Structure of Tempered Homogeneous Spaces
III. Limit Algebras

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Minicourses

Institut Henri Poincaré, France, 17–21 February 2025

References

The theme of the mini-course is joint with Yves Benoist.

Tempered Homogeneous Spaces:

——— I. (J. Euro Math., 2015)

Method (Dynamical System)

——— II. (Margulis Festschrift, 2022, Chicago Univ. Press)

Representation Theory

——— III. (J. Lie Theory, 2021)

Classification Theory (Combinatorics)

——— IV. (J. Inst. Math. Jussieu, 2023)

Limit algebra, geometric quantization

Tensor product of GL_n (J. Algebra, 2023)

Plan of Lectures

- Talk 1: (February 17, 2025) $G \curvearrowright X$

  Tempered homogeneous spaces

  —Dynamical approach : L^q estimate of $\mathrm{vol}(gS \cap S)$

- Talk 2: ~~(February 19, 2025)~~ (February 20, 2025)

  Classification theory of “tempered space” G/H

  —Combinatorics of convex polyhedra

- Talk 3: (February 21, 2025)

  Explore yet another relation of tempered homogeneous spaces with other disciplines.

  1. Topology: Deforming Lie algebras
  2. Geometry: Geometric quantization

Temperedness criterion (generalization)

Lecture 1

			Method
(Theorem A)	<u>Case 1</u>	G (semisimple) $\curvearrowright$ V (linear)	Dynamical approach
(Theorem B′)	<u>Case 2</u>	G (semisimple) $\supset$ H (reductive)	Global geometry + Case 1

Today

(Theorem K)	<u>Case 4</u>	G (any) $\supset$ H (any)	"Limit algebras"

Reminder from Lecture 1

$\mathfrak{a}$: max split abelian subspace of a Lie algebra $\mathfrak{h}$

p_V is defined for a linear action $\mathfrak{h} \curvearrowright V$ by

$$p_V = \max_{Y \in \mathfrak{h} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} = \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\sum |\text{eigenvalues of } Y \curvearrowright \mathfrak{h}|}{\sum |\text{eigenvalues of } Y \curvearrowright V|}.$$

Levi decomposition

Let G be a real algebraic group.

- Levi decomposition

$$\mathfrak{g} = \underset{\text{semisimple}}{\mathfrak{g}_s} \oplus \underset{\text{solvable}}{\mathfrak{u}} \qquad \text{(Levi decomposition)}$$

$$G \supset G_s \qquad \text{(semisimple part)}$$

- For a unitary representation π of a Lie group G, we shall discuss temperedness of π as a representation of the semisimple part G_s.

cf. For solvable Lie groups, all unitary reps are tempered.

Temperedness criterion in the general case

Setting $G \supset H$ real algebraic Lie groups.

We allow G and H to be non-reductive.

Take a maximal semisimple subgroup G_s of G,

$$G_s \subset G \curvearrowright L^2(G/H)$$

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

Temperedness criterion in the general case

Setting $\quad G \supset H \quad$ real algebraic Lie groups.

We allow G and H to be non-reductive.

Take a maximal semisimple subgroup G_s of G,

$$G_s \subset G \curvearrowright L^2(G/H)$$

Question When is $G_s \curvearrowright L^2(G/H)$ tempered?

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

Temperedness criterion in the general case

Setting $\quad G \supset H \quad$ real algebraic Lie groups.

Take maximal semisimple subgroups H_s and G_s of H and G, respectively, such that $H_s \subset G_s$. Consider

$$G_s \subset G \curvearrowright L^2(G/H)$$

We define an H_s-module by $V := \mathfrak{g}/\mathfrak{h} + \mathfrak{g}/\mathfrak{g}_s$.

Theorem K* $\quad L^2(G/H)$ is G_s-tempered $\iff p_V \leq 1$.

$$\iff \rho_{\mathfrak{g}_s} \leq 2\rho_{\mathfrak{g}/\mathfrak{h}} \text{ on } \mathfrak{h}_s$$

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

Temperedness criterion in the general case

Setting $G \supset H$ real algebraic Lie groups.

We allow G and H to be non-reductive.
Take maximal semisimple subgroups H_s and G_s of H and G, respectively, such that $H_s \subset G_s$. Consider

$$G_s \subset G \curvearrowright L^2(G/H)$$

We set $V := \mathfrak{g}/\mathfrak{h} + \mathfrak{g}/\mathfrak{g}_s \cdots H_s$-module.

Theorem K* $L^2(G/H)$ is G_s-tempered $\iff p_V \leq 1$.

When G is semisimple, *i.e.*, $G = G_s$, Theorem K implies:

Theorem B′ (Lecture 1, G semisimple case)
$L^2(G/H)$ is G-tempered $\iff \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$ on $\mathfrak{h}$.

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Temperedness criterion (generalization)

Lecture 1

					Method
(Theorem A)	<u>Case 1</u>	G semisimple	$\curvearrowright$	V linear	Dynamical approach
(Theorem B′)	<u>Case 2</u>	G semisimple	$\supset$	H reductive	Global geometry + Case 1
((Theorem D)	<u>Case 3</u>	G semisimple	$\supset$	H any	Domination of G-spaces)*

<u>Today</u>

(Theorem K)	<u>Case 4</u>	G any	$\supset$	H any	"Limit algebras"

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces II, Chicago Univ. Press (2022).

Plan of Lecture 3

0. Temperedness criterion (generalization)

Explore yet another relation of tempered homogeneous spaces with other disciplines.

1. Topology: Deforming Lie algebras
2. Geometry: Geometric quantization

Deforming Lie algebras (1) — Example

Consider two equi-dimensional subalgebras of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$:

$$\mathfrak{k} = \mathfrak{so}(n), \qquad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$$

reductive

nilpotent

<u>Observation</u> $\exists$ sequence $g_j \in SL(n, \mathbb{R})$ such that $\lim_{j \to \infty} \mathrm{Ad}(g_j)\, \mathfrak{k} = \mathfrak{n}$

<u>Remark</u> $\nexists$ sequence $g_j \in SL(n, \mathbb{R})$ such that $\lim_{j \to \infty} \mathrm{Ad}(g_j)\, \mathfrak{n} = \mathfrak{k}$.

Deforming Lie algebras (1) — Example

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<u>Observation</u> $\exists$ sequence $g_j \in SL(n, \mathbb{R})$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\, \mathfrak{k} = \mathfrak{n}$

Proof. $(n = 2)$ Take $g_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{-j} \end{pmatrix}$. Then

$$\mathrm{Ad}(g_j)\, \mathfrak{k} = \mathrm{Ad}(g_j)\mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbb{R}\begin{pmatrix} 0 & -2^{2j} \\ 2^{-2j} & 0 \end{pmatrix} \xrightarrow{j\to\infty} \mathbb{R}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \mathfrak{n}.$$

<u>Remark</u> $\nexists$ sequence $g_j \in SL(n, \mathbb{R})$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\, \mathfrak{n} = \mathfrak{k}$.

Deformation of space forms S^n, $\mathbb{R}^n$, and H^n

Isometry | Curvature

$$= SO(n+1) \curvearrowright S^n \qquad \kappa > 0$$

$$\downarrow$$

$$SO(n) \ltimes \mathbb{R}^n \curvearrowright \mathbb{R}^n \qquad \kappa = 0$$

$$\uparrow$$

$$I = SO(n,1) \curvearrowright H^n \qquad \kappa < 0$$

Deformation of space forms S^n, $\mathbb{R}^n$, and H^n

$$
\begin{array}{ccc}
K = SO(n+1) \curvearrowright S^n & & \mathfrak{k} = \mathfrak{so}(n+1) \\
\downarrow & & \downarrow \ \text{“limit algebra” in } \mathfrak{g} \\
MN = SO(n) \ltimes \mathbb{R}^n \curvearrowright \mathbb{R}^n & & \mathfrak{m} + \mathfrak{n} = \mathfrak{so}(n) \ltimes \mathbb{R}^n \\
\uparrow & & \downarrow \ \text{“limit algebra” in } \mathfrak{g} \\
H = SO(n,1) \curvearrowright H^n & & \mathfrak{h} = \mathfrak{so}(n,1)
\end{array}
$$

View point from transformation groups

$G = SO(n+1,1)$ contains K, MN, and H.

Deformation of space forms S^n, $\mathbb{R}^n$, and H^n

$$K = SO(n+1) \curvearrowright S^n$$
$$\downarrow$$
$$MN = SO(n) \ltimes \mathbb{R}^n \curvearrowright \mathbb{R}^n$$
$$\uparrow$$
$$H = SO(n,1) \curvearrowright H^n$$

$\mathfrak{k} = \mathfrak{so}(n+1)$

$\downarrow$ "limit algebra" in $\mathfrak{g}$

$\mathfrak{m} + \mathfrak{n} = \mathfrak{so}(n) \ltimes \mathbb{R}^n$

$\downarrow$ "limit algebra" in $\mathfrak{g}$

$\mathfrak{h} = \mathfrak{so}(n,1)$

View point from transformation groups

$G = SO(n+1,1)$ contains K, MN, and H.

Quotient

$$G/K = H^{n+1} \underset{\text{deform}}{\dashrightarrow} G/MN \underset{\text{deform}}{\dashleftarrow} G/H = \mathrm{de}S^{n+1}$$

Limit algebras (2) — Formulation

By forgetting the Lie algebra structure of $\mathfrak{g}$, one considers

$$G \overset{\mathrm{Ad}}{\curvearrowright} \mathrm{Gr}(\mathfrak{g}) := \coprod_{m=0}^{\dim \mathfrak{g}} \mathrm{Gr}_m(\mathfrak{g}), \quad \text{(Grassmann variety)}.$$

$\mathfrak{h}$: a subalgebra of $\mathfrak{g}$, with dimension m.

$\rightsquigarrow$ $\mathfrak{h}$ may be regarded as a point of $\mathrm{Gr}_m(\mathfrak{g})$.

$\mathrm{Gr}(\mathfrak{g}) \underset{\text{submanifold}}{\supset} \mathrm{Ad}(G)\mathfrak{h}$, which may or may not be closed.

$$\mathrm{Gr}(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty \quad \text{(limit algebra)}$$

Definition (limit algebra) $\mathfrak{h}_\infty$ $(\subset \mathfrak{g})$ is a limit algebra of $\mathfrak{h}$ in $\mathfrak{g}$ if $\exists$ sequence $g_j \in G$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h} = \mathfrak{h}_\infty$ in $\mathrm{Gr}(\mathfrak{g})$.

Limit algebras (3) — Properties

$\mathfrak{g} \supset \mathfrak{h}$ subalgebra $\rightsquigarrow$ $\mathrm{Gr}(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty$ (limit algebra)

<u>Remark</u> Limit algebra is not unique.

<u>Basic properties</u>

0) $\mathfrak{h}$ itself is a limit algebra of $\mathfrak{h}$.

Limit algebras (3) — Properties

$\mathfrak{g} \supset \mathfrak{h}$ subalgebra $\rightsquigarrow$ $\mathrm{Gr}(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty$ (limit algebra)

Remark Limit algebra is not unique.

Basic properties

0) $\mathfrak{h}$ itself is a limit algebra of $\mathfrak{h}$.

1) Any limit algebra $\mathfrak{h}_\infty$ is an equi-dimensional Lie algebra.

2) If $\mathfrak{h}$ is {abelian / nilpotent / solvable} then any limit algebra $\mathfrak{h}_\infty$ is also {abelian / nilpotent / solvable}.

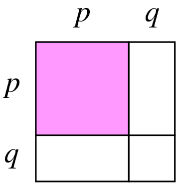
"Semisimple" $\mathfrak{h}$ may collapse to "solvable" $\mathfrak{h}_\infty$, but not vice versa.

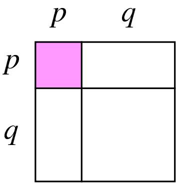
Limit algebras (4) — Example

$\mathfrak{g} \supset \mathfrak{h}$ subalgebra $\rightsquigarrow$ $Gr(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty$ (limit algebra)

Remark $\mathfrak{h}_\infty$ is determined not only by $\mathfrak{h}$ itself
but by how $\mathfrak{h}$ is embedded in $\mathfrak{g}$.

Exercise Fix p, and consider $\mathfrak{h} = \mathfrak{sl}_p \hookrightarrow \mathfrak{g} = \mathfrak{sl}_{p+q}$
Is it possible to "deform" $\mathfrak{h} = \mathfrak{sl}_p$ to a solvable subalgebra in $\mathfrak{g}$?
Namely, does $\overline{\mathrm{Ad}(G)\mathfrak{h}}$ contain solvable $\mathfrak{h}_\infty$?

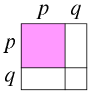




Deforming Lie algebras to solvable ones

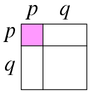
Example $\mathfrak{h} = \mathfrak{sl}_p \hookrightarrow \mathfrak{g} = \mathfrak{sl}_{p+q}$

$q \leq p$



does not have a solvable limit.

$q \geq p + 1$



has a solvable limit.

Definition (solvable limit algebra) $\mathfrak{h} \subset \mathfrak{g}$ Lie algebras

We say $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$ if

$^{\exists} g_j \in G$ such that $\lim_{j \to \infty} \mathrm{Ad}(g_j)\mathfrak{h}$ is a solvable Lie algebra.

Variety of all Lie algebras $\mathcal{L}$ and its subset $\mathcal{S}$

Formulation: Consider the variety of all subalgebras in $\mathfrak{g}$.

$$Gr(\mathfrak{g}) \equiv \coprod_{N=0}^{\dim \mathfrak{g}} \mathrm{Gr}_N(\mathfrak{g}) \qquad \cdots \text{ algebraic variety}$$

$$\cup$$

$$\mathcal{L} := \{\text{subalgebras of } \mathfrak{g}\} \qquad \cdots \text{ algebraic variety}$$

$$\cup$$

$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty \text{ solvable}\}$$

$$\cup$$

$$\{\text{solvable subalgs}\} \qquad \cdots \text{ algebraic variety}$$

Question What does $\mathcal{S}$ look like in $\mathcal{L}$?

Variety of all Lie algebras $\mathcal{L}$ and its subset $\mathcal{S}$

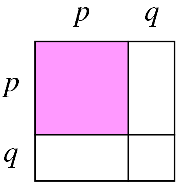
$\mathfrak{g}$: Lie algebra.

$$\mathcal{L} := \{\text{subalgebras of } \mathfrak{g}\}$$

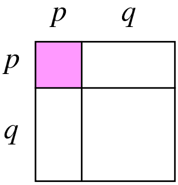
$$\cup$$

$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty \text{ solvable}\}$$

Question What does $\mathcal{S}$ look like in $\mathcal{L}$?



$\notin \mathcal{S} \ni$



$p \geq q$ $\qquad$ $p \leq q - 1$

Topology of $\mathcal{S} = \{\mathfrak{h} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty$ solvable$\}$

Suppose $\mathfrak{g}$ is an algebraic Lie algebra $/\mathbb{C}$.

<u>Open Problem L</u> Is $\mathcal{S}$ open in $\mathcal{L}$?

Recall

$$\mathcal{L} := \{\text{subalgebras of } \mathfrak{g}\}$$

$$\cup$$

$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty \text{ solvable}\}$$

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

Topology of $\mathcal{S} = \{\mathfrak{h} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty$ solvable$\}$

Suppose $\mathfrak{g}$ is an algebraic Lie algebra $/\mathbb{C}$.

<u>Open Problem L</u> Is $\mathcal{S}$ open in $\mathcal{L}$?

<u>Theorem M</u>*

(1) $\mathcal{S}$ is closed in $\mathcal{L}$.

(2) $\mathcal{S}$ is open and closed in $\mathcal{L}$ if $\mathfrak{g}$ is semisimple.

Recall

$$\mathcal{L} := \{\text{subalgebras of } \mathfrak{g}\}$$

$$\cup$$

$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty \ \text{solvable}\}$$

Our proof for Theorem M uses unitary representation theory.

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$\mathcal{S}$ and temperedness of $L^2(G/H)$

G : complex algebraic Lie group,

H : algebraic subgroup.

We recall

$\mathcal{L}$:= {subalgebras of $\mathfrak{g}$}

$\cup$

$\mathcal{S}$:= $\{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty$ solvable $\}$

Theorem N*

$\rho_{\mathfrak{g}_s} \le 2\rho_{\mathfrak{g}/\mathfrak{h}}$ on $\mathfrak{h}_s \iff \mathfrak{h} \in \mathcal{S}$.

Since $\rho_{\mathfrak{g}_s} \le 2\rho_{\mathfrak{g}/\mathfrak{h}}$ is a closed condition, $\mathcal{S}$ is closed in $\mathcal{L}$, showing Theorem M (1).

Recall

Theorem K* $L^2(G/H)$ is G_s-tempered $\iff \rho_{\mathfrak{g}_s} \le 2\rho_{\mathfrak{g}/\mathfrak{h}}$ on $\mathfrak{h}_s$.

Sketch of Proof of Theorem M (easier part)

We explain an easier part of the implication in Theorem N.

$$\rho_{\mathfrak{g}_s} \leq 2\rho_{\mathfrak{g}/\mathfrak{h}} \text{ on } \mathfrak{h}_s \implies \mathfrak{h} \in \mathcal{S}.$$

Take $\mathfrak{h}_\infty \in \overline{\mathrm{Ad}(G)\mathfrak{h}}$ such that $\mathrm{Ad}(G)\mathfrak{h}_\infty$ is closed. We show

$$\rho_{\mathfrak{g}_s} \leq 2\rho_{\mathfrak{g}/\mathfrak{h}} \text{ on } \mathfrak{h}_s \implies \mathfrak{h}_\infty \text{ is solvable.}$$

- Can assume $\mathfrak{h} = \mathfrak{h}_\infty$.
- Want to show $\mathfrak{h}$ is solvable if $\mathrm{Ad}(G)\mathfrak{h}$ is closed.
- Can find a parabolic $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{h}$ is an ideal of $\mathfrak{q}$

$\Downarrow$ some elementary computation

$$\rho_{\mathfrak{g}_s} \leq 2\rho_{\mathfrak{g}/\mathfrak{h}} \text{ on } \mathfrak{h}_s \text{ implies } \mathfrak{h}_s = 0.$$

Hence, $\mathfrak{h}$ is solvable.

Plan of Lecture 3

0. Temperedness criterion (generalization)

Explore yet another relation of tempered homogeneous spaces with other disciplines.

1. Topology: Deforming Lie algebras
2. Geometry: Geometric quantization

Geometric quantization and temperedness

$\mathrm{Ad}\colon G \to GL_{\mathbb{R}}(\mathfrak{g})$ adjoint representation.

$\mathrm{Ad}^*\colon G \to GL_{\mathbb{R}}(\mathfrak{g}^*)$ coadjoint representation.

Coadjoint orbit $\mathcal{O}_\lambda := \mathrm{Ad}^*(G)\lambda$ for $\lambda \in \mathfrak{g}^*$.

Lemma (Kostant–Kirillov–Souriau)
Every coadjoint orbit $\mathcal{O}_\lambda$ carries a natural symplectic structure.

"Geometric quantization":

$$\mathfrak{g}^* \supset \mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \overset{?}{\rightsquigarrow} \pi_\lambda \in \widehat{G}$$

symplectic mfd   unitary rep

Expect

$$\mathfrak{g}^* / \mathrm{Ad}^*(G) \fallingdotseq \widehat{G}$$

Geometric quantization and temperedness

"Geometric quantization": $\mathfrak{g}^* \supset \mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \overset{?}{\rightsquigarrow} \pi_\lambda \in \widehat{G}$

symplectic mfd — unitary rep

$$\begin{array}{ccc} \mathrm{Ad}^*(G)\mathfrak{h}^\perp / \mathrm{Ad}^*(G) & \fallingdotseq & \mathrm{Supp}(L^2(G/H)) \\ \cap & & \cap \\ \mathfrak{g}^* / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G} \\ \cup & & \cup \\ \mathfrak{g}^*_{\mathrm{reg}} / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G}_{\mathrm{temp}} \end{array}$$

$\mathfrak{h}^\perp := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}$

$\mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G) \cdot \lambda$ is of maximal dimension$\}$

We may ask:

<u>Question</u>*

Suppose G is a real reductive Lie group,
and H a connected closed subgroup. Is (i) ⇔ (ii)?
(i) $G \curvearrowright L^2(G/H)$ is tempered.
(ii) $\mathfrak{g}^*_{\mathrm{reg}} \cap \mathfrak{h}^\perp$ is dense in $\mathfrak{h}^\perp$.

From orbit philosophy by Kirillov–Kostant

We assume now G is a complex reductive Lie group.

$$\mathfrak{g}^* \supset \mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G)\lambda \text{ is of maximal dimension}\},$$
$$\mathfrak{g}^* \supset \mathfrak{h}^{\perp} := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}.$$

Orbit philosophy by Kirillov–Kostant

$$\begin{array}{ccc} \mathrm{Ad}^*(G)\mathfrak{h}^{\perp} / \mathrm{Ad}^*(G) & \fallingdotseq & \mathrm{Supp}(L^2(G/H)) \\ \cap & & \cap \\ \mathfrak{g}^* / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G} \\ \cup & & \cup \\ \mathfrak{g}^*_{\mathrm{reg}} / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G}_{\mathrm{temp}} \end{array}$$

Remark $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset \iff \mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \underset{\text{dense}}{\subset} \mathfrak{h}^{\perp}$

Geometric quantization and temperedness

"Geometric quantization": $\mathfrak{g}^* \supset \mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \overset{?}{\rightsquigarrow} \pi_\lambda \in \widehat{G}$

symplectic mfd — unitary rep

$$
\begin{array}{ccc}
\mathrm{Ad}^*(G)\mathfrak{h}^\perp / \mathrm{Ad}^*(G) & \fallingdotseq & \mathrm{Supp}(L^2(G/H)) \\
\cap & & \cap \\
\mathfrak{g}^* / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G} \\
\cup & & \cup \\
\mathfrak{g}^*_{\mathrm{reg}} / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G}_{\mathrm{temp}}
\end{array}
$$

$\mathfrak{h}^\perp := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}$

$\mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G)\cdot\lambda$ is of maximal dimension$\}$

<u>Theorem O</u>*

Suppose G is a complex reductive Lie group, and H a connected closed subgroup. Then (i) $\Leftrightarrow$ (ii).

(i) $G \curvearrowright L^2(G/H)$ is tempered.

(ii) $\mathfrak{g}^*_{\mathrm{reg}} \cap \mathfrak{h}^\perp \neq \emptyset$.

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

Further interactions for “tempered spaces”

Theorem P Let $\mathfrak{g}$ be a complex reductive Lie algebra.
The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

(i)	(Analysis)	$L^2(G/H)$ is tempered.
(ii)	(combinatorics)	$2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.
(iii)	(Geometric quantization)	$\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.
(iv)	(Topology)	$\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$.

Application   Representation theory $\Longrightarrow$ Topology

Corollary Q (Topology) The property “having solvable limit” is an open and closed condition for subalgebras in a complex reductive Lie algebra $\mathfrak{g}$, namely, $\mathcal{S}$ is open and closed in $\mathcal{L}$.

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Sketch of Proof for Theorem P: Tempered homogeneous spaces

Thm P Let $\mathfrak{g}$ be a complex reductive Lie algebra.

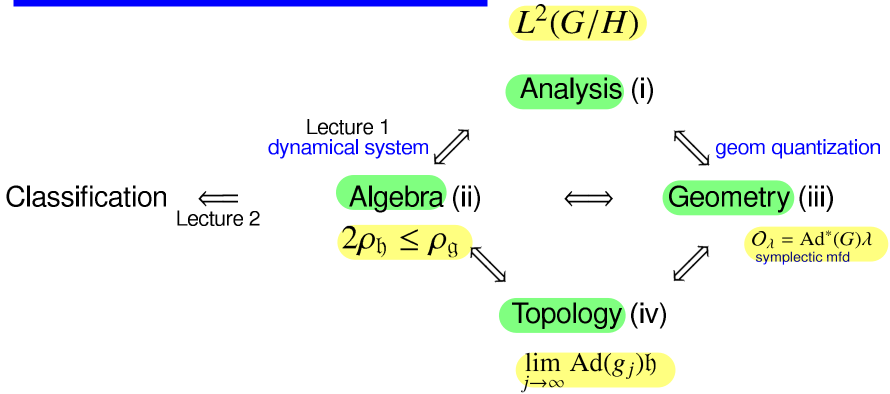
The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

(i) (unitary rep) $L^2(G/H)$ is tempered.

(ii) (combinatorics) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.

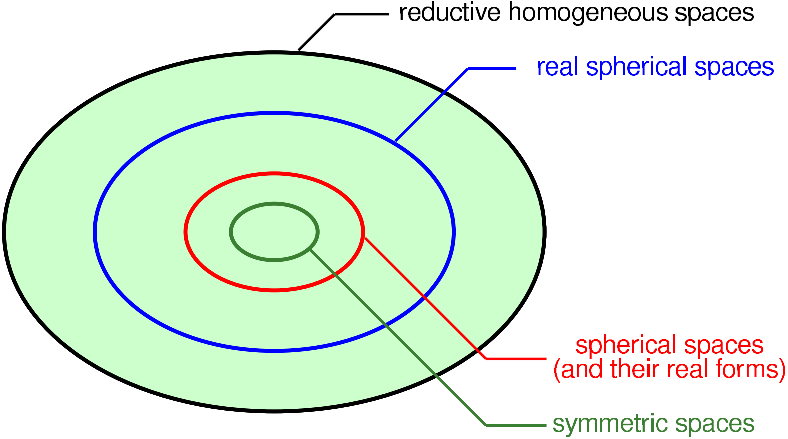
(iii) (orbit method) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.

(iv) (limit algebra) $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$.



Reductive homogeneous space G/H

G: real reductive groups
H: reductive subgroup



We extend the case where G and H are not necessarily reductive.

Basic Questions in Group-Theoretic Analysis on Manifolds

$$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright L^2(X), \cdots$$

Geometry | Function Space

Basic Question (Lectures 1–3)

- What is the spectrum of $L^2(X)$?
- Are matrix coefficients almost L^q?
- Can we decompose $L^2(X)$ by irreducible tempered reps?

Use ideas of dynamical system, combinatorics, unitary reps, limit algebras, and more.

Thank you very much!

References

The theme of the mini-course is joint with Yves Benoist.

Tempered Homogeneous Spaces:

- I. (J. Euro Math., 2015)
  Method (Dynamical System)
- II. (Margulis Festschrift, 2022, Chicago Univ. Press)
  Representation Theory
- III. (J. Lie Theory, 2021)
  Classification Theory (Combinatorics)
- IV. (J. Inst. Math. Jussieu, 2023)
  Limit algebra, geometric quantization

Tensor product of GL_n (J. Algebra, 2023)