Geometric Analysis on Minimal Representations

Representation Theory of Real Reductive Groups

University of Utah, Salt Lake City, USA, 27–31 July 2009

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Minimal representations

Oscillator rep. ($=$ Segal–Shale–Weil rep.)
Minimal rep. of $Mp(n, \mathbb{R})$ (= double cover of $Sp(n, \mathbb{R})$)
$\cdots$ split simple group of type C

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Minimal rep. of $O(p, q)$, $p + q$: even
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(Ambitious) Project:
Use minimal reps as a guiding principle to find new interactions with other fields of mathematics.

If possible, try to formulate a theory in a wide setting without group, and prove it without representation theory.

Minimal rep of reductive groups

Minimal representations of a reductive group G
(their annihilators are the Joseph ideal in $U(\mathfrak{g})$)

Loosely, minimal representations are

- one of 'building blocks' of unitary reps.
- 'smallest' infinite dimensional unitary rep.
- 'isolated' (finitely many) among the unitary dual (continuously many)
- 'attached to' the minimal nilpotent orbit
- matrix coefficients are of bad decay

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Viewpoint:

Minimal representation ($\Leftarrow$ group)

$\approx$ Maximal symmetries ($\Leftarrow$ rep. space)

Geometric analysis on minimal reps of $O(p, q)$

[1] Laguerre semigroup and Dunkl operators $\cdots$
preprint, 74 pp. arXiv:0907.3749

[2] Special functions associated to a fourth order differential equation $\cdots$
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[3] Generalized Fourier transforms $\mathcal{F}_{k,a}$ $\cdots$ C.R.A.S. Paris (to appear)

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Collaborated with

S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers and B. Ørsted

Indefinite orthogonal group $O(p+1, q+1)$

Throughout this talk, $p, q \geq 1$, $p+q$: even > 2

$$\begin{aligned} G &= O(p+1, q+1) \\ &= \{g \in GL(p+q+2, \mathbb{R}) : {}^t g \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix} g = \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix} \end{aligned}$$

$\cdots$ real simple Lie group of type D

Minimal representation of $G = O(p+1, q+1)$

- $q = 1$
  highest weight module $\oplus$ lowest weight module
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  - $p = q = 3$ case: Kostant (1990)
- p, q: general
  non-highest, non-spherical
  - subrepresentation of most degenerate principal series (Howe–Tan, Binegar–Zierau)
  - dual pair correspondence
    $(Sp(1, \mathbb{R}) \times O(p+1, q+1)$ in $Sp(p+q+2, \mathbb{R}))$ (Huang–Zhu)

Two constructions of minimal reps.

1. Conformal model

    Theorem B

2. L^2 model

(Schrödinger model)

    Theorem D

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal model		
Theorem B	Clear	?
v.s.		
2. L^2 model (Schrödinger model)	?	Clear
Theorem D		

Clear $\cdots$ advantage of the model

Two constructions of minimal reps.

	Group action	Hilbert structure
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Theorem B	Clear	Theorem C
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3. Deformation of Fourier transforms (Theorems F, G, H)
(interpolation, Dunkl operators, special functions)

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§1 Conformal construction of minimal reps.

Idea: Composition of holomorphic functions

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$\Longrightarrow$ Try to modify the definition!

$\mathrm{Conf}(X, g) \supset \mathrm{Isom}(X, g)$

(X, g) Riemannian manifold
$\varphi \in \mathrm{Diffeo}(X)$

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<u>Def.</u>

φ is isometry $\iff \varphi^* g = g$

φ is conformal $\iff$ $^\exists$positive function $C_\varphi \in C^\infty(X)$ s.t.

$$\varphi^* g = C_\varphi^2 \, g$$

C_φ : conformal factor

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$$\mathrm{Diffeo}(X) \supset \underset{\text{Conformal group}}{\mathrm{Conf}(X, g)} \supset \underset{\text{isometry group}}{\mathrm{Isom}(X, g)}$$

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(X, g) pseudo-Riemannian manifold
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$$\mathrm{Diffeo}(X) \supset \mathrm{Conf}(X, g) \supset \mathrm{Isom}(X, g)$$

Conformal group isometry group

Harmonic $\circ$ conformal $\neq$ harmonic

Modification

$\varphi \in \mathrm{Conf}(X^n, g)$, $\varphi^* g = C_\varphi^2 g$

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$\varphi \in \mathrm{Conf}(X^n, g)$, $\varphi^* g = C_\varphi^2 g$

- pull-back $\rightsquigarrow$ twisted pull-back

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- $\mathcal{S}ol(\Delta_X) = \{f \in C^\infty(X) : \Delta_X f = 0\}$ (harmonic functions)

  $\rightsquigarrow \mathcal{S}ol(\widetilde{\Delta_X}) = \{f \in C^\infty(X) : \widetilde{\Delta_X} f = 0\}$

  $$\widetilde{\Delta_X} := \Delta_X + \frac{n-2}{4(n-1)} \kappa$$

  Yamabe operator / Laplacian / scalar curvature

Distinguished rep. of conformal groups

harmonic $\circ$ conformal $\doteqdot$ harmonic

$\Downarrow$ Modification

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$\Downarrow$ Modification

Theorem A ([6, Part I]) (X^n, g) Riemannian mfd

$\Longrightarrow \mathrm{Conf}(X, g)$ acts on $\mathcal{S}ol(\widetilde{\Delta_X})$ by $f \mapsto C_\varphi^{-\frac{n-2}{2}} f \circ \varphi$

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Point $\widetilde{\Delta_X} = \Delta_X + \frac{n-2}{4(n-2)} \kappa$

$\widetilde{\Delta_X}$ is not invariant by $\mathrm{Conf}(X, g)$.

But $\mathcal{S}ol(\widetilde{\Delta_X})$ is invariant by $\mathrm{Conf}(X, g)$.

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Conformal group — isometry group

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Application of Theorem A

$$(X, g) := (S^p \times S^q, \underbrace{+ \cdots +}_{p} \underbrace{- \cdots -}_{q})$$

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Theorem B ([6, Part II])

0) $\mathrm{Conf}(X,g) \simeq O(p+1,q+1)$

1) $\mathcal{S}ol(\widetilde{\Delta_X}) \neq \{0\} \iff p+q$ even

2) If $p+q$ is even and > 2, then
$\mathrm{Conf}(X,g) \curvearrowright \mathcal{S}ol(\widetilde{\Delta_X})$ is irreducible,
and for $p+q > 6$ it is a minimal rep of $O(p+1,q+1)$.

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1) (conformal geometry) $\iff$ (represntation theory)
characterizing subrep in $\mathrm{Ind}_{P_{\max}}^{G}(\mathbb{C}_\lambda)$ (K-picture)
by means of differential equations

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$\uparrow$

$^\exists$a $\mathrm{Conf}(X,g)$-invariant inner product, and take the Hilbert completion

Flat model

Stereographic projection

$$S^n \to \mathbb{R}^n \cup \{\infty\} \qquad \text{conformal map}$$

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$$\underset{+\cdots+}{S^p} \times \underset{-\cdots-}{S^q} \hookrightarrow \underset{ds^2=dx_1^2+\cdots+dx_p^2-dx_{p+1}^2-\cdots-dx_{p+q}^2}{\mathbb{R}^{p+q}} \quad \text{conformal embedding}$$

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Functoriality of Theorem A

$$\begin{array}{ccc} \mathcal{S}ol(\widetilde{\Delta}_{S^p \times S^q}) & \subset & \mathcal{S}ol(\widetilde{\Delta}_{\mathbb{R}^{p,q}}) \\ \circlearrowright & & \circlearrowright \\ \mathrm{Conf}(S^p \times S^q) & \hookleftarrow & \mathrm{Conf}(\mathbb{R}^{p,q}) \end{array}$$

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction		
Theorem B	Clear	?
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Clear $\cdots$ advantage of the model

Conservative quantity for ultra-hyperbolic eqn.

$$\mathbb{R}^{p,q} = \mathbb{R}^{p+q},\ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$$

$$\widetilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q}$$

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Unitarization of subrep (representation theory)

$\Longleftrightarrow$

Conservative quantity (differential eqn)

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- Easy formulation
- Challenging formulation

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<u>Problem</u> Find an 'intrinsic' inner product on (a 'large' subspace of) $\mathcal{S}ol(\Box_{p,q})$ if exists.

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<u>Problem</u> Find an 'intrinsic' inner product on (a 'large' subspace of) $\mathcal{S}ol(\Box_{p,q})$ if exists.

Easy: if allowed to use the integral representation of solutions

Cf. (representation theory) by using the Knapp–Stein intertwining formula

Challenging: to find the intrinsic formula

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$q = 1$ wave operator

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energy $\cdots$ conservative quantity for wave equations
w.r.t. time translation $\mathbb{R}$

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$\Downarrow$

? $\cdots$ conservative quantity for ultra-hyperbolic eqs
w.r.t. conformal group $O(p+1, q+1)$

Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

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For $f \in \mathcal{S}ol(\Box_{p,q})$

$$(f, f) := \int_\alpha Q_\alpha f \qquad \cdots\cdots ①$$

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Theorem C ([6, Part III]+ε)

1) ① is independent of hyperplane α.

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Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

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Theorem C ([6, Part III]+ε)

1) ① is independent of hyperplane α.

2) ① gives the unique inner product (up to scalar) which is invariant under $O(p+1, q+1)$.

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$$O(p, q) \curvearrowright \mathbb{R}^{p,q} \quad \text{(linear)}$$

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~~$O(p,q)$~~ $\curvearrowright$ $\mathbb{R}^{p,q}$ ~~(linear)~~

$O(p+1, q+1)$ (Möbius transform)

Parametrization of non-characteristic hyperplane

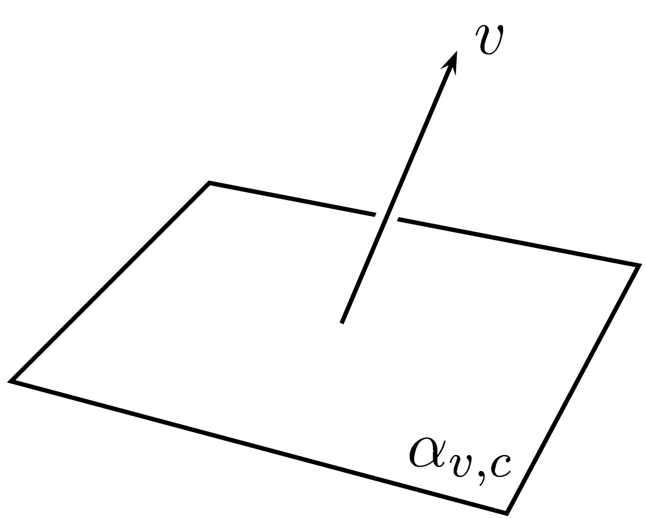
Fix $v \in \mathbb{R}^{p,q}$ s.t. $(v, v)_{\mathbb{R}^{p,q}} = \pm 1$

$c \in \mathbb{R}$

$\Downarrow$

$\mathbb{R}^{p,q} \supset \alpha \equiv \alpha_{v,c} := \{x \in \mathbb{R}^{p+q} : (x, v)_{\mathbb{R}^{p,q}} = c\}$

non-characteristic hyperplane



‘Intrinsic’ inner product

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)

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For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at ∞

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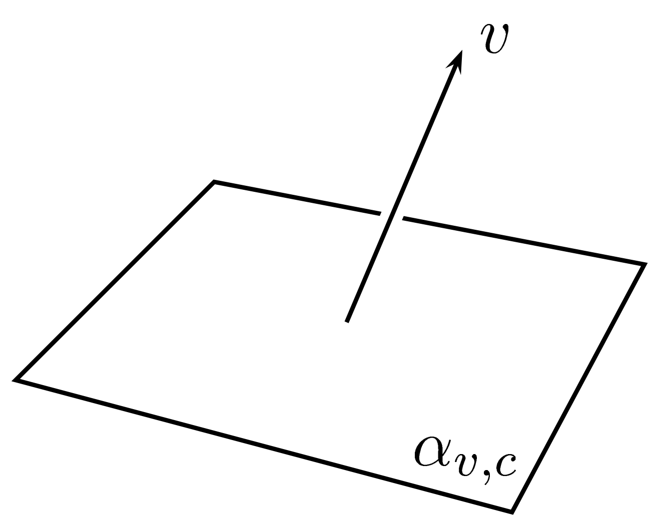
$$Q_\alpha f := \frac{1}{i}\left(f_+\overline{f'_+} - f_-\overline{f'_-}\right)$$



Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha = \alpha_{v,c} \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\Box_{p,q})$

$$(f, f) := \int_\alpha Q_\alpha f \qquad \cdots\cdots ①$$

Theorem C ([6, Part III]$+\varepsilon$)

1) ① is independent of hyperplane α.

2) ① gives the unique inner product (up to scalar) which is invariant under $O(p+1, q+1)$.

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1) ① is independent of hyperplane α.

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non-trivial even for $q = 1$ (wave equation)
In space-time,
average in space (i.e. time $t =$ constant)
$=$ average in (any hyperplane in space) $\times \mathbb{R}_t$ (time)

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction		
Theorem B	Clear	?
v.s.		
2. L^2 construction (Schrödinger model)	?	Clear
Theorem D		

Clear $\cdots$ advantage of the model

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Clear $\cdots$ advantage of the model

Conformal model $\Longrightarrow$ L^2-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

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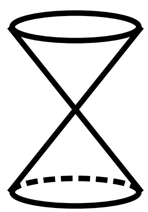
$=$  (figure for $(p,q) = (2,1)$)

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$\overline{\cdot}$ denotes the closure with respect to the inner product.

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$G = O(p+1, q+1) \quad \curvearrowright \quad L^2(\Xi)$ unitary rep.

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Inversion element

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Unitary inversion operator $\mathcal{F}_\Xi$

$p+q$: even > 2

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Problem Find the unitary operaotr $\mathcal{F}_\Xi$ explicitly.

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Problem Find the unitary operaotr $\mathcal{F}_\Xi$ explicitly.

Easy: express it as a composition of integral transforms and a known formula for other models (e.g. conformal model)

Challenging: to find a single and explicit formula in L^2 model

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Problem Find the unitary operaotr $\mathcal{F}_\Xi$ explicitly.

Cf. Analogous operator for the oscillator rep.

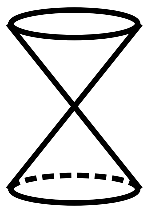
$$Mp(n, \mathbb{R}) \curvearrowright L^2(\mathbb{R}^n)$$

unitary inversion operator coincides with

Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ (up to scalar)!

Fourier transform $\mathcal{F}_\Xi$ on Ξ

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$=$  (figure for $(p,q) = (2,1)$)

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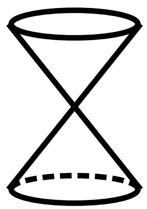
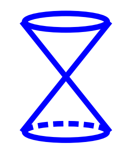
$=$  (figure for $(p,q) = (2,1)$)

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi =$ 

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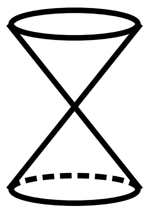
$=$  (figure for $(p, q) = (2, 1)$)

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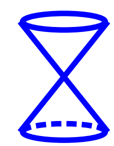
$\mathcal{F}_\Xi$ on $\Xi =$ 

Problem Define $\mathcal{F}_\Xi$ and find its formula.

‘Fourier transform’ $\mathcal{F}_{\Xi}$ on Ξ

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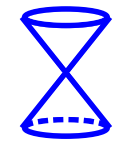
$\mathcal{F}_{\Xi}$ on $\Xi =$ 

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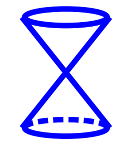
$$\mathcal{F}^4 = \mathrm{id}$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

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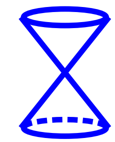
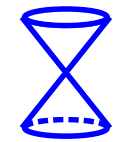
$\mathcal{F}_\Xi$ on $\Xi =$ 

$$\mathcal{F}_\Xi^2 = \mathrm{id}$$

'Fourier transform' $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$Q_j \mapsto -P_j$$
$$P_j \mapsto Q_j$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

$Q_j = x_j$ (multiplication by coordinate function)

$$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$$

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

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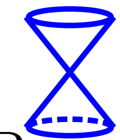
$\mathcal{F}_\Xi$ on $\Xi =$ 

$$Q_j \mapsto R_j$$
$$R_j \mapsto Q_j$$

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$R_j = {}^{\exists}$second order differential op. on Ξ

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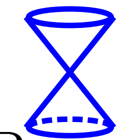
$\mathcal{F}_\Xi$ on $\Xi =$ 

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Bargmann–Todorov’s operators

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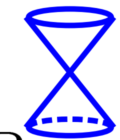
$\mathcal{F}_\Xi$ on $\Xi =$ 

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$R_j = {}^{\exists}$second order differential op. on Ξ

Notice
$$\left.\begin{aligned} {Q_1}^2 + \cdots + {Q_p}^2 - {Q_{p+1}}^2 - \cdots - {Q_{p+q}}^2 = 0 \\ {R_1}^2 + \cdots + {R_p}^2 - {R_{p+1}}^2 - \cdots - {R_{p+q}}^2 = 0 \end{aligned}\right\} \text{on } \Xi$$

Unitary inversion operator $\mathcal{F}_\Xi$

$p+q$: even >2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

P-action $\cdots$ translation and multiplication on $L^2(\Xi)$

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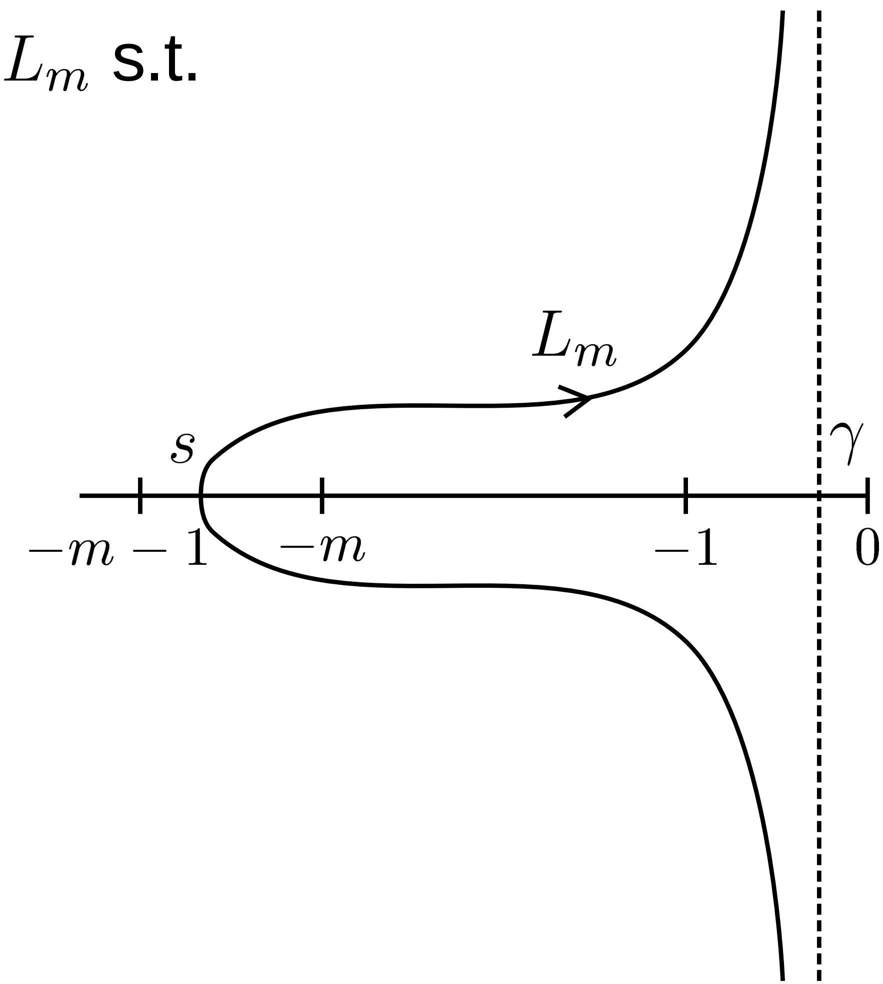
Mellin–Barnes type integral

Idea: Apply Mellin–Barnes type integral to distributions.

Fix $m \in \mathbb{N}$. Take a contour L_m s.t.



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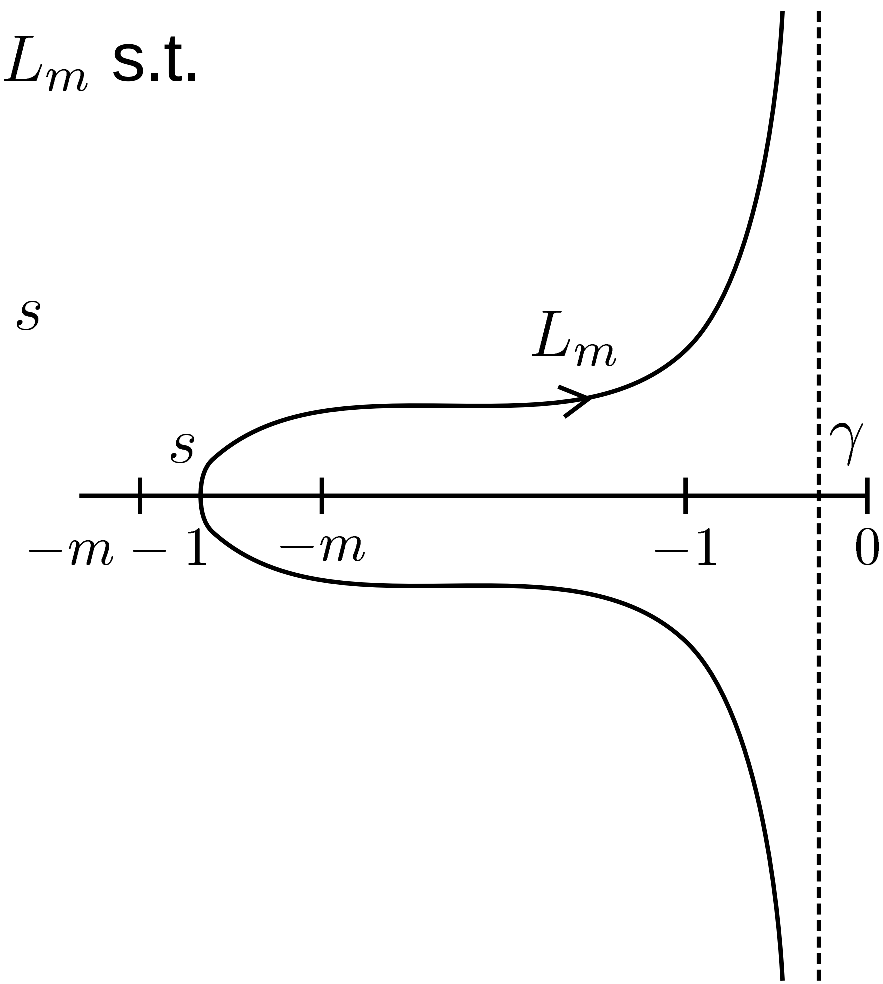
Fix $m \in \mathbb{N}$. Take a contour L_m s.t.

1) L_m starts at $\gamma - i\infty$
2) passes the real axis at s
3) ends at $\gamma + i\infty$

where

$$-m-1 < s < -m$$

$$-1 < \gamma < 0$$



Explicit formula of $\mathcal{F}_\Xi$ on Ξ

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$$\Phi^\varepsilon_m(t) = \begin{cases} \displaystyle\int_{L_0} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)} (2t)^\lambda_+ d\lambda & (\varepsilon = 0) \\ \displaystyle\int_{L_m} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)} (2t)^\lambda_+ d\lambda & (\varepsilon = 1) \\ \displaystyle\int_{L_m} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)} \left(\frac{(2t)^\lambda_+}{\tan(\pi\lambda)} + \frac{(2t)^\lambda_-}{\sin(\pi\lambda)} \right) d\lambda & (\varepsilon = 2) \end{cases}$$

Regularity of $\Phi_m^\varepsilon(t)$

Cf. Euclidean Fourier transform $e^{-it} \in \mathcal{A}(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R}) \cap \cdots$

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Cf. Euclidean Fourier transform $e^{-it} \in \mathcal{A}(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R}) \cap \cdots$

Recall two distributions on $\mathbb{R}$

$\delta(t)$: Dirac's delta function

t^{-1}: Cauchy's principal value

$$= \lim_{s \to 0} \left(\int_{-\infty}^{-s} + \int_{s}^{\infty} \right) \langle \frac{1}{t}, \cdot \rangle dt$$

these are not in $L^1_{\mathrm{loc}}(\mathbb{R})$

Regularity of $\Phi^{\varepsilon}_{m}(t)$

Cf. Euclidean Fourier transform $e^{-it} \in \mathcal{A}(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R}) \cap \cdots$

Prop. ([4]) We have the identities $\mod L^1_{\mathrm{loc}}(\mathbb{R})$

$$\Phi^{\varepsilon}_{m}(t) \equiv \begin{cases} 0 & (\varepsilon = 0) \\ -\pi i \displaystyle\sum_{l=0}^{m-1} \frac{(-1)^l}{2^l (m-l-1)!} \delta^{(l)}(t) & (\varepsilon = 1) \\ -i \displaystyle\sum_{l=0}^{m-1} \frac{l!}{2^l (m-l-1)!} t^{-l-1} & (\varepsilon = 2) \end{cases}$$

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Cor. $\mathcal{F}_{\Xi}$ has a locally integrable kernel if and only if G is $O(p+1,2)$, $O(2,q+1)$, or $O(3,3)$ $(\doteq SL(4,\mathbb{R}))$.

Bessel distribution

Prop. ([4]) $\Phi_m^\varepsilon(t)$ solves the differential equation

$$(\theta^2 + m\theta + 2t)u = 0$$

where $\theta = t\frac{d}{dt}$.

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$$\Phi_m^2(t) = 2\pi i (2t)_+^{-\frac{m}{2}} Y_m(2\sqrt{2t_+}) + 4(-1)^{m+1} i\,(2t)_-^{-\frac{m}{2}} K_m(2\sqrt{2t_-})$$

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	conservative quantity
v.s.		
2. L^2 construction (Schrödinger model) Theorem D	'Fourier transform' $\mathcal{F}_\Xi$	Clear

Clear $\cdots$ advantage of the model

3. Deformation of Fourier transforms (Theorems F, G, H)

Two constructions of minimal reps.

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Theorems A, B	Clear	Theorem C
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(Schrödinger model)	Theorem E	Clear
Theorem D		

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3. Deformation of Fourier transforms (Theorems F, G, H)

Deformation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_{\Xi}$	$\cdots$	'Fourier transform' on $\Xi \quad \subset \mathbb{R}^{p,q}$
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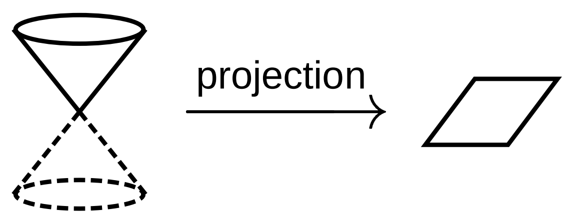
$\mathbb{R}^{N,1} \supset \Xi =$  $= \mathbb{R}^N$

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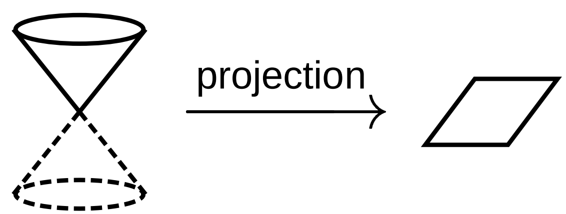

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$\mathcal{F}_{\Xi}$	$\mathcal{F}_{\mathbb{R}^N}$

$O(N+1,2)$ $\qquad$ $Mp(N,\mathbb{R})$

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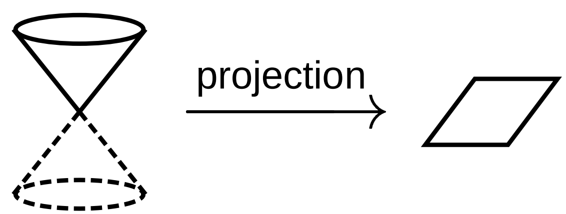
$\mathbb{R}^{N,1} \supset \Xi =$  $= \mathbb{R}^N$

$\mathcal{F}_{\Xi}$ .... deform .... $\mathcal{F}_{\mathbb{R}^N}$

$a = 1$ $a = 2$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\mathcal{F}_{\mathbb{R}^N} = \quad c \quad \exp\left(\frac{\pi i}{4}(\quad \Delta \quad -|x|^2)\right)$$

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self-adjoint op. on $L^2(\mathbb{R}^N)$

$$\mathcal{F}_{\mathbb{R}^N} = c \exp\Big(\frac{\pi i}{4}(\Delta - |x|^2)\Big)$$

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Hermite semigroup

$$I(t) := \exp\frac{t}{2}(\Delta - |x|^2)$$

R. Howe (oscillator semigroup, 1988)

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Hankel-type transform on Ξ

$$\mathcal{F}_\Xi = \underbrace{c}_{\text{phase factor}} \exp\Bigl(\frac{\pi i}{2}(\overbrace{|x|\Delta}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N, \frac{dx}{|x|})}-|x|)\Bigr)$$

phase factor $= e^{\frac{\pi i(N-1)}{2}}$ Laplacian

"Laguerre semigroup" ([5], 2007 Howe 60th birthday volume)

$$\mathcal{I}(t) := \exp t(|x|\Delta - |x|)$$

(k, a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

(k, a)-generalized Fourier transform

self-adjoint op. on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

$$\mathcal{F}_{k,a} = c \exp\left(\frac{\pi i}{2a}(|x|^{2-a}\Delta_k - |x|^a)\right)$$

phase factor $= e^{i\frac{\pi(N+2\langle k\rangle+a-2)}{2a}}$

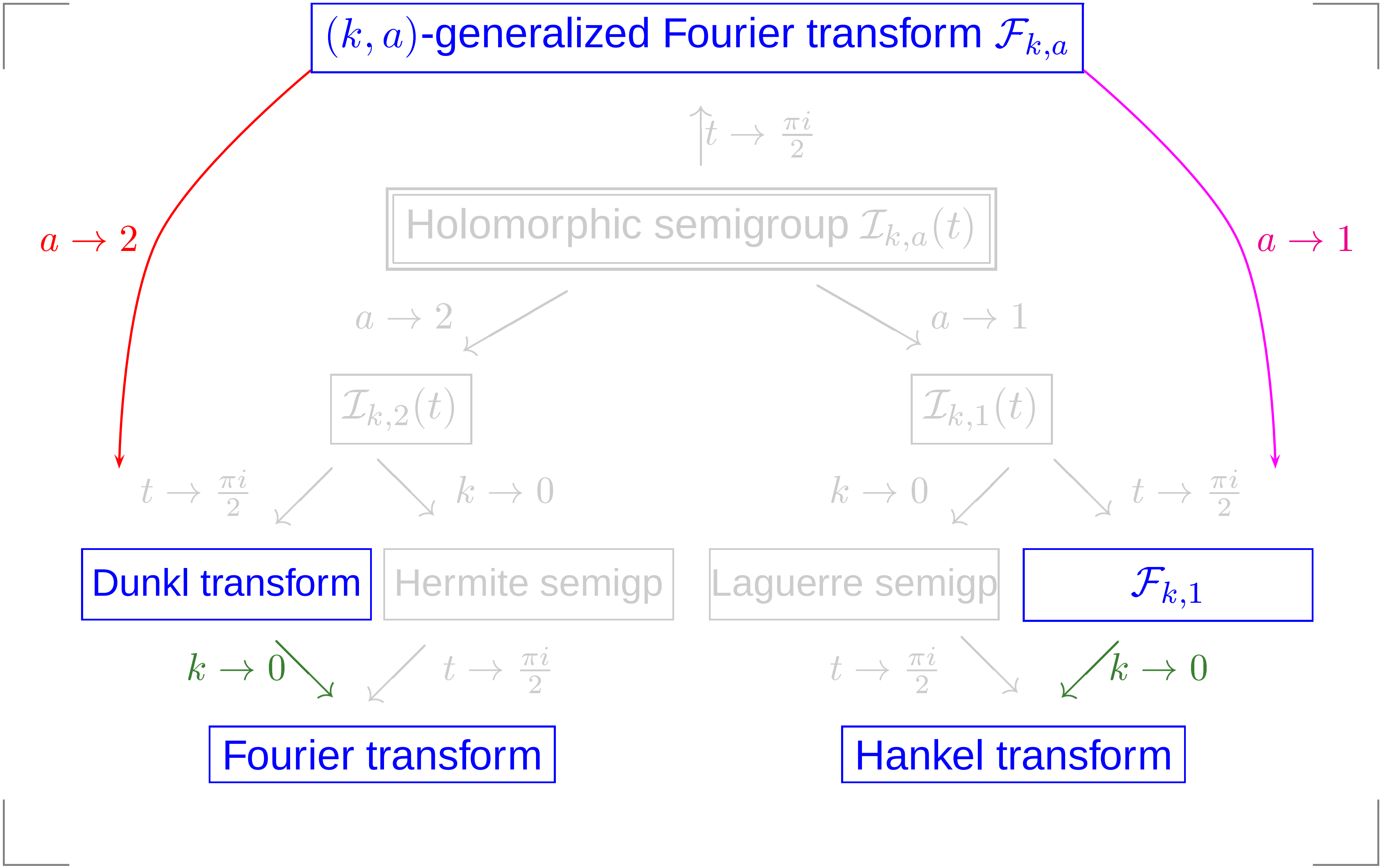
Dunkl Laplacian

(k, a)-deformation of Hermite semigroup ([1], 2009)

$$\mathcal{I}_{k,a}(t) := \exp \frac{t}{a}(|x|^{2-a}\Delta_k - |x|^a)$$

k: multiplicity on root system $\mathcal{R}$, $a > 0$

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



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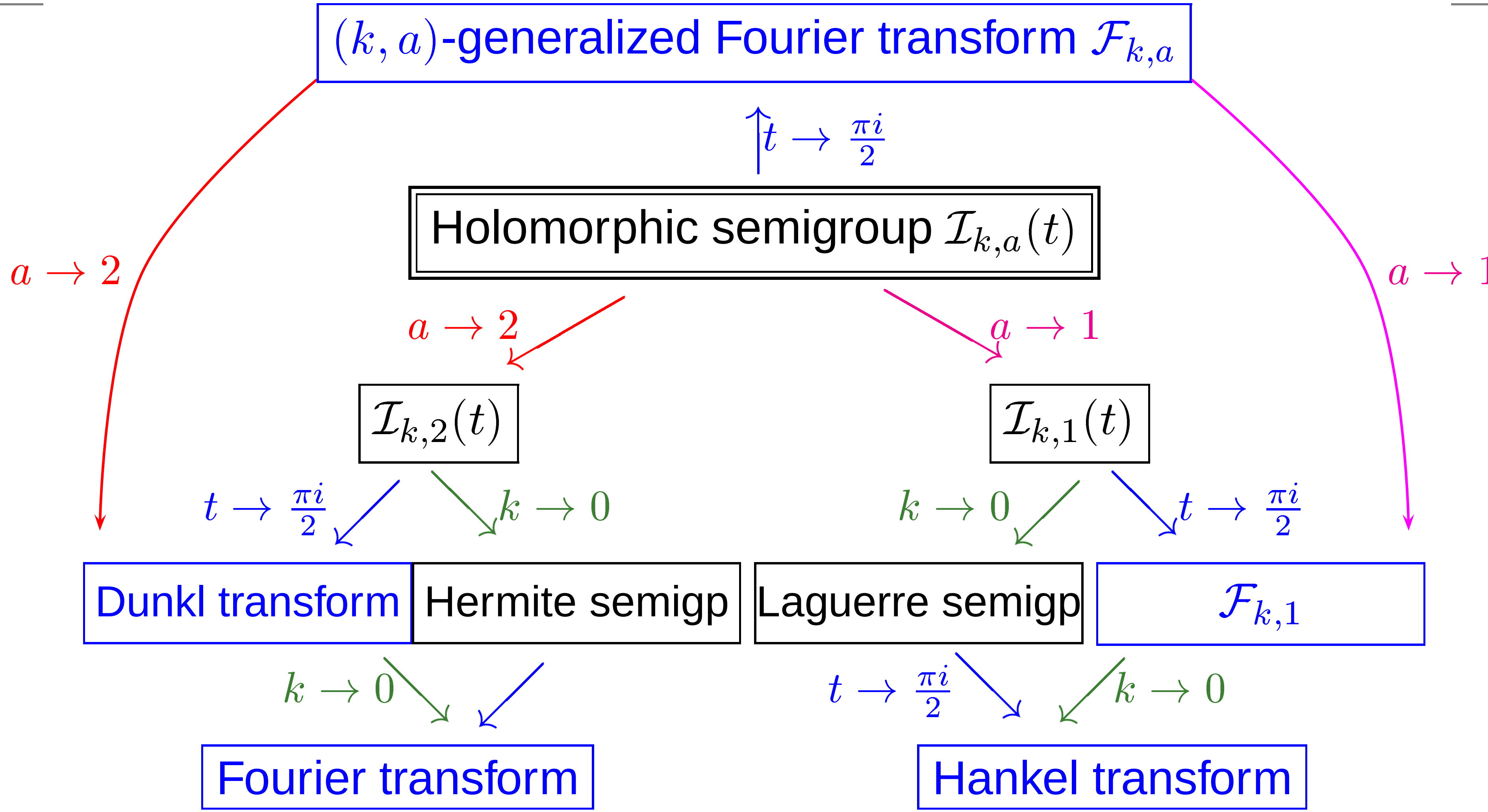

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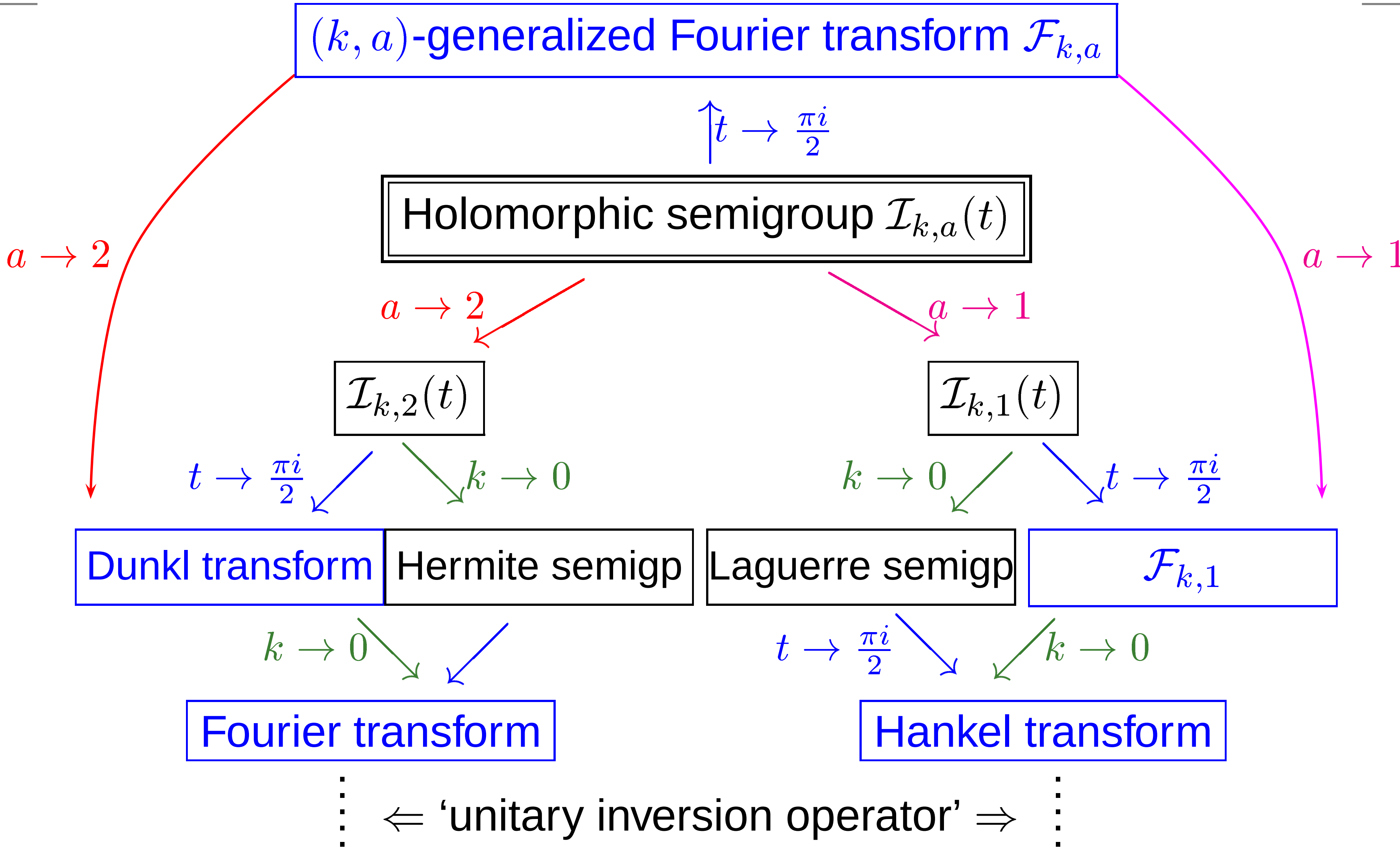

the Weil representation of
the metaplectic group $Mp(N,\mathbb{R})$

the minimal representation of
the conformal group $O(N+1,2)$

(k, a)-deformation of Hermite semigp

$k = (k_\alpha)$: multiplicity of root system $\mathcal{R}$ in $\mathbb{R}^N$

$$\mathcal{H}_{k,a} := L^2(\mathbb{R}^N, \, |x|^{a-2} \prod_{\alpha \in \mathcal{R}} |\langle x, \alpha \rangle|^{k_\alpha} dx)$$

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$$\mathcal{I}_{k,a}(t_1) \circ \mathcal{I}_{k,a}(t_2) = \mathcal{I}_{k,a}(t_1 + t_2) \quad \text{for } \operatorname{Re} t_1, t_2 \geq 0$$

$(\mathcal{I}_{k,a}(t) f, g)$ is holomorphic for $\operatorname{Re} t > 0$, for ${}^\forall f, {}^\forall g$

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$\Longrightarrow$ $^\forall$ Spectrum of $|x|^{2-a}\Delta_k - |x|^a$ is discrete and negative

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$\Longrightarrow$ automorphisms of the ring of operators.
$a = 1 \Longrightarrow SL(2, \mathbb{Z})$ action on degenerate DAHA (Cherednik)

(k, a)-deformation of Hermite semigp

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Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c\,\mathcal{I}_{k,a}\left(\frac{\pi i}{2}\right)$$

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Thm G 1) $\mathcal{F}_{k,a}$ is a unitary operator

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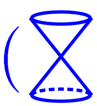
$F_{k,a} =$ Dunkl transform on $\mathbb{R}^N$

$\mathcal{F}_{0,1} =$ Hankel transform on $L^2($  $)$

3) $\mathcal{F}_{k,a}$ is of finite order $\iff a \in \mathbb{Q}$

4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a}\Delta_k$

Generalized Fourier transform $\mathcal{F}_{k,a}$

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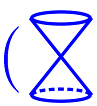
   $\mathcal{F}_{0,1} =$ Hankel transform on $L^2($  $)$
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$\Longrightarrow$ generalization of classical identities such as Hecke identity, Bochner identity, Parseval–Plancherel formulas, Weber's second exponential integral, etc.

Application to special functions

Minimal reps ($\Leftarrow$ group)

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$\approx$ Maximal symmetries ($\Leftarrow$ space)

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with 4 parameters

$$(\underbrace{p, \quad q}_{\text{dimension}} \; ; \; \underbrace{l, \quad m}_{\text{branching laws (multiplicity-free)}})$$

Special case $q = 1$: Laguerre polynomials $4 = 2 \times 2$

Heisenberg-type inequality

Thm H (Heisenberg inequality)

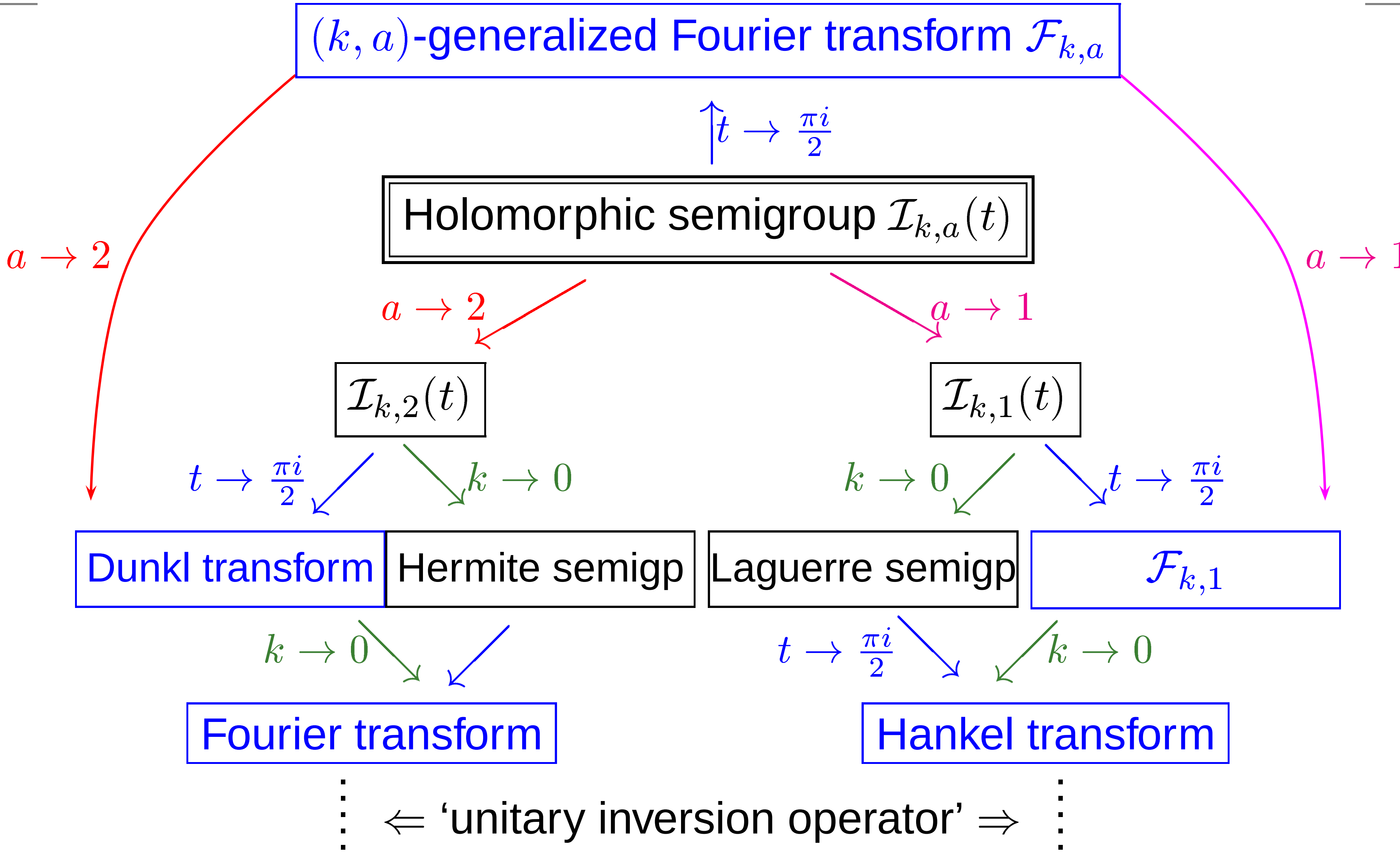
$$\| |x|^{\frac{a}{2}} f(x) \|_k \, \| |\xi|^{\frac{a}{2}} (\mathcal{F}_{k,a} f)(\xi) \|_k \geq \frac{2\langle k \rangle + N + a - 2}{2} \| f(x) \|_k^2$$

$k \equiv 0$, $a = 2$ $\cdots$ Weyl–Pauli–Heisenberg inequality for Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

k: general, $a = 2$ $\cdots$ Heisenberg inequality for Dunkl transform $\mathcal{D}_k$ (Rösler, Shimeno)

$k \equiv 0$, $a = 1$, $N = 1$ $\cdots$ Heisenberg inequality for Hankel transform

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



the Weil representation of
the metaplectic group $Mp(N,\mathbb{R})$

the minimal representation of
the conformal group $O(N+1,2)$

Hidden symmetries in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

$$O(N+1,2)^{\sim}$$

$a \to 1$ (↗)

Coxeter group

$$\boxed{\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})}} \xrightarrow{k \to 0} \boxed{O(N) \times \widetilde{SL(2,\mathbb{R})}}$$

$(k, a$: general$)$

$a \to 2$ (↘)

$$Mp(N, \mathbb{R})$$

Bessel functions

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{z}{2}\right)^{2j}}{j!\,\Gamma(j+\nu+1)}$$

$$I_\nu(z) := e^{-\frac{\sqrt{-1}\nu\pi}{2}} J_\nu\left(e^{\frac{\sqrt{-1}\pi}{2}} z\right)$$

$$Y_\nu(z) := \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \qquad \text{(second kind)}$$

$$K_\nu(z) := \frac{\pi}{2\sin\nu\pi}\left(I_{-\nu}(z) - I_\nu(z)\right) \qquad \text{(third kind)}$$

Geometric analysis on minimal reps of $O(p, q)$

[1] Laguerre semigroup and Dunkl operators $\cdots$
preprint, 74 pp. arXiv:0907.3749

[2] Special functions associated to a fourth order differential equation $\cdots$
preprint, 45 pp. arXiv:0907.2608, arXiv:0907.2612

[3] Generalized Fourier transforms $\mathcal{F}_{k,a}$ $\cdots$ C.R.A.S. Paris (to appear)

[4] Schrödinger model of minimal rep. $\cdots$
Memoirs of Amer. Math. Soc. (in press), 171 pp. arXiv:0712.1769

[5] Inversion and holomorphic extension $\cdots$
R. Howe 60th birthday volume (2007), 65 pp.

[6] Analysis on minimal representations $\cdots$
Adv. Math. (2003) I, II, III, 110 pp.

Collaborated with

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