Existence Problem of Compact Locally Symmetric Spaces

Colloquium, Harvard University

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Naive question

Discontinuous groups for
homogeneous spaces

(e.g. symmetric spaces)

Riemannian $\Longrightarrow$ non-Riemannian ?

Representation theory

Reps of Lie groups/algebras
Non-commutative harmonic analysis

Great trends of developments through 20th cent.

compact	$\Longrightarrow$	non-compact
Riemannian	$\Longrightarrow$	non-Riemannian
finite dim'l rep	$\Longrightarrow$	∞ dim'l rep

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A fruitful theory?

Riemannian $\to$ pseudo-Riemannian

Isometry gp for pseudo-Riemannian mfd

discrete subgp $\underset{\Longleftarrow}{\not\Rightarrow}$ properly discont actions

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$$n \qquad (x, y) \mapsto (2^n x, 2^{-n} y)$$

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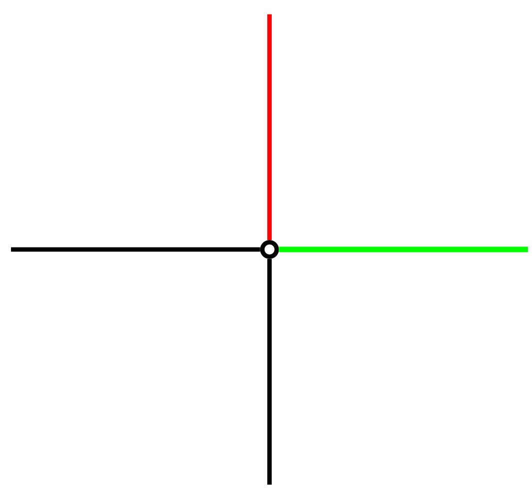

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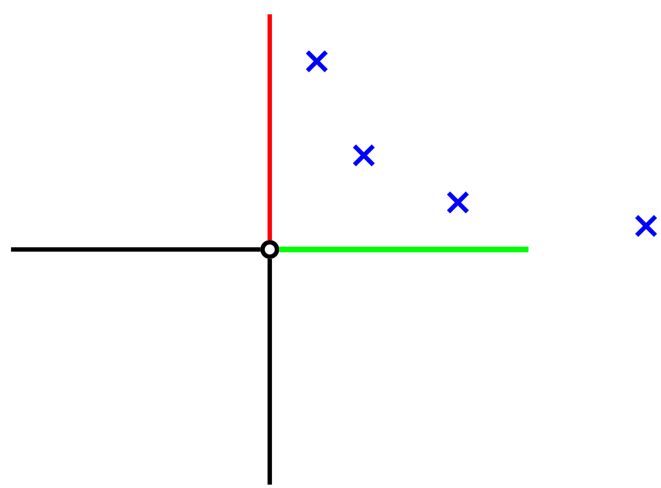

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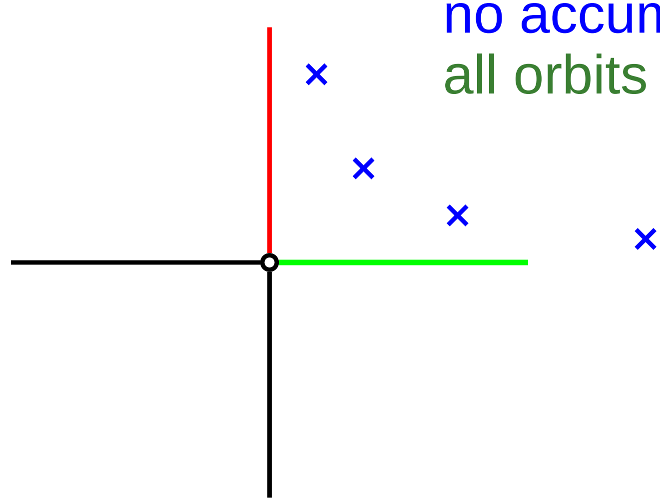

no accumulation points

all orbits are closed

$\mathbb{Z} \overset{\text{isometry}}{\curvearrowright} X$

$n \qquad (x,y)$

$\qquad\quad \downarrow$

$(2^n x, 2^{-n} y)$

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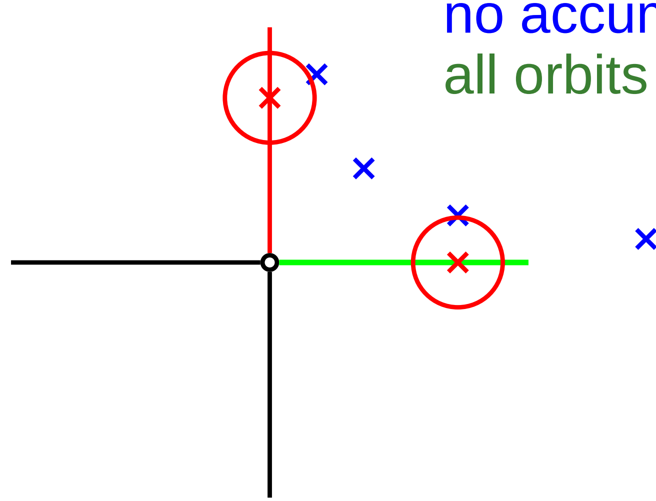

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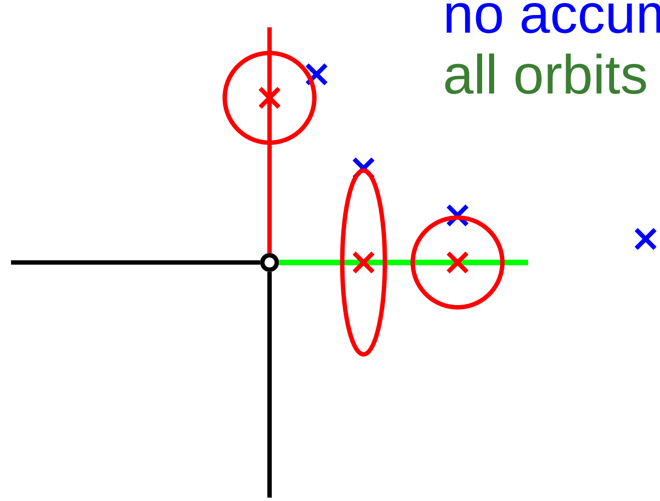

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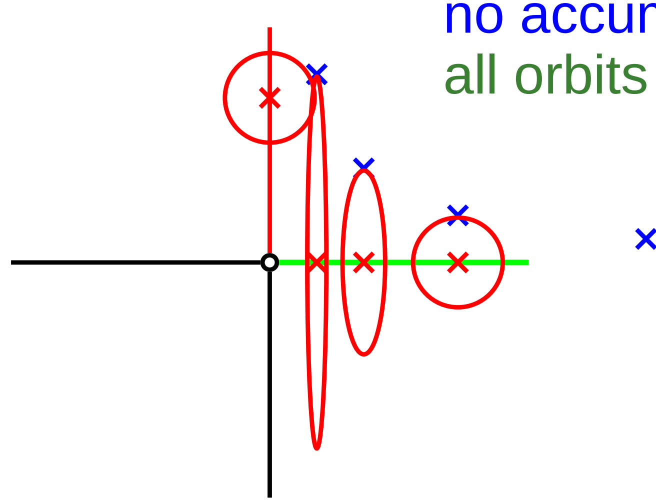

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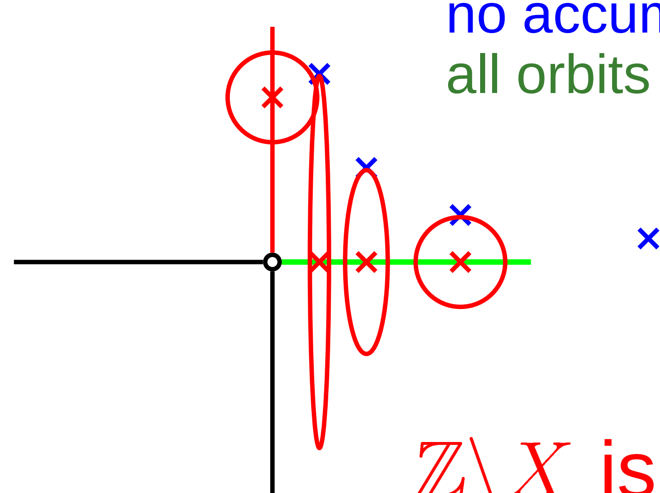


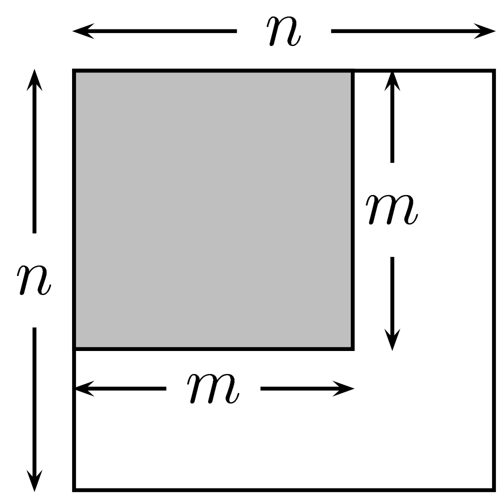
$\mathbb{Z}\backslash X$ is not Hausdorff

Compact quotients for $SL(n)/SL(m)$

Problem: Does there exist compact Hausdorff quotients of

$$SL(n,\mathbb{F})/SL(m,\mathbb{F}) \quad (n > m,\ \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$$

by discrete subgps of $SL(n,\mathbb{F})$?



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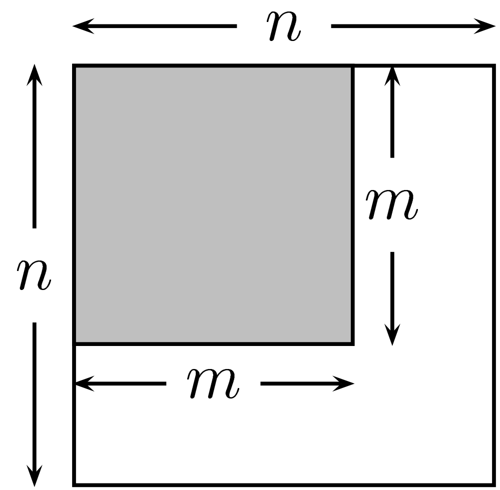
Conjecture: No for any $n > m$.



$SL(n)/SL(m)$ **case**

Cf. Space Form Conjecture (mentioned later)

<u>Conjecture 1</u> $SL(n)/SL(m)$ $(n > m > 1)$
has no uniform lattice.

K–	criterion of proper actions	$\frac{n}{3} > [\frac{m+1}{2}]$
Zimmer	orbit closure thm (Ratner)	$n > 2m$
Labourier–Mozes–Zimmer		
	ergodic action	$n \geq 2m$
Benoist	criterion of proper actions	$n = m + 1, \ m$ even
Margulis	unitary rep	$(n \geq 5, m = 2)$
Shalom	unitary rep	$n \geq 4, m = 2$

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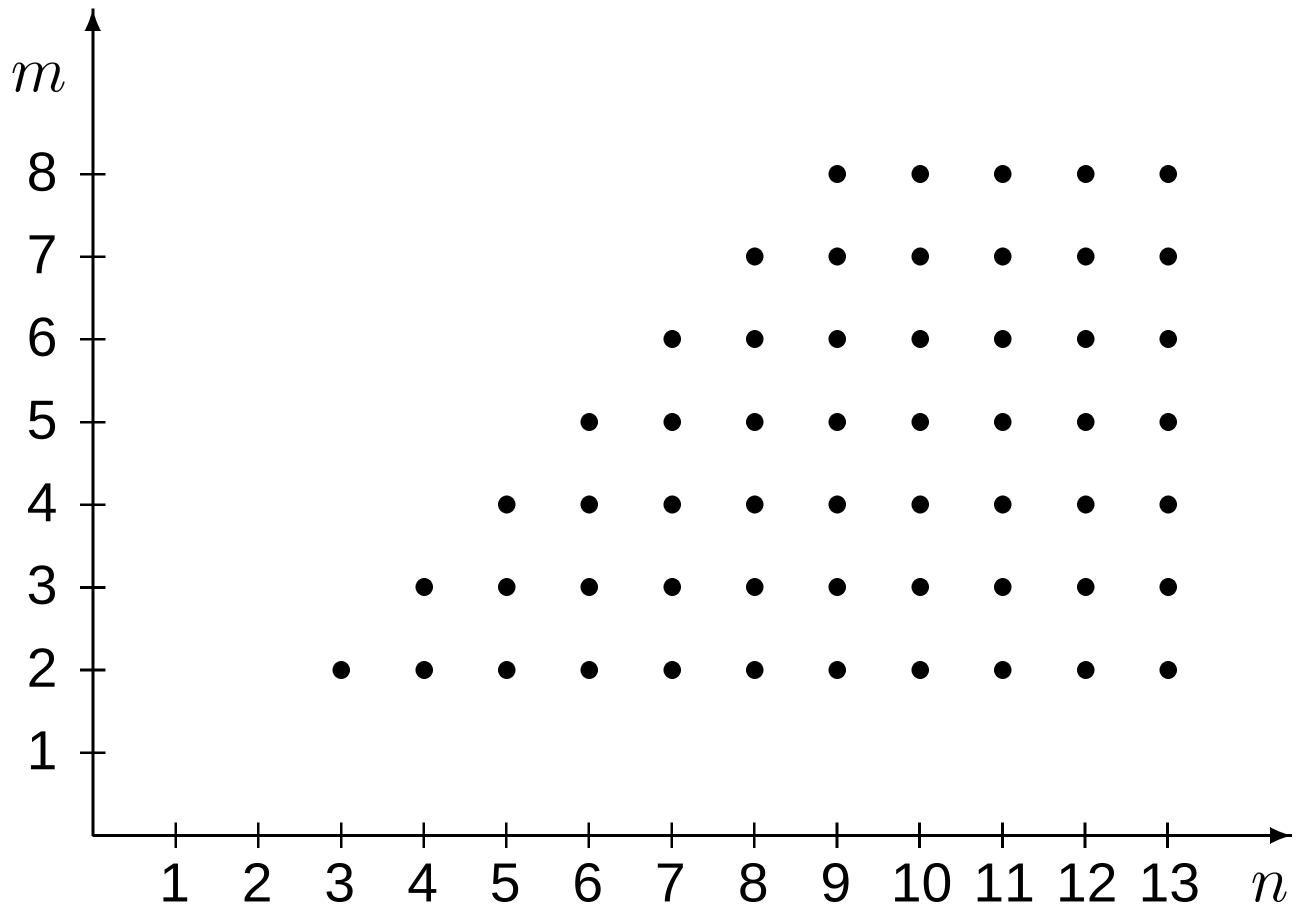

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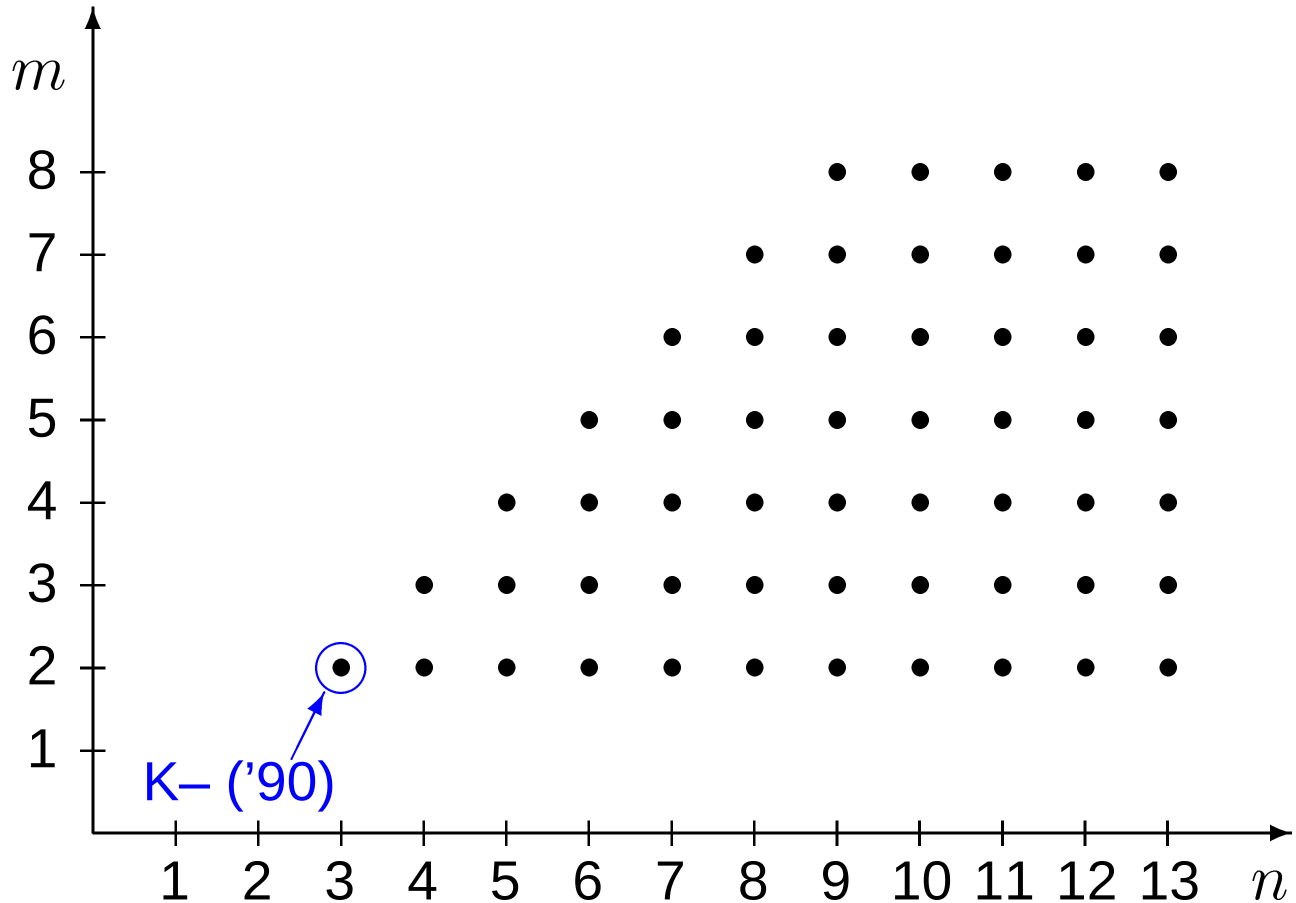

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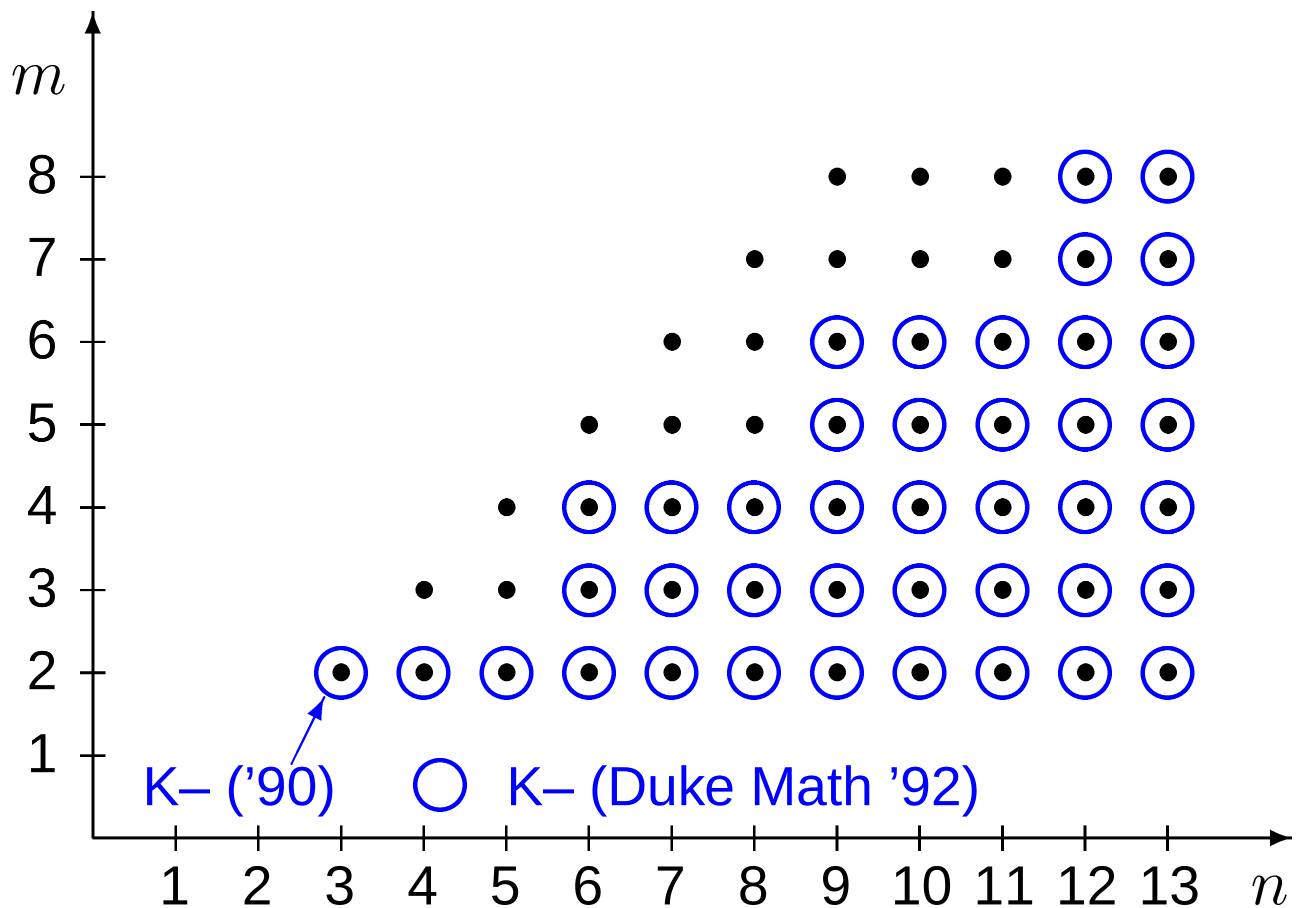

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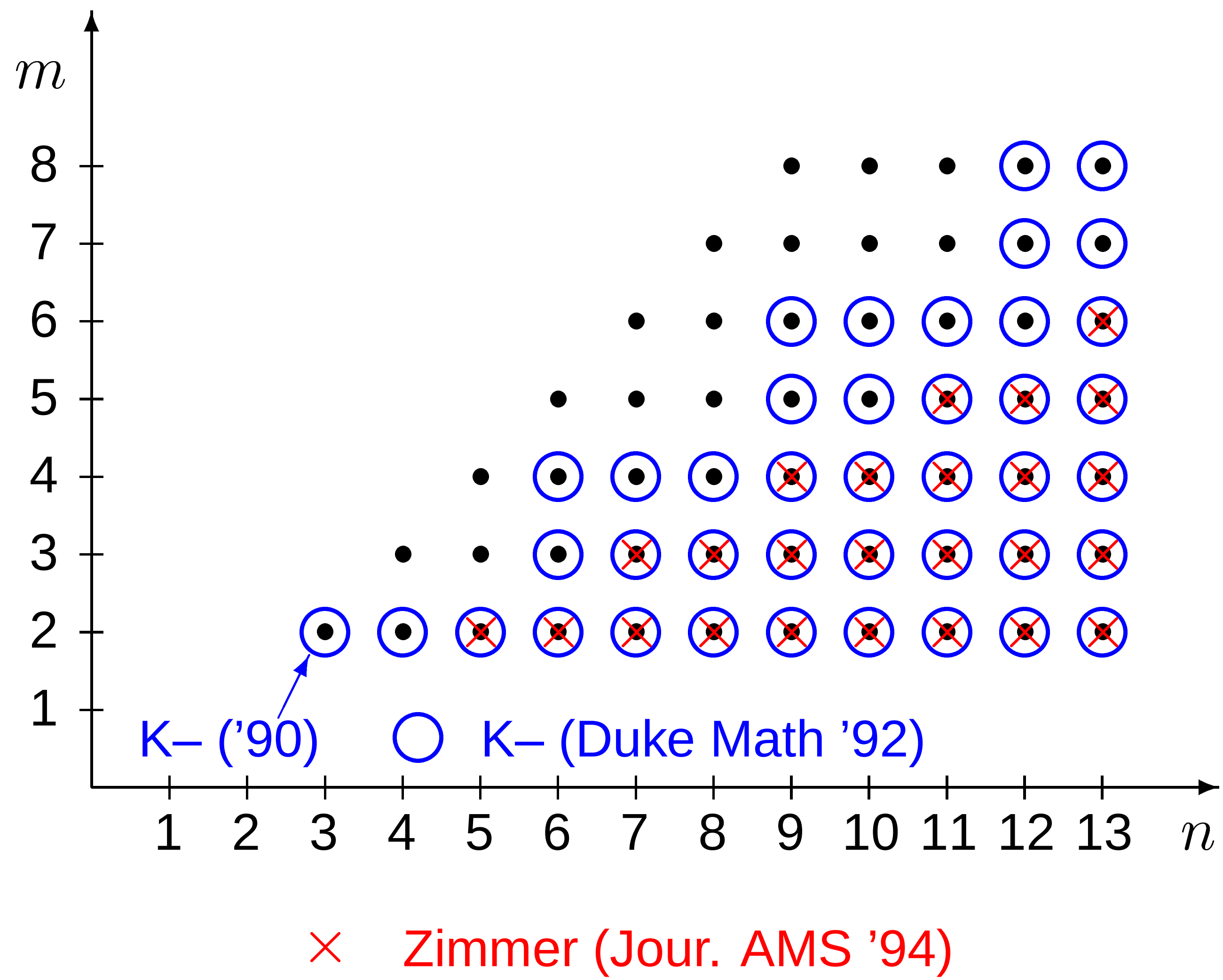

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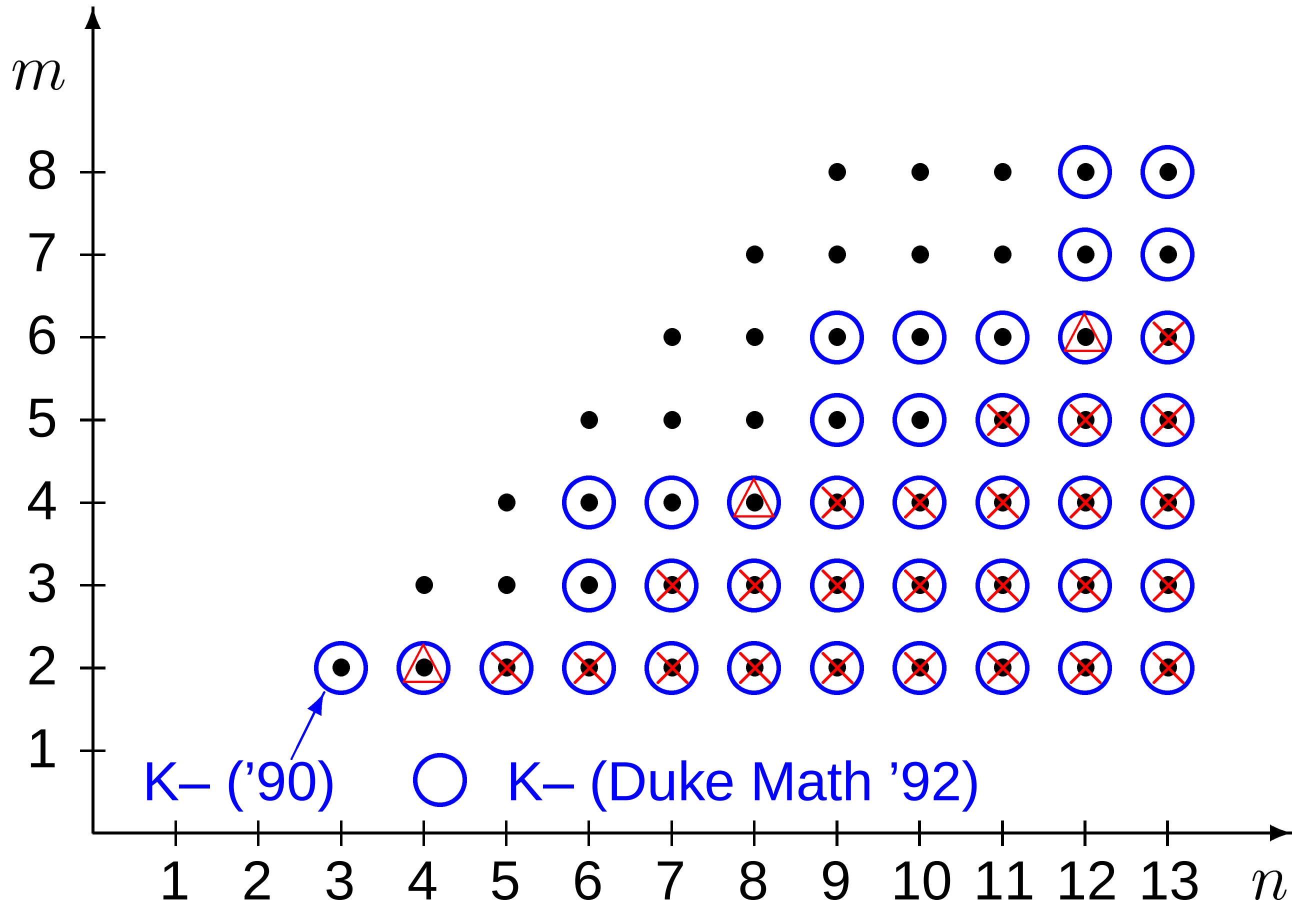

× Zimmer (Jour. AMS '94)

△ Labourie–Mozes–Zimmer (GAFA '95)

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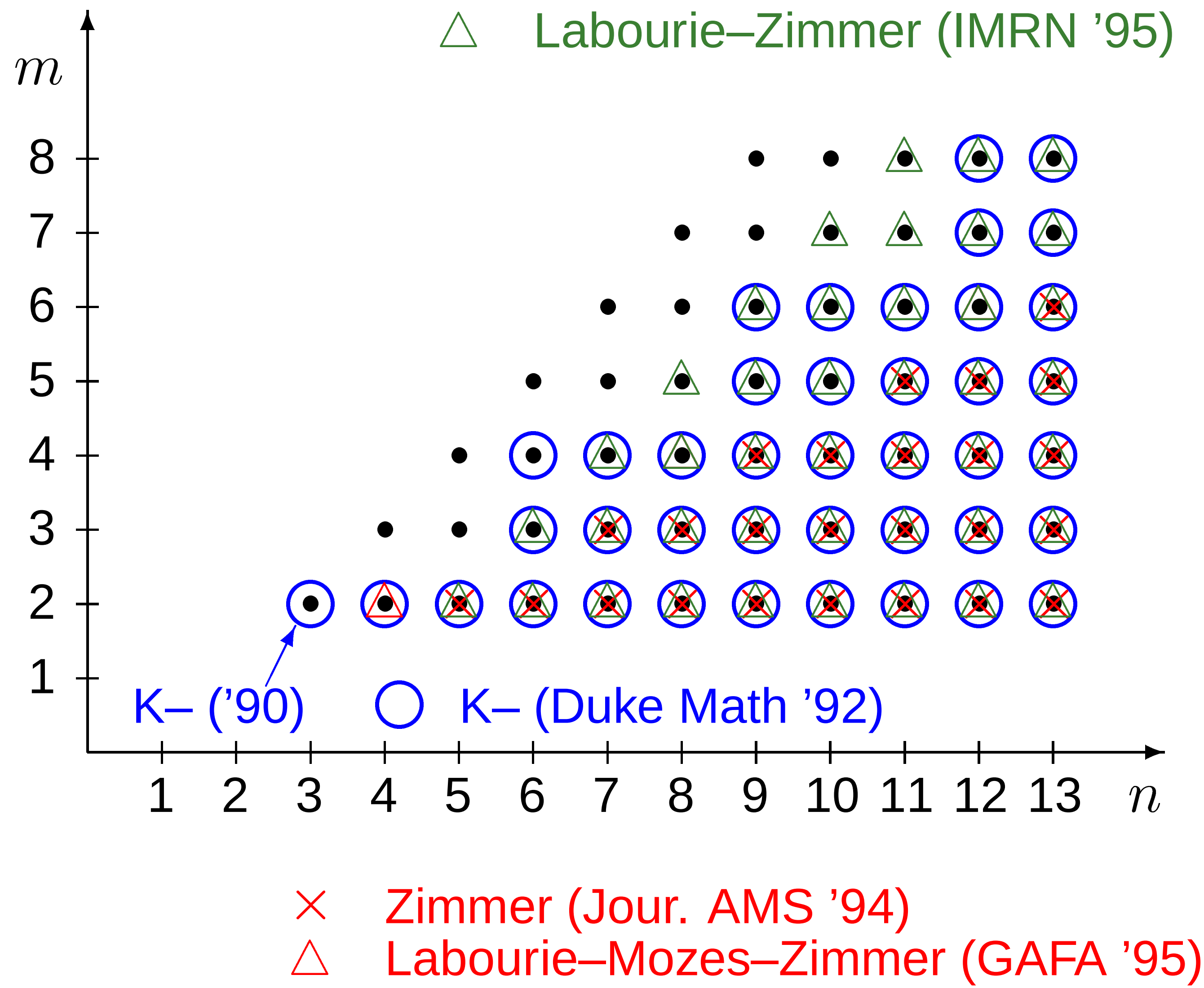

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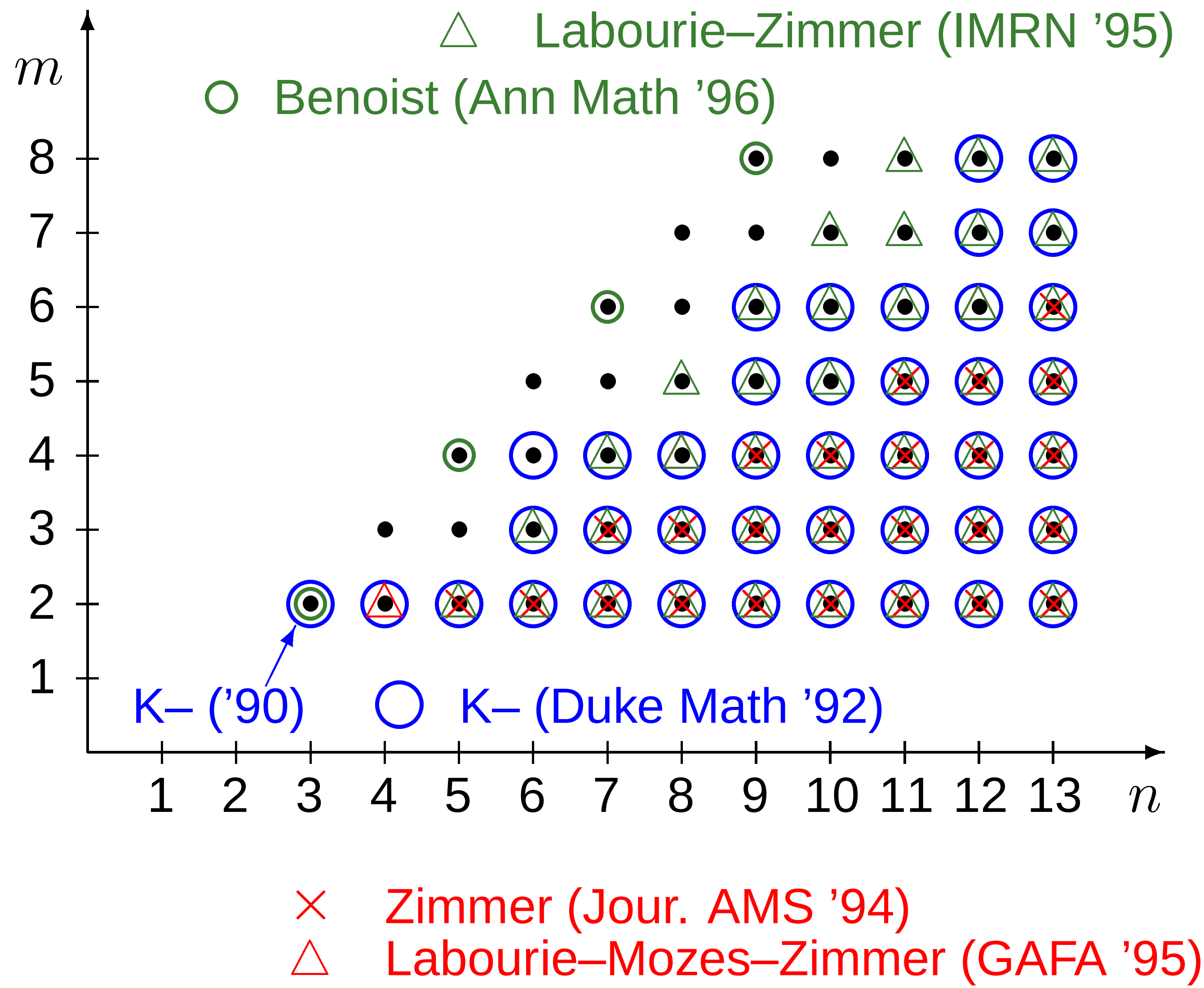

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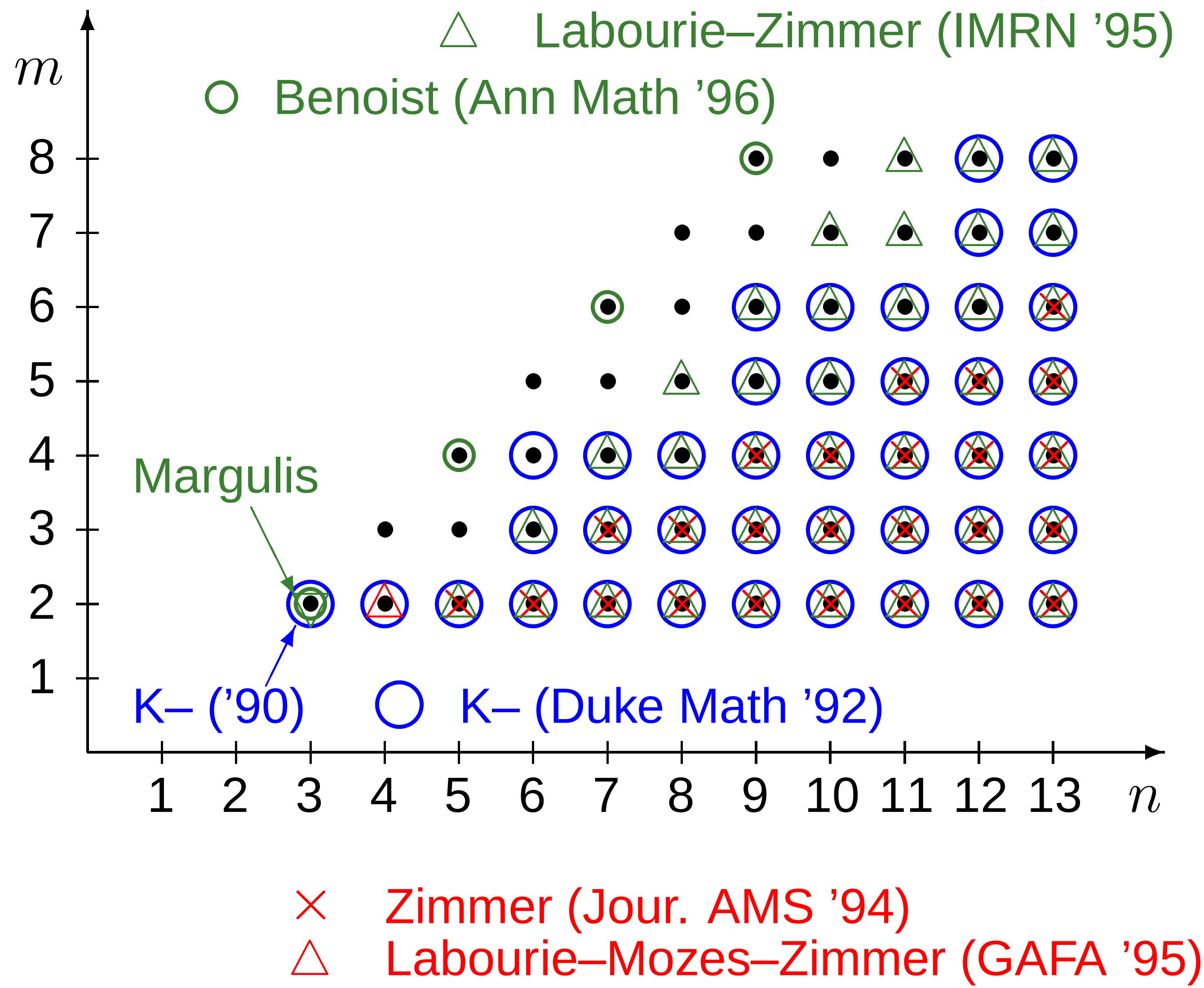

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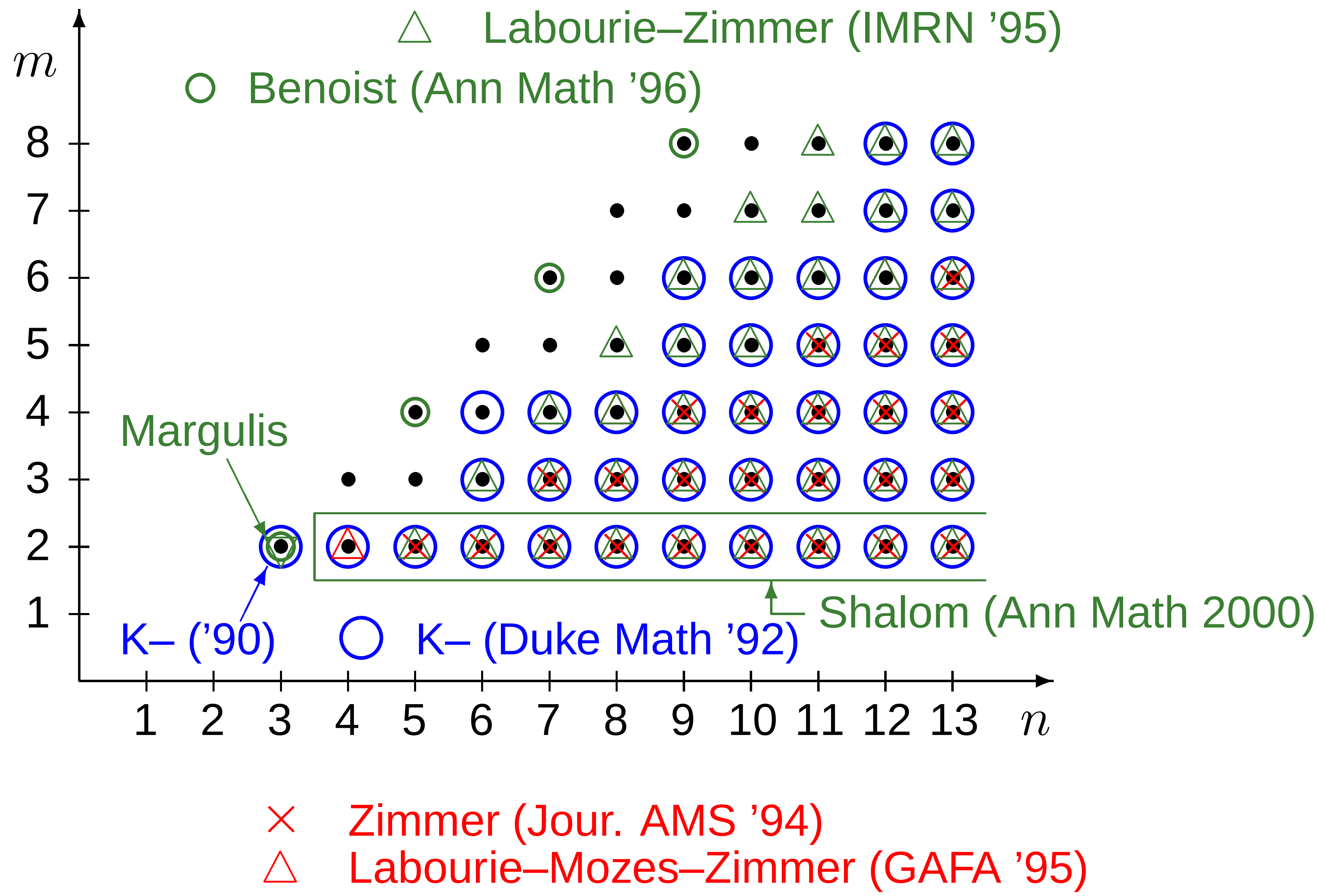

Non-Riemannian homo. spaces

Discrete subgp $\substack{\not\Rightarrow \\ \Leftarrow}$ Discontinuous gp

for non-Riemannian homo. spaces

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$$\text{Discrete subgp} \ \begin{smallmatrix} \not\Rightarrow \\ \Leftarrow \end{smallmatrix} \ \text{Discontinuous gp}$$

for non-Riemannian homo. spaces

How does a local geometric structure affect the global nature of manifolds?

New phenomena & methods?

Non-Riemannian homo. spaces

$$\text{Discrete subgp} \ \begin{matrix}\not\Rightarrow \\ \Leftarrow\end{matrix} \ \text{Discontinuous gp}$$

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Fundamental problems

- Are there many discont. gps?
  (cf. Calabi–Markus phenomenon)

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- Are there many discont. gps?
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- Existence problem of compact quotients
  (unsolved even for space forms)

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Fundamental problems

- Are there many discont. gps?
  (cf. Calabi–Markus phenomenon)
- Existence problem of compact quotients
  (unsolved even for space forms)
- Rigidity and deformation
  (rigidity may fail even for high dim.)

Existence Problem of Compact Locally Symmetric Spaces

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Contents

1. Space form of signature (p, q)

(M, g) : pseudo-Riemannian mfd,
geodesically complete

Def. (M, g) is a space form
$\iff$ sectional curvature κ is constant

Space forms (examples)

Space form $\cdots$ $\begin{cases} \text{Signature } (p,q) \text{ of pseudo-Riemannian metric } g \\ \text{Curvature } \kappa \in \{+,0,-\} \end{cases}$

<u>E.g.</u> $q = 0$ (Riemannian mfd)

sphere S^n	$\mathbb{R}^n$	hyperbolic sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

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E.g. $q = 1$ (Lorentz mfd)

de Sitter sp	Minkowski sp	anti-de Sitter sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

Space form problem

Space form problem for pseudo-Riemannian mfds

Local Assumption

signature (p, q), curvature $\kappa \in \{+, 0, -\}$

$$\Downarrow$$

Global Results

- Do compact quotients exist?
- What groups can arise as their fundamental groups?

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Global Results

- Do compact quotients exist?

  Is the universe closed?

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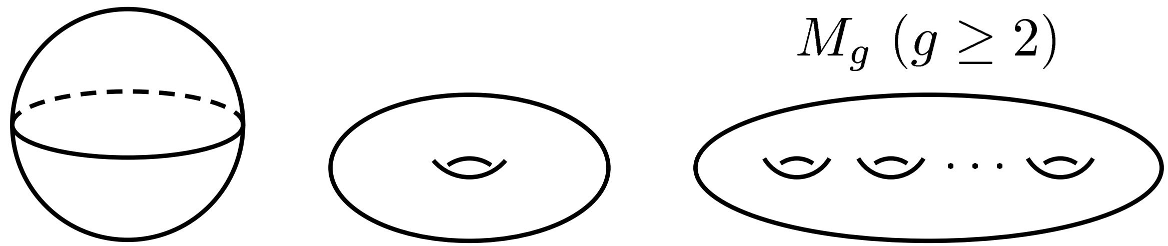
2-dim'l compact space forms

Riemannian case ($\Longleftrightarrow$ signature $(2, 0)$)

$M_g \ (g \geq 2)$



curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

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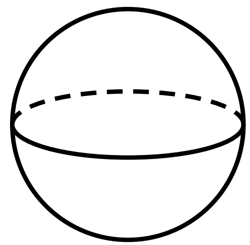
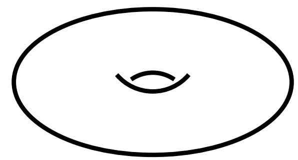
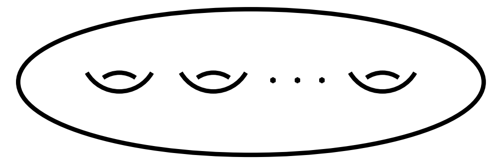
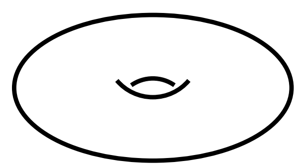




curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

Lorentz case ($\Longleftrightarrow$ signature $(1,1)$)



no compact form for $\kappa > 0$ or $\kappa < 0$

Compact space forms

(p, q): signature of metric, curvature $\kappa \in \{+, 0, -\}$

Assume $p \geq q$ (without loss of generality).

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(Geometry) Compact space forms exist
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$\Longleftrightarrow$ (Group theoretic formulation)
Cocompact discontinuous gps exist
for symmetric space $O(p, q+1)/O(p, q)$

Compact space forms ($\kappa < 0$)

- Riemannian case $\cdots$ hyperbolic space

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  (uniform lattice)

Compact space forms ($\kappa < 0$)

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Compact hyperbolic spaces

$\iff$ Cocompact discrete subgp of $O(n,1)$

(uniform lattice)

Exist by $\underbrace{\text{Siegel, Borel}}_{\text{arithmetic}}$, $\underbrace{\text{Vinberg, Gromov–Piateski-Shapiro}}_{\text{non-arithmetic}}$ $\cdots$

Space form conjecture $\kappa < 0$

- Pseudo-Riemannian mfd of signature (p, q)

Thm 2 Compact space forms of $\kappa < 0$ exist

$\Longleftarrow$

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$\Longleftarrow$ ① q any, $p = 0$ $(\leftrightarrow \kappa > 0)$

② $q = 0$, p any (hyperbolic sp)

$\Longleftarrow$ True (Proved (1950–2005))

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Space form conjecture $\kappa < 0$

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Thm 2 Conjecture 3 Compact space forms of $\kappa < 0$ exist

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$\Longrightarrow$

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$\Longrightarrow$ Partial answers:

$q = 1$, $p \leq q$, or pq is odd

Methods

Understanding of proper actions
as "coarse geometry" ($\pitchfork, \sim$)
$\Rightarrow$ criterion for proper actions (§3)
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Construction of lattice

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Understanding of proper actions
as "coarse geometry" ($\pitchfork, \sim$)
$\Rightarrow$ criterion for proper actions (§3)

$\Downarrow$

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Obstruction of lattice

- Topological obstructions
- Comparison theorem: $\Gamma \curvearrowright X \iff \Gamma \curvearrowright Y$

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If $\Gamma \curvearrowright X$ properly discontinuously and freely, then

$$\begin{array}{c} X \\ \downarrow \\ \Gamma \backslash X \end{array} \quad \text{covering}$$

2. Locally homogeneous spaces

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If $\Gamma \curvearrowright X$ properly discontinuously and freely, then

$$\begin{array}{cl} X & \\ \downarrow & \text{covering} \\ \Gamma \backslash X & \text{manifold (Hausdorff)} \end{array}$$

Clifford–Klein forms

discrete subgp		Lie gp		closed subgp
Γ	$\subset$	G	$\supset$	H

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Clifford–Klein form

(Local) geometric structures on $\Gamma\backslash G/H$ inherit from G/H.

Locally symmetric sp.

$$G/H \quad \xrightarrow{\text{covering}} \quad \Gamma \backslash G/H$$

Locally symmetric sp.

$$\underbrace{G/H}_{\substack{\uparrow \\ \text{local geometric str.}}} \xrightarrow{\text{covering}} \underbrace{\Gamma}_{\substack{\uparrow \\ \text{global}}} \backslash G/H$$

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$$\underbrace{G/H}_{\uparrow} \xrightarrow{\text{covering}} \underbrace{\Gamma}_{\uparrow}\backslash G/H$$

local geometric str. global

Ex. M: complete, locally symmetric sp.

Locally symmetric sp.

$$\underbrace{G/H}_{\substack{\uparrow \\ \text{local geometric str.}}} \xrightarrow{\text{covering}} \underbrace{\Gamma\backslash G/H}_{\substack{\uparrow \\ \text{global}}}$$

Ex. M: complete, locally symmetric sp.

i.e. M: C^∞ manifold with affine connection

s.t. • M is geodesically complete

• geodesic symmetry at every point is affine

Locally symmetric sp.

$$\underbrace{G/H}_{\uparrow} \xrightarrow{\text{covering}} \Gamma\backslash \underbrace{G/H}_{\uparrow}$$

local geometric str. global

Ex. M: complete, locally symmetric sp.

$\Longrightarrow$ $M \simeq \Gamma\backslash G/H$ for some triple Γ, G, H s.t.

$\Gamma \simeq \pi_1(M)$

$\cap$

G: Lie group

$\cup$

H: an open subgp of G^σ $(\sigma \in \mathrm{Aut}(G),\ \sigma^2 = \mathrm{id})$

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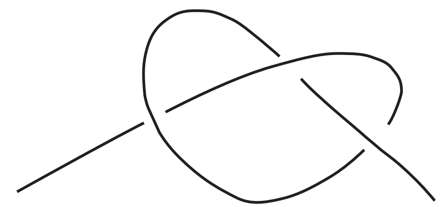
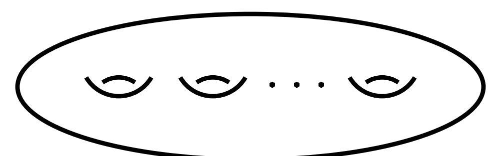
Examples of Clifford–Klein forms

(G, Γ, H)	$\Gamma \backslash G / H$
$(\mathbb{R}^n, \mathbb{Z}^n, \{0\})$	$\mathbb{T}^n$ (n-torus)
$(SL(2,\mathbb{R}), SL(2,\mathbb{Z}), \{e\})$	 (non-compact, finite volume)
$(PSL(2,\mathbb{R}), PSO(2), \pi_1(M_g))$	$M_g \simeq$  $(g \geq 2)$
$(O(p, q+1), O(p,q), \Gamma)$	Space form (signature (p,q), $\kappa < 0$)
$(GL(n,\mathbb{R}) \ltimes \mathbb{R}^n, GL(n,\mathbb{R}), \Gamma)$	affinely flat

Discrete $\Rightarrow$ Continuous

$$\underset{\text{top. gp}}{\Gamma} \overset{\text{action}}{\curvearrowright} \underset{\text{top. sp (locally compact)}}{X}$$

$$\begin{array}{rccc} & X & & \Gamma \\ \text{subset} & \cup & \rightsquigarrow & \cup \\ & S & & \Gamma_S := \{\gamma \in \Gamma : \gamma S \cap S \neq \phi\} \end{array}$$

$S = \{p\} \quad \Longrightarrow \quad \Gamma_S =$ stabilizer of p

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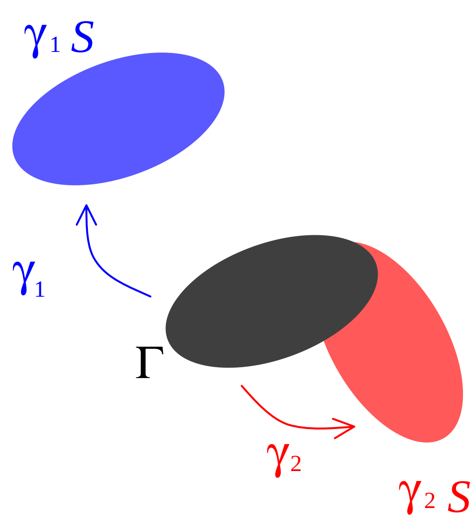

$\Longrightarrow$ $\gamma_1 \not\in \Gamma_S$, $\gamma_2 \in \Gamma_S$

Discrete $\Rightarrow$ Continuous

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top. gp $\qquad$ top. sp (locally compact)

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Def. $\Gamma \curvearrowright X$ is proper $\iff$ Γ_S is compact ($^\forall S$: compact)

$\Gamma \curvearrowright X$ is free $\iff$ $\#\Gamma_{\{p\}} = 1 \ (\forall p \in X)$

proper $+$ discrete $=$ properly discont.

properly discont. action

$\|$

proper action

$+$

group is discrete

proper $+$ discrete $=$ properly discont.

action	properly discont. action
	$\|\|$
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3. Criterion for proper discontinuity

Setting

$$\Gamma \subset G \supset H$$

discrete subgp (Γ) closed subgp (H)

Problem A Find effective methods to determine whether $\Gamma \curvearrowright G/H$ is properly discont.

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Problem A   Find effective methods to determine whether $\Gamma \curvearrowright G/H$ is properly discont.

Idea: Forget that Γ and H are group

$\pitchfork$ and $\sim$ (definition)

$$L \quad \subset \quad G \quad \supset \quad H$$

Forget even that L and H are group

Def. (K–)

1) $L \pitchfork H \iff \overline{L \cap SHS}$ is compact
for $^{\forall}$ compact $S \subset G$

2) $L \sim H \iff {}^{\exists}$ compact $S \subset G$
s.t. $L \subset SHS$ and $H \subset SLS$.

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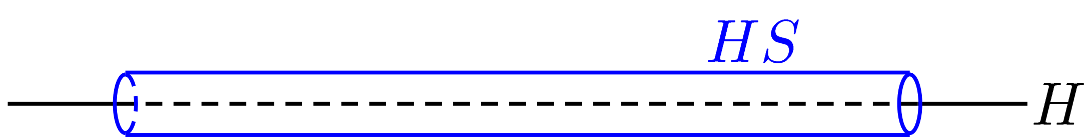

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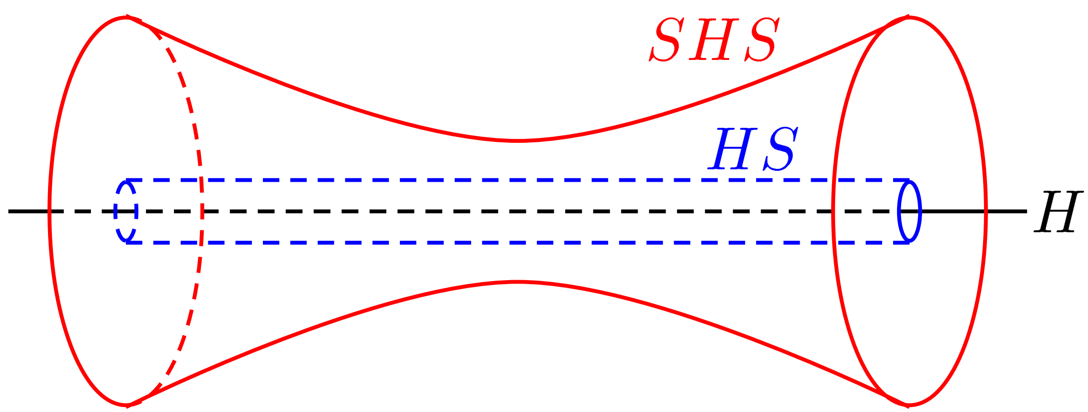

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E.g. $G = \mathbb{R}^n$; L, H subspaces

$$L \pitchfork H \iff L \cap H = \{0\}.$$

$$L \sim H \iff L = H.$$

$\pitchfork$ and $\sim$

$$L \quad \subset \quad G \quad \supset \quad H$$

Forget even that L and H are group

1) $L \pitchfork H \iff$ generalization of proper actions
2) $L \sim H \iff$ economy in considering

$\pitchfork$ means in special case that
L is discrete subgp & H is closed subgp

$L \pitchfork H \iff L \curvearrowright G/H$ properly discont.

$\sim$ provides economies in considering $\pitchfork$

$H \sim H' \Longrightarrow$ $H \pitchfork L \iff H' \pitchfork L$

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Discontinuous duality theorem

$G \supset H$ subset $\Longrightarrow H^{\pitchfork} := \{L : L \pitchfork H\}$ discont. dual

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Thm 4 (Discontinuous duality; K– , Yoshino)

G: loc. compact top. gp, separable

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Then, H is recovered up to $\sim$ from $H^{\pitchfork}$.

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cf. $G \quad \Longrightarrow \quad \widehat{G}$ (unitary dual)

Fact 5 (Pontrjagin–Tannaka–Tatsuuma duality theorem)

G: loc. compact top. gp

Then, G is recovered from the unitary dual $\widehat{G}$.

Criterion for $\pitchfork$ and $\sim$

G: real reductive Lie group
$G = K \exp(\mathfrak{a}_+)K$: Cartan decomposition
ν: $G \to \mathfrak{a}_+$: Cartan projection

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E.g. ν : $GL(n, \mathbb{R}) \to \mathbb{R}^n$

$$g \mapsto \frac{1}{2}(\log \lambda_1, \cdots, \log \lambda_n)$$

Here, $\lambda_1 \geq \cdots \geq \lambda_n \, (> 0)$ are the eigenvalues of ${}^t gg$.

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Thm 6 (K– , Benoist)

1) $L \sim H$ in $G \iff \nu(L) \sim \nu(H)$ in $\mathfrak{a}$.

2) $L \pitchfork H$ in $G \iff \nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$.

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Special cases include

(1)'s $\Rightarrow$: Uniform bounds on errors in eigenvalues when a matrix is perturbed.

(2)'s $\Leftrightarrow$: Criterion for properly discont. actions.

Calabi–Markus phenomenon (criterion)

$G \supset H$ reductive Lie groups

$\Longrightarrow$ G/H pseudo-Riemannian homo. sp

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Application (space form of signature (p, q), $\kappa < 0$)

Exists a space form M s.t. $|\pi_1(M)| = \infty$

$\Longleftrightarrow p > q$ or $(p, q) = (1, 1)$

(Calabi, Markus, Wolf, Kulkarni, Wallach)

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$p > q + 1$

$\Longrightarrow {}^{\exists}M$ with free non-commutative $\pi_1(M)$

Criterion of $\pitchfork$ and $\sim$ (general case)

E.g. G: reductive Lie group $\Longrightarrow$ Solved

G: general Lie gp $\Longrightarrow$ Unsolved

Not known an effective criterion for $\pitchfork$ even in the case

$$(G, H) = (GL(n, \mathbb{R}) \ltimes \mathbb{R}^n, GL(n, \mathbb{R}))$$

cf. Auslander conjecture (unsolved)

Goldman–Kamishima, Tomanov, Milnor, Margulis, Abels, Soifer, $\cdots$

Criterion of $\pitchfork$ and $\sim$ (nilpotent case)

G : nilpotent Lie group

Criterion for $\pitchfork$ for connected H, L (Lipsman conjecture)

Does criterion analogous to reductive case hold for nilpotent case?

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G : nilpotent Lie group

Criterion for $\pitchfork$ for connected H, L (Lipsman conjecture)

Does criterion analogous to reductive case hold for nilpotent case?

1-step (abelian)	OK
2-step	OK (Nasrin $_{2001}$)
3-step	OK (Baklouti–Khlif $_{2005}$, Yoshino $_{2007}$, A. Püttemann $_{2008}$)
4-step	No (Yoshino $_{2005}$)

↓ more non-commutative

4. Existence problem of compact quotients

$$\Gamma \subset G \supset H$$

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Remark $\Gamma \subset G$ uniform lattice, torsion free
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e.g.

$$G/H = SL(n,\mathbb{R})/SO(n),\ SL(n,\mathbb{C})/SU(n),\ \ldots$$

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Fact 8 (Borel 1963)
G/H is a Riemannian symmetric sp. $\Longrightarrow$ Yes

i.e. Compact forms exist for $^{\forall}$ Riemannian symmetric sp.

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$$G/H = SL(n,\mathbb{R})/SO(n),\ SL(n,\mathbb{C})/SU(n),\ \ldots$$

Existence problem of compact quotients

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- H is non-compact

$\Longrightarrow$?

Ex. $G/H = SL(n,\mathbb{R})/SL(m,\mathbb{R}), SL(n,\mathbb{R})/SO(p,n-p)$

Existence problem of compact quotients

Problem B Does there exist a uniform lattice for G/H?

- $M = G/H$ is para-Hermitian symmetric sp

  $TM = TM_+ + TM_-$ (Whitney direct sum)
  $TM_\pm$: completely integrable, equi-dimensional
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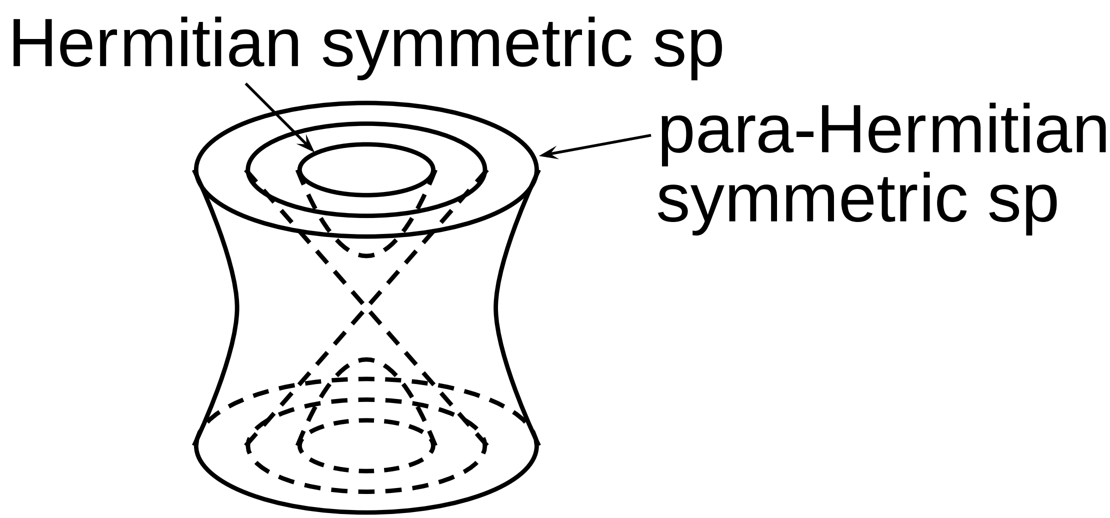

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Thm 9 G/H is a para-Hermitian symmetric sp. $\Longrightarrow$ No

Ex. $M = GL(p+q,\mathbb{R})/GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$,
$GL(n,\mathbb{C})/GL(n,\mathbb{R})$, $Sp(n,\mathbb{R})/GL(n,\mathbb{R})$, ...

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Proof: use Cor 7 (criterion for Calabi–Markus phenomenon)

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Problem B Does there exist a uniform lattice for G/H?

- G/H is complex sphere $S^n_{\mathbb{C}}$, i.e.

$$G/H := SO(n+1,\mathbb{C})/SO(n,\mathbb{C})$$
$$= \{(z_1,\ldots,z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 1\}$$

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Thm 10 (2005)

$$G/H = SO(n+1,\mathbb{C})/SO(n,\mathbb{C})$$
$$n = 1,3,7 \quad \Longrightarrow \quad \text{Yes}$$

There exist closed complex manifolds that are locally isomorphic to complex spheres if its dimension $= 1,3$ or 7.

Existence problem of compact quotients

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Thm 10 **Conjecture 11**

$$G/H = SO(n+1,\mathbb{C})/SO(n,\mathbb{C})$$

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<u>Thm 10</u> <u>Conjecture 11</u>

$$G/H = SO(n+1,\mathbb{C})/SO(n,\mathbb{C})$$

$$n = 1, 3, 7 \quad \overset{\Longrightarrow}{\Longleftarrow} \quad \text{Yes}$$

Evidence: n: odd $\Leftarrow$ Yes (K–)

Existence problem of compact quotients

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Thm 10 **Conjecture 11**

$$G/H = SO(n+1,\mathbb{C})/SO(n,\mathbb{C})$$

$$n = 1, 3, 7 \quad \overset{\Longrightarrow}{\Longleftarrow} \quad \text{Yes}$$

Evidence: n: odd $\Leftarrow$ Yes (K–)

$n = 4k+3$ (or $n = 1$) $\Leftarrow$ Yes (Benoist)

Existence problem of compact quotients

Problem B Does there exist a uniform lattice for G/H?

- G/H is complex sphere, i.e.
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Infinitesimal version: $n = 1,3,7 \Leftrightarrow$ Yes

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$M \approx S^{11}_{\mathbb{C}}, S^{15}_{\mathbb{C}}, \dots$ not known

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Methods: criterion of $\pitchfork$, F_2 action, comparison thm

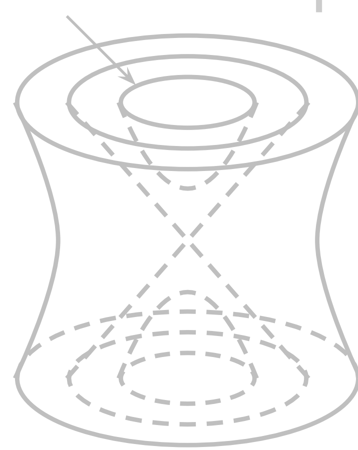
Real form of complex spheres $S^n_{\mathbb{C}}$

$$n = p + q$$

$O(p, q+1)/O(p, q)$ two viewpoints

$\cdots$ "real form" of $O(n+1, \mathbb{C})/O(n, \mathbb{C}) \simeq S^n_{\mathbb{C}}$

$\cdots$ space form: pseudo-Riemannian mfd of signature (p, q) with negative constant sectional curvature



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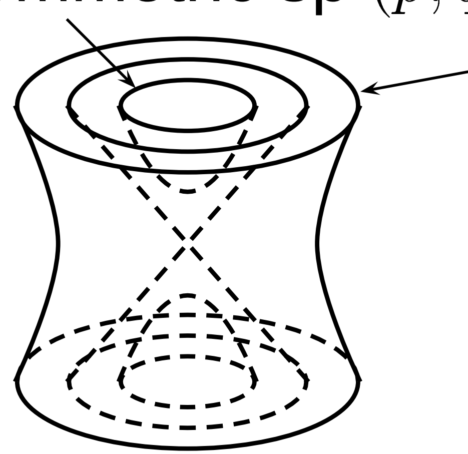

Existence of compact locally symm. sp

Thm 13 Exists a uniform lattice for the following G/H:

	G/H	
1	$SU(2,2n)/Sp(1,n)$	
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7	$SO(4,3)/SO(4,1)\times SO(2)$	
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5	$SO(4,4n)/SO(3,4n)$	$Sp(1,n)$
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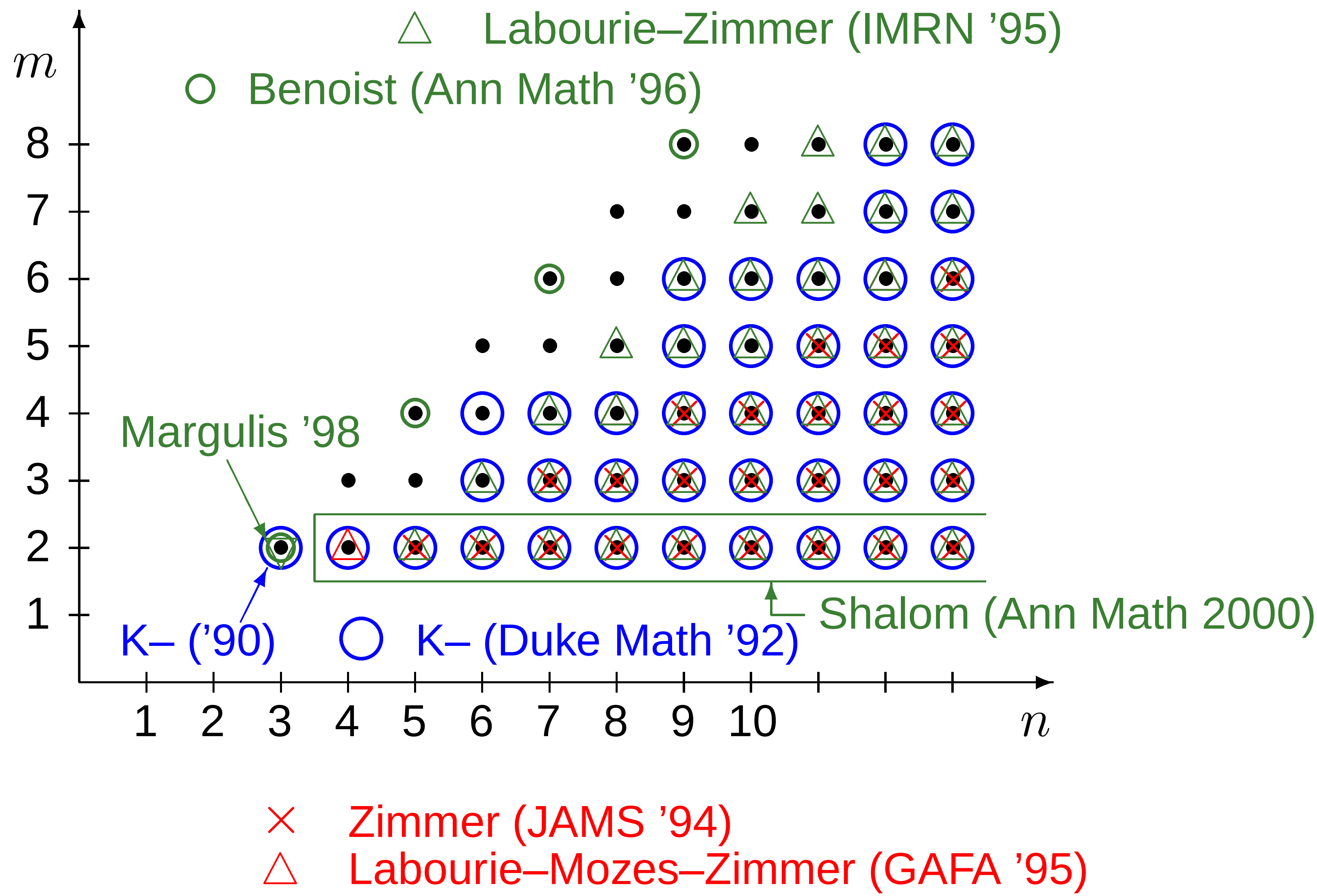
Compact quotients for $SL(n)/SL(m)$

There is no compact quotients if $n > m$ satisfies:



Rigidity/deformation

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proper discontinuity

may break down

5. Rigidity, stability, and deformation

$$G \curvearrowright X$$

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Γ properly discont. $\curvearrowright X$

$\Gamma \subset G \curvearrowright X$

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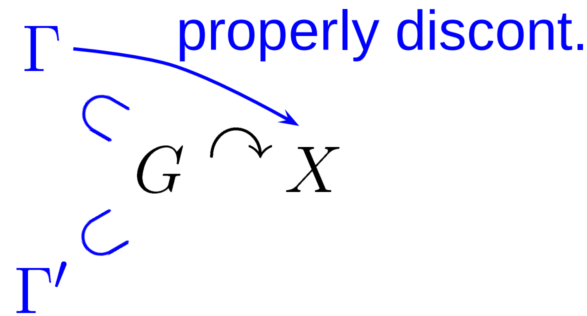

Suppose Γ' is ‘close to’ Γ

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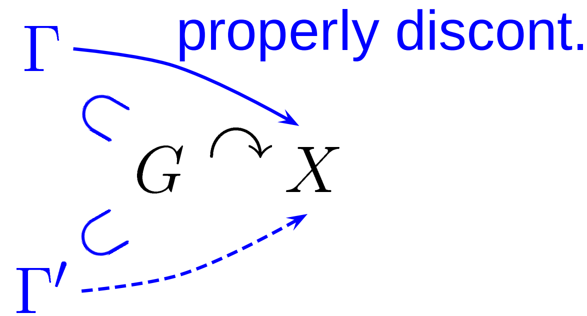

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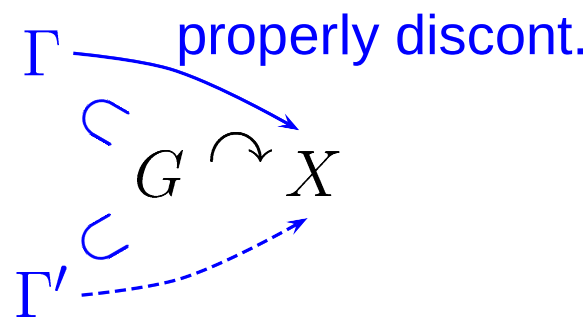

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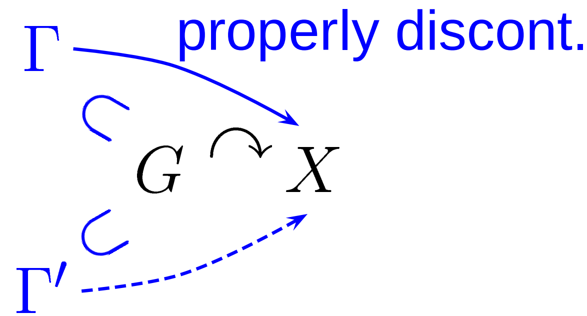

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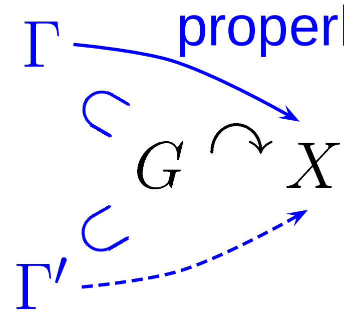

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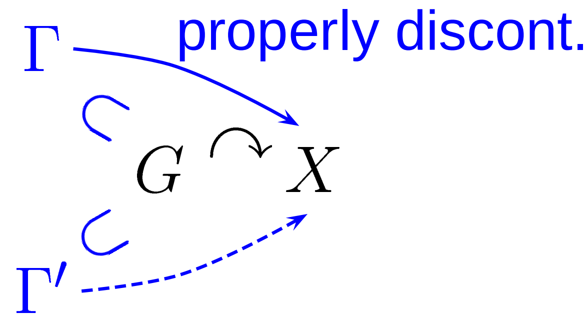

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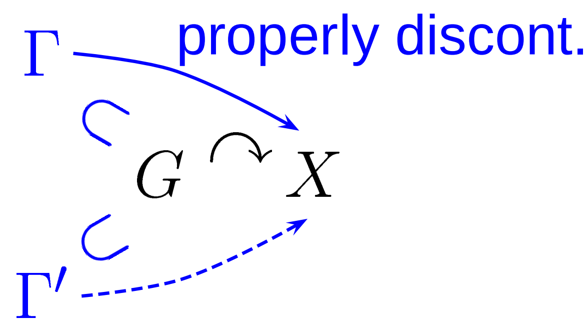

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Local rigidity and deformation

$\Gamma \subset G \curvearrowright X = G/H$ uniform lattice

<u>Problem C</u>

1. When does local rigidity (R) fail?
2. Does stability (S) still hold?

Local rigidity and deformation

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Problem C

1. When does local rigidity (R) fail?
2. Does stability (S) still hold?

Point: for non-compact H

1. There may be large room for deformation of Γ itself.
2. Properly discontinuity may fail under deformation.

Local rigidity (formulation)

Γ : finitely generated, G

$$\mathrm{Hom}(\Gamma, G)$$

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Def. $u \in R(\Gamma, G; X)$ is locally rigid as a discontinuous gp for X if $\{[u]\}$ is open in $\mathrm{Hom}(\Gamma, G)/G$.

Group manifold case

	$G/\{e\}$	$\simeq$	$(G \times G)/\Delta G$	
①	Riemannian		pseudo-Riemannian	②
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$\Gamma \subset G$ simple Lie gp

$$\Gamma \curvearrowright G \iff (\Gamma \times 1) \curvearrowright (G \times G)/\Delta G$$

Rigidity Theorem for pseudo-Riem. case

① $G/\{e\} \simeq (G \times G)/\Delta G$ ②

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Kazhdan's property (T) fails
$\iff$ trivial representation is not isolated in the unitary dual

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Method: use the criterion of $\pitchfork$
($\Rightarrow$ criterion for properly discontinuous actions)

Local rigidity and stability

$\Gamma', \Gamma \subset G \curvearrowright X$

$\Gamma \curvearrowright X$ properly discont. & Γ' is ‘close to’ Γ

(R) (local rigidity)	$\Gamma' = g\Gamma g^{-1}$ $({}^{\exists}g \in G)$
(S) (stability)	$\Gamma' \curvearrowright X$ properly discont.

In general,

- (R) $\Rightarrow$ (S).
- (S) may fail (so does (R)).
- Goldman's theorem and conjecture (1985)
  $X =$ 3-dim'l Lorentz space form
  (R) fails. It is likely that (S) holds.

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<u>Thm 16</u> (Kulkarni–Raymond, K–)

~~$G = SL(2, \mathbb{R})$~~ G: semisimple Lie gp, $\operatorname{rank}_{\mathbb{R}} G = 1$

$\Longrightarrow$ ~~$\Longleftrightarrow$~~ Any discontinuous gp for $G = (G \times G)/\Delta G$ is virtually of the form Γ_ρ for some Γ and ρ up to switch of factors.

Low dimensional case

$G = SL(2,\mathbb{R})$ $(\approx SO(2,1) \approx SU(1,1))$

Deformations for ①
$\cdots$ deformation of complex structure of Riemann surface

Deformations for ②
$\cdots$ negatively curved 3-dim'l Lorentz space forms
(Goldman, K– , Salein, . . .)

$G = SL(2,\mathbb{C})$ $(\approx SO(3,1))$

Deformation for ②
$\cdots$ 3-dimensional complex mfd
(Ghys, . . .)

Criterion for proper action

Discrete	properly discontinuous	Benoist, K– (Thm 6)
↕		
Continuous analog	proper action	K–
↕ ?		
Representations	discretely decomposable restriction	K– (Invent Math 94, Ann Math 98, Invent Math 98)

Discontinuous gps — Unitary reps

M : topological space

Γ: discontinuous gp $\curvearrowright M$

$\cdots$ Γ behaves nicely in $\mathrm{Homeo}(M)$
as if it were a finite group

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$L \curvearrowright \mathcal{H}$ "nice" unitary representations

$\cdots$ L behaves nicely in $U(\mathcal{H})$
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$\mathcal{H} = L^2(G/H)$, $L^2(G/\Gamma)$: Hilbert space

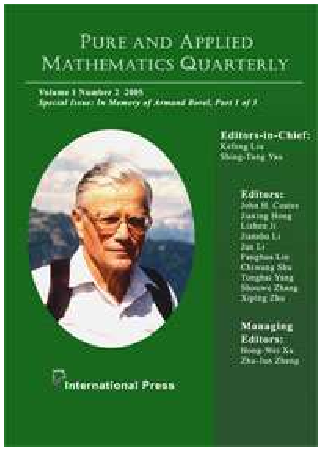
$L \curvearrowright \mathcal{H}$ "nice" unitary representations

$\cdots$ L behaves nicely in $U(\mathcal{H})$
as if it were a compact group

decay of matrix coefficients (Margulis, Oh)
discretely decomposable restrictions (K–)

References

1) PAMQ vol.1 (2005) Borel Memorial Volume



2) math.DG/0603319 (survey paper, translated by M. Reid)

3) work in progress (with T. Yoshino)

For more references:

http://www.math.harvard.edu/~toshi

Existence problem of compact quotients

Various approaches including

- criterion for proper actions
- Hirzebruch's proportionality principle
- cohomology of discrete groups
- symplectic geometry
- ergodic actions
- unitary representation theory
- …