[rr, leftrightarrow, "\text{Kostant–Sekiguchi}"'] \arrow[d, rightsquigarrow, "\text{Lagrangian}"'] & & \mathbb{O}^{K_{\mathbb{C}}}_{\min} \arrow[ul, phantom, "\supset" sloped] \arrow[d, rightsquigarrow, "\text{Brylinski–Kostant}"] \\
L^2(\Xi) \arrow[rr, phantom, "\text{+Kronheimer–Vergne}", "\text{‘Geometric quantization’}"'] & & \mathcal{O}(\mathbb{O}^{K_{\mathbb{C}}}_{\min})
\end{tikzcd}$$

L^2-model of min rep
(Thm C)

(Lie algebra acts
by ψDO)

Geometric quantization of Kostant–Sekiguchi correspondence

$$\begin{tikzcd}
 & \mathbb{O}_{\min}^{G_{\mathbb{C}}} & \\
\Xi \subset \mathbb{O}_{\min}^{G} \arrow[ur, phantom, sloped, "\subset"] \arrow[rr, leftrightarrow, "\text{Kostant–Sekiguchi}"'] & & \mathbb{O}_{\min}^{K_{\mathbb{C}}} \arrow[ul, phantom, sloped, "\supset"] \arrow[dd, rightsquigarrow, "\text{HKMO arXiv:1203.5462}"] \\
\text{Lagrangian} \arrow[d, rightsquigarrow] & \text{`Geometric quantization'} & \\
L^2(\Xi) & & \mathcal{O}(\mathbb{O}_{\min}^{K_{\mathbb{C}}}) \cap L^2(\mathbb{O}_{\min}^{K_{\mathbb{C}}}, d\mu)
\end{tikzcd}$$

L^2-model of min rep
(Thm C)

holomorphic model of min rep

(G/K tube domain)

Geometric quantization of Kostant–Sekiguchi correspondence

$$\begin{tikzcd}
 & \mathbb{O}_{\min}^{G_{\mathbb{C}}} & \\
\Xi \subset \mathbb{O}_{\min}^{G} \arrow[ur, phantom, sloped, "\subset"] \arrow[rr, leftrightarrow, "\text{Kostant–Sekiguchi}"'] & & \mathbb{O}_{\min}^{K_{\mathbb{C}}} \arrow[ul, phantom, sloped, "\supset"] \\
L^2(\Xi) \arrow[u, leftsquigarrow, "\text{Lagrangian}"'] \arrow[rr, "\sim", "\text{unitary transform}"'] & & \mathcal{O}(\mathbb{O}_{\min}^{K_{\mathbb{C}}}) \cap L^2(\mathbb{O}_{\min}^{K_{\mathbb{C}}}, d\mu) \arrow[u, leftsquigarrow, "\text{HKMO arXiv:1203.5462}"']
\end{tikzcd}$$

‘Geometric quantization’

L^2-model of min rep $\mathbb{B}_{\Xi}$ holomorphic model of min rep

(Thm C)

↺ $(G/K$ tube domain)

$\mathcal{F}_{\Xi}$

(Thm D)

Geometric quantization of Kostant–Sekiguchi correspondence

$$\begin{tikzcd}[column sep=large]
 & \mathbb{O}^{G_\mathbb{C}}_{\min} & \\
\Xi \subset \mathbb{O}^{G}_{\min} \arrow[ur, phantom, sloped, "\subset"] \arrow[rr, leftrightarrow, "\text{Kostant–Sekiguchi}"'] & & \mathbb{O}^{K_\mathbb{C}}_{\min} \arrow[ul, phantom, sloped, "\supset"] \\
\text{Lagrangian} \arrow[d, squiggly] & \text{‘Geometric quantization’} & {} \arrow[d, squiggly, "\text{HKMO arXiv:1203.5462}"] \\
L^2(\Xi) \arrow[rr, "\sim", "\text{unitary transform}"'] & & \mathcal{O}(\mathbb{O}^{K_\mathbb{C}}_{\min}) \cap L^2(\mathbb{O}^{K_\mathbb{C}}_{\min}, d\mu)
\end{tikzcd}$$

L^2-model of min rep (Thm C)

$\mathbb{B}_\Xi$

holomorphic model of min rep

$\circlearrowleft$ $\mathcal{F}_\Xi$

(G/K tube domain)

Thm E (HKMO, arXiv:1203.5462) We construct explicitly:

1) $d\mu$: K-Bessel, $\mathbb{B}_\Xi$: I-Bessel, $\mathcal{F}_\Xi$: J-Bessel

2) $G \curvearrowright \mathcal{O}(\mathbb{O}^{K_\mathbb{C}}_{\min}) \cap L^2(\mathbb{O}^{K_\mathbb{C}}_{\min}, d\mu)$ minimal rep

3) $\mathbb{B}_\Xi$ intertwines G-actions on two models of minimal reps

Geometric quantization of Kostant–Sekiguchi correspondence

$$\begin{tikzcd}
 & \mathbb{O}_{\min}^{G_{\mathbb{C}}} & \\
\Xi \subset \mathbb{O}_{\min}^{G} \arrow[ur, phantom, sloped, "\subset"] \arrow[rr, leftrightarrow, "\text{Kostant–Sekiguchi}"'] & & \mathbb{O}_{\min}^{K_{\mathbb{C}}} \arrow[ul, phantom, sloped, "\supset"] \arrow[dd, squiggly, "\text{HKMO arXiv:1203.5462}"] \\
\text{Lagrangian} \arrow[d, squiggly] & \text{`Geometric quantization'} & \\
L^2(\Xi) \arrow[rr, "\sim", "\text{unitary transform}"'] & & \mathcal{O}(\mathbb{O}_{\min}^{K_{\mathbb{C}}}) \cap L^2(\mathbb{O}_{\min}^{K_{\mathbb{C}}}, d\mu)
\end{tikzcd}$$

L^2-model of min rep (Thm C) $\qquad \mathbb{B}_{\Xi} \qquad$ holomorphic model of min rep

$\circlearrowleft$

$\mathcal{F}_{\Xi}$ $\qquad$ (G/K tube domain)

Classical case: G/K = Siegel upper half space

$L^2(\Xi) \cdots$ Schrödinger model of Weil rep

$\mathcal{O}(\mathbb{O}_{\min}^{K_{\mathbb{C}}}) \cap L^2(\mathbb{O}_{\min}^{K_{\mathbb{C}}}, d\mu) \cdots$ Fock model of Weil rep

$\mathbb{B}_{\Xi} \cdots$ Bargman–Segal transform

§1 What are minimal representations?

§2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?

§3 L^2 model of minimal representations

§4 Fock model and Schrödinger model

§5 Deformation of Fourier transform

- §1 What are minimal representations?
- §2 $\mathrm{Limit} \circ Q = Q \circ \mathrm{Limit}$?
- §3 L^2 model of minimal representations
- §4 Fock model and Schrödinger model
- §5 Deformation of Fourier transform

Deformation theory of Fourier transform

- Laguerre semigroup and Dunkl operators, 74 pp.
  Compositio Math (2012) (to appear)
  joint with S. Ben Saïd and B. Ørsted
- Generalized Fourier transform $\mathscr{F}_{k,a}$ C.R.A.S. Paris (2009)
- Inversion and holomorphic extension, 65 pp.
  R. Howe 60th birthday volume (2007) with G. Mano

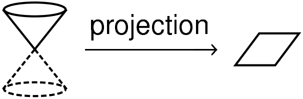
Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_{\Xi}$	$\cdots$	unitary inversion on $\Xi \quad \subset \mathbb{R}^{p,q}$
$\mathcal{F}_{\mathbb{R}^N}$	$\cdots$	Fourier transform on $\mathbb{R}^N$

Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_{\Xi}$	$\cdots$	unitary inversion on $\Xi \quad \subset \mathbb{R}^{p,q}$
$\mathcal{F}_{\mathbb{R}^N}$	$\cdots$	Fourier transform on $\mathbb{R}^N$

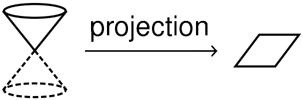
Assume $q = 1$. Set $p = N$.

$\mathbb{R}^{N,1} \supset \Xi =$  $= \mathbb{R}^N$

Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_\Xi$	$\cdots$	unitary inversion on $\Xi \quad \subset \mathbb{R}^{p,q}$
$\mathcal{F}_{\mathbb{R}^N}$	$\cdots$	Fourier transform on $\mathbb{R}^N$

Assume $q = 1$. Set $p = N$.



$\mathbb{R}^{N,1} \supset \Xi =$ $\xrightarrow{\text{projection}}$ $= \mathbb{R}^N$

$\mathcal{F}_\Xi$ $\mathcal{F}_{\mathbb{R}^N}$

$O(N+1,2)$ $Mp(N,\mathbb{R})$

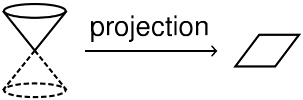
Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_\Xi$ $\cdots$ unitary inversion on $\Xi \subset \mathbb{R}^{p,q}$

$\mathcal{F}_{\mathbb{R}^N}$ $\cdots$ Fourier transform on $\mathbb{R}^N$

Assume $q = 1$. Set $p = N$.

$\mathbb{R}^{N,1} \supset \Xi =$



$= \mathbb{R}^N$

$$\mathcal{F}_\Xi \overset{\text{interpolate}}{\cdots\cdots\cdots} \mathcal{F}_{\mathbb{R}^N}$$

$a = 1$ $\qquad$ $a = 2$

a $\cdots$ deformation parameter > 0

$k = (k_\alpha)$ multiplicity on root system $\cdots$ Dunkl operator

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\boxed{\mathcal{F}_{\mathbb{R}^N} = \quad c \quad \exp\left(\frac{\pi i}{4}(\quad \Delta \quad - |x|^2)\right)}$$

(k,a)-deformation of $\exp\frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\boxed{\mathcal{F}_{\mathbb{R}^N} = \underbrace{c}_{\substack{\text{phase factor}\\ = e^{\frac{\pi i N}{4}}}} \exp\Big(\frac{\pi i}{4}(\overbrace{\underbrace{\Delta}_{\text{Laplacian}} - |x|^2}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N)})\Big)}$$

(k, a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\boxed{\mathcal{F}_{\mathbb{R}^N} = \underbrace{c}_{\substack{\text{phase factor} \\ = e^{\frac{\pi i N}{4}}}} \exp\left(\frac{\pi i}{4}\big(\overbrace{\underbrace{\Delta}_{\text{Laplacian}} - |x|^2}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N)}\big)\right)}$$

Hermite semigroup (oscillator semigp $\cdots$ R. Howe)

$$\boxed{I(t) := \exp \frac{t}{2}(\Delta - |x|^2)}$$

Mehler kernel using $\exp(-x^2)$

(k, a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Unitary inversion on Ξ

$$\boxed{\mathcal{F}_\Xi = \underbrace{c}_{\substack{\text{phase factor} \\ = e^{\frac{\pi i(N-1)}{2}}}} \exp\Big(\frac{\pi i}{2}\big(\overbrace{\underbrace{|x|\Delta}_{\text{Laplacian}} - |x|}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N, \frac{dx}{|x|})}\big)\Big)}$$

“Laguerre semigroup”

$$\boxed{\mathcal{I}(t) := \exp t(|x|\Delta - |x|)} \qquad \mathrm{Re}\, t > 0$$

closed formula using Bessel function ([K–Mano], 2007)

(k, a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

$(0, a)$-generalized Fourier transform

$$\boxed{\mathcal{F}_{0,a} = c \quad \exp\Big(\frac{\pi i}{2a}\overbrace{(|x|^{2-a}\Delta \; - |x|^a)}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N, |x|^{a-2}dx)}\Big)}$$

phase factor Laplacian

$= e^{i\frac{\pi(N+a-2)}{2a}}$

$(0, a)$-deformation of Hermite semigroup

$$\boxed{\mathcal{I}_{0,a}(t) := \exp \frac{t}{a}(|x|^{2-a}\Delta - |x|^a)}$$

Deformation parameter

$a > 0$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

(k,a)-generalized Fourier transform

$$\mathcal{F}_{k,a} = c \exp\Big(\frac{\pi i}{2a}\overbrace{(|x|^{2-a}\Delta_k - |x|^a)}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)}\Big)$$

phase factor

$= e^{i\frac{\pi(N+2\langle k\rangle+a-2)}{2a}}$

Dunkl Laplacian

(k,a)-deformation of Hermite semigroup

$$\boxed{\mathcal{I}_{k,a}(t) := \exp \frac{t}{a}(|x|^{2-a}\Delta_k - |x|^a)}$$

Deformation parameter

k: multiplicity on root system $\mathcal{R}$

$a > 0$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

(k,a)-generalized Fourier transform

$$\boxed{\mathcal{F}_{k,a} = c \quad \exp\Big(\frac{\pi i}{2a}(\overbrace{|x|^{2-a}\Delta_k \quad -|x|^a}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)})\Big)}$$

phase factor $= e^{i\frac{\pi(N+2\langle k\rangle+a-2)}{2a}}$

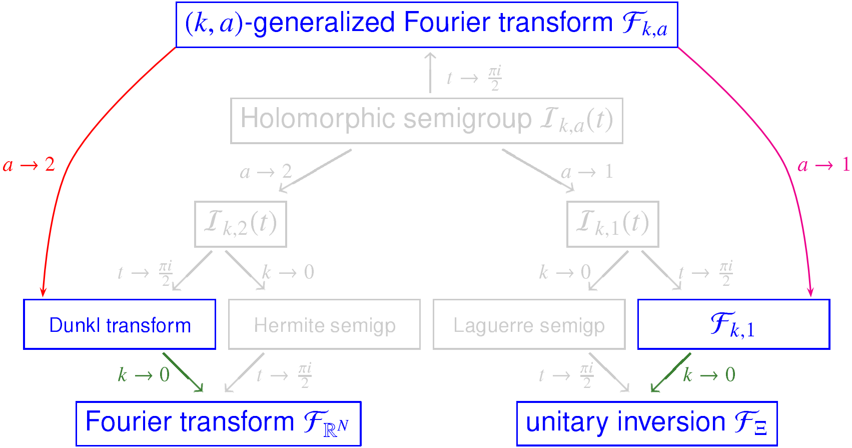
Dunkl Laplacian

(k,a)-deformation of Hermite semigroup (Compositio Math (2012) joint with Ben Saïd and Ørsted)

$$\boxed{\mathcal{I}_{k,a}(t) := \exp \frac{t}{a}(|x|^{2-a}\Delta_k - |x|^a)} \qquad \operatorname{Re} t > 0$$

Deformation parameter k: multiplicity on root system $\mathcal{R}$, $a > 0$

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



(k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$

$t \to \frac{\pi i}{2}$

Holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$a \to 2$

$a \to 2$

$a \to 1$

$a \to 1$

$\mathcal{I}_{k,2}(t)$

$\mathcal{I}_{k,1}(t)$

$t \to \frac{\pi i}{2}$

$k \to 0$

$k \to 0$

$t \to \frac{\pi i}{2}$

Dunkl transform

Hermite semigp

Laguerre semigp

$\mathcal{F}_{k,1}$

$k \to 0$

$t \to \frac{\pi i}{2}$

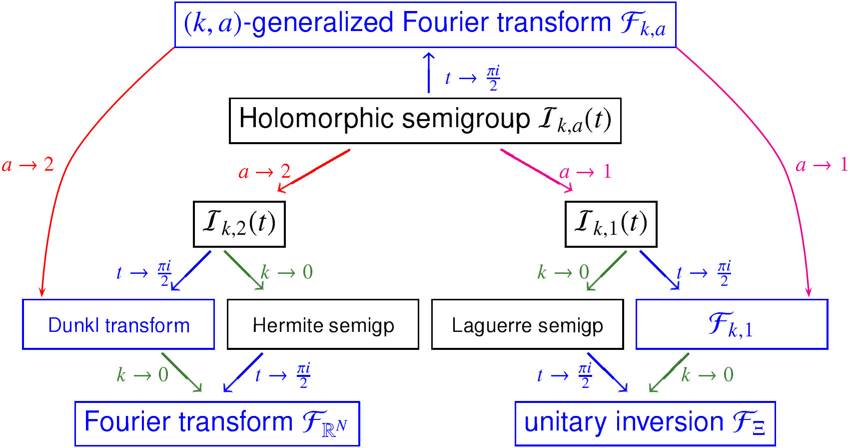
$t \to \frac{\pi i}{2}$

$k \to 0$

Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

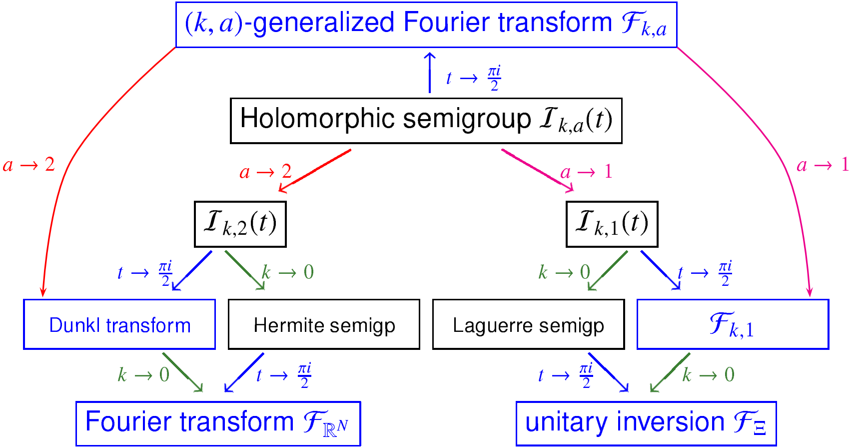
unitary inversion $\mathcal{F}_{\Xi}$

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



$$\begin{tikzcd}
 & & & \text{(k,a)-generalized Fourier transform } \mathcal{F}_{k,a} \arrow[dddlll, bend right, "a \to 2"'] \arrow[dddrrr, bend left, "a \to 1"] & & & \\
 & & & \text{Holomorphic semigroup } \mathcal{I}_{k,a}(t) \arrow[u, "t \to \frac{\pi i}{2}"'] \arrow[dll, "a \to 2"'] \arrow[drr, "a \to 1"] & & & \\
 & \mathcal{I}_{k,2}(t) \arrow[dl, "t \to \frac{\pi i}{2}"'] \arrow[dr, "k \to 0"] & & & & \mathcal{I}_{k,1}(t) \arrow[dl, "k \to 0"'] \arrow[dr, "t \to \frac{\pi i}{2}"] & \\
\text{Dunkl transform} \arrow[dr, "k \to 0"'] & & \text{Hermite semigp} \arrow[dl, "t \to \frac{\pi i}{2}"] & & \text{Laguerre semigp} \arrow[dr, "t \to \frac{\pi i}{2}"'] & & \mathcal{F}_{k,1} \arrow[dl, "k \to 0"] \\
 & \text{Fourier transform } \mathcal{F}_{\mathbb{R}^N} & & & & \text{unitary inversion } \mathcal{F}_{\Xi} &
\end{tikzcd}$$

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



⇐ 'unitary inversion operator' ⇒

the Weil representation of
the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of
the conformal group $O(N+1, 2)$

Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c\,\mathcal{I}_{k,a}(\frac{\pi i}{2})$$

Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c\,\mathcal{I}_{k,a}(\frac{\pi i}{2}) = c\exp\left(\frac{\pi i}{2a}(|x|^{2-a}\Delta_k - |x|^a)\right)$$

Thm F (Compositio Math (2012) joint with Ben Saïd and Ørsted)

1) $\mathcal{F}_{k,a}$ is a unitary operator

Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c\,\mathcal{I}_{k,a}(\frac{\pi i}{2}) = c\exp\left(\frac{\pi i}{2a}(|x|^{2-a}\Delta_k - |x|^a)\right)$$

Thm F (Compositio Math (2012) joint with Ben Saïd and Ørsted)

1) $\mathcal{F}_{k,a}$ is a unitary operator

2) $\mathcal{F}_{0,2}$ = Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^N}$ on $\mathbb{R}^N$

$F_{k,2}$ = Dunkl transform $\mathcal{D}_k$ on $\mathbb{R}^N$

$\mathcal{F}_{0,1}$ = unitary inversion $\mathcal{F}_{\Xi}$ on $L^2($$)$ for $O(2, N+1)$

3) $\mathcal{F}_{k,a}$ is of finite order $\Longleftrightarrow$ $a \in \mathbb{Q}$

4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a}\Delta_k$

Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c\,\mathcal{I}_{k,a}(\frac{\pi i}{2}) = c\exp\left(\frac{\pi i}{2a}(|x|^{2-a}\Delta_k - |x|^a)\right)$$

Thm F (Compositio Math (2012) joint with Ben Saïd and Ørsted)

1) $\mathcal{F}_{k,a}$ is a unitary operator

2) $\mathcal{F}_{0,2}$ = Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^N}$ on $\mathbb{R}^N$
   $F_{k,2}$ = Dunkl transform $\mathcal{D}_k$ on $\mathbb{R}^N$
   $\mathcal{F}_{0,1}$ = unitary inversion $\mathcal{F}_{\Xi}$ on $L^2($$)$ for $O(2, N+1)$

3) $\mathcal{F}_{k,a}$ is of finite order $\Longleftrightarrow a \in \mathbb{Q}$

4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a}\Delta_k$

$\Longrightarrow$ generalization of classical identities such as Hecke identity, Bochner identity, Parseval–Plancherel formulas, Weber's second exponential integral, etc.

Heisenberg-type inequality

Thm G ([3]) (Heisenberg inequality)

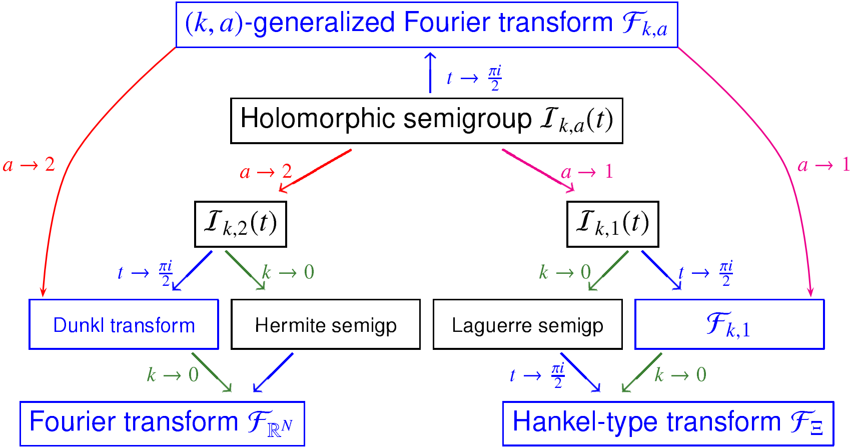
$$\||x|^{\frac{a}{2}} f(x)\|_k \, \||\xi|^{\frac{a}{2}} (\mathcal{F}_{k,a} f)(\xi)\|_k \geq \frac{2\langle k\rangle+N+a-2}{2} \|f(x)\|_k^2$$

$k \equiv 0$, $a = 2$ $\cdots$ Weyl–Pauli–Heisenberg inequality for Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

k: general, $a = 2$ $\cdots$ Heisenberg inequality for Dunkl transform $\mathcal{D}_k$ (Rösler, Shimeno)

$k \equiv 0$, $a = 1$, $N = 1$ $\cdots$ Heisenberg inequality for Hankel transform

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



(k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$

$t \to \frac{\pi i}{2}$

Holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$a \to 2$

$a \to 2$

$a \to 1$

$a \to 1$

$\mathcal{I}_{k,2}(t)$

$\mathcal{I}_{k,1}(t)$

$t \to \frac{\pi i}{2}$

$k \to 0$

$k \to 0$

$t \to \frac{\pi i}{2}$

Dunkl transform

Hermite semigp

Laguerre semigp

$\mathcal{F}_{k,1}$

$k \to 0$

$t \to \frac{\pi i}{2}$

$k \to 0$

Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

Hankel-type transform $\mathcal{F}_{\Xi}$

⇐ ‘unitary inversion operator’ ⇒

the Weil representation of
the metaplectic group $Mp(N,\mathbb{R})$

the minimal representation of
the conformal group $O(N+1,2)$

Hidden symmetries in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

Coxeter group

$$\begin{tikzcd}
 & & \boxed{O(N+1,2)^{\sim}} \\
\boxed{\mathfrak{C} \times S\widetilde{L(2,\mathbb{R})}} \arrow[r, "k\to0"] & \boxed{O(N) \times S\widetilde{L(2,\mathbb{R})}} \arrow[ur, "a \to 1"'] \arrow[dr, "a \to 2"] & \\
 & & \boxed{Mp(N,\mathbb{R})}
\end{tikzcd}$$

(k, a : general)

Hidden symmetries in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

Coxeter group

$$\begin{tikzcd}
 & & \boxed{O(N+1,2)^{\sim}} \arrow[dl, bend right=60] \\
\boxed{\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})}} \arrow[r, "k \to 0"] & \boxed{O(N) \times \widetilde{SL(2,\mathbb{R})}} \arrow[ur, "a \to 1"'] \arrow[dr, "a \to 2"] & \\
 & & \boxed{Mp(N,\mathbb{R})} \arrow[ul, bend left=60]
\end{tikzcd}$$

$(k, a$: general$)$

Idea

discretely decomposable and multiplicity-one branching laws of minimal reps

Minimal reps ⇔ Maximal symmetries

My wish:

Dig out some interesting and (potentially) rich
geometric analysis
inspired by minimal reps.

Study of minimal reps

⇓

Try to forget (a part of) rep theory!