# Branching in Representation Theory

## Lecture 2. Discrete Decomposability and Admissible Restriction


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Minicourses: branching problems and symmetry-breaking

Thematic trimester Representation Theory and Noncommutative Geometry
Organizers: Alexandre Afgoustidis, Anne-Marie Aubert, Pierre Clare, Haluk Şengün

Institut Henri Poincaré, Paris, France, 15 January 2025

# Branching problems in the general setting

$$\begin{array}{ccc} G & \xrightarrow[\text{irreducible}]{\pi} & GL(V) \\ \cup & \nearrow_{\pi|_{G'}} & \\ G' & & \end{array}$$

<u>Branching problem</u> (in a broader sense than the usual)

$\cdots$ wish to understand

how the <u>restriction $\pi|_{G'}$</u> behaves as a $G'$-module.

# A Program: Stage ABC for Branching Problem

**Stage A.** **A**bstract Feature of Restriction

- spectrum: discrete or continuous?/ support?
- multiplicities: infinite, finite, bounded, or one, . . . ?

**Stage B.** **B**ranching Laws

- (irreducible) decomposition of representations

**Stage C.** **C**onstruction of SBOs/HOs

SBO $\cdots$ Symmetry Breaking Operator
HO $\cdots$ Holographic Operator

- decomposition of vectors

Ref. T. Kobayashi, "A program for branching problems in rep theory. . . ", Progress in Mathematics, **312**, (2015).

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# Notation

Throughout this talk, $G$: real reductive Lie gp

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$G \supset H$: reductive symmetric pair $\quad GL(n, \mathbb{R}) \supset O(p, n-p)$

i.e. $G$ reductive Lie gp

$\sigma \in \mathrm{Aut}(G),\ \sigma^2 = \mathrm{id}$

$H =$ any open subgp of $G^\sigma$

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$\Downarrow$ more general

$G \supset G'$: real reductive subgp — $GL(n_1 + n_2 + n_3, \mathbb{R})$

$\supset GL(n_1, \mathbb{R}) \times GL(n_2, \mathbb{R}) \times GL(n_3, \mathbb{R})$

# Decomposition into irreducible reps

$G$: locally compact group

$\widehat{G}$ = unitary dual

$\quad = \{$irreducible unitary reps$\}/\sim$

Fact (Mautner '50–Teleman '76)

Any unitary rep $\pi$ can be decomposed into a direct integral of irreducible unitary reps:

$$\pi \simeq \underbrace{\int_{\widehat{G}}^{\oplus}}_{\text{unitary dual}} \underbrace{n(\tau)}_{\text{multiplicity}} \tau \underbrace{d\mu(\tau)}_{\text{Borel measure}}$$

$$n(\tau) \in \mathbb{N} \cup \{\infty\}$$

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$\rightsquigarrow$ strengthen

<u>Def</u> Restriction $\pi|_{G'}$ is $G'$-admissible if

$$\begin{cases} \pi \text{ is dicretely decomposable, and} \\ n_\pi(\tau) < \infty \quad ({}^\forall \tau \in \widehat{G'}). \quad \text{(finite mult.)} \end{cases}$$

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$G = Mp(n, \mathbb{R})$, $G' = G_1' \cdot G_2'$ : dual pair, $G_2'$ compact

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Ex. 4 (S. Martens, Jakobsen, Vergne, 1970s)
$\pi$ : holomorphic discrete series, $G'/K' \subset G/K$
Hermitian symm sp

# When is the restriction $G'$-admissible?

Strange example:

Ex. 5 (K– 1988) $\quad (G, G') = (SO(4,2),\ SO(4,1))$

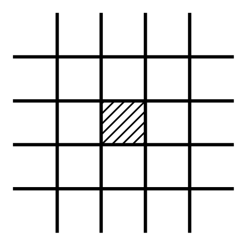
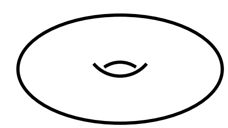
$\pi$: discrete series, Gelfand–Kirillov dim $= 5$

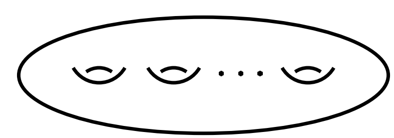
(neither holomorphic nor anti-holomorphic disc. ser.)

$\Longrightarrow \pi|_{G'}$ is $G'$-admissible

Idea: Tessellation of indefinite Kähler mfd $X$

$$X = SO(4,2)/U(2,1) \quad (\underset{\text{open}}{\subset} \mathbb{P}^3\mathbb{C})$$

Tessellation of $\mathbb{R}^2$  $\rightarrow$ 

Tessellation of $SL(2,\mathbb{R})/SO(2)$ $\rightarrow$ 

# Analytic approach

Let $\pi \in \widehat{G}$ and $G' \underset{\text{reductive}}{\subset} G$.

Question

When does the restriction $\pi|_{G'}$ become $G'$-admissible?

i.e. When does the restriction $\pi|_{G'}$ decompose discretely with finite multiplicities ?

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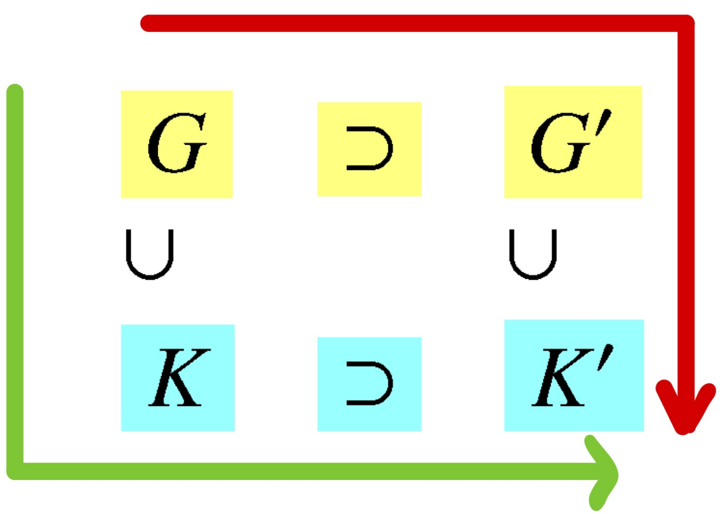


reductive groups $/\mathbb{R}$

max compact subgroups

# Two closed cones

$$G \quad \supset \quad K \quad \supset \quad T$$

max compact    max torus

Example

$$SL(n,\mathbb{R}) \supset SO(n) \supset \mathbb{T}^{\lfloor \frac{n}{2} \rfloor}.$$

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$\widehat{G} \ni \pi$

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$G \supset G'$

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Define two closed cones in $\sqrt{-1}\mathfrak{t}^*$:

$$\begin{array}{c} \widehat{G} \ni \pi \rightsquigarrow \ \mathrm{AS}_K(\pi) \\ \cap \\ \sqrt{-1}\mathfrak{t}^* \end{array}$$

asymptotic $K$-support (invariant of $\pi$)

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# Two closed cones

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$G \supset G'$ reductive groups $/\mathbb{R}$

$\cup \quad \cup$

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$$\cup$$

$$G \supset G' \rightsquigarrow \operatorname{C}_K(K')$$

Hamiltonian action $T \subset K \curvearrowright T^*(K/K')$

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Define two closed cones in $\sqrt{-1}\mathfrak{t}^*$:

| | |
|---|---|
| $\widehat{G} \ni \pi \rightsquigarrow \mathrm{AS}_K(\pi)$ | asymptotic $K$-support |
| $\cap$ | (invariant of $\pi$) |
| $\sqrt{-1}\mathfrak{t}^*$ | |
| $\cup$ | |
| $G \supset G' \rightsquigarrow \mathrm{C}_K(K')$ | momentum image |
| | $T^*(K/K') \to \sqrt{-1}\mathfrak{t}^*$ |

# Asymptotic cone

$$\mathbb{R}^n$$

$$\cup$$

$$S$$

# Asymptotic cone

$$\begin{array}{cccl} \mathbb{R}^n & & \mathbb{R}^n & \\ \cup & & \cup & \text{closed cone} \\ S & \Longrightarrow & S\infty & \text{(asymptotic cone)} \end{array}$$

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$$S\infty := \{y \in \mathbb{R}^n : {}^{\exists}y_j \in S,\ {}^{\exists}\varepsilon_j \downarrow 0 \text{ s.t. } \lim_{j\to\infty} \varepsilon_j y_j = y\}$$

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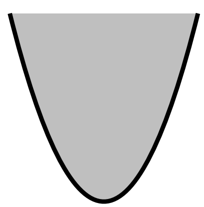
$S$ $\qquad\qquad\qquad\qquad$ $S\infty$



$y > x^2$

$\Longrightarrow$

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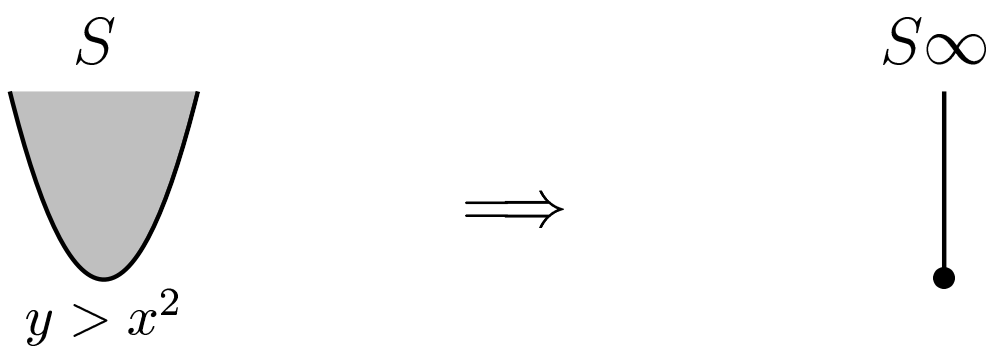

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closed cone

(asymptotic cone)

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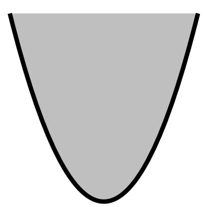

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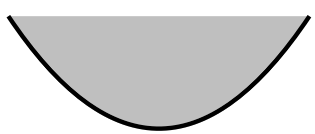
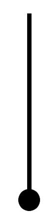


$y^2 - x^2 > 1,\ y > 0$



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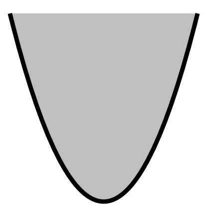

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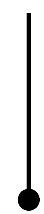
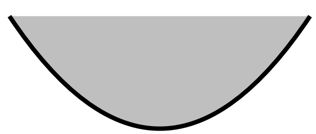
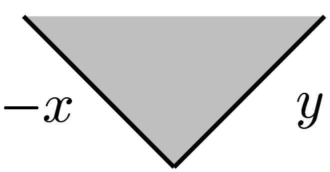




$y^2 - x^2 > 1,\ y > 0$

$\Longrightarrow$

$y = -x$



$y = x$

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$$
\begin{array}{ccccc}
\widehat{K} & \simeq & \Lambda_+ := \widehat{T} \cap & C_+ & \subset \sqrt{-1}\mathfrak{t}^* \\
\cup\!\!\!\!\cdot & & \cup\!\!\!\!\cdot & & \\
\tau_\lambda & \longleftarrow\!\!\!\mid & \lambda & &
\end{array}
$$

dominant chamber

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$G \supset K \supset T$

max compact, max torus

$\pi$ rep of $G$

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$$\rightsquigarrow \operatorname{Supp}_K(\pi) := \{\lambda \in \sqrt{-1}\mathfrak{t}^* : \operatorname{Hom}_K(\tau_\lambda, \pi|_K) \neq 0\}$$

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$\rightsquigarrow \mathrm{AS}_K(\pi) := \mathrm{Supp}_K(\pi)\infty$ (Asymptotic $K$-support)

# Example of $SL(2, \mathbb{R})$ = G

$\hat{K} \simeq \mathbb{Z}$ for $K = SO(2)$

| $\pi$ | $\mathrm{Supp}_K(\pi)$ | $\mathrm{AS}_K(\pi)$ | Series |
|---|---|---|---|
| **1** | | | trivial rep. |
| $\pi_\lambda$ | | | principal ser. |
| $\pi_n^+$ | | | holo. discrete ser. |
| $\pi_n^-$ | | | anti-holo discrete ser. |

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| $\pi_\lambda$ | $2\mathbb{Z}$ | $\mathbb{R}$ | principal ser. |
| $\pi_n^+$ | $\{n, n+2, n+4, \cdots\}$ | $\mathbb{R}_{\geq 0}$ | holo. discrete ser. |
| $\pi_n^-$ | $\{-n, -n-2, -n-4, \cdots\}$ | $\mathbb{R}_{\leq 0}$ | anti-holo discrete ser. |

# Criterion of admissible restriction

Theorem 2 (Criterion) (K– Ann Math '98 )

Let $G' \underset{\text{reductive}}{\subset} G$ and $\pi \in \widehat{G}$. If

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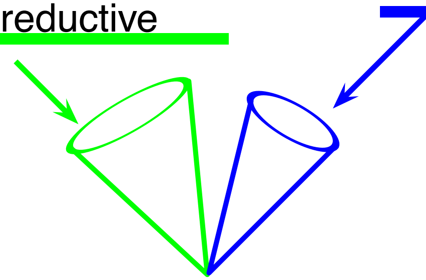

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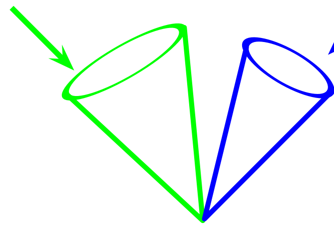

$\mathbb{R}^n$ || 

$(\star)$ $\qquad \mathrm{C}_K(K') \cap \mathrm{AS}_K(\pi) = \{0\}$ in $\sqrt{-1}\mathfrak{t}^*$,

then $\pi|_{K'}$ is $K'$-admissible.

| | | | |
|---|---|---|---|
| $G$ | $\supset$ | $G'$ | reductive groups $/\mathbb{R}$ |
| $\cup$ | | $\cup$ | |
| $K$ | $\supset$ | $K'$ | max compact subgroups |

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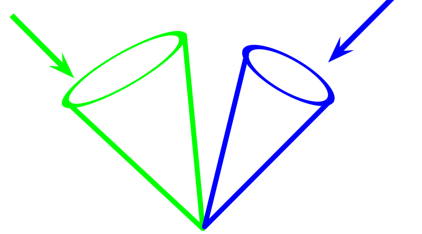

(⋆) $\quad \mathrm{C}_K(K') \cap \mathrm{AS}_K(\pi) = \{0\} \quad$ in $\sqrt{-1}\mathfrak{t}^* = \mathbb{R}^n$,

then $\pi|_{K'}$ is $K'$-admissible.

The converse also holds.

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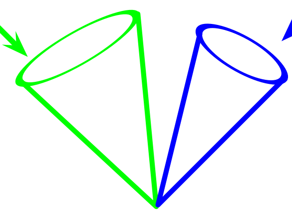


$\mathbb{R}^n$
||

($\star$) $\quad C_K(K') \cap AS_K(\pi) = \{0\}$ in $\sqrt{-1}\mathfrak{t}^*$,

then $\pi|_{K'}$ is $K'$-admissible.

The converse also holds.

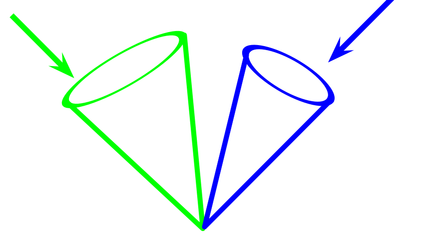
In particular, the restriction $\pi|_{G'}$ is $G'$-admissible

| $G$ | $\supset$ | $G'$ | reductive groups $/\mathbb{R}$ |
|---|---|---|---|
| $\cup$ | | $\cup$ | |
| $K$ | $\supset$ | $K'$ | max compact subgroups |

# Criterion of admissible restriction

Theorem 2 (Criterion) (K– Ann Math '98, Progr Math '05 )

Let $G' \underset{\text{reductive}}{\subset} G$ and $\pi \in \widehat{G}$. If



$(\star) \qquad \mathrm{C}_K(K') \cap \mathrm{AS}_K(\pi) = \{0\} \quad$ in $\sqrt{-1}\mathfrak{t}^* \; (= \mathbb{R}^n)$,

then $\pi|_{K'}$ is $K'$-admissible.

The converse also holds.

In particular, the restriction $\pi|_{G'}$ is $G'$-admissible

No continuous spectrum & finite multiplicity

in the irred. decomp. of $\pi|_{G'}$.

# Special cases of Theorem 2

Ex.1 $\quad \mathrm{C}_K(K') = \{0\} \Longleftrightarrow G' \supset K$

$\Longrightarrow$ Harish-Chandra's admissibility thm

Ex.2 $\quad \mathrm{AS}_K(\pi) = \{0\} \Longleftrightarrow \dim \pi < \infty$

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Ex.3 (S. Martens, Jakobsen, Vergne, 1970s)

$\pi$: holomorphic discrete series

$G'/K' \subset G/K$ holomorphic embedding

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Ex.3 (Jakobsen–Vergne 1970)

$\pi$: holomorphic discrete series

$G'/K' \subset G/K$ holomorphic embedding

Ex.4 $\pi = A_{\mathfrak{q}}(\lambda)$ (Zuckerman's module)

(may be non-tempered/non-highest wt module)

$\Longrightarrow$ $\mathrm{AS}_K(\pi) \subset \mathbb{R}_+$-span of $\Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t})$

$(\mathfrak{q} = \mathfrak{l} + \mathfrak{u}, \ \ \mathfrak{g} = \mathfrak{k} + \mathfrak{p})$

# Idea of Proof

Restriction of a **representation** $\pi$ to a subgroup $G'$

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$\Uparrow$

Restriction of its **character** Trace $\pi$ to a submfd

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$\Uparrow$ wave front set/singularity spectrum of Trace $\pi$

$\Uparrow$

$\mathrm{AS}_K(\pi)$

Restriction of a **holomorphic function** to a complex submfd

# Example of $SL(2,\mathbb{R})$

| $\pi$ | $\mathrm{Supp}_K(\pi)$ | $\mathrm{AS}_K(\pi)$ | $\mathrm{Trace}\,\pi = \sum_{k \in \mathrm{Supp}_K(\pi)} z^k$ |
|---|---|---|---|
| **1** | | | |
| $\pi_\lambda$ | | | |
| $\pi_n^+$ | | | |
| $\pi_n^-$ | | | |

# Example of $SL(2, \mathbb{R})$

$\hat{K} \simeq \mathbb{Z}$ for $K = SO(2)$

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|---|---|---|---|
| **1** | $\{0\}$ | | |
| $\pi_\lambda$ | $2\mathbb{Z}$ | | |
| $\pi_n^+$ | $\{n, n+2, n+4, \cdots\}$ | | |
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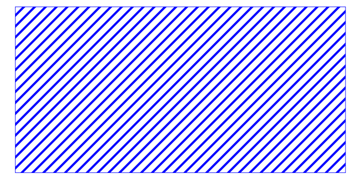
| **1** | $\{0\}$ | $\{0\}$ |  |
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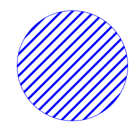
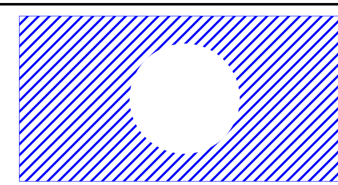
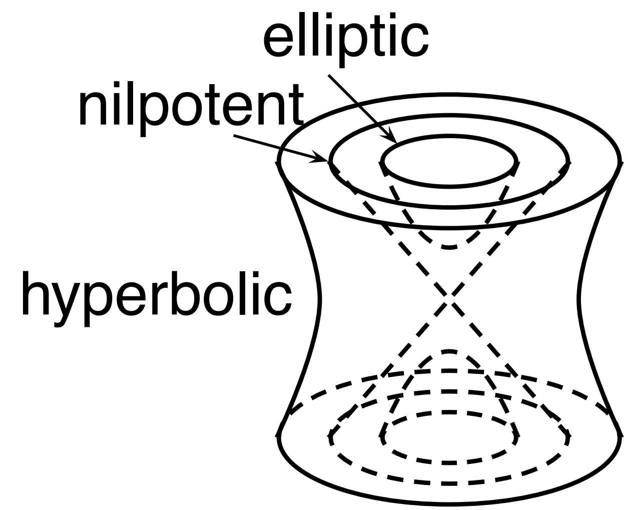
| $\pi_n^+$ | $\{n, n+2, n+4, \cdots\}$ | $\mathbb{R}_{\geq 0}$ |  |
| $\pi_n^-$ | $\{-n, -n-2, -n-4, \cdots\}$ | $\mathbb{R}_{\leq 0}$ |  |

# Geometric quantization of elliptic orbits

$$\underset{\text{reductive Lie gp}}{G} \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \simeq \mathfrak{g}$$



# Geometric quantization of elliptic orbits

$\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ elliptic orbit

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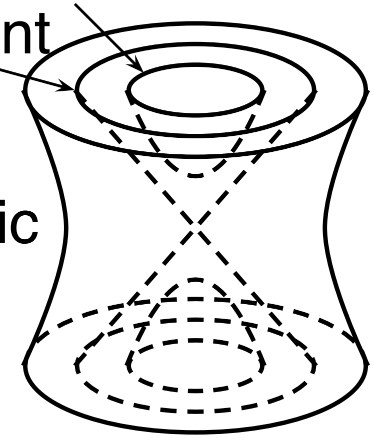
reductive
Lie gp

$\rightsquigarrow$




$\pi_\lambda$ unitary rep of $G$

(Vogan, Wallach '84)

Geometric aspect of Zuckerman's derived functor modules

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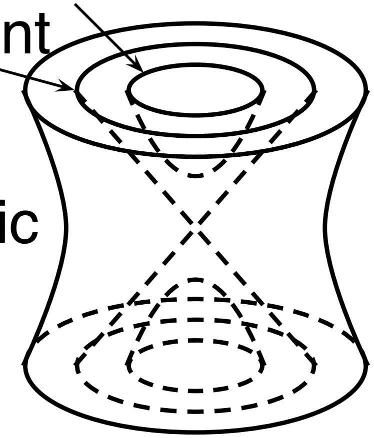
$\mathcal{L}_\lambda \to \mathcal{O}_\lambda$ $G$-equiv. holo. line b'dle

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maximal globalization
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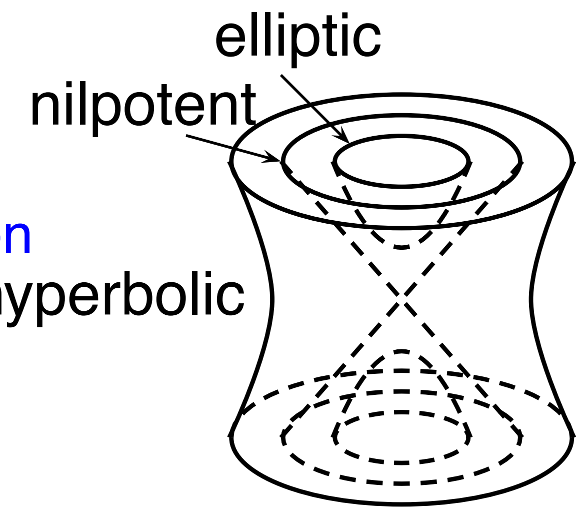


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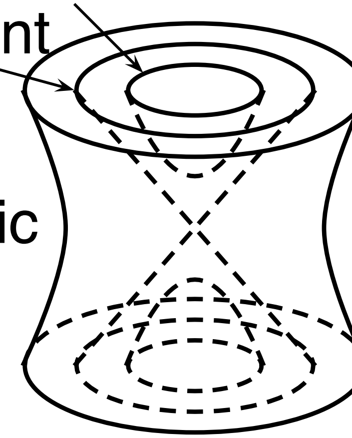


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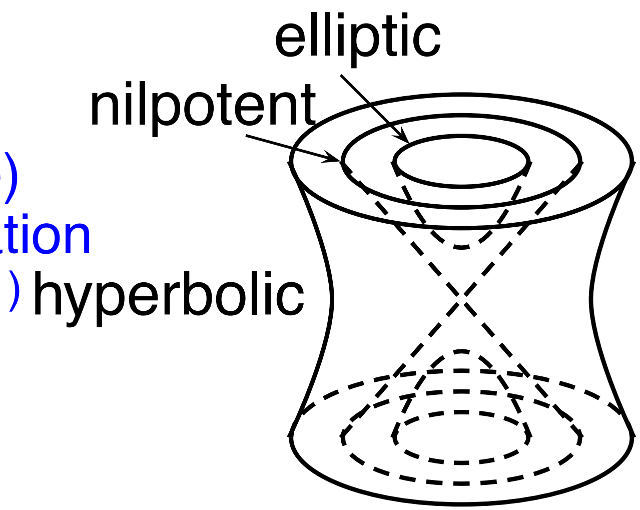
$A_\mathfrak{q}(\lambda)$




Geometric aspect of Zuckerman's derived functor modules

# Geometric quantization of elliptic orbits

$\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ elliptic orbit

$\Downarrow$ complex structure

$\mathcal{L}_{\lambda+\rho} \to \mathcal{O}_\lambda$ $G$-equiv. holo. line b'dle

$\Downarrow$

$H^*_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_{\lambda+\rho})$ $G$-module (Fréchet space)

maximal globalization

(Schmid, Wong '91)

$\cup$ dense

$\exists 1 \, \pi_\lambda$ unitary rep of $G$

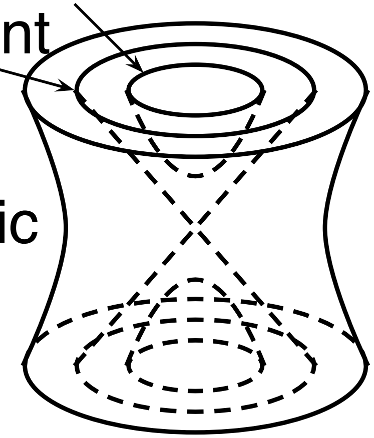
(Vogan, Wallach '84)

$\cup$ dense

$A_{\mathfrak{q}}(\lambda - \rho)$

$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^* \simeq \mathfrak{g}$

reductive Lie gp




Geometric aspect of Zuckerman's derived functor modules

# Basic properties of $\pi_\lambda$

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$G_\lambda = \{g : \mathrm{Ad}^*(g)\lambda = \lambda\}$

| | | |
|---|---|---|
| $G$ compact | $\cdots$ | Borel–Weil–Bott construction |
| $G_\lambda$ compact torus | $\cdots$ | $\pi_\lambda =$ discrete series |
| $G_\lambda$ abelian | $\cdots$ | $\pi_\lambda =$ fundamental series |

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Geometry

$$\mathfrak{g}^* \ni \lambda \rightsquigarrow Q \underset{\text{parabolic}}{\subset} G_{\mathbb{C}}$$

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$$G_{\mathbb{C}}/Q \underset{\text{closed}}{\supset} K_{\mathbb{C}} \cdot o$$

$$\underset{\text{open}}{\cup} \quad G \cdot o$$

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$G_{\mathbb{C}}/Q \underset{\text{closed}}{\supset} K_{\mathbb{C}}\cdot o \rightsquigarrow \mathcal{D}$-module $i_+(\mathcal{O}_{K_{\mathbb{C}}\cdot o}(\lambda))$

$\updownarrow$ Matsuki duality $\updownarrow$ HMSW duality

$\underset{\text{open}}{\supset}$ $G\cdot o \simeq \mathcal{O}_\lambda \rightsquigarrow H^S_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_{\lambda+\rho})$

geometric quantization

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$\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\cdot\lambda$ elliptic orbit $\rightsquigarrow$ $\pi_\lambda$ unitary rep of $G$

Geometry

$\mathfrak{g}^* \ni \lambda \rightsquigarrow Q \subset G_{\mathbb{C}}$ (parabolic)

$G_{\mathbb{C}}/Q \supset K_{\mathbb{C}}\cdot o$ (closed) $\rightsquigarrow$ $\mathcal{D}$-module $i_+(\mathcal{O}_{K_{\mathbb{C}}\cdot o}(\lambda))$

$\updownarrow$ Matsuki duality $\qquad \updownarrow$ HMSW duality

$G_{\mathbb{C}}/Q \supset G\cdot o \simeq \mathcal{O}_\lambda$ (open) $\rightsquigarrow$ $H^S_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_{\lambda+\rho})$

geometric quantization

$\updownarrow$

$A_{\mathfrak{q}}(\lambda)$

algebraic construction
(Zuckerman '76–'77)

# Reductive symmetric pair

$(G, G')$: reductive symm. pair

$$\sigma^2 = \mathrm{id}, \quad G_0^\sigma \subset G' \subset G^\sigma$$

$$\begin{array}{ccccc}
\mathfrak{g} & = & \mathfrak{g}^\sigma & + & \mathfrak{g}^{-\sigma} \\
\cup & & \cup & & \cup \\
\mathfrak{k} & = & \mathfrak{k}^\sigma & + & \mathfrak{k}^{-\sigma} \\
\cup & & \cup & & \cup \ \text{maximal} \\
\mathfrak{t} & = & \mathfrak{t}^\sigma & + & \mathfrak{t}^{-\sigma}
\end{array}$$

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\end{array}$$

$$\Sigma^+(\mathfrak{k}, \mathfrak{t}^{-\sigma}) \cup \{0\} \overset{\text{rest.}}{\longleftarrow} \Delta^+(\mathfrak{k}, \mathfrak{t}) \supset \Delta(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{t})$$

# Discrete decomposability of $\pi_\lambda$

$\mathcal{O}_\lambda = \mathrm{Ad}^*(G) \cdot \lambda$ elliptic orbit $\rightsquigarrow \pi_\lambda$ unitary rep of $G$

$$\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u} \underset{\theta\text{-stable parabolic}}{\subset} \mathfrak{g}_{\mathbb{C}}$$

$(G, G')$: reductive symmetric pair $\Leftarrow \sigma$

$$\mathfrak{g} \supset \mathfrak{k} \supset \mathfrak{t}$$

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$\theta$-stable parabolic

$(G, G')$: reductive symmetric pair $\Leftarrow \sigma$

$$\mathfrak{g} \supset \mathfrak{k} \supset \mathfrak{t}$$

Theorem* Equivalent:

(1) $\pi_\lambda$ is $K'$-admissible.

(2) $\pi_\lambda|_{G'}$ is alg. discretely decomposable.

(3) $\mathbb{R}_+$-span $\Delta(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}) \cap \sqrt{-1}(\mathfrak{t}^*)^{-\sigma} = \{0\}$.

(4) $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}})) \subset \mathcal{N}^*_{\mathfrak{g}'}$.

*TK, Invent Math (1998)

# Approaches to admissible restrictions

Analytic Approaches*

- Wavefront set / singularity spectrum of characters.

Algebraic Approaches**

- $\mathrm{gr}\, U(\mathfrak{g}_{\mathbb{C}}) \qquad \simeq \qquad S(\mathfrak{g}_{\mathbb{C}})$
  (nonn-commutative $\rightsquigarrow$ commutative).

Geometric Approaches

- Complex geometry, Orbit philosophy†.
- Symplectic geometry, $\mathcal{D}$-modules††.

* TK, Ann Math (1998);

** TK, Invent Math (1998); Gross–Wallach (2000);

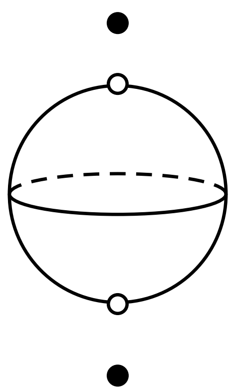
† TK, Invent Math (1994); Duflo–Vargas, Proc. Japan Acad (2010);

†† TK, Kostant Memorial Volume (2011); Y. Oshima, Progr. Math (2025); M. Kitagawa, Progr. Math. (2025)

# Beilinson–Bernstein localization

$$G = U(2,2)$$
$$\cup$$
$$K = U(2) \times U(2)$$

# $GL(1,\mathbb{C}) \times GL(1,\mathbb{C}) \curvearrowright GL(2,\mathbb{C})/B$



$\bullet \quad = X$

$= U$

$\bullet \quad = Y$

$\mathbb{P}^1\mathbb{C} \simeq GL(2,\mathbb{C})/B$

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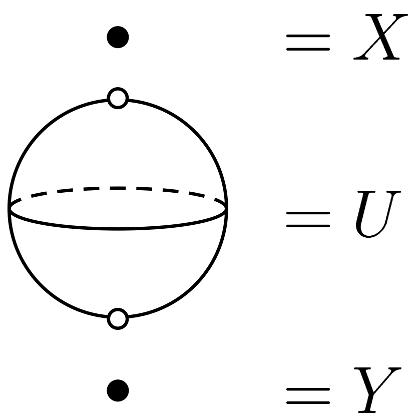
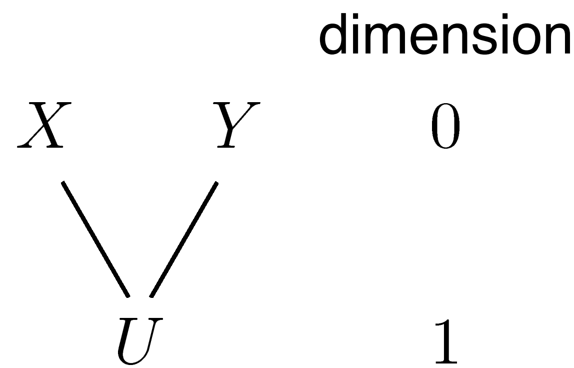
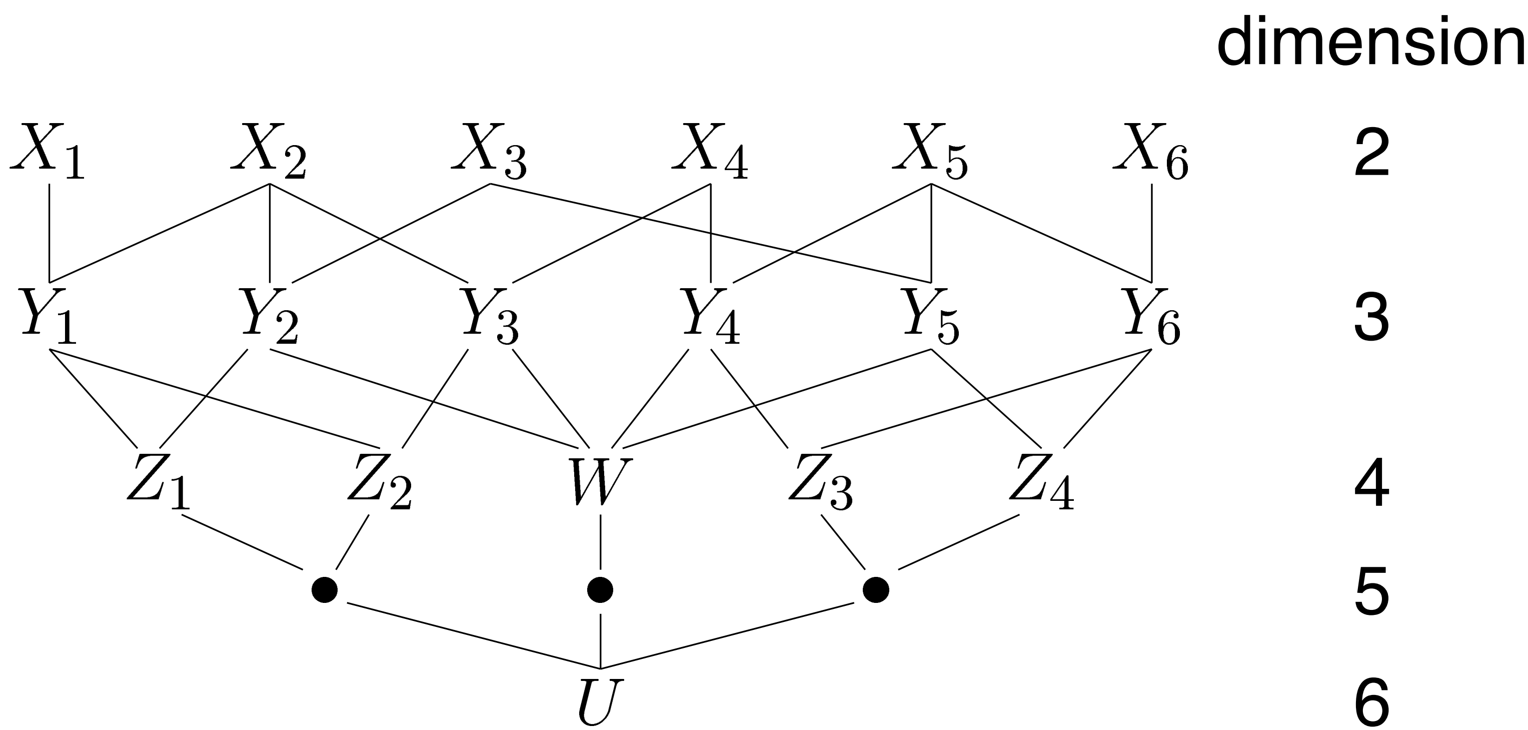



$\rightsquigarrow$

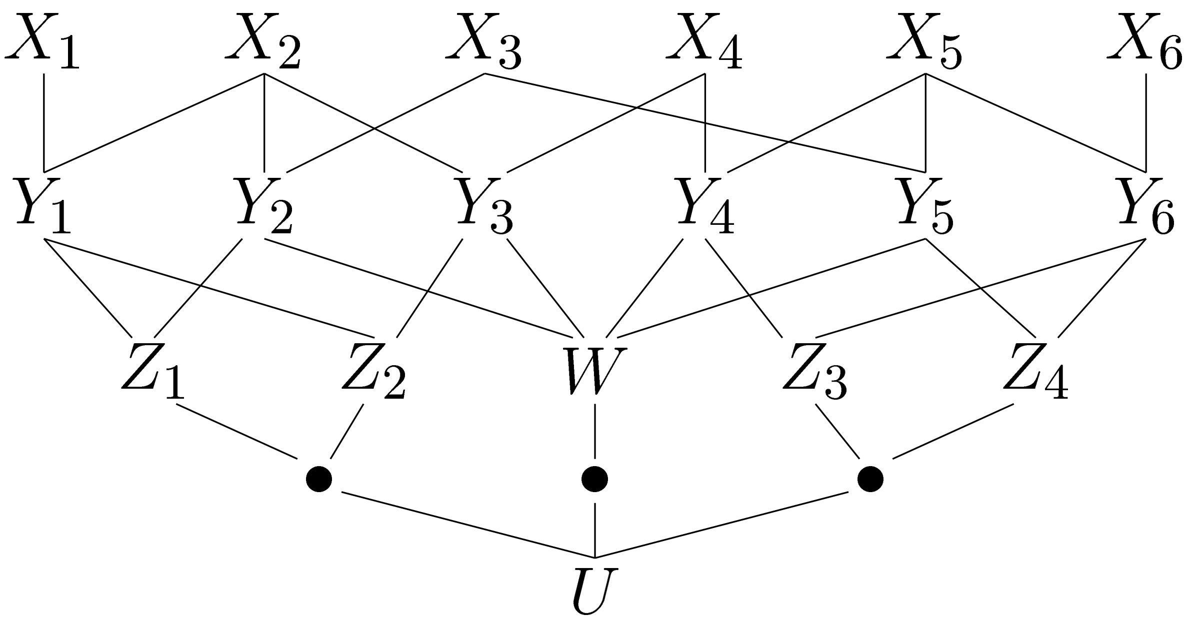



$$\mathbb{P}^1\mathbb{C} \simeq GL(2,\mathbb{C})/B$$

# $GL(2,\mathbb{C}) \times GL(2,\mathbb{C}) \curvearrowright GL(4,\mathbb{C})/B$



# $K_{\mathbb{C}} \curvearrowright G_{\mathbb{C}}/B \Leftarrow$ **reps of** $G = U(2,2)$



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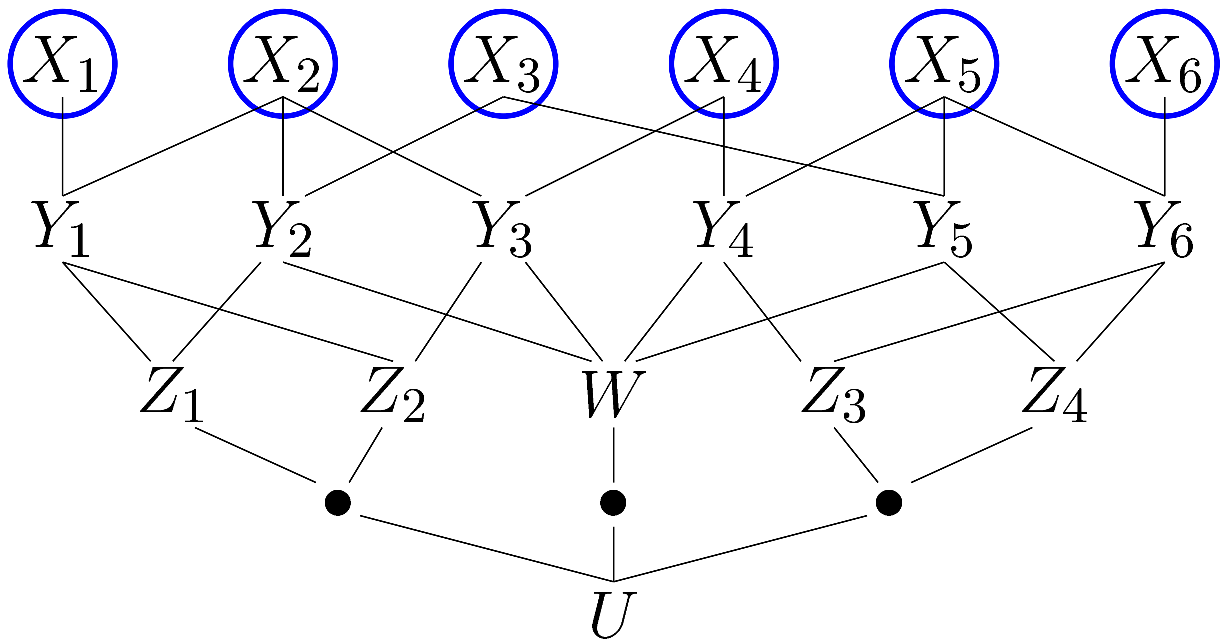


discrete ser.

trivial rep

○ discrete series for $L^2(G)$

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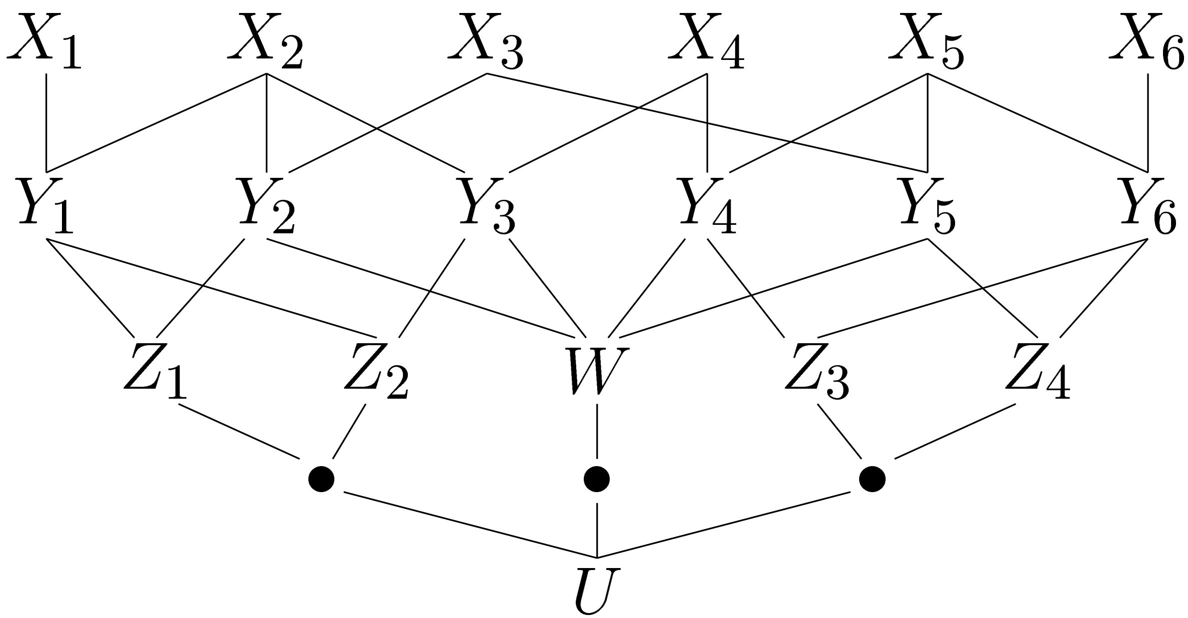

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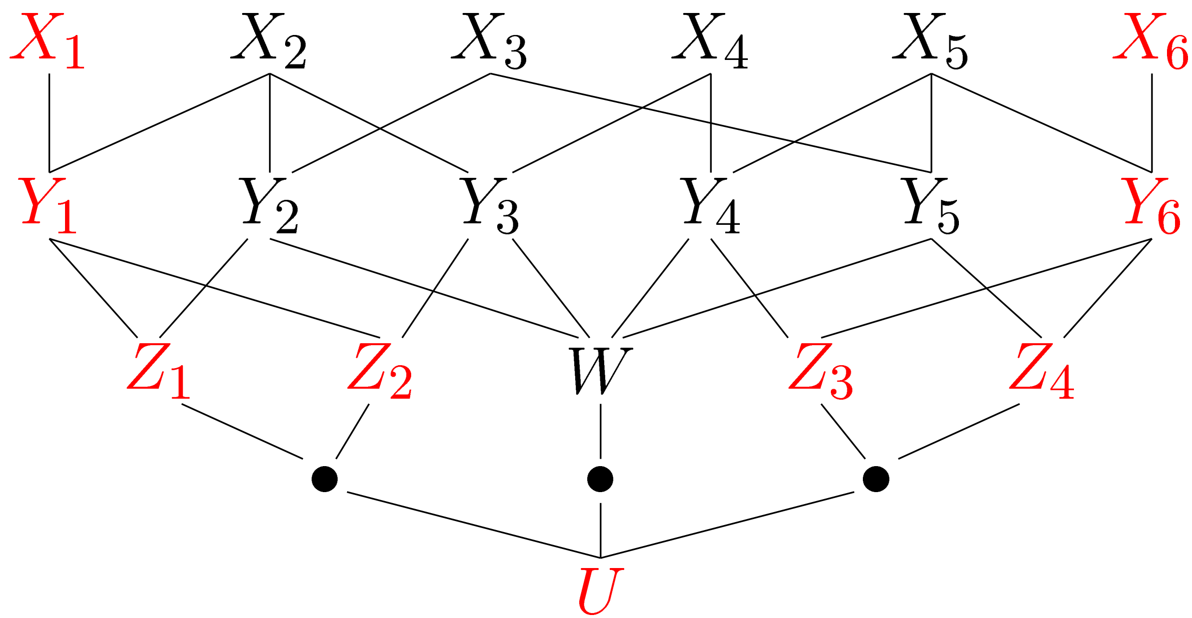
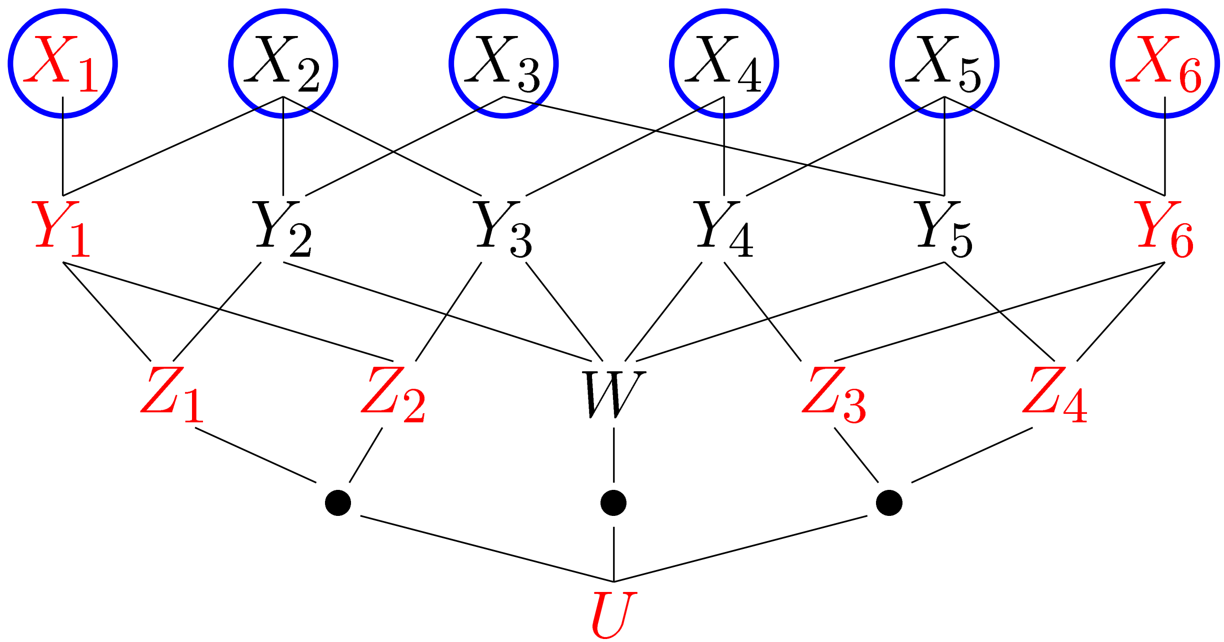


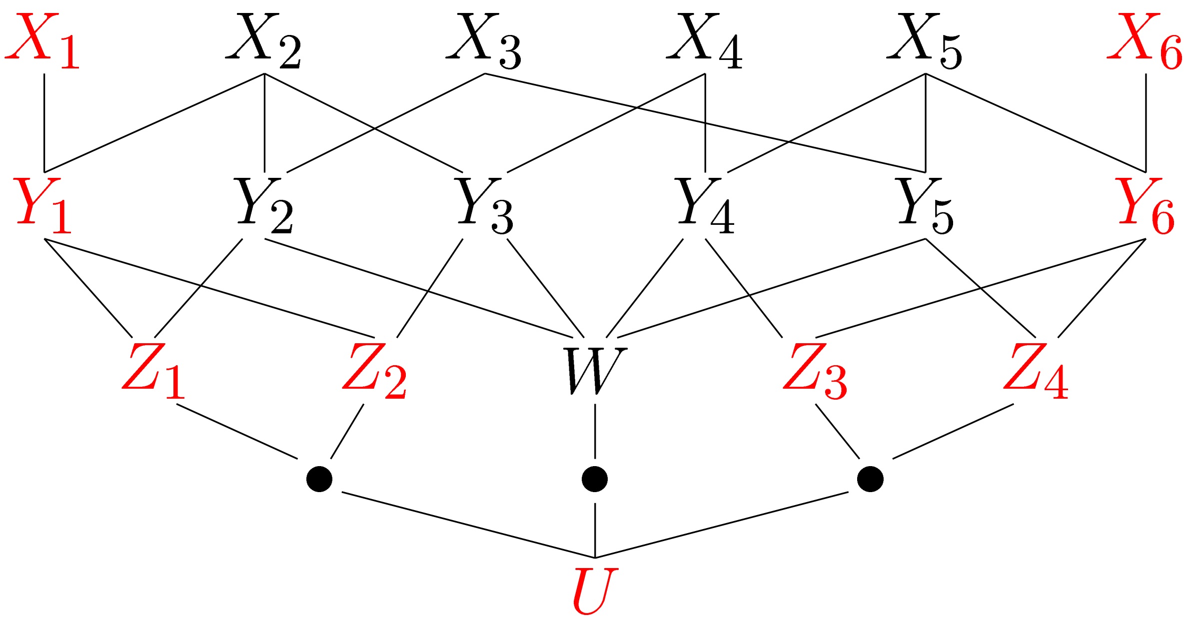

- unitary highest weight modules

# $K_{\mathbb{C}} \curvearrowright G_{\mathbb{C}}/B \Rightarrow$ reps of $G = U(2,2)$




- ○ discrete series representations
- • unitary highest weight modules
- ⊙ holomorphic (anti-holomorphic) discrete series

# $U(2,2) \downarrow U(2,1) \times U(1)$

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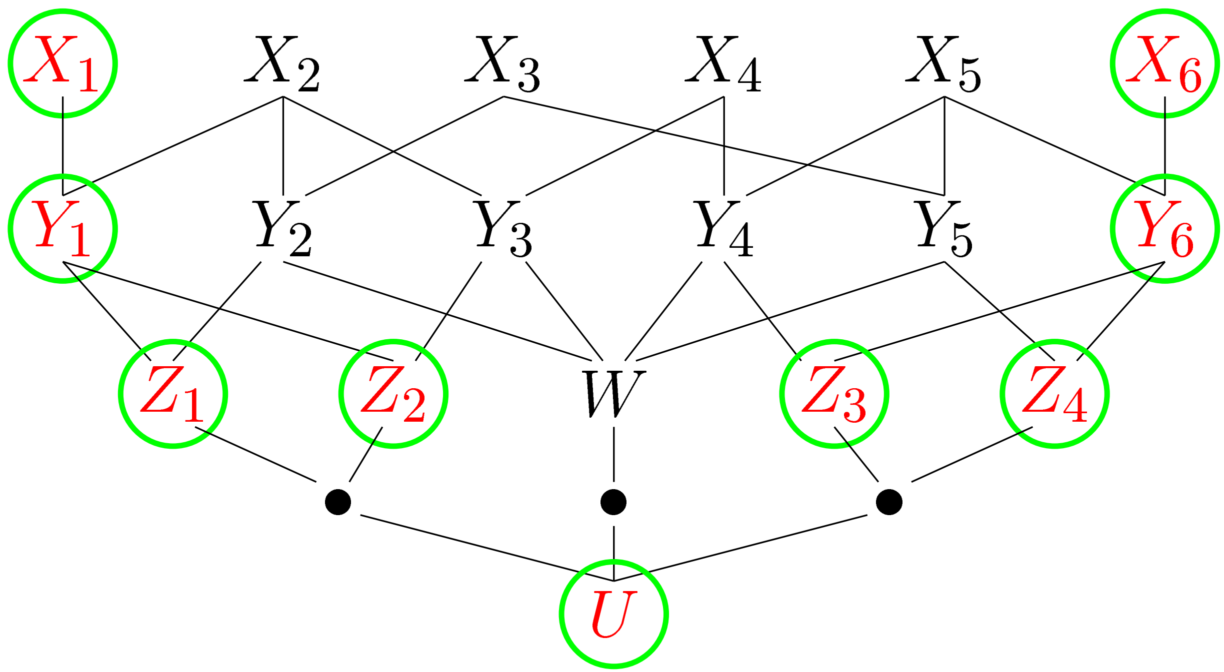
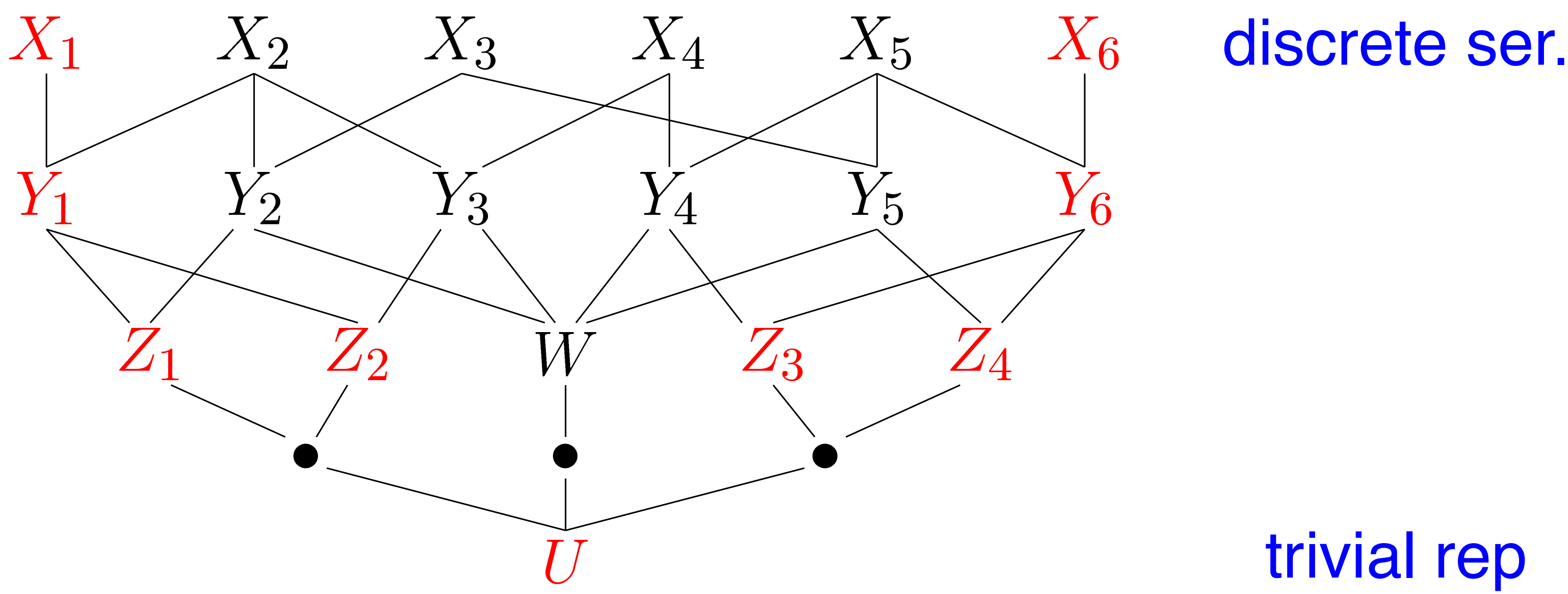



○ restriction to $U(2,1) \times U(1)$ is admissible

- unitary highest weight modules

# $U(2,2) \downarrow Sp(1,1)$ ($\simeq Spin(4,1)$)

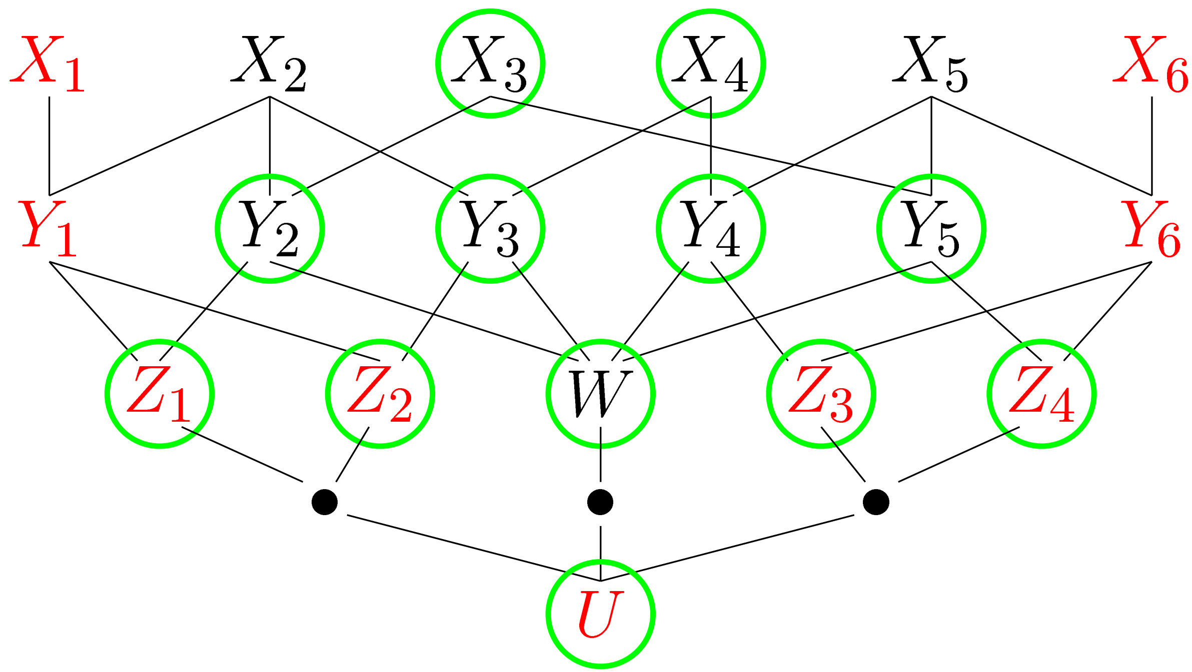
$\sim$ ($SO(4,2) \downarrow SO(4,1)$ up to covering, abelian factor)




Note • unitary highest weight modules

# $U(2,2) \downarrow Sp(1,1)$

~ ($SO(4,2) \downarrow SO(4,1)$ up to covering, abelian factor)



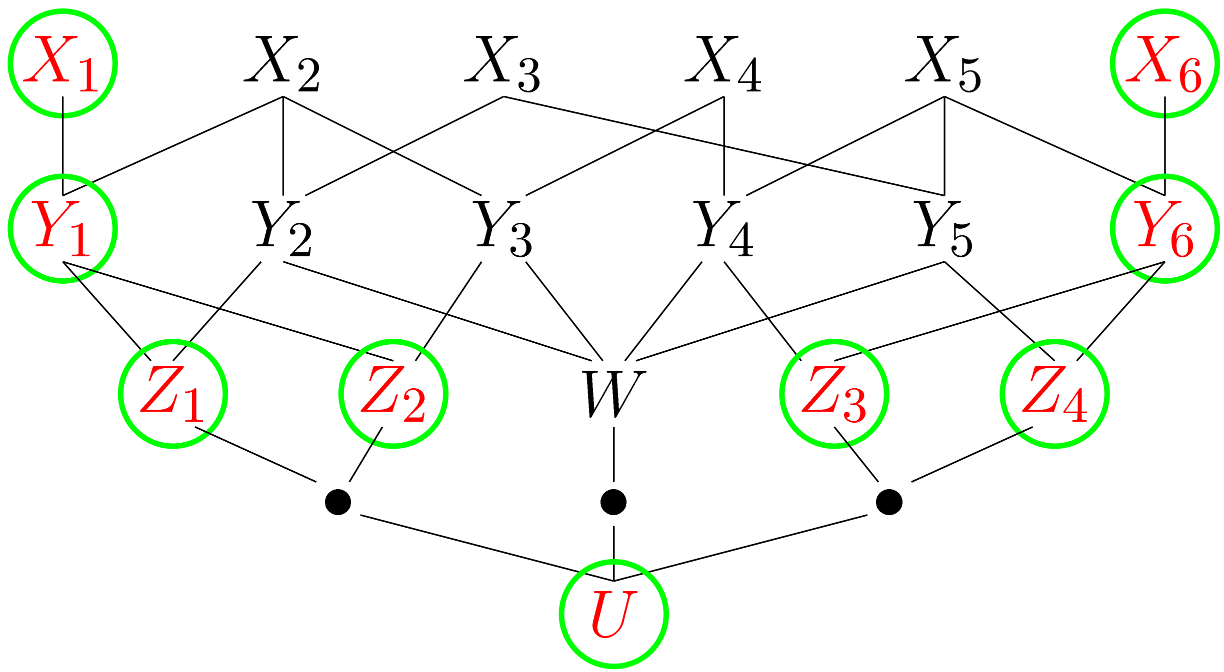

discrete ser.

trivial rep

○ restriction to $Sp(1,1)$ $(\simeq Spin(4,1))$ is admissible

Note • unitary highest weight modules

# $U(2,2) \downarrow U(2,1) \times U(1)$




discrete ser.

trivial rep

○ restriction to $U(2,1) \times U(1)$ is admissible

- unitary highest weight modules

## Admissible restriction $\Pi|_{G'}$ — classification

Theorem (+ algebraic criterion*) provides a family of the triples

$$\Pi \in \widehat{G} \quad \text{and} \quad G \supset G'$$

for which the restriction $\Pi|_{G'}$ is $G'$-admissible (discretely decomposable with finite multiplicity).

Some classification results**,***

- (K–Y. Oshima, 2012) $G \supset G'$ symmetric pair, $\Pi_K = A_{\mathfrak{q}}(\lambda)$,
  *e.g.*, $\Pi =$ Harish-Chandra's discrete series rep.
- ( — , 2015) $G \supset G'$ symmetric pair, $\Pi =$ minimal rep,
- ( — , 2015) $\Pi_1 \otimes \Pi_2$ for any $\Pi_1, \Pi_2$,
- (Duflo–Galina–Vargas, 2017) $G' = SL(2, \mathbb{R})$, $\Pi =$ discrete series.

New geometric examples (will be discussed next week)
$\cdots$ arisen from locally symmetric spaces $\Gamma \backslash G/H$.

* T. Kobayashi, "Discrete decomposability of the restriction$\cdots$ III" Invent. Math. (1998);

** Kobayashi–Y. Oshima, "Classification of $\cdots$" Adv Math (2012) 2013–2047; Crelles (2015) 201–223;

*** M. Duflo–E. Galina–J. Vargas, J. Lie Theory (2017), 1033–1056.

Thank you very much!

# Branching in Representation Theory

# Mini Courses (January 13-17, IHP, 2025)

## Branching in Representation Theory

## References for Lecture 2:

## Discrete Decomposability and Admissible Restriction

The general theory of the main results are in

T.Kobayashi, Invent Math 1994, Annals of Mathematics, 1998, Invent Math 1998.

Duflo-Vargas, Proc. Japan Academy, 2010.

T.Kobayashi, Pure and Applied Mathematics Quarterly, 2021. (special issue: in memory of Prof. Bertram Kostant).

Classification theory for admissible restrictions

T.Kobayashi-Y.Oshima, Adv Math 2013, Crelles 2015.

Surveys, see also references thereis:

T. Kobayashi. Advanced Study in Pure Mathematics vol. 26, pages 98-126, 2000.

T. Kobayashi. Recent advances in branching problems of representations. Sugaku Expositions 37 (2024), 129-177, Amer Math Soc.

Further Readings, M.Kitagawa, Y.Oshima In: Symmetry in Geometry and Analysis, Volumes 3, Progress in Mathematics, Birkhäuser, 2025.