Branching in Representation Theory
Lecture 1

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Minicourses: branching problems and symmetry-breaking

Institut Henri Poincaré, Paris, France, 13–17 January 2025

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Branching laws — examples in the finite-dim'l case

$$GL(n,\mathbb{C}) \curvearrowright \mathbb{C}^n$$

$$\rightsquigarrow$$

$$GL(n,\mathbb{C}) \curvearrowright S^k(\mathbb{C}^n) \quad (k = 0, 1, 2, \ldots)$$

irreducible representation

Branching laws — examples in the finite-dim'l case

$$
\begin{array}{ccc}
GL(n,\mathbb{C}) & \curvearrowright & \mathbb{C}^n \\
\rightsquigarrow & & \\
SL(n,\mathbb{C}) \subset GL(n,\mathbb{C}) & \curvearrowright & S^k(\mathbb{C}^n) \quad (k = 0, 1, 2, \dots)
\end{array}
$$

irreducible representation

Branching laws — examples in the finite-dim'l case

- Tensor product $\pi_1 \otimes \pi_2$ for $G = SL(2)$

- Restriction $\pi|_{G'}$ for $(G, G') = (GL(3),\ GL(2) \times GL(1))$

Branching laws — examples in the finite-dim'l case

- Tensor product $\pi_1 \otimes \pi_2$ for $G = SL(2)$

A special case of Clebsch–Gordan formula:

$$SL(2) \times SL(2) \supset SL(2)$$

$$S^3(\mathbb{C}^2) \otimes S^4(\mathbb{C}^2) \simeq S^7(\mathbb{C}^2) + S^5(\mathbb{C}^2) + S^3(\mathbb{C}^2) + S^1(\mathbb{C}^2)$$

- Restriction $\pi|_{G'}$ for $(G, G') = (GL(3),\ GL(2) \times GL(1))$

A special case of Littlewood–Richardson's rule:

$$GL(3) \supset GL(2) \times GL(1)$$

$$S^3(\mathbb{C}^3) \simeq S^3(\mathbb{C}^2) + S^2(\mathbb{C}^2) + S^1(\mathbb{C}^2) + S^0(\mathbb{C}^2)$$

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Dimension $4 \times 5 = 8 + 6 + 4 + 2$

- Restriction $\pi|_{G'}$ for $(G, G') = (GL(3), \ GL(2) \times GL(1))$

A special case of Littlewood–Richardson's rule:

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$$S^3(\mathbb{C}^3) \simeq S^3(\mathbb{C}^2) + S^2(\mathbb{C}^2) + S^1(\mathbb{C}^2) + S^0(\mathbb{C}^2)$$

Dimension $10 = 4 + 3 + 2 + 1$

$$\langle x^3, x^2y, xy^2, y^3 \rangle \quad \langle x^2, xy, y^2 \rangle \otimes z \quad \langle x, y \rangle \otimes z^2 \quad z^3$$

Branching problems in the general setting

$$G \xrightarrow[\text{irreducible}]{\pi} GL(V)$$

$$\cup$$

$$G' \dashrightarrow \pi|_{G'}$$

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$$G' \quad \dashrightarrow \quad \pi|_{G'}$$

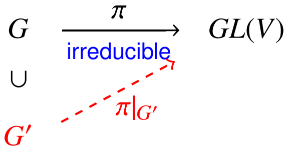
Example (tensor product of two representations)

$$G_1 \times G_1 \xrightarrow[\text{outer tensor product}]{\pi' \boxtimes \pi''} GL(V' \otimes V'')$$

$$\cup$$

$$\operatorname{diag} G_1 \quad \dashrightarrow \quad \pi' \otimes \pi''$$

Branching problems in the general setting



- Branching law = Irreducible decomposition of $\pi|_{G'}$.

  Fusion rule is the special case of the branching law, that is, the irreducible decomposition of the tensor product rep $\pi' \otimes \pi''$.

  - There exists an "algorithm" when G is a compact Lie group.
  - Challenging when G is non-compact.

Branching law of unitary representations

$\widehat{G}$:= {irreducible unitary representations of G} (unitary dual).

Mautner: Any unitary rep Π of a locally compact group can be disintegrated into irreducibles.

$$\Pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi \pi \, d\mu(\pi) \qquad \text{(direct integral)}$$

$$m \colon \widehat{G} \to \mathbb{N} \cup \{\infty\}, \quad \pi \mapsto m_\pi \quad \text{(multiplicity).}$$

$$m_\pi \pi = \underbrace{\pi \oplus \cdots \oplus \pi}_{m_\pi}$$

Branching Law (unitary case)

For $G \supset G'$ and $\Pi \in \widehat{G}$,

$$\Pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} m_\pi \pi \, d\mu(\pi) \quad \text{(direct integral)}$$

Branching problems in the general setting

$$G \xrightarrow[\text{irreducible}]{\pi} GL(V)$$

$$\cup$$

$$G' \quad \pi|_{G'}$$

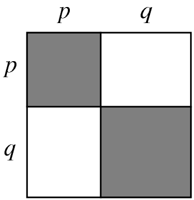
Branching problem (in a broader sense than the usual)

$\cdots$ wish to understand
how the restriction $\pi|_{G'}$ behaves as a G'-module.

Nice and bad features in the infinite dim'l rep

- Tensor product $\pi_1 \otimes \pi_2$ for $G = SL(n, \mathbb{R})$

- Restriction $\pi|_{G'}$ for $(G, G') = (GL(p+q, \mathbb{R}),\ GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$



Nice and bad features in the infinite dim'l rep

'Multiplicities'
= the number of times that the same irreducible reps
occur in the decomposition
(to be precise, later)

Nice and bad features in the infinite dim'l rep

- Tensor product $\pi_1 \otimes \pi_2$ for $G = SL(n, \mathbb{R})$

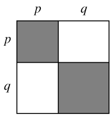
  'Multiplicities' of irreducible unitary reps in the decomposition

  Nice case $\cdots$ $n = 2$
  (concrete formula: Pukánszky '61, Williams, Repka '78)
  at most 2 for any $\pi_1, \pi_2 \in \widehat{G}$

- Restriction $\pi|_{G'}$ for $(G, G') = (GL(p+q, \mathbb{R}),\ GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$

  'Multiplicities' of irreducible reps for the restriction

  Nice case $\cdots$ $q = 1$
  (abstract results: K–T. Oshima; Sun–Zhu 2012)
  uniformly bounded, at most 1 for any $\pi \in \widehat{G}_{\mathrm{adm}}$



Nice and bad features in the infinite dim'l rep

- Tensor product $\pi_1 \otimes \pi_2$ for $G = SL(n, \mathbb{R})$

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  Bad case $\cdots$ $n \geq 3$ (K– '86)
  ∞ or 0 for any tempered $\pi_1, \pi_2 \in \widehat{G}$

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  'Multiplicities' of irreducible unitary reps in the branching laws

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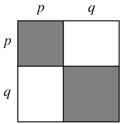

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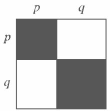

- Even worse example $\cdots$ in the discrete spectrum

$$(G, G') = (SO(5, \mathbb{C}),\ SO(3, 2))$$

(K– 2000)

A Program: Stage ABC for Branching Problem

Stage A.

Stage B.

Stage C.

Ref. T. Kobayashi, “A program for branching problems in rep theory. . . ”, Progress in Mathematics, **312**, (2015).

A Program: Stage ABC for Branching Problem

Stage A. **A**bstract Feature of Restriction

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- spectrum: discrete or continuous?/ support?

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Branching Law (unitary case)

For $G \supset G'$ and $\Pi \in \widehat{G}$,

$$\Pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} m_\pi \pi \, d\mu(\pi).$$ (direct integral)

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- (irreducible) decomposition of representations

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Stage C. **C**onstruction of SBOs/HOs

SBO $\cdots$ Symmetry Breaking Operator
HO $\cdots$ Holographic Operator

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Symmetry Breaking Ops/Holographic Ops

$$G \supset G'$$

$$\Pi \in \mathrm{Irr}(G), \qquad \pi \in \mathrm{Irr}(G').$$

A G'-homomorphism

$$T : \Pi \to \pi$$

is called a symmetry breaking operator (SBO).

A G'-homomorphism

$$S : \pi \to \Pi$$

is called a holographic operator (HS).

A Program: Stage ABC for Branching Problem

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Stage B. **B**ranching Laws

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SBO · · · Symmetry Breaking Operator
HO · · · Holographic Operator

- decomposition of vectors

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Example. Holomorphic discrete rep of $SL(2,\mathbb{R})$

$$SL(2,\mathbb{R}) \curvearrowright \mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

$$z \mapsto \frac{az+b}{cz+d}$$

$$SL(2,\mathbb{R}) \curvearrowright (L^2_\lambda \cap \mathcal{O})(\mathcal{H}) \ (\subset \mathcal{O}(\mathcal{H}))$$

$$f(z) \mapsto (\pi_\lambda(g)f)(z) := (cz+d)^{-\lambda} f(\tfrac{az+b}{cz+d})$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$

$L^2_\lambda(\mathcal{H}) := \{f(z) : \int_{\mathcal{H}} |f(x+iy)|^2 y^{\lambda-2} dy < \infty\}$

π_λ: irreducible unitary rep of G if $\lambda = 2, 3, 4, \cdots$

(holomorphic discrete series rep of lowest weight λ)

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Ex. Tensor product rep $\cdots$ branching for $G \times G \downarrow \operatorname{diag} G$

π_λ : holomorphic discrete rep of $G = SL(2,\mathbb{R})$, lowest weight $\lambda \geq 2$

Abstract feature

$\pi_{\lambda'} \otimes \pi_{\lambda''}$: decomposes discretely and multiplicity-freely

Branching law (Repka, Molchanov)

$$\pi_{\lambda'} \otimes \pi_{\lambda''} \simeq \bigoplus_{a \in \mathbb{N}} \pi_{\lambda'+\lambda''+2a}$$

Construction of SBOs (Rankin–Cohen bidifferential operator)

$RC^{\lambda'''}_{\lambda',\lambda''} : \pi_{\lambda'} \otimes \pi_{\lambda''} \to \pi_{\lambda'''}$ when $\lambda''' - \lambda' - \lambda'' =: 2a \in 2\mathbb{N}$

$$RC^{\lambda'''}_{\lambda',\lambda''}(f_1 \otimes f_2)(z) = \sum_{\ell=0}^{a} \frac{(-1)^\ell \Gamma(\lambda'+a)\Gamma(\lambda''+a)}{\ell!\,(a-\ell)!\,\Gamma(\lambda'+a-\ell)\Gamma(\lambda''+\ell)} \frac{\partial^{a-\ell} f_1}{\partial^{a-\ell} z} \frac{\partial^\ell f_2}{\partial z^\ell}$$

Realization of π_λ : $SL(2,\mathbb{R}) \curvearrowright (L^2_\lambda \cap \mathcal{O})(\mathcal{H}) \subset \mathcal{O}(\mathcal{H})$

by $f(z) \mapsto (cz+d)^{-\lambda} f(\frac{az+b}{cz+d})$ for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathcal{H} = \{\operatorname{Im} z > 0\}$

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Mini Courses (January 13-17, IHP, 2025)

Branching in Representation Theory

References for Lecture 1: Overview

Here are a few survey papers from various perspectives. See also references therein.

T. Kobayashi, Harmonic analysis on homogeneous manifolds of reductive type and unitary representation theory, Translations, Series II, vol. 183, Amer. Math. Soc., 1998, pp. 1-31, (Original article was published in Sugaku 46 (1994), 124-143, Math Soc. Japan.)

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