Branching in Representation Theory

Lecture 3. Multiplicity in Branching of the Restriction

Toshiyuki Kobayashi

The Graduate School of Mathematical Sciences
The University of Tokyo
http://www.ms.u-tokyo.ac.jp/~toshi/

Minicourses: branching problems and symmetry-breaking

Thematic trimester Representation Theory and Noncommutative Geometry
Organizers: Alexandre Afgoustidis, Anne-Marie Aubert, Pierre Clare, Haluk Şengün
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A Program: Stage ABC for Branching Problem

Wednesday

Lecture 2. Discrete Decomposability and Admissible Restriction

Stage A. **A**bstract Feature of Restriction

- spectrum: discrete or continuous?/ support?
- multiplicities: infinite, finite, bounded, or one, . . . ?

Stage B. **B**ranching Laws

- (irreducible) decomposition of representations

Stage C. **C**onstruction of SBOs/HOs

SBO $\cdots$ Symmetry Breaking Operator
HO $\cdots$ Holographic Operator

- decomposition of vectors

Ref. T. Kobayashi, “A program for branching problems in rep theory. . . ”, Progress in Mathematics, **312**, (2015).

A Program: Stage ABC for Branching Problem

Today

Lecture 3. Multiplicity in Branching of the Restriction

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- spectrum: discrete or continuous?/ support?
- multiplicities: infinite, finite, bounded, or one, . . . ?

Stage B. **B**ranching Laws

- (irreducible) decomposition of representations

Stage C. **C**onstruction of SBOs/HOs

SBO · · · Symmetry Breaking Operator
HO · · · Holographic Operator

- decomposition of vectors

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Notation

Throughout this talk, G: real reductive Lie gp

Ex.

$G \supset K$: max compact subgp $\quad GL(n, \mathbb{R}) \supset O(n)$

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$\Downarrow$ more general

$G \supset H$: reductive symmetric pair $GL(n,\mathbb{R}) \supset O(p,n-p)$

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$G \supset K$: max compact subgp — $GL(n,\mathbb{R}) \supset O(n)$

$\Downarrow$ more general

$G \supset H$: reductive symmetric pair — $GL(n,\mathbb{R}) \supset O(p,n-p)$

$\Downarrow$ more general

$G \supset G'$: real reductive subgp — $GL(n_1+n_2+n_3,\mathbb{R})$

$\supset GL(n_1,\mathbb{R})\times GL(n_2,\mathbb{R})\times GL(n_3,\mathbb{R})$

Multiplicities

$$G/K \qquad G \downarrow K$$

$$G/H \qquad G \downarrow H$$

Notation

$G \supset K$: max compact subgp $\quad GL(n,\mathbb{R}) \supset O(n)$

$\Downarrow$ more general

$G \supset H$: reductive symmetric pair $\quad GL(n,\mathbb{R}) \supset O(p, n-p)$

Multiplicities

G/K $\qquad$ $G \downarrow K$

G/H $\qquad$ $G \downarrow H$

multiplicities

- $L^2(G/K)$ 0 or 1 (Cartan '29, Gelfand '50)

Notation $G \supset K$: max compact subgp $\quad GL(n,\mathbb{R}) \supset O(n)$

Multiplicities

$$G/K \qquad G \downarrow K$$

$$G/H \qquad G \downarrow H$$

multiplicities

- $L^2(G/K)$ — 0 or 1 — (Cartan '29, Gelfand '50)
- $G \downarrow K$ — finite — (Harish-Chandra's admissibility thm)

Notation

$G \supset K$: max compact subgp $\quad GL(n, \mathbb{R}) \supset O(n)$

Multiplicities

$$G/K \qquad G \downarrow K$$

$$G/H \qquad G \downarrow H$$

multiplicities

- $L^2(G/K)$ — 0 or 1 — (Cartan '29, Gelfand '50)
- $G \downarrow K$ — finite — (Harish-Chandra's admissibility thm)
- $L^2((G_1 \times G_1)/G_1)$ — 0 or 1 — (Harish-Chandra)

Multiplicities

G/K $\quad G \downarrow K$

G/H $\quad G \downarrow H$

multiplicities

- $L^2(G/K)$ $\quad 0$ or 1 $\quad$ (Cartan '29, Gelfand '50)
- $G \downarrow K$ $\quad$ finite $\quad$ (Harish-Chandra's admissibility thm)
- $L^2((G_1 \times G_1)/G_1)$
  0 or 1 $\quad$ (Harish-Chandra)
- $L^2(G/H)$ uniformly bounded

Notation

$G \supset K$: max compact subgp $\quad GL(n,\mathbb{R}) \supset O(n)$

$\Downarrow$ more general

$G \supset H$: reductive symmetric pair $\quad GL(n,\mathbb{R}) \supset O(p, n-p)$

Multiplicities

$G/K \qquad G \downarrow K$

$G/H \qquad G \downarrow H$

multiplicities

- $L^2(G/K)$ — 0 or 1 — (Cartan '29, Gelfand '50)
- $G \downarrow K$ — finite — (Harish-Chandra's admissibility thm)
- $L^2(G/H)$ uniformly bounded
- $G \downarrow H$ — can be ∞ (usually) bad feature — but . . . (unexpectedly) nice feature

Notation

$G \supset H$: reductive symmetric pair

Definition of "Multiplicity" in the branching

Various inequivalent definition of multiplicities can be considered for the branching of the restriction $G \downarrow G'$

- Dimension of $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ (Symmetry Breaking Operators), or
- Dimension of $\mathrm{Hom}_{G'}(\pi, \Pi|_{G'})$ (Holographic Operators).
  - in the category of $(\mathfrak{g}, K)$-modules,
  - in the category of unitary representations,
  - in the category of smooth representations,

$\cdots$

Various definitions of “multiplicities” for the restriction

$G \supset G'$ reductive Lie groups

$\Pi \in \widehat{G}, \quad \pi \in \widehat{G'}$

$\dim \mathrm{Hom}_{(\mathfrak{g}',K')}(\pi_{K'}, \Pi_K|_{(\mathfrak{g}',K')})$ (holographic op)

$\wedge|$

$\dim \mathrm{Hom}_{G'}(\pi, \Pi|_{G'})$ (holographic op)

$||$

$\dim \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ (symmetry breaking op)

$\wedge|$

$\dim \mathrm{Hom}_{G'}(\Pi^\infty|_{G'}, \pi^\infty)$ (symmetry breaking op)

Various definitions of “multiplicities” for the restriction

$G \supset G'$ reductive Lie groups

$\Pi \in \widehat{G}, \quad \pi \in \widehat{G'}$

$\dim \mathrm{Hom}_{(\mathfrak{g}', K')}(\pi_{K'}, \Pi_K\|_{(\mathfrak{g}', K')})$	(holographic op)
$\wedge\!\mid$	
$\dim \mathrm{Hom}_{G'}(\pi, \Pi\|_{G'})$	(holographic op)
$\parallel$	
$\dim \mathrm{Hom}_{G'}(\Pi\|_{G'}, \pi)$	(symmetry breaking op)
$\wedge\!\mid$	
$\dim \mathrm{Hom}_{G'}(\Pi^{\infty}\|_{G'}, \pi^{\infty})$	(symmetry breaking op)

Non-unitary representation and smooth representations

G: Lie group
V: complete, locally convex top. space (*e.g.*, Banach space)
(π, V): continuous representation of G, that is,

$$G \times V \to V, \qquad (g, u) \mapsto \pi(g)u \text{ is continuous}$$

u is a smooth vector $\Leftrightarrow G \to V$, $g \mapsto \pi(g)u$ is of C^∞-class

$$\rightsquigarrow \quad V^\infty := \{\text{smooth vectors}\} \underset{\text{dense}}{\subset} V.$$

Definition (π^∞, V^∞) is called the smooth representation of G.
We note that V^∞ is endowed with Fréchet topology.

Example $G = SU(1,1)\ (\simeq SL(2,\mathbb{R}))$

$V := L^p(S^1) \quad (1 \le p \le \infty)$

For $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$, a principal series representation π_λ is defined by

$$(\pi_\lambda(g)f)(z) := |cz+d|^\lambda f(\frac{az+b}{cz+d}).$$

$$\Rightarrow \quad \begin{cases} L^p(S^1) \not\simeq L^{p'}(S^1) & \text{if } p \ne p', \\ (L^p(S^1))^\infty \text{ does not depend on } 1 \le p \le \infty. \end{cases}$$

(It is isomorphic to $C^\infty(S^1)$.)

What does this mean?

$\mathrm{Irr}(G)$ in the admissible smooth category

G: real reductive Lie group
V: Banach space
(π, V): Irreducible, K-admissible, continuous representation of G

$\rightsquigarrow$

Fact (π^∞, V^∞) does not depend on the original topology on V.

Definition $\mathrm{Irr}(G) := \{(\pi^\infty, V^\infty)\}/\sim$.

(Irreducible objects in the Casselman–Wallach category)

Multiplicity of the restriction $\Pi|_{G'}$ including non-unitary case

G : real reductive Lie group

$\mathrm{Irr}(G)$: irreducible, smooth, admissible reps

$$\underset{\text{unitary dual}}{\widehat{G}} \hookrightarrow \mathrm{Irr}(G), \quad \Pi \mapsto \Pi^{\infty}.$$

$G \supset G'$: real reductive groups

Definition (multiplicity) For $\Pi \in \mathrm{Irr}(G)$ and $\pi \in \mathrm{Irr}(G')$, we set

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) := \{\text{symmetry breaking operators}\}$$

$$[\Pi|_{G'} : \pi] := \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}$$

Various definitions of “multiplicities” for the restriction

$G \supset G'$ reductive Lie groups

$\Pi \in \widehat{G}, \quad \pi \in \widehat{G'}$

$\dim \mathrm{Hom}_{(\mathfrak{g}',K')}(\pi_{K'}, \Pi_K|_{(\mathfrak{g}',K')})$ (holographic op)

$\wedge|$

$\dim \mathrm{Hom}_{G'}(\pi, \Pi|_{G'})$ (holographic op)

$\|$

$\dim \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ (symmetry breaking op)

$\wedge|$

$\dim \mathrm{Hom}_{G'}(\Pi^\infty|_{G'}, \pi^\infty)$ (symmetry breaking op)

Spherical vs real spherical

$G_{\mathbb{C}}$ complex reductive $\curvearrowright X_{\mathbb{C}}$ complex manifold (connected)

Definition $X_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.

G real reductive $\curvearrowright X$ C^{∞} manifold (connected)

Definition (TK– '89) We say X is real spherical if a minimal parabolic P of G has an open orbit in X.

— in search of a broader framework for global analysis on homogeneous spaces than the usual (e.g. group manifolds, reductive symmetric spaces)

Example of spherical spaces

$G_{\mathbb{C}}$ complex reductive $\curvearrowright X_{\mathbb{C}}$ complex manifold (connected)

Definition $X_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit $X_{\mathbb{C}}$.

1. When G is a simple compact Lie group, $X_{\mathbb{C}}$ is $G_{\mathbb{C}}$-spherical if $X = G/H$ is a symmetric space. There are also a few number of non-symmetric, reductive, spherical spaces such as $SO(2n+1,\mathbb{C})/GL(n,\mathbb{C})$, $G_{2,\mathbb{C}}/SL(3,\mathbb{C})$, ... (classified by Krämer*).

2. (Triple space) When $G_{\mathbb{C}}$ is a complex simple Lie group, $(G_{\mathbb{C}} \times G_{\mathbb{C}} \times G_{\mathbb{C}})/\operatorname{diag} G_{\mathbb{C}}$ is spherical if and only if $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$ **.

* M. Krämer, Arch Math (1976).

Example of real spherical spaces

G real reductive $\curvearrowright$ X manifold (connected)

Definition (TK– '89) We say X is real spherical if
a minimal parabolic P of G has an open orbit X.

1. When G is compact, a minimal parabolic subgroup is the whole group G, hence any homogeneous space $X = G/H$ is real spherical.

2. (Triple space)* For a non-compact simple Lie group G,

$$(G \times G \times G)/\operatorname{diag} G$$

is real spherical if and only if $\mathfrak{g} \simeq \mathfrak{so}(n, 1)$.

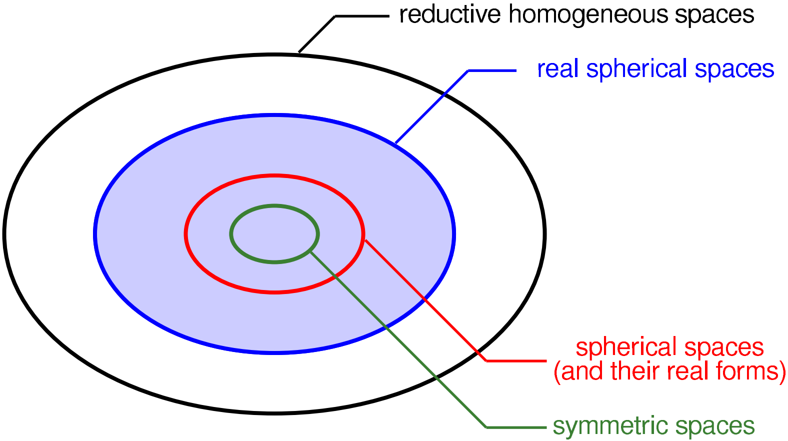
cf. Kazhdan's property (T) fails if $\mathfrak{g} \simeq \mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$.

* T. Kobayashi, Introduction to Real Spherical Space, Proc. Number Theory, (1995), pp. 22–41.

Reductive homogeneous space G/H

G: real reductive groups

G': reductive subgroup



Spherical vs real spherical

$G_{\mathbb{C}}$ complex reductive $\supset H_{\mathbb{C}}$ complex subgroup

Definition $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/H_{\mathbb{C}}$.

G real reductive $\supset H$ subgroup

Definition (TK– '89) We say G/H is real spherical if a minimal parabolic P of G has an open orbit in G/H.

G/H symmetric space $\overset{\Rightarrow}{\not\Leftarrow}$ $G_{\mathbb{C}}/H_{\mathbb{C}}$ spherical $\overset{\Rightarrow}{\not\Leftarrow}$ G/H real spherical

Spherical vs real spherical

$G_{\mathbb{C}}$ complex reductive $\supset H_{\mathbb{C}}$ complex subgroup

<u>Definition</u> $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/H_{\mathbb{C}}$.

$\iff \#(B\backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty$ (Brion, Vinberg) (~1986)

G real reductive $\supset H$ subgroup

<u>Definition</u> (TK– '89) We say G/H is real spherical if a minimal parabolic P of G has an open orbit in G/H.

$\iff \#(P\backslash G/H) < \infty$ (Kimelfeld, Matsuki, Bien) (~1990s)

G/H symmetric space $\underset{\not\Leftarrow}{\Rightarrow}$ $G_{\mathbb{C}}/H_{\mathbb{C}}$ spherical $\underset{\not\Leftarrow}{\Rightarrow}$ G/H real spherical

Finite multiplicity criterion for restriction $G \downarrow G'$

$G \supset G'$: algebraic real reductive Lie groups

Theorem 1[*] (i) and (ii) are equivalent.

(i) (Finite multiplicity property)

$$\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$$

for any $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(G')$.

(ii) $(G \times G')/\operatorname{diag}(G')$ is real spherical.

[*] T. Kobayashi–T. Oshima, “Finite multiplicity theorems for inductionand restriction”, Adv. Math., (2013), 921–943;

T. Kobayashi, Sintani functions, real spherical manifolds, . . . , Perspective Math., (2014).

Finite multiplicity criterion for restriction $G \downarrow G'$

$G \supset G'$: algebraic real reductive Lie groups

Theorem 1* (i) and (ii) are equivalent.

(i) (Finite multiplicity property)

$$\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$$

for any $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(G')$.

(ii) $(G \times G')/\operatorname{diag}(G')$ is real spherical.

(ii) $\Leftrightarrow$ $^{\exists}$open $P'_{\min}$-orbit on $G/P_{\min}$

Classification Theory

Theorem 2 (Classification)* Let (G, G') be a reductive symmetric pair. Then, (ii) $\Leftrightarrow$ (iii).

(ii) $(G \times G')/\operatorname{diag}(G')$ is real spherical.

(iii) $(\mathfrak{g}, \mathfrak{g}')$ is a direct sum of the following pairs

- (A) (Easy case)
- (B) (Strong Gelfand pairs and real forms)
- (C) (Split rank one case)
- (D) (Other cases)

* T. Kobayashi, Introduction to Real Spherical Space, (1995); T. Kobayashi–T. Matsuki, Transf. Group, (2014).

Classification Theory

Theorem 2 (Classification)* Let (G, G') be a reductive symmetric pair.
Then, (ii) ⇔ (iii).

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- (A) (Easy case)
- (B) (Strong Gelfand pairs and real forms)
- (C) (Split rank one case)
- (D) (Other cases)

(A–1) (Trivial case) $\mathfrak{g} = \mathfrak{g}'$ ←Bruhat decomposition

(A–2) (Abelian case) $\mathfrak{g} = \mathbb{R}$

(A–3) (Compact case) G is compact

(A–4) (Riemannian symmetric pair) $G' = K$ ←Iwasawa decomposition

* T. Kobayashi, Introduction to Real Spherical Space, (1995); T. Kobayashi–T. Matsuki, Transf. Group, (2014).

Classification Theory

<u>Theorem 2</u> (Classification)* Let (G, G') be a reductive symmetric pair.
Then, (ii) $\Leftrightarrow$ (iii).

(ii) $(G \times G')/\operatorname{diag}(G')$ is real spherical.

(iii) $(\mathfrak{g}, \mathfrak{g}')$ is a direct sum of the following pairs

- (A) (Easy case)
- (B) (Strong Gelfand pairs and real forms)
- (C) (Split rank one case)
- (D) (Other cases)

(B–1) $(\mathfrak{sl}(n+1, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$ $(n \geq 2)$.

(B–2) $(\mathfrak{o}(n+1, \mathbb{C}), \mathfrak{o}(n, \mathbb{C}))$ $(n \geq 2)$.

(B–3) $(\mathfrak{sl}(n+1, \mathbb{R}), \mathfrak{gl}(n, \mathbb{R}))$ $(n \geq 1)$.

(B–4) $(\mathfrak{su}(p+1, q), \mathfrak{u}(p, q))$ $(p + q \geq 1)$.

(B–5) $(\mathfrak{o}(p+1, q), \mathfrak{o}(p, q))$ $(p + q \geq 2)$.

* T. Kobayashi, Introduction to Real Spherical Space, (1995); T. Kobayashi–T. Matsuki, Transf. Group, (2014).

Classification Theory

Theorem (Classification)* Let (G, G') be a reductive symmetric pair.
Then, (ii) $\Leftrightarrow$ (iii).

(ii) $(G \times G')/\operatorname{diag}(G')$ is real spherical.

(iii) $(\mathfrak{g}, \mathfrak{g}')$ is a direct sum of the following pairs

- (A) (Easy case)
- (B) (Strong Gelfand pairs and real forms)
- (C) (Split rank one case)
- (D) (Other cases)

(C) $\operatorname{rank}_{\mathbb{R}} G' = 1$

(neither necessary nor sufficient; but fairly large families)

Classification Theory

Theorem (Classification)* Let (G, G') be a reductive symmetric pair.
Then, (ii) ⇔ (iii).

(ii) $(G \times G')/\operatorname{diag}(G')$ is real spherical.

(iii) $(\mathfrak{g}, \mathfrak{g}')$ is a direct sum of the following pairs

- (A) (Easy case)
- (B) (Strong Gelfand pairs and real forms)
- (C) (Split rank one case)
- (D) (Other cases)

(C–1) $(\mathfrak{o}(n,1) + \mathfrak{o}(n,1), \operatorname{diag} \mathfrak{o}(n,1)) \quad (n \geq 2)$.

(C–2) $(\mathfrak{o}(p+q,1), \mathfrak{o}(p) + \mathfrak{o}(q,1)) \quad (p+q \geq 2)$.

(C–3) $(\mathfrak{su}(p+q,1), \mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(q,1))) \quad (p+q \geq 1)$.

(C–4) $(\mathfrak{sp}(p+q,1), \mathfrak{sp}(p) + \mathfrak{sp}(q,1)) \quad (p+q \geq 1)$.

(C–5) $(\mathfrak{f}_{4(-20)}, \mathfrak{o}(8,1))$.

(C–6) $(\mathfrak{o}(2n,2), \mathfrak{u}(n,1))$.

* TK, Introduction to Real Spherical Space, (1995); TK–T. Matsuki, Transf. Group, (2014), Dynkin Volume

Classification Theory

Theorem (Classification)* Let (G, G') be a reductive symmetric pair.
Then, (ii) ⇔ (iii).
(ii) $(G \times G')/\operatorname{diag}(G')$ is real spherical.
(iii) $(\mathfrak{g}, \mathfrak{g}')$ is a direct sum of the following pairs

- (A) (Easy case)
- (B) (Strong Gelfand pairs and real forms)
- (C) (Split rank one case)
- (D) (Other cases)

(D–1) $(\mathfrak{su}^*(2n+2), \mathfrak{su}(2) + \mathfrak{su}^*(2n) + \mathbb{R})$ $(n \geq 1)$.
(D–2) $(\mathfrak{o}^*(2n+2), \mathfrak{o}(2) + \mathfrak{o}^*(2n))$ $(n \geq 1)$.
(D–3) $(\mathfrak{sp}(p+1, q), \mathfrak{sp}(p, q) + \mathfrak{sp}(1))$.
(D–4) $(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9, 1) + \mathbb{R})$.

* TK, Introduction to Real Spherical Space, (1995); TK–T. Matsuki, Transf. Group, (2014), Dynkin Volume.

Classification of finite-multiplicity restriction

Corollary 1 Let (G, G') be a reductive symmetric pair. Then, (i) ⇔ (iii).

(i) $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ for any $\pi \in \widehat{G}_{\mathrm{adm}}$ and any $\tau \in \widehat{G'}_{\mathrm{adm}}$.

(iii) $(\mathfrak{g}, \mathfrak{g}')$ is a direct sum of the following pairs

(A) (Easy case)

- (A–1) $\mathfrak{g} = \mathfrak{g}'$
- (A–2) $\mathfrak{g} = \mathbb{R}$
- (A–3) G is compact
- (A–4) $G' = K$

(B) (Strong Gelfand pairs and real forms)

- (B–1) $(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}))$ $(n \geq 2)$.
- (B–2) $(\mathfrak{o}(n+1,\mathbb{C}), \mathfrak{o}(n,\mathbb{C}))$ $(n \geq 2)$.
- (B–3) $(\mathfrak{sl}(n+1,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}))$ $(n \geq 1)$.
- (B–4) $(\mathfrak{su}(p+1,q), \mathfrak{u}(p,q))$ $(p+q \geq 1)$.
- (B–5) $(\mathfrak{o}(p+1,q), \mathfrak{o}(p,q))$ $(p+q \geq 2)$.

(C) (Split rank one case)

- (C–1) $(\mathfrak{o}(n,1)+\mathfrak{o}(n,1), \mathrm{diag}\,\mathfrak{o}(n,1))$ $(n \geq 2)$.
- (C–2) $(\mathfrak{o}(p+q,1), \mathfrak{o}(p)+\mathfrak{o}(q,1))$ $(p+q \geq 2)$.
- (C–3) $(\mathfrak{su}(p+q,1), \mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(q,1)))$ $(p+q \geq 1)$.
- (C–4) $(\mathfrak{sp}(p+q,1), \mathfrak{sp}(p)+\mathfrak{sp}(q,1))$ $(p+q \geq 1)$.
- (C–5) $(\mathfrak{f}_{4(-20)}, \mathfrak{o}(8,1))$.
- (C–6) $(\mathfrak{o}(2n,2), \mathfrak{u}(n,1))$.

(D) (Other cases)

- (D–1) $(\mathfrak{su}^*(2n+2), \mathfrak{su}(2)+\mathfrak{su}^*(2n)+\mathbb{R})$ $(n \geq 1)$.
- (D–2) $(\mathfrak{o}^*(2n+2), \mathfrak{o}(2)+\mathfrak{o}^*(2n))$ $(n \geq 1)$.
- (D–3) $(\mathfrak{sp}(p+1,q), \mathfrak{sp}(p,q)+\mathfrak{sp}(1))$.
- (D–4) $(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1)+\mathbb{R})$.

Restriction $G \downarrow G'$ with uniformly bounded multiplicity property

Theorem 3 (Uniformly bounded multiplicity criterion)

For a pair $G \supset G'$ of real reductive groups, (i)⇔(ii) (also (ii)′ or (ii)″).

(i) (Rep) $\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty$.

(ii) (Geometry) $(G_{\mathbb{C}} \times G'_{\mathbb{C}}) / \mathrm{diag}(G'_{\mathbb{C}})$ is spherical.

(ii)′ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is commutative.

(ii)″ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is a polynomial ring.

- The equivalence (i) ⇔ (ii) is proved in (TK–T. Oshima)*.
- A stronger estimate for (ii) ⇒ (i), namely, multiplicity-free theorem holds for most of (not all of) the cases (Sun–Zhu)**.
- Classification for (ii): $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is $(\mathfrak{sl}(n, \mathbb{C}), \mathfrak{gl}(n-1, \mathbb{C}))$, $(\mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(n-1, \mathbb{C}))$, or up to direct product, abelian factors, or automorphisms (Kostant, Krämer).

* T. Kobayashi–T. Oshima, "Finite multiplicity theorems for induction and restriction", Adv. Math., (2013), 921–943.

** Sun–Zhu, "Multiplicity one theorems: the Archimedean case", Ann. of Math., (2012), 23–44.

Good Control of Restriction $G \downarrow G'$

Theorem B (Uniformly bounded multiplicity criterion)

For a pair $G \supset G'$ of real reductive groups, (i)⇔(ii) (also (ii)′ or (ii)′′).

(i) (Rep) $\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$

(ii) (Geometry) $(G_{\mathbb{C}} \times G'_{\mathbb{C}}) / \operatorname{diag}(G'_{\mathbb{C}})$ is spherical.

(ii)′ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is commutative.

(ii)′′ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is a polynomial ring.

Geometry | Representation

$G_{\mathbb{C}} \times G'_{\mathbb{C}} / \operatorname{diag}(G'_{\mathbb{C}}) \rightsquigarrow \Pi|_{G'}$

$U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$

Algebra

Tri-linear invariant forms

G : a simple Lie group

Corollary 3 (Finite multiplicity) Equivalent on $\mathfrak{g}$:

(1) For any triple of irred reps π_1, π_2, and $\pi_3 \in \operatorname{Irr}(G)$
$\dim \operatorname{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) < \infty$

(1)′ For any triple of irred reps π_1, π_2, and $\pi_3 \in \operatorname{Irr}(G)$
$\dim \operatorname{Hom}_G(\pi_1 \otimes \pi_2, \pi_3) < \infty$

(2) $\mathfrak{g} \simeq \mathfrak{o}(n, 1) \ (n \geq 2)$

Tri-linear invariant forms

G : a simple Lie group

Corollary 3 (Finite multiplicity) Equivalent on $\mathfrak{g}$:

(1) For any triple of irred reps π_1, π_2, and $\pi_3 \in \mathrm{Irr}(G)$
$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) < \infty$

(1)′ For any triple of irred reps π_1, π_2, and $\pi_3 \in \mathrm{Irr}(G)$
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(2) $\mathfrak{g} \simeq \mathfrak{o}(n, 1) \ (n \geq 2)$

Cf. For $\mathfrak{g} = \mathfrak{su}(n, 1)\ (n \geq 2),\ \mathfrak{sp}(n, 1),\ \mathfrak{f}_{4(-20)}$

$$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3,\ \mathbb{C}) = \infty$$

for some π_1, π_2 and π_3.

Tri-linear invariant forms

G : a simple Lie group

Corollary 3 (Finite multiplicity) Equivalent on $\mathfrak{g}$:

(1) For any triple of irred reps π_1, π_2, and $\pi_3 \in \mathrm{Irr}(G)$

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Corollary 4 (Uniformly bounded multiplicity)

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$\Longleftrightarrow$ $\mathfrak{g} \simeq \mathfrak{o}(2, 1)$ or $\mathfrak{o}(3, 1)$

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- Pukánszky, Williams, Repka (Decomposition of $\pi_1 \otimes \pi_2$ for $SL(2,\mathbb{R})$) Note: $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{o}(2,1)$
- Bernstein–Rezhikov integral (Clerc–K–Ørsted–Pevzner 2011)
  $\mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C})$ for $G = O(n,1)$

Symmetry breaking operators

A nice framework of branching $G \downarrow G'$ in the noncompact case:

- Discrete decomposability of the restriction $\pi|_{G'}$
- Finiteness/boundedness of $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$

⇝ expect a simple and detailed study

Analysis of Branching Problems

Multiplicities

G/K $\quad G \downarrow K$

G/H $\quad G \downarrow H$

	multiplicities	
$L^2(G/K)$	0 or 1	(Cartan '29, Gelfand '50)
$G \downarrow K$	finite	(Harish-Chandra's admissibility thm)
$L^2(G/H)$	uniformly bounded	
$G \downarrow H$	can be ∞ — (usually) bad feature; but . . . — (unexpectedly) nice feature	

Notation

$G \supset H$: reductive symmetric pair

Bounded multiplicity theorem for Π with small GK dim.

Let G be a 1-connected real non-compact semisimple Lie group.

Theorem 5 (K–, 23)* There exist $C \equiv C(G) > 0$ and an infinite-dim'l irreducible rep Π of G such that

$$\sup_{\pi \in \mathrm{Irr}(H)} [\Pi|_H : \pi] \leq C$$

for *all* reductive symmetric pairs $G \supset H$.

* T. Kobayashi, Bounded multiplicity branching for symmetry pairs, J. Lie Theory, (2023) pp. 305–328.

Bounded multiplicity theorem for Π with small GK dim.

Theorem 6 (K–, 2022)* Suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple.

If the associated variety of $\Pi \in \mathrm{Irr}(G)$ is the minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}^*$, then $^{\exists}C > 0$ such that

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* T. Kobayashi, Multiplicity in restricting minimal representations, PROMS, (2022).

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Example** (KØ 2003; Lecture 2 of Bent Ørsted, this morning)
$(G, G') =$ (Conformal group, "Isometry group") for $X = S^{p-1} \times S^{q-1}$.
$\Pi = \mathrm{Ker}(\widetilde{\Delta})$, $\widetilde{\Delta}$ is the Yamabe operator on X ($p + q$ even).
$\rightsquigarrow$ The restriction for $O(p, q) \downarrow O(p', q') \times O(p'', q'')$ has a uniform bounded multiplicity.

* T. Kobayashi, Multiplicity in restricting minimal representations, PROMS, (2022).

** T. Kobayashi–B. Ørsted, Analysis on minimal Reps, I, II, III, Adv. Math. (2003).

Symmetry breaking operators

A nice framework of branching $G \downarrow G'$ in the noncompact case:

Stage A

- Discrete decomposability of the restriction $\pi|_{G'}$
- Finiteness/boundedness of $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$

⇝ expect a simple and detailed study

Stage B.

Stage C.

. . .

Analysis of Branching Problems

A Program: Stage ABC for Branching Problem

Stage A. **A**bstract Feature of Restriction

- spectrum: discrete or continuous?/ support?
- multiplicities: infinite, finite, bounded, or one, . . . ?

Stage B. **B**ranching Laws

- (irreducible) decomposition of representations

Stage C. **C**onstruction of SBOs/HOs

SBO · · · Symmetry Breaking Operator
HO · · · Holographic Operator

- decomposition of vectors

Construction and classification of SBOs for $G \downarrow G'$

Assumption Suppose the pair $G \supset G'$ satisfies
$\dim \operatorname{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ ($^{\forall}\pi \in \operatorname{Irr}(G)$, $^{\forall}\tau \in \operatorname{Irr}(G')$).

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General Problem*

Construct and classify SBOs between principal series reps,

$$T: \ \mathrm{Ind}_P^G(V) \to \mathrm{Ind}_{P'}^{G'}(W)$$

for finite dimensional $V \in \mathrm{Irr}(P)$ and $W \in \mathrm{Irr}(P')$

- Special case $\cdots$ when T is a local operator

$\Rightarrow$ This is an important and challenging case **.

* T. Kobayashi–B. Speh, (Memoirs of AMS 2015, Lect. Notes in Math., 2018),

** Rankin, Cohen, Juhl (conformal geometry 2009), K–Kubo–Pevzner (2016), ...

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Assumption $\underset{\text{Theorem 1*}}{\Longleftrightarrow}$ $\sharp(P'_{\min} \backslash G / P_{\min}) < \infty$

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Geometric invariants of SBOs for $G \downarrow G' \cdots \operatorname{Supp} K_T \subset P'\backslash G/P$

The distribution kernel K_T of a symmetry breaking operator (SBO)

$$T\colon\ \operatorname{Ind}_P^G(V) \to \operatorname{Ind}_{P'}^{G'}(W)$$

is a P'-invariant distribution on G/P

$\rightsquigarrow \operatorname{Supp} K_T \cdots$ a closed P'-invariant subset in G/P

$$\begin{array}{ccc} \operatorname{Hom}_{G'}(\operatorname{Ind}_P^G(V), \operatorname{Ind}_{P'}^{G'}(W)) & \to & \{\text{Closed subsets of } P'\backslash G/P\} \\ T & \mapsto & \operatorname{Supp} K_T \end{array}$$

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T is a differential SBO iff $\operatorname{Supp} K_T$ is a singleton.
$\updownarrow$ opposite extremal case
We say T is regular SBO if $\operatorname{Supp} K_T$ contains inner points in G/P.

Strategy: Induction by the closure relation of $P'\backslash G/P$.

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Point $\sharp(P'\backslash G/P) < \infty$ from finite-multiplicity assumption

Thank you very much!

Branching in Representation Theory